The Meskhidze-Weatherall theory in context

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Abstract

We situate a recent theory of non-relativistic torsionful gravity developed by Meskhidze and Weatherall (2023) within the context of the broader philosophical and physics literature; we also discuss the philosophical significance of that theory.

Contents

1 Introduction

Newton-Cartan theory (NCT) was developed initially by Cartan (1925) and Friedrichs (1928) as a curved spacetime model of Newtonian gravity. The theory went through a classical phase of investigation in the 1960s and 70s (see in particular Dautcourt 1964; Dixon 1975; Havas 1964; Künzle 1972, 1976;

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Trautman 1965), two of the many fruits of which were the Trautman (1965) geometrisation and recovery theorems, which together establish a precise sense in which NCT is 'equivalent' to standard Newtonian gravity. More recently, NCT has undergone a renaissance in which it and its torsionful generalizations have been applied to non-relativistic holography (see e.g. Christensen et al. 2014a,b) and condensed matter physics, especially the fractional quantum Hall effect (see e.g. Geracie et al. 2016; Son 2013; Wolf, Read, and Teh 2023).

More recently, philosophers have also begin to study torsion in the classical spacetime context. Motivated by questions raised by Knox (2011), Read and Teh (2018) explore the extent to which the mappings between NCT and ungeometrised, potential-based Newtonian gravity (henceforth NG)—made precise in Trautman geometrisation/recovery—can be understood exactly as a case of 'teleparallelisation'—i.e., the map relating general relativity to its torsionful equivalent, teleparallel gravity (TPG)—finding an affirmative answer: NG just is the teleparallel equivalent of NCT, and the gravitational potential of that former theory can be understood as a (gauge-fixed) 'mass torsion', associated with the mass gauge field which arises once one gauges the Bargmann algebra (Andringa et al. 2011; Read and Teh 2018; Teh 2018; Wolf, Read, and Teh 2023). Read and Teh (2018) also show that this NCT–NG correspondence is the non-relativistic limit of the GR–TPG correspondence, when said 'limit' is implemented via null reduction; an alternative non-relativistic limit (now implemented via a $1/c$ expansion) is undertaken by Schwartz (2023), from which the same results are obtained. Building on this, Read and Teh (2022) exploit these connections in order to explore the status of 'Newtonian equivalence principles'; Wolf and Read (2023b) use these results to motivate the construction of a purely non-metric equivalent to NCT, thereby completing a 'non-relativistic geometric trinity'[1]; and March et al. (2023) identify that the 'common core' of this non-relativistic trinity is Maxwell gravitation (on which see Chen (2023), Dewar (2018), and March (2023)).[2]

To this by-now quite mature physics literature, and still-blossoming philosophical literature, Meskhidze and Weatherall (2023) have recently added their own contribution.[3] In their article, they seek to construct a non-relativistic theory of gravitation which (in some sense) is equivalent to NCT, yet the gravitational effects in which are manifestations only of (spatiotemporal) torsion. This theory is certainly interesting and worthy of study; however, in our view there remains much to be said about it, especially with respect to the following questions:

[1]This of course is the non-relativistic analogue of the 'relativistic geometric trinity' (Jiménez et al. 2019), which is a collection of three empirically equivalent gravitational theories which are formulated using different geometric degrees of freedom: curvature for GR, torsion for TPG, and non-metricity for 'symmetric teleparallel gravity' (STGR). See Wolf and Read (2023a) and Wolf, Sanchioni, et al. (2023) for further philosophical discussion on issues concerning theory equivalence and underdetermination in this context.

[2]Here, 'common core' is meant in the sense of Le Bihan and Read (2018).

[3]To be perfectly clear on the chronology: Meskhidze and Weatherall (2023) appeared as an online preprint a couple of months before March et al. (2023) and Wolf and Read (2023b).

1. How is the Meskhidze-Weatherall theory best situated with respect to the existing physics and philosophy literature on non-relativistic torsionful theories of gravitation: what in particular is lacking in that literature; where (if anywhere) does that literature go wrong; and what (if anything) does the Meskhidze-Weatherall theory add to this literature which was not already known?

2. What is the best interpretation of the Meskhidze-Weatherall theory *qua* theory, and how best to eke out its philosophical significance?

Our goal in this discussion note is to undertake a systematic exploration of the above two questions. Accordingly, the structure of the note is as follows. In §2, we introduce the technical details of the Meskhidze-Weatherall theory; in §3 we answer question (1) by situating this theory with respect to the existing literature; in §4, we answer question (2) by engaging in a thoroughgoing interpretation of this theory; in §5, we conclude.

2 The Meskhidze-Weatherall theory

Let's first recall the details of the Meskhidze-Weatherall theory of non-relativistic torsionful gravitation (henceforth MWT). Kinematical possibilities of this theory are tuples $\langle M, t_a, h^{ab}, \nabla, \rho \rangle$, where the first four elements denote a classical (i.e., non-relativistic) spacetime (assumed to be temporally orientable) in the sense of (Malament 2012, ch. 4), and ρ is a scalar field denoting the matter density content. In this theory, ∇ is a derivative operator with torsion (which, recall, encodes the antisymmetry of the connection—see Wald (1984, p. 53)) generically decomposable as

$$T^a{}_{bc} = 2F^a{}_{[b}\, t_{c]}; \tag{1}$$

one can treat this as a kinematical restriction on the content of this theory. Dynamical possibilities of MWT are picked out by the field equation

$$\delta^n{}_a \nabla_{[n} F^a{}_{b]} = 2\pi\rho t_b; \tag{2}$$

gravitating but otherwise force-free test bodies with velocity vectors ξ^a are subject to

$$\xi^n \nabla_n \xi^a = -F^a{}_n\, \xi^n; \tag{3}$$

hence, such bodies experience torsion-dependent forces and thereby exhibit non-geodesic motion. Meskhidze and Weatherall (2023) prove a 'recovery' theorem *à la* Trautman (1965) relating the models of Newton-Cartan theory (NCT) to (orbits of) models of MWT—we return to this in §4.

3 Situating the Meskhidze-Weatherall theory in the literature

With the technical details of MWT on the table, we now consider and assess some claims made by Meskhidze and Weatherall (2023) with respect to the existing physics/philosophy literature on torsion in non-relativistic gravity—claims which they use to motivate the construction of MWT. There are, in particular, four claims made with respect to that literature upon which we here focus: (i) claims regarding the relationship between the existence of torsion and the closed nature of t_a in a(n orientable) classical spacetime model (§3.1); (ii) claims regarding the literature's (supposedly) blinkered attention upon a (supposedly) restricted form of the connection (§3.2); (iii) a particular terminological choice regarding 'spatial torsion' and 'temporal torsion' which clashes with the existing literature (§3.3); and (iv) claims made regarding the purposes of certain articles on the non-relativistic limits of relativistic theories (§3.4).

3.1 Does a closed clock form imply vanishing torsion?

Meskhidze and Weatherall (2023) assert that physicists often claim "that taking $\partial_\mu t_\nu = 0$, where ∂ is a (torsion-free) coordinate derivative operator *will always result in a torsion-free spacetime*" (p. 9, emphasis in original). The first thing to say about this passage is that we don't find any evidence that the claim is true, when $\partial_\mu t_\nu = 0$ is not antisymmetrised (thereby making t_a a closed form). Since all of the important action (as it were) with respect to the above quote from Meskhidze and Weatherall (2023) concerns the relationship between (a) the existence of torsion and (b) the closed nature of t_a, we'll focus our considerations upon said relationship in the remainder of this subsection.

On this relationship, earlier on in their article, Meskhidze and Weatherall (2023) write that "it is widely claimed that a classical spacetime with torsion cannot have a temporal metric that is closed [...] this is not true" (p. 2). Denoting schematically all torsion by T, Meskhidze and Weatherall (2023), in other words, impute to the literature the claim that

$$T \neq 0 \quad \overset{?}{\Longrightarrow} \quad dt \neq 0, \tag{4}$$

which of course by contraposition is equivalent to the claim that

$$dt = 0 \quad \overset{?}{\Longrightarrow} \quad T = 0. \tag{5}$$

Now, on the one hand, Meskhidze and Weatherall (2023) are completely correct that this claim is false (hence our oversetting with '?' above)—one need only look to the expressions for torsion in terms of exterior derivatives of gauge fields found given in Andringa et al. (2011, §4.1) to see that $dt = 0$ does *not* imply that all components of the torsion vanish. One can make this point more incisively by deriving the relationship[4]

$$t_a T^a{}_{bc} = (dt)_{bc}, \tag{6}$$

[4] See e.g. Bekaert and Morand (2014, proposition 3.2).

from which we see that, although the torsion T need not vanish when $dt = 0$, it is nonetheless severely constrained by this condition: in particular, the upper (vector) index of T must lie in the kernel of t, and is thus 'spacelike'. We note that this general consideration thus constrains the form of the torsion that Meskhidze and Weatherall introduce.

All of the above is well-known. Thus, although Meskhidze and Weatherall (2023) are correct that the implication (5) fails, it is not correct to impute to the physics literature a widespread failure to recognise this.

Perhaps more worrying on this front, though, is an apparent logical error which occurs when Meskhidze and Weatherall (2023) seek to impute (5) to the literature. As a way of making their case, Meskhidze and Weatherall (2023) quote the following passage from a widely-cited physics article on this topic:

> The absence of torsion implies that the temporal vielbein τ_μ corresponds to a closed one-form and that it can be used to define an absolute time in the space–time [...] TTNC geometry is characterized by the fact that the temporal vielbein is hypersurface orthogonal but not necessarily closed. (Bergshoeff et al. 2014, p. 3)

But note that the content of the first sentence of this passage is the implication

$$T = 0 \quad \Longrightarrow \quad dt = 0 \tag{7}$$

—i.e., exactly the *converse* of the claim which Meskhidze and Weatherall (2023) impute to the literature! Moreover, (7) is clearly true, as again evident from e.g. Andringa et al. (2011, §4.1). So, to summarise: Meskhidze and Weatherall (2023) are (i) correct that (5) is false, but (ii) incorrect to impute (5) to the literature; moreover (iii) commit an error in reading Bergshoeff et al. (2014) in this way, for that article asserts only (7), which is true.

3.2 Justifying the form of the connection considered in the literature

In any case, moving on from the above, Meskhidze and Weatherall (2023) further target the form of the connection used in the 'TTNC' literature ('twistless torsionful Newton-Cartan theory'—i.e., most recent physics work on this topic)—i.e., one with coefficients

$$\Gamma^\lambda{}_{\mu\nu} = v^\lambda \partial_{(\mu} t_{\nu)} + \frac{1}{2} h^{\lambda\rho} \left(\partial_\rho h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu} \right) \tag{8}$$

(Geracie et al. 2015, eq. 2.45)—claiming that the only reason the torsion of their connection vanishes when the clock form t_a is closed is because "*they have adopted such a strict definition for their connection*" (p. 10, emphasis in original).

Taken at face value, this claim is true—but it fails to appreciate the full reasons underlying *why* physicists in the TTNC community have used a connection of this form as their starting point. One motivation for using such a connection is stated clearly by e.g. Hansen et al. (2020, §2) (cf. Geracie et al. (2015, §4)) when they emphasize that they are specifically concerned with expanding GR in powers of $1/c$. Accordingly, they begin with the Levi-Civita connection of GR, rewrite it in terms of what they call 'pre-non-relativistic' variables (which are convenient for the subsequent expansion which they perform), and then implement a $1/c$ expansion in order to find the non-relativistic limit of general relativity. When one imposes the condition that $dt = 0$, one arrives at 'Type I' NCT (i.e., NCT *à la* Malament (2012)) or its teleparallel equivalent (Read and Teh 2018; Schwartz 2023) or its non-metric equivalent (Wolf and Read 2023b); when one relaxes this condition in favour of the condition $t \wedge dt = 0$, one arrives at the 'Type II' NCT found in Hansen et al. (2019a, 2020).[5]

Both Type I and Type II NCT are interesting in their own rights. Type I NCT is an empirically equivalent geometric reformulation of Newtonian gravity in which gravitational effects are manifestations of curvature. Type II NCT is a novel theory with both curvature and torsion which exhibits a remarkable overlap with GR in terms of its empirical content, as it can also account for the strong field gravitational physics of perihelion precession, gravitational redshift, and the bending of light that was previously thought to be the exclusive purview of relativistic physics (see e.g. Hansen et al. (2019a,b), Van den Bleeken (2017), and Wolf, Sanchioni, et al. (2023)).

So yes: the particular form of the connection is responsible for the vanishing of torsion when we have a closed clock; however, this is simply because the Levi-Civita connection is a very special object. These physicists are (of course!) aware of the fact that it is possible to write down connections that are more general than the Levi-Civita connection and manifest all different kinds of geometric qualities (see e.g. Geracie et al. (2015, eq. 2.27)). Within the physics literature, however, these authors are interested primarily in the particular relationship between general relativity and theories which can be understood as its non-relativistic limit. Put another way: there are two 'ingredients' which together can justify focusing on a connection of the form (8): (a) the non-relativistic limit of the Levi-Civita connection of GR, and (b) a closed clock form, i.e. $dt = 0$. Of course, there are more general connections which manifest torsion while still being compatible with a closed clock form, but those will be ones for which (a) does not obtain.

3.3 The meaning of 'temporal torsion' and 'spatial torsion'

Our next point pertains to a discrepancy between the use of the terms 'temporal torsion' and 'spatial torsion' in the hands of Meskhidze and Weatherall (2023) when compared with the rest of the existing literature. Typically in the physics literature, Newtonian theories are treated as gauge theories of the Bargmann algebra, with generators $\{M, H, P, G, J\}$, and associated torsions and curvatures

[5] Indeed, this condition is derived in those latter works as part of the expansion; it need not be imposed 'by hand'.

given by the Cartan equations:

$$(\boldsymbol{f})_{\mu\nu} := T_{\mu\nu}(M) = 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}{}^{\mathsf{a}}e_{\nu]\mathsf{a}}, \tag{9}$$

$$T_{\mu\nu}(H) = 2\partial_{[\mu}t_{\nu]}, \tag{10}$$

$$T_{\mu\nu}{}^{\mathsf{a}}(P) = 2\partial_{[\mu}e_{\nu]}{}^{\mathsf{a}} - 2\omega_{[\mu}{}^{\mathsf{ab}}e_{\nu]\mathsf{b}} - 2\omega_{[\mu}{}^{\mathsf{a}}t_{\nu]}, \tag{11}$$

$$R_{\mu\nu}{}^{\mathsf{a}}(G) = 2\partial_{[\mu}\omega_{\nu]}{}^{\mathsf{a}} - 2\omega_{[\mu}{}^{\mathsf{ab}}\omega_{\nu]\mathsf{b}}, \tag{12}$$

$$R_{\mu\nu}{}^{\mathsf{ab}}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{\mathsf{ab}}. \tag{13}$$

(For a thorough review of this material, see Andringa et al. (2011).) One then defines (now suppressing indices) $T(H)$ (i.e., the torsion associated with time translations) as the 'temporal torsion', and $T(P)$ (i.e., the torsion associated with spatial translations) as 'spatial torsion'. Together with the 'mass torsion' $\boldsymbol{f}$, one then defines the 'extended torsion' $(T(H), T(P), \boldsymbol{f})$.

This terminology is different from that of Meskhidze and Weatherall (2023), who use 'vanishing spatial torsion' to refer to the condition $T^{abc} = 0$ (Meskhidze and Weatherall 2023, p. 6). This is *not* equivalent to the requirement that $T(P) = 0$, but rather that $T(P)|_S = 0$ for any spacelike hypersurface S. Meskhidze and Weatherall also assume that t_a is closed, from which it follows (along with metric compatibility) that $T(H) = 0$. So in the more usual Cartan terminology, (a) MWT has no temporal torsion, and (b) MWT has spatial torsion, but such that it vanishes when restricted to any spacelike hypersurface.

3.4 The aims of Schwartz

Our final point in this section is not major, but is perhaps nevertheless one worth making; it regards the purposes of Schwartz (2023). On this article, Meskhidze and Weatherall (2023) write:

> A cursory review suggests that the theory described in [(Schwartz 2023)] differs from what we describe here, but since there is not sufficient time before the submission deadline, we leave an analysis of the relationship to future work. (Meskhidze and Weatherall 2023, p. 1)

This might imply that Schwartz is in the business of building a torsionful equivalent to NCT. In a sense, this is true, but in fact his main aim is to take the non-relativistic limit of TPG via a $1/c$ expansion (as discussed in the introduction to this paper); that he obtains a torsionful theory is in some sense a corollary of this. Moreover, the theory which Schwartz (2023) obtains is in fact standard Newtonian gravity—i.e. a theory without spacetime torsion (but still with mass torsion—see again Read and Teh (2018)). We will clarify in §4.2 how this theory relates to MWT.

There is one final—more positive—point to be made here. As Schwartz (2023, p. 20) writes, after taking a $1/c$ expansion of TPG, one must 'gauge fix' the connection to vanishing spatial torsion in order to obtain NGT. However, Meskhidze (2023) considers the convergence of derivative operators in the non-relativistic limit and argues that this result thereby follows automatically. If correct, we agree that this is a genuine and helpful contribution to the literature on non-relativistic limits of physical theories.

4 Analysing the Meskhidze-Weatherall theory

Having completed our discussion of the claims made by Meskhidze and Weatherall (2023) regarding the existing literature on torsional non-relativistic gravity, we turn now to an analysis of MWT itself. Our focus in this section will be threefold: (i) the relationship between the models of MWT and NCT (§4.1); (ii) the role of torsion in MWT, particularly with reference to claims in Meskhidze and Weatherall (2023) that this differs significantly from the situation in existing theories of torsional non-relativistic gravity (§4.2); and (iii) whether MWT is empirically equivalent to NCT/NGT (§4.3).

4.1 Geometrisation and recovery

Meskhidze and Weatherall (2023) prove a 'recovery' theorem to the effect that any (compatible, torsion-free) non-relativistic spacetime $\langle M, t_a, h^{ab}, \tilde{\nabla}, \rho\rangle$ which satisfies $\tilde{R}^{ab}{}_{cd} = 0$ and $\tilde{R}_{ab} = 4\pi\rho t_a t_b$ (i.e., NCT without explict commitment to the 'Newtonian' curvature condition $\tilde{R}^a{}_b{}^c{}_d = \tilde{R}^c{}_d{}^a{}_b$) gives rise, non-uniquely, to a model of MWT (Meskhidze and Weatherall 2023, theorem 1). However, they do not similarly prove a 'geometrisation' theorem linking models of MWT to models of (this version of) NCT. In the absence of such a theorem, the relationship between MWT and NCT remains somewhat unclear, so we begin by filling in this gap on Meskhidze and Weatherall's behalf:

Proposition 1. *Let $\langle M, t_a, h^{ab}, \nabla, \rho\rangle$ be a model of MWT such that $F^n{}_m F^m{}_n = 0$. Then there exists a unique torsion-free derivative operator $\tilde{\nabla}$ compatible with the metrics such that $\tilde{R}^{ab}{}_{cd} = 0$, $\tilde{R}_{ab} = 4\pi\rho t_a t_b$, and for all unit timelike vector fields on M, $\xi^n \tilde{\nabla}_n \xi^a = 0 \Leftrightarrow \xi^n \nabla_n \xi^a = -F^a{}_n \xi^n$.*

Proof. Let $\tilde{\nabla} = (\nabla, -F^a{}_b t_c)$. We claim that it satisfies the required conditions. First, note that $\tilde{\nabla}$ is compatible with the metrics since $\tilde{\nabla}_a h^{bc} = \nabla_a h^{bc} + F^b{}_a t_n h^{nc} + F^c{}_a t_n h^{bn} = 0$ and $\tilde{\nabla}_a t_b = \nabla_a t_b - F^n{}_a t_b t_n = 0$, where we have used that ∇ is compatible and $F^a{}_b$ is spacelike in the a index. $\tilde{\nabla}$ is also torsion-free since $-2F^a{}_{[b}t_{c]} = \tilde{T}^a{}_{bc} - T^a{}_{bc} = \tilde{T}^a{}_{bc} - 2F^a{}_{[b}t_{c]} \Leftrightarrow \tilde{T}^a{}_{bc} = 0$. Furthermore, if ξ^a is a unit timelike vector field on M such that $\xi^n \nabla_n \xi^a = -F^a{}_n \xi^n$ then $\xi^n \tilde{\nabla}_n \xi^a = \xi^n \nabla_n \xi^a + F^a{}_n t_m \xi^n \xi^m = 0$ (conversely, if ξ^a is geodesic with respect to $\tilde{\nabla}$ then $\xi^n \nabla_n \xi^a = -F^a{}_n \xi^n$). $\tilde{\nabla}$ is clearly unique in this regard, since an arbitrary derivative operator $(\nabla, C^a{}_{bc})$ will satisfy the above condition just in case $C^a{}_{nm}\xi^n\xi^m = -F^a{}_n \xi^n$ for *any* unit timelike vector field ξ^a, from which it follows that $C^a{}_{bc} = -F^a{}_b t_c$.

It remains to verify that $\tilde{R}^{ab}{}_{cd} = 0$ and $\tilde{R}_{ab} = 4\pi\rho t_a t_b$. First, using the expression relating two Riemann tensors

$$\begin{aligned}\tilde{R}^a{}_{bcd} &= R^a{}_{bcd} - 2\nabla_{[c}F^a{}_{d]}t_b + 2t_b F^n{}_{[c}F^a{}_{d]}t_n + 2F^n{}_{[c}t_{d]}F^a{}_n t_b \\ &= -2\nabla_{[c}F^a{}_{d]}t_b + 2F^n{}_{[c}t_{d]}F^a{}_n t_b,\end{aligned}$$

where we have used that ∇ is flat and again that $F^a{}_b$ is spacelike in the a index. It follows immediately that $\tilde{R}^{ab}{}_{cd} = 0$ since ∇ is compatible. Meanwhile, using (2) and that $F^a{}_b$ is spacelike in the a index we have

$$\begin{aligned}\tilde{R}_{ab} &= -2\delta^n{}_m\nabla_{[b}F^m{}_{n]}t_a + 2\delta^n{}_m F^r{}_{[b}t_{n]}F^m{}_r t_a \\ &= 2\delta^n{}_m\nabla_{[n}F^m{}_{b]}t_a - F^r{}_n F^n{}_r t_a t_b \\ &= 4\pi\rho t_a t_b - F^r{}_n F^n{}_r t_a t_b \\ &= 4\pi\rho t_a t_b,\end{aligned}$$

where we have used that the last equality holds just in case $F^n{}_m F^m{}_n = 0$. $\square$

So MWT, as presented in Meskhidze and Weatherall (2023), is *not* equivalent to NCT without the Newtonian condition; however, this may straightforwardly be rectified with the additional assumption that $F^n{}_m F^m{}_n = 0$. Since this condition also holds with respect to the recovered models of MWT considered in Meskhidze and Weatherall's theorem 1, we will assume it in what follows.

4.2 Torsion in the Meskhidze-Weatherall theory

With the relationship between MWT and NCT on a firmer footing, we turn now to the analysis of torsion in MWT. After introducing MWT, Meskhidze and Weatherall (2023) note that there already exists discussion of teleparallellisation in the Newtonian context, such as the proposal developed in Read and Teh (2018). These theories, naturally understood as gauge theories of the Bargmann algebra, feature a mass torsion term $\boldsymbol{f}$ (see (9)) which plays the role of the Newtonian gravitational potential. However, such proposals are almost immediately dismissed with the claim that

> [MWT], insofar as it features *spacetime* torsion instead of mass torsion, is a stronger analog to a classical TPG. (Meskhidze and Weatherall 2023, p. 10, emphasis in original)

One can distinguish here two (closely related) claims:

1. MWT features spacetime torsion, whereas other torsional theories of non-relativistic gravity do not.

2. MWT does not feature mass torsion, whereas other torsional theories of non-relativistic gravity do.

Our aim in this section is to (a) isolate precisely what sort of torsion features in MWT, and thereby (b) assess the above two claims—*viz.*, whether MWT is relevantly different to previous torsionful reformulations of Newtonian gravitation such as those of Read and Teh (2018) and Schwartz (2023) in this respect.

We have already seen that the central object in MWT is a tensor $F^a{}_b$ which is supposed to "play the role of a torsional force term" (Meskhidze and Weatherall 2023, p. 8). In particular, $F^a{}_b$ encodes the difference between their torsional connection ∇ and curvature-based connection $\tilde{\nabla}$ via $\nabla = (\tilde{\nabla}, F^a{}_b t_c)$. However, Meskhidze and Weatherall are not explicit about how their tensor $F^a{}_b$ relates to those standardly considered in the literature, where the difference tensor between two such connections is encoded in the (spatiotemporal) contorsion and a 2-form Ω_{ab}, via $\nabla = (\tilde{\nabla}, 1/2(T^a{}_{bc} + T_b{}^a{}_c + T_c{}^a{}_b) + t_{(b}\Omega_{c)}{}^a)$ (see e.g. Bekaert and Morand (2014)).[6] To bridge this gap, we note the following points:

- Given the assumptions that t_a is closed and $T^{abc} = 0$, choosing a difference tensor of the form $F^a{}_b t_c$ is equivalent to imposing the requirement $t_b\Omega^{ac} = 2T^{[a}{}_b{}^{c]}$.
- It follows that $F^a{}_b = 1/2(T^a{}_{bn} + T_b{}^a{}_n + T_n{}^a{}_b)\xi^n + t_{(b}\Omega_{n)}{}^a\xi^n$, where ξ^a is an arbitrary unit timelike vector field.
- In particular, $F^a{}_b$ is related to Ω_{ab} via $\Omega^a{}_b = -2h^{an}F_{[nb]}$.

On (1), as Meskhidze and Weatherall (2023, theorem 1) in fact show, we are always free to 'gauge fix' $F^a{}_b$ so that the spatiotemporal torsion vanishes in the recovered models of MWT. In this respect, MWT is precisely analogous to the theories considered by Read and Teh (2018) and Schwartz (2023), in which the spatiotemporal torsion is generically non-vanishing, but nevertheless can always consistently be *chosen* to vanish via a suitable gauge fixing of the torsion and frame.

So much for the situation regarding (1); what of the situation regarding (2)? This is complicated somewhat by the fact that Meskhidze and Weatherall (2023) do not impose the Newtonian condition in their analysis; however, if one *does* impose this condition, then $\tilde{\nabla}$ is the unique extended torsion-free connection for some Bargmann structure, fixed up to $U(1)$ gauge symmetry (for details, see e.g. Geracie et al. (2015) and Schwartz (2023)). Given that choice of Bargmann structure, for a model of MWT with vanishing spacetime torsion, the components of the 2-form Ω_{ab} appearing in the difference tensor are coextensive with the components of the mass torsion $\boldsymbol{f}$ in ungeometrised NG, on which see Read and Teh (2018).

4.3 Empirical equivalence

Our final point has to do with empirical equivalence. Meskhidze and Weatherall (2023) motivate the construction of MWT as follows:

[6] We remind the reader that one must be very careful raising and lowering indices in the context of non-relativistic spacetime theories, since the degenerate spatial metric h^{ab} does not have a unique inverse.

> [T]o better understand the forces of TPG, a natural place to begin is another gravitational theory that employs forces, namely, Newtonian Gravity. Newtonian Gravity, however, is a non-relativistic theory and gravitational force does not involve torsion. A classical theory of gravity with torsional forces will prove to be a more informative comparison. (Meskhidze and Weatherall 2023, p. 2)

(Here, let us set aside what we have noted already above: that the gravitational potential of Newtonian gravity admits an interpretation in terms of mass torsion.) One might imagine, from reading this passage, that a necessary condition for MWT to fulfil the above-stated aims would be that it is empirically equivalent to standard Newtonian gravity, if Meskhidze and Weatherall are hoping to use MWT to draw morals from our understanding of forces in standard Newtonian gravity into the teleparallel context. However, MWT is *not* empirically equivalent to 'standard' Newtonian gravity.

To understand this, note that the version of NCT for which Meskhidze and Weatherall (2023) prove their recovery theorem is actually a slightly generalised version of standard NCT, which drops the Newtonian condition. It is well known that the Newtonian condition is equivalent to the requirement that there exists, at least locally, a unit timelike vector field which is geodesic and twist-free, thereby ensuring that there always exist non-rotating inertial observers (Malament 2012, propositions 4.3.3, 4.3.7). By contrast, if one drops the Newtonian condition, then there will be models in which all inertial trajectories are twisted, which amounts to the gravitational field having a non-vanishing curl. This means that MWT cannot be empirically equivalent to NCT with the Newtonian condition imposed—although MWT can still be empirically equivalent to a version of Newtonian gravity without the assumption of conservative forces (e.g., ungeometrised NG with a gravitational field G^a, as presented by Dewar (2018)).

5 Close

When should a given physical theory be of interest, whether that be to physicists or to philosophers (or both)? To answer this question, let us distinguish between what we might call 'theoretical interest' in a theory from 'practical interest':

Theoretical interest: A theory fits into a known 'web' of theories, and can shed light on that web.[7]

Practical interest: A theory has application to furthering our understanding of certain empirical phenomena.

The question to be considered here is: does MWT have either theoretical or practical interest?

[7] Presumably, it is principally (although perhaps not exclusively) theoretical interest which underlies Lehmkuhl's (2017) call for philosophers to explore the "space of spacetime theories".

On the former: a theory can satisfy this criterion in many ways—e.g., by being the (non-relativistic) limit of some other known theory, or by being the geometrised/recovered version (where the empirical content is retained!), etc. For example, both physics and philosophy communities have had substantial historical and current interest in exploring the geometric languages (curvature, torsion, and non-metricity) in which both Newtonian gravity and general relativity can be expressed. Such reformulated theories, indeed, invite questions concerning fundamental ontology, the interpretation of theoretical constructs, underdetermination, conventionalism, etc. Moreover, these theories are all interconnected in the sense that the non-relativistic versions of the relativistic theories can all be understood straightforwardly to proceed from their relativistic counterparts in the appropriate limits. Investigating these relationships sheds light on and increases understanding of the natures of the theories involved—see e.g. Jiménez et al. (2019), March et al. (2023), and Wolf and Read (2023b).

By contrast, MWT neither retains the empirical content of the paradigmatic non-relativistic spacetime theories which are Newtonian gravity and NCT (§4.3), nor is the non-relativistic limit of any known relativistic theory. For these reasons, it doesn't seem to us (at least at present!) that MWT has a great deal of theoretical interest, at least in the above sense.

Now on the latter (i.e., practical interest): consider e.g. that the different relativistic theories in the 'geometric trinity' (Jiménez et al. 2019) actually have practical relevance in terms of their calculation facility,[8] in addition to suggesting potentially different routes towards quantisation, while some of the non-relativistic theories considered here have found important applications in condensed matter systems and non-relativistic holography, as recapped in the introduction. When it comes MWT, again by contrast, we don't yet see any application of this theory—although of course we would welcome such applications, were they to be identified.

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[8] For example, TPG deals with spacetime boundaries in a way that is arguable more elegant and straightforward than GR. See Oshita and Wu (2017) and Wolf and Read (2023a) for further discussion.

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