MATHEMATICAL DETERMINACY AND INTERNAL CATEGORICITY

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Contents

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1 Introduction

1.1 Mathematical determinacy

Some mathematical concepts (such as “natural number” or “real number”) are commonly understood to be about a single mathematical domain *(the* natural numbers, *the* real numbers). In contrast, a mathematical concept such as that of “group” is not intended to be about one particular mathematical domain, but rather to describe a class of different structures which share certain properties but also exhibit important differences (some groups are finite, some infinite, some abelian, some non-abelian etc.). The mathematical concept of “set” has a more controversial status, with some mathematicians believing that the set concept determines a unique universe of sets, while others believe that the set concept delineates a range of different set universes (a set-theoretic “multiverse”).[1]

[1]See D.A. Martin, “Multiple Universes of Sets and Indeterminate Truth Values,” *Topoi* 20 (2001): 5–16; Joel David Hamkins, “The Set-Theoretic Multiverse,” *The Review of*

In general terms, the issue of *mathematical determinacy* may be stated as follows: do certain of our mathematical concepts (such as number and set) manage to define a unique mathematical structure, and if so, how? More specifically, we may state the issue of the *determinacy of arithmetic* as follows: does our concept of natural number define a unique mathematical structure (the natural numbers) and, if so, how? The issue of *set-theoretic determinacy* may be stated as: does our concept of set define a unique mathematical structure (the universe of sets) and, if so, how?

In recent work, Tim Button and Sean Walsh have argued that arithmetical and set-theoretic determinacy follow from certain "internal categoricity" results proved in second-order logic.[2] In this paper I critically evaluate this claim and argue that such internal categoricity results fail to entail determinacy as claimed. In order to concentrate on the key issues in some depth, I focus on arithmetical determinacy. The rest of this introduction gives a high-level overview of the material covered in this paper.

1.2 Referential determinacy and truth-value determinacy

We can identify two distinct but closely related concepts of mathematical determinacy: referential determinacy and truth-value determinacy.[3]A mathematical concept is taken to be *referentially determinate* if it manages to somehow define ("pick-out" or "pin-down") a unique mathematical domain. For example, the concept of the natural numbers is referentially determinate if it manages in some sense to "pin-down" a unique natural-number domain. The notions of "domain" and "pinning-down" here are pretty vague — ways in which they can be made more precise using the notion of categoricity are discussed in section 3.3 below.

Symbolic Logic, Volume 5, Number 3 (September 2012): 416-449. Note that Hamkins suggests that the variety of different set structures in a multiverse of sets may also imply a variety of different natural number structures within that multiverse – see Hamkins, "The Set-Theoretic Multiverse": 427-8.

[2]Tim Button and Sean Walsh, *Philosophy and Model Theory* (Oxford : Oxford University Press, 2018); Tim Button, "Mathematical Internal Realism," in J. Conant and S. Chakraborty (eds), *Engaging Putnam*, Harvard University Press, forthcoming.

[3]See Tim Button and Sean Walsh, "Structure and Categoricity: Determinacy of Reference and Truth Value in the Philosophy of Mathematics," *Philosophia Mathematica* Vol. 24 No. 3 (2016): 283-307.

A mathematical concept is taken to be *truth-value determinate* if it determines answers to all mathematical questions pertaining to the concept. For example, the truth-value determinacy of the natural number concept would imply that Goldbach's conjecture (that every even number greater than two is the sum of two primes), which has to-date been neither proved nor disproved, nevertheless is either true or false: there is a "fact of the matter" about it which obtains whether we know the answer or not. Similarly, the truth-value determinacy of the set concept would imply that there is a "fact of the matter" about the Continuum Hypothesis (that there is no set whose cardinality is strictly between that of the natural numbers and the real numbers) even though we do not currently know what that fact is; even though, indeed, the Continuum Hypothesis is known to be independent of the standardly accepted axioms of set theory. As noted above, referential determinacy and truth-value determinacy are related: if a mathematical concept is referentially determinate then we might expect that it is also truth-value determinate, since if the concept defines a unique mathematical domain, then we would expect questions about that unique domain to have uniquely defined answers. On the other hand, if a mathematical concept is truth-value determinate then we would expect it to define at least an equivalence class of mathematical domains which respect those truth-values.

1.3 Categoricity and determinacy

Very broadly understood, a mathematical theory is "categorical" if all interpretations (models) of the theory have the same mathematical structure. A categorical theory of arithmetic can be seen as implying both referential and truth-value determinacy since the theory defines a single mathematical structure, and all arithmetical statements have a determinate truth-value in that structure. Unfortunately the mathematical theories in which we are interested (such as theories of arithmetic and sets) are not categorical when expressed in first-order logic.

These failures of categoricity motivate a move to second-order logic, since the second-order theory of arithmetic is categorical (when second-order logic is given its "standard semantics") and second-order set theory is "quasi-categorical" (again given the standard semantics). However the standard semantics for second-order logic relies on a very powerful concept – the power-set (set of all subsets) of a given domain (for example, the set of all subsets of the natural numbers). This casts doubt on whether second-order theories can really show mathematical determinacy, since the determinacy of the natural-number concept then depends on a much stronger concept – the set of all subsets of the natural numbers. As Button and Walsh put it, we are "out of the frying pan into another frying pan."[4] I term this the "determinacy regress" problem. Furthermore, the standard semantics for second-order logic is so strong that no deductive system of logic can capture that semantics. Reliance on the standard semantics to ensure determinacy thus violates what I call "the naturalistic constraint", which I discuss in detail in section 2 below. The naturalistic constraint ("moderation" in Button and Walsh's terminology) requires that mathematical determinacy must be understood as arising through characteristics of publicly articulable and communicable practices, such as an explicit deductive theory.

1.4 Internal categoricity and determinacy

Both the determinacy regress and violation of the naturalistic constraint motivate Button and Walsh's moving to a different type of categoricity – internal categoricity – as a basis for determinacy. Internal categoricity is a version of categoricity in which the models in question are in a certain sense "internal" to a theory, in this case second-order logic. Internal categoricity of second-order arithmetic can be proved within second-order logic itself and does not require any explicit reference to the semantics of second-order arithmetic. Button and Walsh's proposal is that the internal categoricity of second-order arithmetic can be used to show the determinacy of arithmetic, while avoiding both the determinacy regress and violation of the naturalistic constraint.

I argue, contrary to Button and Walsh, that internal categoricity of second-order arithmetic does not entail the determinacy of arithmetic in a way which satisfies the naturalistic constraint. To satisfy that constraint, Button and Walsh rely on characterising arithmetical practice as deduction in second-order arithmetic. However, I show that their argument that arithmetical determinacy follows from internal categoricity depends on an additional assumption which does not itself follow from that characterisation of arithmetical practice.

[4] Button and Walsh, *Philosophy and Model Theory*: 160.

1.5 Outline of this paper

Section 2 motivates the naturalistic constraint as a constraint on possible explanations of mathematical determinacy. Section 3 reviews basic definitions and results concerning first-order theories, their models, and failures of categoricity for first-order theories of arithmetic and set-theory. Section 4 provides an overview of second-order logic, its semantics and the relevant categoricity and incompleteness results, and discusses in detail the determinacy regress and the failure of the naturalistic constraint in relation to second-order logic. Section 5 sets out the basic definitions and results concerning internal categoricity and the way in which internal categoricity overcomes the determinacy regress and satisfies the naturalistic constraint. Finally, section 6 sets out in detail my argument that the internal categoricity of second-order arithmetic does not entail the determinacy of arithmetic, and considers a number of possible objections and replies.

With regard to mathematical prerequisites, this paper assumes only familiarity with first-order logic. Further mathematical definitions and results are introduced as required. No proofs are included, but references to where they may be found are provided.

2 Mathematical determinacy and the naturalistic constraint

The problem of mathematical determinacy includes a "how" component. If some of our mathematical concepts achieve either referential or truth-value determinacy, we seek to understand *how* that is achieved. As McGee puts it, in discussing truth-value determinacy:

> there must be something we think, do, or say that fixes the intended meaning of mathematical terms. How are we able to do this?
>
> ... The problem before us is to understand how our thoughts and practices can fix the meanings of mathematical terms with sufficient precision to ensure that each sentence has a determinate truth value.[5]

[5]Vann McGee, "How we learn mathematical language," *Philosophical Review,* Vol. 106 No. 1 (1997): 36.

One possible explanation of how a mathematical concept achieves determinacy is that we can just "see" or "intuit" that the concept determines a unique mathematical domain, or can just "see" or "apprehend" the unique mathematical domain defined by the concept.[6] However, such an explanation does not shed much light on how the concept achieves that determination. Further, if there is controversy over whether a mathematical concept is determinate, then one party to the controversy simply claiming to be able to "see" or "intuit" its determinacy is not likely to advance the debate. Consequently, it is usual to seek explanations of mathematical determinacy which eschew such explanations based on insight or intuition. Hilary Putnam, for example, rules out explanations of mathematical determinacy based on "mysterious faculties of 'grasping concepts' (or 'perceiving mathematical objects')" which he sees as not capable of naturalistic explanation.[7] In like vein, Warren and Waxman reject "platonistic and heavily metaphysical explanations of mathematical determinacy", which are based on "an explanatorily freestanding and fully determinate mathematical realm."[8]

Drawing on the views of Putnam, Button and Walsh introduce the idea of *moderation* as a particular kind of naturalistic constraint on explanations of mathematical determinacy. The "moderate", they say

> accepts that we cannot fix reference to mathematical entities by seeing them, pointing to them, or interacting with them in any way ...
>
> A better thought is that we come to refer to mathematical entities after some process of mathematical education. But, for a moderate, ... if learning some mathematical theory is what allows us to refer to specific mathematical entities, then ... the theories themselves must precisely pin down the mathematical entities. In a brief slogan: for the moderate ... there can be no 'reference by acquaintance' to mathematical objects; 'reference by description'

[6] For example, Kurt Gödel argues for a form of "perception" of mathematical objects; see Kurt Gödel, "What is Cantor's Continuum Problem?" in: Paul Benacerraf and Hilary Putnam (eds.) *Philosophy of Mathematics, Selected Readings*, Second Edition (Cambridge: Cambridge University Press, 1983): 483-4.

[7] Hilary Putnam, "Models and Reality," *The Journal of Symbolic Logic*, Vol. 45, No. 3 (Sep., 1980): 466.

[8] Jared Warren and Daniel Waxman, "A metasemantic challenge for mathematical determinacy," *Synthese* 197 (2020): 486.

> is her only hope.[9]

In essence, this naturalistic constraint rules out an explanation of determinacy by some kind of direct apprehension of an already-determinate mathematical reality. Button emphasizes the importance of the *acquisition* and *manifestation* of mathematical concepts:

> We have to *acquire* our mathematical concepts. Even if we are born with the capacity to acquire mathematical concepts, we are not born with the concepts themselves ...
>
> Equally, we must be able to *manifest* our mathematical concepts. Whilst mathematicians sometimes work alone, mathematical practice is fundamentally communal. Mathematicians present each other with proofs and projects, ...
>
> In our early steps towards acquiring the NUMBER concept , we learn how to recite sequences like "1, 2, 3, 4 , 5 " , ... Later, we master algorithms for adding and multiplying numbers in decimal notation. And so it goes. But my interest here is ... in the NUMBER concept itself, as used in serious mathematics. And, ... we qualify as having acquired it fully, only when we have grasped some full-blown arithmetical theory, such as Peano Arithmetic. Equally, we fully manifest our grasp of the concept, only by articulating and using some such theory.[10]

We are seeking explanations of mathematical determinacy which satisfy a *naturalistic constraint* (*moderation* in Button and Walsh's terminology): determinacy must be understood as arising through characteristics of publicly articulable and communicable practices.

[9] Button and Walsh, *Philosophy and Model Theory*: 43.

[10] Button, "Mathematical Internal Realism": 1-2. See also Warren and Waxman's "cognitive constraint" in "A metasemantic challenge for mathematical determinacy": 485.

3 Model Theory, mathematical determinacy and external categoricity

3.1 The role of mathematical logic and model theory

The naturalistic constraint leads to the idea that if mathematical determinacy can be achieved at all, it can only be through the adoption and articulation of a mathematical theory. Such a theory may be, and indeed commonly is, informally expressed (that is, without the explicit use of a formal logical language and inference rules). However, it is commonly believed by mathematicians that informal mathematical theories can be fully formalised. Further, mathematical logic provides a framework within which mathematical languages, theories and their interpretations (models) can be understood in a precise way. Consequently many explorations of mathematical determinacy, including those discussed in this paper, are couched in the framework of formal logical languages and their models. I turn now to review that framework and its role in discussions of mathematical determinacy.

In the next section I first review some of the basic terminology and results concerning mathematical theories expressed in first-order logic and models of those theories. While I assume that the reader is familiar with the basic concepts of first-order logic, this review serves to fix some terminology and notation as a basis for further discussion.

3.2 First-order mathematical theories and their models

Recall that the language of first order logic consists of three kinds of symbols:

1. *logical symbols*: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists$, *(* and *);*
2. *non-logical symbols*, which come in three types
   (a) *constant symbols - a; b; c; ...*
   (b) *relation symbols - P; Q;R; ...*
   (c) *function symbols*: *f; g; h; ...*; and
3. *individual variables*: v_1,v_2,..

The relation and function symbols each have an arity which defines how many arguments they take.

Definition 1. A *language* $\mathscr{L}$ within first-order logic is defined by a set of non-logical symbols $\mathscr{L} = \{a, b, c, ..., P, Q, R, ..., f, g, h\}$.

Definition 2. A *model* $\mathscr{M}$ for a language $\mathscr{L}$ consists of

- a set M which is the underlying domain of individuals for the model $\mathscr{M}$
- an element $c^{\mathscr{M}} \in M$ for each constant symbol c of $\mathscr{L}$
- a function $f^{\mathcal{M}} : M^n \rightarrow M$ for each n-place function symbol f of $\mathscr{L}$
- a relation $R^{\mathscr{M}} \subseteq M^n$ for each n-place relation symbol R of $\mathscr{L}$.

Intuitively, a model of a language provides a particular interpretation of the language. Note that generally a language may admit many different interpretations (models).

Given these definitions, it is possible to formally define the relation of *satisfaction* between a model of a language and a sentence of the language. The formal definition, which goes back to Tarski, is not given here.[11] The intuitive idea is that a model $\mathscr{M}$ of $\mathscr{L}$ *satisfies* (alternatively, *is a model of*) a sentence φ of $\mathscr{L}$, denoted $\mathscr{M} \models \varphi$, if φ is true in the model.

If Γ is a set of sentences, we say that $\mathscr{M}$ is a model of Γ if $\mathscr{M}$ is a model of every sentence $\varphi \in \Gamma$. We also write $\Gamma \vDash \varphi$ (Γ *entails* φ, or φ is a *logical consequence* of Γ) if every model of Γ is also a model of φ.

We are interested in *theories* expressed in a particular language, which specify a set of sentences required to be true in all models of the theory. To specify a theory in practice, we write down a set of sentences which are the axioms of the theory. The resulting theory is defined by the *deductive closure* of the set of axioms, that is, the set of all sentences which are logical consequences of the axioms. For example, to specify the first-order theory of Peano Arithmetic we write down a set of axioms such as the following.

Definition 3. *First-order Peano Arithmetic* is defined by the following axioms:

$\forall x\,(S(x) \neq 0)$
$\forall x \forall y\,(S(x) = S(y) \rightarrow x = y)$
$\forall x(x + 0 = x)$

[11] See, for example, Button and Walsh, *Philosophy and Model Theory*: 12-13.

$$\forall x \forall y\,(x + S(y) = S(x + y))$$
$$\forall x(x \times 0 = 0)$$
$$\forall x \forall y\,(x \times S(y) = (x \times y) + x)$$
$$[\varphi(0) \wedge \forall y(\varphi(y) \rightarrow \varphi(S(y)))] \rightarrow \forall y \varphi(y)$$

The last axiom (the induction axiom) is actually an axiom schema, defining an axiom for every formula φ of the language in which the variable y occurs free.

The resulting theory of arithmetic consist of all sentences in the language which are logical consequences of the axioms.

First-order logic has a number of well-known deductive systems (for example, natural deduction or tableau). We write $\Gamma \vdash \varphi$ if the sentence φ can be deduced from the set of sentences Γ in one of those systems of deduction. The standard deductive systems are known to be both *sound* ($\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$) and *complete* ($\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$). As a result of these soundness and completeness properties, the set of sentences which are logical consequences of a set of sentences Γ is the same as the set of sentences which can be deduced from Γ in a suitable system of deduction for first-order logic. This means that in first-order logic, logical consequence is equivalent to deductive consequence. As we will see below, this is not necessarily the case for stronger systems of logic.

3.3 "Pinning-down" a unique structure – categoricity

We are interested in whether a theory can "pick-out" or "pin-down" a unique mathematical domain. We could try to make this idea precise through the requirement that a theory has just one model. However this is too strong. Two models of a language (such as the language of arithmetic) may differ from each other in ways which make no difference from the point of view of the language.

It is well known, for example, that the natural numbers can be represented within set theory in a number of different ways. The two most common are the representations by the von Neumann ordinals (in which the sequence 0, 1, 2, ... is represented by the sequence $\emptyset$,$\{\emptyset\}$, $\{\emptyset,\{\emptyset\}\}$,...) and the Zermelo ordinals (in which the sequence 0,1,2, ... is represented by the sequence $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, ...$). However, when doing arithmetic it makes no difference which particular representation is used.[12]

[12]Paul Benacerraf, "What Numbers Could not Be," *The Philosophical Review*, Vol. 74,

The idea that two different models of a language may differ in ways which make no difference from the point of view of the language can be captured in the notion of an *isomorphism* between models, as follows.

Definition 4. Two models $\mathscr{M}$ and $\mathscr{N}$ of a language $\mathscr{L}$ are said to be *isomorphic* ($\mathscr{M} \cong N$) if there is a mapping σ between $\mathscr{M}$ and $\mathscr{N}$ such that

- σ is bijective (one-to-one and onto)
- for every constant symbol c of $\mathscr{L}$, $\sigma(c^{\mathscr{M}}) = c^{\mathscr{N}}$
- for every n-ary relation R in $\mathscr{L}$ and any m_1,..., m_n from M

$$<m_1,..., m_n> \in R^{\mathscr{M}} \Leftrightarrow < \sigma(m_1),..., \sigma(m_n) > \in R^{\mathscr{N}}$$

- for every n-ary function symbol f of $\mathscr{L}$and any m_1,..., m_n from M

$$\sigma(f^{\mathcal{M}}(m_1, ...,m_n)) = f^{\mathscr{N}}(\sigma(m_1), ..., \sigma(m_n)).$$

Two isomorphic models are the same from the point of view of the language $\mathscr{L}$: they have exactly the same structure with respect to the functions and relations of the language.

Definition 5. A theory is *categorical* if all of its models are isomorphic.

The notion of categoricity of a theory seems to provide a good formalisation of the referential determinacy of a theory, since there is a sense in which a categorical theory "pins down" a unique mathematical structure, namely the structure shared by all its (isomorphic) models.

Furthermore, if a theory is categorical then it is also *semantically complete* in the following sense.

Definition 6. A theory Σ of a language $\mathscr{L}$ is *semantically complete* if for every sentence σ of $\mathscr{L}$ either $\Sigma \models \sigma$ or $\Sigma \models \neg\sigma$.

Semantic completeness can be seen as a formalisation of the idea of truth-value determinacy, since in a sense the theory itself determines that each sentence is either true or false (the truth or falsity of the sentence does not vary from model to model). Further, the completeness of first order logic means that a first-order theory which is semantically complete is also *deductively complete* – that is, for every sentence σ either $\Sigma \vdash \sigma$ or $\Sigma \vdash \neg\sigma$.

No. 1 (Jan., 1965): 47-73.

3.4 Failures of categoricity and completeness for first order theories

Unfortunately, the first order theories in which we are interested, such as Peano Arithmetic and Zermelo-Frankel set theory are known *not* to be categorical. The Löwenheim-Skolem Theorem shows that any (countable) first-order theory with an infinite model has a model of every infinite size.[13] Consequently theories of arithmetic and set theory, which have infinite models, cannot be categorical.

In addition, Gödel's incompleteness theorems show that first-order Peano Arithmetic and Zermelo-Frankel set theory are each both semantically and deductively incomplete (provided they are consistent). Furthermore this incompleteness cannot be remedied by adding additional axioms (provided the resulting systems continue to be consistent).[14]

Consequently, in the context of first-order theories we are unable to demonstrate either the referential or truth-value determinacy of our theories of arithmetic and sets by relying on the formal properties of categoricity or completeness. In fact, if we were to take our concepts of number or set to be fully captured by our standard first-order theories, the failures of categoricity and completeness give us reason to *doubt* both the referential and truth-value determinacy of those concepts.

4 The move to second-order logic

The situation just outlined has motivated a number of mathematicians and philosophers to move to a stronger-logic, specifically second-order logic, in a bid to formulate mathematical theories of numbers and sets which are categorical and/or complete, and so vindicate the referential or truth-value determinacy of those theories. I turn now to briefly review second-order logic and its properties before going on to consider whether it can deliver the hoped-for determinacy results.

[13] Button and Walsh, *Philosophy and Model Theory*: 154; Dirk van Dalen, *Logic and Structure* (London: Springer-Verlag, 2013): 105-6.

[14] Button and Walsh, *Philosophy and Model Theory*: 130-2; Peter Smith, *An Introduction to Gödel's Theorems* (Cambridge: Cambridge University Press, 2007).

4.1 Second-order logic

Second-order logic extends first-order logic by allowing quantification over relations and functions. As a simple example, first-order logic can express the statement $\exists x(P(x) \to \forall x P(x))$ whereas second-order logic permits in addition quantification over the predicate P, as in $\forall P \exists x(P(x) \to \forall x P(x))$.[15]

Second-order logic introduces variables which can range over n-ary relations and functions (for any n) and permits quantification over those variables. The arity of the function and relation variables is typically indicated by superscripts. For example X^n is used as an n-place relation variable which can appear in quantifiers as in $\exists X^n \varphi$ or $\forall X^n \varphi$ where φ is second-order formula.

A deductive system for second-order logic is defined by extending the rules for first order quantifiers with analogous rules for second-order quantifiers. In addition, deductive systems for second-order logic typically include two additional axiom schemes as follows.[16]

Definition 7. Comprehension Schema. $\exists X^n \forall \vec{v}(\varphi(\vec{v}\,) \leftrightarrow X^n(\vec{v}\,))$, for every formula $\varphi(\vec{v}\,)$ which does not contain X^n free.

This axiom schema requires that for each n-ary relation definable by a formula φ in the language, there exists an n-ary relation which is equivalent to that relation. For example, given a two-place relation symbol R in a language $\mathscr{L}$, an instance of the Comprehension Schema yields an axiom $\exists X^2 \forall v_1 \forall v_2(\neg R(v_1, v_2) \leftrightarrow X^2(v_1, v_2))$. This requires that in any model $\mathcal{M}$ satisfying the axiom, there must exist a relation consisting of the set of all pairs not in $R^{\mathcal{M}}$.

Definition 8. Choice Schema. This axiom schema requires that whenever there is a relation which ensures the existence of an element standing in that relation, there is a function which picks such an element out. Formally:
$\forall X^{n+1}(\forall \vec{v}\, \exists y X^{n+1}(\vec{v}\,, y) \to \exists f^n \forall \vec{v}\, X^{n+1}(\vec{v}, f^n(\vec{v}\,))$.

[15] For details on the syntax, semantics and deductive systems for second-order logic see Button and Walsh, *Philosophy and Model Theory*: 21-33; Stewart Shapiro, *Foundations Without Foundationalism* (Oxford: Oxford University Press, 1991): 61-96; van Dalen, *Logic and Structure*: 145-153.

[16] See Button and Walsh, *Philosophy and Model Theory*: 25-6; Shapiro, *Foundations Without Foundationalism:* 66-7.

For example, if S is a two-place relation such that for any x there is a y such that *S(x,y)* then the choice axiom requires that in any model satisfying the schema there must be a one-place function f such that for each x, *S(x,f(x))*.

A second-order axiomatisation of Peano Arithmetic is given as follows.

Definition 9. *Second-Order Arithmetic* (PA_2) is defined by the following axioms.

$\forall x\,(S(x) \neq 0)$
$(\forall x, y(S(x) = S(y) \rightarrow x = y)$
$\forall X[(X(0) \wedge (\forall x[X(x) \rightarrow X(S(x)]) \rightarrow (\forall x X(x)]$

Note that in the second-order setting addition and multiplication are explicitly definable,[17] and, for simplicity, we omit those definitions from the axioms.

Unlike the situation in first-order logic, two possible semantics can be given for second-order logic: *full (or standard)* semantics on the one hand and *general (or Henkin)* semantics on the other. Because second-order logic permits quantification over relation and function variables, the semantics must specify the set of possible relations and functions which such quantifications range over. The full and Henkin semantics specify those sets differently.

The more general semantics is the Henkin semantics, defined as follows (for simplicity this definition deals with relation symbols and variables only: it can be readily extended to include functions).

Definition 10. A *Henkin model* $\mathscr{M}$ for a second-order language $\mathscr{L}$ consists of

- a set M which is the underlying domain of individuals for the model $\mathscr{M}$
- an element $c^{\mathscr{M}} \in M$ for each constant symbol c of $\mathscr{L}$
- a relation $R^{\mathscr{M}} \subseteq M^n$ for each n-place relation symbol R of $\mathscr{L}$
- a set $M_n^{rel} \subseteq \mathscr{P}(M^n)$ for each n.

[17]See Shapiro, *Foundations Without Foundationalism*: 120.

The first three components of this definition are the same as those for first-order models. The set M_n^{rel} defines the domain of quantification for the n-place relation variables. It is a subset of $\mathscr{P}(M^n)$, the set of all subsets of M^n, the n-fold cartesian product of M. That is, M_n^{rel} is a *subset* of all the n-place relations on M.

An *admissible Henkin Model* is a Henkin model which satisfies the axiom schemas of Comprehension and Choice

Definition 11. A *full model* is a Henkin Model in which $M_n^{rel} = \mathscr{P}(M^n)$, that is, the domain of quantification for the n-place relation variables is the set of *all* n-place relations on M.

Definition 12. The *Henkin semantics* for a second-order language admits all admissible Henkin models, including those in which the domain of quantification for the n-place relation variables is a restricted subset of the n-place relations on M.

In contrast, the *full semantics* restricts the possible models to the full models, in which n-place relation variables always range over *all* n-place relations on M.

Note that the standard deductive systems for second-order logic are the same for both full and Henkin semantics.

4.2 Categoricity and completeness in second-order logic

This section sets out important properties of second-order logic relevant to the issue of mathematical determinacy. As we will see, the relevant properties depend critically on whether the full or Henkin semantics is chosen for the logic.

Proposition 13. *Second-order Peano Arithmetic (PA_2) is categorical and semantically complete with respect to the full semantics,*[18] *but fails to be either categorical or semantically complete with respect to the Henkin semantics.*[19]

[18] Button and Walsh, Philosophy and Model Theory: 155; Shapiro, *Foundations Without Foundationalism*: 82-3. Corresponding "quasi-categoricity" results hold for second-order versions of set theory, see Button and Walsh, *Philosophy and Model Theory*: 192; Shapiro, *Foundations Without Foundationalism*: 86.

[19] Shapiro, *Foundations Without Foundationalism*, 92-5.

Proposition 14. *The standard deductive systems for second-order logic are not complete with respect to the full semantics but are complete with respect to the Henkin semantics. Further, there is no deductive system for second-order logic which is complete with respect to the full semantics.*[20]

Proposition 15. *PA_2 is not deductively complete provided it is consistent. Further, no strengthening of PA_2 by the addition of further axioms is deductively complete (provided that the strengthened system remains consistent).*[21]

Note that the last result concerns only the deductive consequences of second-order PA_2 and holds whether the full or the Henkin semantics is adopted.

4.3 Second-order logic and determinacy

What implications do these properties of second-order logic have for the mathematical determinacy of second-order theories?

The categoricity and semantical completeness of PA_2 (with respect to the full semantics) at first sight look like good news for the mathematical determinacy of arithmetic. These results appear to imply that second-order PA_2 "pins down" a unique mathematical structure (up to isomorphism) and also determines the truth or falsity of all arithmetical propositions. However, these results rely on adoption of the full semantics, which raises problems for the claim that they can secure arithmetical determinacy, as we now discuss.[22]

4.3.1 The determinacy regress

The first problem may be termed a *determinacy regress*.[23] Categoricity (and semantical completeness) only hold for PA_2 with the full semantics. That is,

[20]Shapiro, *Foundations Without Foundationalism*: 87, 89.

[21]Smith, *An Introduction to Gödel's Theorems*, in particular pages 186-197. Second-order versions of set theory are likewise not deductively complete

[22]Again, similar considerations apply in relation to second-order set theory.

[23]This determinacy regress have been discussed in a variety of guises by a number of authors including Thomas Weston, "Kreisel, the Continuum Hypothesis and Second Order Set Theory," *Journal of Philosophical Logic*, Vol. 5, No. 2 (May, 1976): 288; Button and Walsh, *Philosophy and Model Theory*: 158-160; Toby Meadows, "What can a categoricity theorem tell us?" *The Review of Symbolic Logic,* Volume 6, Number 3 (September 2013): 537; Warren and Waxman, "A metasemantic challenge for mathematical determinacy": 488.

categoricity only holds when variables for relations on numbers range over the full set of all possible relations on the numbers. In particular, variables for number predicates range over all possible subsets of the numbers, that is, the full powerset of the numbers. This introduces a regress, since to ensure the categoricity of PA_2 we need to assume the determinacy of the full powerset of the natural numbers. If the powerset is not itself determinate then we cannot conclude the determinacy of the natural numbers.

As Toby Meadows points out, we are trying to show that there is a unique (up to isomorphism) structure corresponding to our arithmetic practice, but in relying on the full semantics of second-order logic to do so, we are relying on the uniqueness (again up to isomorphism) of the structure of all subsets of the natural numbers, a vastly more complex structure than the structure of the natural numbers. As Meadows puts it "we are using the uniqueness claim about a complex structure to lever a result about a comparatively simple one."[24]

Button and Walsh make a similar point, noting that categoricity depends on reference to the full powerset of the underlying domain of numbers. But, as they point out

> the full powerset of this domain has the same cardinality as the real numbers. Moreover, via familiar coding mechanisms, we can view each real number as a certain subset of natural numbers. So, to explain how [we] can pick out the natural numbers (up to isomorphism), it seems [we] must first explain how [we] can pick out the real numbers (up to isomorphism). [Our] problems have only worsened: whatever problems arise in referring to the naturals will pale in comparison to the problems which arise in referring to the reals. Either way, then, in appealing to full second-order logic, [we are] simply out of the frying pan, and into another frying pan. And, if anything, the second frying pan is slightly larger and hotter than the first.[25]

[24] Meadows, "What can a categoricity theorem tell us?": 537.

[25] Button and Walsh, *Philosophy and Model Theory*: 159-160.

4.3.2 Failure of the naturalistic constraint

The second, related problem is a *failure of the naturalistic constraint.*[26] Reliance on the full semantics means that the full powerset concept (the set of all subsets of the natural numbers) is "hard-wired" into the logic. However, nothing in our explicit (publicly articulable and communicable) practice of second-order logic manifests that concept. Our explicit practice of second-order logic is captured by the deductive system for second-order logic. However, this deductive system also admits the Henkin semantics, and categoricity (and semantical completeness) of PA_2 does not hold in that semantics. As Putnam puts the problem, the "intended" (full) interpretation of the second-order formalism:

> is not fixed by the use of the formalism (the formalism itself admits "Henkin models", i.e., models in which the second-order variables fail to range over the full power set of the universe of individuals), and it becomes necessary to attribute to the mind special powers of "grasping second-order notions".[27]

Button and Walsh further elaborate on the issue as follows:

> There are sound and complete inference rules for first-order logic and for second order logic with a Henkin semantics. So the moderate can plausibly demonstrate her grasp of those logical ideas, just by demonstrating her mastery of the inference rules. By contrast, full second-order logic has no sound and complete deductive system. Consequently, to claim a grasp of [the] supposedly categorical theory [PA_2], we must claim to grasp either a theory or a semantics which cannot be laid down in any finitary fashion. Given the [moderate's] self-conscious naturalism, it is hard to see how she could sustain such a claim.[28]

These problems have lead to a focus in recent literature on *internal categoricity* results which some argue are able to deliver determinacy without

[26] This problem has also been discussed in a variety of guises by a number of authors including Button and Walsh, *Philosophy and Model Theory*: 158-60; Warren and Waxman, "A metasemantic challenge for mathematical determinacy": 485, 488.

[27] Putnam, "Models and Reality": 481.

[28] Tim Button and Sean Walsh, "Ideas and Results in Model Theory: Reference, Realism, Structure and Categoricity": 27.

the problems outlined above. I turn now to examine these results and then go on to critically evaluate their significance for the problem of mathematical determinacy.

5 Internal (or relative) categoricity

5.1 Internal categoricity in second-order logic

Second-order logic permits arithmetic to be finitely axiomatised (for example, through the axioms of PA_2). Further, because it allows quantification over predicates and functions, second-order logic can refer "internally" to "packages" of functions and relations which satisfy the axioms of PA_2 and which can be viewed as "internal" models of PA_2. More specifically, we can "package up" the axioms of PA_2 as a conjunction of assertions in second-order logic as follows.[29]

Definition 16. Let *PA(NzS)* abbreviate the following conjunction, in which the variables N (a predicate representing "is a natural number"), z (a variable representing "zero") and S (a function representing "successor") occur free.

$N(z) \wedge (\forall x : N)(\exists! y : N)S(x) = y \ \wedge$
$(\forall x : N)S(x) \neq z \wedge$
$(\forall x, y : N)(S(x) = S(y) \rightarrow x = y) \wedge$
$\forall X[(X(z) \wedge (\forall x : N)[X(x) \rightarrow X(S(x))]) \rightarrow (\forall x : N)X(x)]$

Note that the above formula utilises the following abbreviations: $(\exists x : N)\psi$ abbreviates $\exists x(N(x) \wedge \psi)$; $(\forall x : N)\psi$ abbreviates $\forall x(N(x) \rightarrow \psi)$ and $\exists x!\varphi(x)$ (there is a unique x such that $\varphi(x)$) abbreviates $\exists x(\varphi(x) \wedge \forall v(\varphi(v) \rightarrow v = x))$.

Intuitively, the first conjunct says that z is a natural number, the second that every natural number has a unique natural number successor (that is, S is a function from N to N), the third that zero is not the successor of any natural number, the fourth that no two natural numbers have the same successor, and the final conjunct states the induction axiom. Note that $PA(NzS)$ is effectively a parameterized version of PA_2 where the underlying domain N, the constant *0* and the successor function S are replaced by variables.

[29] See Button and Walsh, *Philosophy and Model Theory*: 225.

Now second-order logic can refer "internally" to multiple "copies" of the above conjunction which refer to different variables. In that way $PA(N_1 z_1 S_1)$ and $PA(N_2 z_2 S_2)$ can be understood as referring to two possible "internal structures" satisfying PA_2. It is then possible to define in second-order logic the property that a relation R is an isomorphism between two such "internal arithmetical structures" as follows.[30]

Definition 17. The property that a relation R is an isomorphism between $(N_1 z_1 S_1)$ and $(N_2 z_2 S_2)$ can be defined as follows:

$$
\begin{aligned}
&IsoN_{1\triangleright 2}(R) \coloneqq \\
&\forall v \forall y (R(v,y) \rightarrow [N_1(v) \wedge N_2(y)]) \wedge \\
&(\forall v : N_1) \exists ! y R(v,y) \wedge \\
&(\forall y : N_2) \exists ! v R(v,y) \wedge \\
&R(z_1, z_2) \wedge \forall v \forall y (R(v,y) \rightarrow R(S_1(v), S_2(y))
\end{aligned}
$$

The first of the above conjuncts says that R maps N_1 to N_2, the second that R is functional, the third that R is a bijection, and the last says that R "preserves arithmetical structure" (R maps z_1 to z_2 and, if R maps v to y, then R maps the successor of v to the successor of y).

It is now possible to prove within second-order logic that for every two "internal arithmetical structures" $(N_1 z_1 S_1)$ and $(N_2 z_2 S_2)$ satisfying $PA(NzS)$ there is a relation R which is an isomorphism between these internal arithmetical structures.[31] This proposition (proved deductively within second-order logic) is referred to as the internal categoricity of PA_2.

Proposition 18. *Internal Categoricity of* PA_2 :

$$\vdash \forall N_1 z_1 S_1 N_2 z_2 S_2 ([PA(N_1 z_1 S_1) \wedge PA(N_2 z_2 S_2)] \rightarrow \exists R\, IsoN_{1\triangleright 2}(R))$$

This proposition provides a kind of "internal" counterpart to the model-theoretic categoricity result discussed in section 4.3 above.

Furthermore, it is also possible to prove the following result, which can be understood informally as saying that for every arithmetical claim φ, either φ holds in every internal arithmetical structure or $\neg\varphi$ holds in every internal arithmetical structure.[32]

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[30] Ibid., 228.

[31] Button and Walsh, *Philosophy and Model Theory* (Chapter 10): 228, 243-5; Jouko Väänänen and Tong Wang, "Internal categoricity in arithmetic and set theory," *Notre Dame Journal of Formal Logic Vol.* 56 No. 1 (2015): 124.

[32] Button and Walsh, *Philosophy and Model Theory* (Chapter 10): 232, 245-6.

Proposition 19. *Intolerance of PA_2. For any formula $\varphi(NzS)$ whose quantifiers are N-restricted and whose free variables are all displayed:*
$\vdash \forall NzS(PA(NzS) \to \varphi(NzS)) \vee \forall NzS(PA(NzS) \to \neg\varphi(NzS))$

This result gives a kind of "internal" version of semantical completeness since it shows that all internal arithmetical structures satisfying the axioms embodied in *PA(NzS)* agree on the truth (or falsity) of every arithmetical claim φ.

5.2 Internal categoricity and incompleteness

While Proposition 19 delivers a kind of "internal completeness", it does not imply that PA_2 is deductively complete. We can define a version of PA_2 which is closely related to the formula *PA(NzS)* as follows.[33]

Definition 20. *Let PA_{int}* abbreviate the following conjunction.
$Num(0) \wedge (\forall x : Num)(\exists! y : Num)\, Succ(x) = y \wedge$
$(\forall x : Num)Succ(x) \neq z \wedge$
$(\forall x, y : Num)(Succ(x) = Succ(y) \to x = y) \wedge$
$\forall X[(X(0) \wedge (\forall x : Num)[X(x) \to X(Succ(x)]) \to (\forall x : Num)X(x)]$

PA_{int} is the theory obtained by replacing the free variables of *PA(NzS)* (see Definition 16) with the constants *Num*, 0 and *Succ*. Now Godel's incompleteness theorems imply that PA_{int} is arithmetically incomplete. In particular, PA_{int} does not prove its own consistency sentence $Con(PA_{int})$.[34]

At first sight the relationship between the acknowledged deductive incompleteness of PA_{int} and Proposition 19 is somewhat puzzling. Proposition 19 seems to be saying that for any arithmetical proposition φ either every instantiation of the axioms of PA_{int} must entail φ or every instantiation of the axioms of PA_{int} must entail $\neg\ \varphi$. Thus all instantiations of the axioms of PA_{int} must agree on the status of φ. However, from incompleteness, we know that there are certain arithmetical propositions φ for which neither φ nor $\neg\varphi$ can be deduced from PA_{int}.

The initially puzzling relationship between intolerance and incompleteness can be illuminated by considering the Henkin semantics for second order logic. Väänänen and Wang provide an informal characterization of internal categoricity in terms of Henkin models as follows: "a theory T is

[33]Button and Walsh, *Philosophy and Model Theory*: 230.
[34]Smith, *An Introduction to Gödel's Theorems*, in particular pages 186-197.

internally categorical if all [internal] models of T within a common Henkin model are witnessed to be isomorphic by the model."[35] In terms of the internal structures satisfying PA_{int}, for example, internal categoricity means that in every Henkin model any two internal arithmetical structures $(N_1 z_1 S_1)$ and $(N_2 z_2 S_2)$ satisfying *PA(NzS)* within that Henkin model are isomorphic: there exists within that Henkin model a 1-1 mapping between N_1 and N_2 which preserves arithmetical structure in the sense of Definition 17. Note in particular that in Proposition 18 (internal categoricity) the bound predicate variable X in the induction axioms of $PA(N_1 z_1 S_1)$ and $PA(N_2 z_2 S_2)$ ranges over the *same* subsets of the underlying domain of the Henkin model. Both internal categoricity and arithmetic intolerance hold in this sense on a "per Henkin model" basis.

Consequently, as Button and Walsh themselves point out, "although all internal-structures are alike within a single Henkin interpretation, they need not be alike across different Henkin interpretations."[36] As Väänänen explains, internal categoricity is called "internal" because it is internal to each Henkin model. Nothing is claimed about interpretations outside the Henkin model. PA_{int} is not categorical across Henkin models. For every fixed Henkin model there is a unique model of PA_{int}, but different Henkin models may give rise to non-isomorphic models.[37]

Returning to the relationship between incompleteness and arithmetical intolerance, deduction within second order logic proceeds across *all* Henkin models; that is, if a proposition is deduced from PA_{int}, it holds for all Henkin models of PA_{int}. Arithmetical intolerance tells us that *within* a common Henkin model, all arithmetical claims are determinate (that is, have the same truth value across internal arithmetical structures within that Henkin model). *Across* Henkin models they are not, which accords with PA_{int} being deductively incomplete.

5.3 Internal categoricity and the naturalistic constraint

Button and Walsh argue that the internal categoricity and arithmetical intolerance results can be used to show the determinacy of arithmetic, without

[35] Väänänen and Wang, "Internal categoricity in arithmetic and set theory": 122.

[36] Button and Walsh, *Philosophy and Model Theory*: 230; see also Jouko Väänänen, "Categoricity and Consistency in Second-Order Logic", *Inquiry* Vol. 58, no. 1 (2015): 26; Jouko Väänänen, "Tracing Internal Categoricity": 8.

[37] Väänänen, J, "Tracing Internal Categoricity": 8.

the problems identified in section 4.3 above.

To see that internal categoricity does not suffer from those problems, note firstly that the internal categoricity of PA_{int} is not affected by a determinacy regress. The internal categoricity result can be deduced in second-order logic, but that does not depend on any prior understanding of the set of all subsets of the natural numbers. The categoricity result depends only on the deductive system for second-order logic, which has, in addition to the full semantics, the Henkin semantics, which does not involve any commitment to a particular interpretation of the range of second-order quantifiers.

Secondly, the internal categoricity of PA_{int} does not involve a failure of the naturalistic constraint. The internal categoricity result can be deduced in second-order logic. In order to manifest a sufficient grasp of the internal categoricity result, it thus suffices to manifest a grasp of the deductive rules of second-order logic, which poses no challenges for the naturalistic constraint.

I turn now to critically examine arguments which purport to show that the internal categoricity of PA_{int} implies the determinacy of arithmetic.

6 What can internal categoricity show?

6.1 The argument from internal categoricity to determinacy

Button and Walsh argue that Internal Categoricity of PA_{int} ensures the determinacy of arithmetical practice as follows.[38] Imagine two agents Kurt

[38]This type of argument was first developed to argue that theories incorporating "open-ended schemas" ensure the determinacy of arithmetical practice, where an open-ended schema is a first-order induction schema in which substitutions for the schematic variable can include formulas in any expansion of the vocabulary of the language; see Charles Parsons, "The Uniqueness of the Natural Numbers," *Iyyun* 39 (January 1990): 35-38; Charles Parsons, *Mathematical Thought and Its Objects* (Cambridge : Cambridge University Press, 2008): 283-4. Similar types of argument have been advanced for forms of set-theoretic determinacy, for example, Shaughan Lavine, *Understanding the Infinite* (Cambridge, M.A. : Harvard University Press, 1994): 235-7. The version presented by Button and Walsh is based on Parsons' argument but in the context of second-order logic rather than open-ended schemas. In this paper I do not specifically consider the open-ended schema version of the argument although I believe it also fails to ensure arithmetical determinacy for essentially the same reasons as presented in this paper. As Warren and Waxman point out: "an open-ended understanding of induction is naturally seen as equivalent to the Π^1_1-fragment of second-order logic, and it is hard to see how a sub-theory of second-order logic could

and Michael who are each doing arithmetic (using the axioms of PA_{int}):

> Kurt has a predicate '... is a natural number', which we can symbolise as N_k, he has a name 'zero', which we can symbolise as 0_k, and he has a notion of function 'the successor of . . . ', which we can symbolise as s_k. We similarly symbolise Michael's arithmetical vocabulary with $N_m, 0_m, s_m$. Let us allow that Kurt and Michael are in communication with one another to the point that both are able to take the other's vocabulary into his own language (and both know this). Both can now prove [internal categoricity] and so, since they both have access to each other's vocabulary, both can prove
>
> $\vdash PA(N_k 0_k s_k) \wedge PA(N_m 0_m s_m) \rightarrow \exists R Iso N_{k \triangleright m}(R))$
>
> Furthermore, both can presumably see that the antecedent obtains: they affirm one of the conjuncts themselves, and their interlocutor happily affirms the other. They therefore obtain the consequent. And this guarantees that, for arithmetical purposes, their languages differ only in the subscripts we have imposed.

Button and Walsh conclude:

> Once Kurt and Michael have established the existence of their second order isomorphism, they can see that if they ever disagree (modulo subscripts) about any arithmetical sentence, then only one of them is right.[39]

Button and Walsh elaborate further that the arithmetical intolerance result (Proposition 19) shows that if the two agents were to advance contradictory arithmetical claims then

> "on pain of deductive inconsistency, both parties must hold that one of them is right and that the other is wrong".[40]

be better at securing determinacy than full second-order logic itself" ("A metasemantic challenge for mathematical determinacy": 489).

[39]Button and Walsh, "Structure and Categoricity: Determinacy of Reference and Truth Value in the Philosophy of Mathematics": 299-300, following the argument presented by Parsons, but placed in the context of second-order logic. The formulation of internal categoricity has been changed slightly for the sake of notational uniformity.

[40]Button and Walsh, *Philosophy and Model Theory*: 271.

Button provides a similar argument and concludes that all of Kurt's arithmetical structure is mirrored in Michael's numbers (and vice versa) and further (appealing to arithmetical intolerance) that their respective number concepts cannot diverge over any arithmetical claim. Button further develops this into the claim that PA_{int} articulates the number concept *precisely*.[41] In particular, Button claims that the intolerance of PA_{int} entails that every mathematical claim φ is *determinate*, where he characterizes determinacy as follows:

> If we can equally well render a claim right or wrong, just by sharpening up the concepts involved in the claim in different ways, then the claim is indeterminate (prior to any sharpening of concepts). Otherwise it is determinate.[42]

Button then claims that arithmetical intolerance entails that for all arithmetical claims φ, either it is determinate that φ or it is determinate that $\neg\varphi$. In summary he says that "thanks to its intolerance PA_{int} articulates our NATURAL NUMBER concept sufficiently precisely that every arithmetical claim is determinate." The idea here is that there is no possibility of divergent sharpenings of the NATURAL NUMBER as characterised by PA_{int} – once a commitment is made to PA_{int}, then, for every mathematical claim φ, that commitment is consistent with affirming only one of φ or $\neg\varphi$.

We can sum up the conclusion of the above argument as the *No Divergence* thesis: Kurt and Michael's commitment to the axioms and inference rules of PA_{int} entails that they cannot consistently diverge on any arithmetical claim — given their commitment to the axioms and inference rules of PA_{int}, there is only one right answer for any arithmetical claim. Call this argument from internal categoricity to No Divergence, the *No Divergence Argument*.

6.2 Analysing the No Divergence Argument

6.2.1 What the No Divergence Argument seeks to establish

Let us consider in greater depth what the No Divergence Argument is trying to establish. We have two agents, each engaged in arithmetic, using the same second order axioms as a basis for reasoning. The practice of the two agents is distinguished by subscripts. Now, since Kurt and Michael

[41] Button, "Mathematical Internal Realism": 11-12.

[42] Ibid., 12.

are using exactly the same axioms (distinguished only by subscripts) it is clear that they can only reach the same conclusions (modulo subscripts) by deduction from those axioms. If we consider an arithmetical sentence φ which is undecided by PA_{int} then *neither* agent will be able to deduce either φ or $\neg\varphi$. So, *if the arithmetical practice of the two agents is assumed to be fully captured in their explicit second-order axioms*, then it is clear, without invoking any internal categoricity results, that neither agent can diverge over any arithmetical claim in the sense just explained. Nevertheless there are arithmetical propositions φ whose truth values are undetermined by both practices.

However, the No Divergence Argument seeks to establish a stronger result. It contemplates a situation in which there is some doubt that the explicit axioms of PA_{int} *fully* capture one or the other of the agent's practices (either now or in the future). Intuitively, the concern is that each agent's practice may be *consistent* with PA_{int} but "in the background" as it were, the two agents may interpret them differently, or may come to interpret them differently, or their practices may somehow evolve, so that at some point in their practice, the two agents may adopt divergent arithmetical propositions which are compatible with PA_{int} but not entailed by it. The No Divergence Argument is designed to show that this is not possible — that the *explicit* commitment by each agent to the axioms of PA_{int} means that they cannot (consistently) adopt divergent arithmetical claims.

So what we are really interested in are two *potentially diverging* practices each (perhaps partially) characterized by the axioms of PA_{int}. The No Divergence Argument purports to rule out any divergence, to show that the practices cannot consistently diverge. Lavine describes the problem situation well (referring here to the issue of set-theoretic determinacy rather than arithmetical determinacy):

> I use the words, say, set and member and you use the words set and member too. Our usage is similar in many respects, and in particular let us assume that we employ the same axioms. Isn't there a legitimate worry about whether we are using the words in the same way? It is, after all, possible that you could come to believe the continuum hypothesis, while I came to believe its negation. (I am assuming, of course, that the continuum hypothesis is independent of the axioms we agree on, as will almost certainly be the case given the present state of knowledge about

sets.) If that were to happen, it would then be clear that we were not using the words set and member in the same way. But then here is the legitimate worry: is our present usage sufficiently determinate that our present agreement guarantees future agreement? So the question we are raising is whether there is more than one way to use set and member compatible with present commitments.[43]

6.2.2 What the No Divergence Argument may assume

It is important to understand that the only premise which the No Divergence Argument is entitled to assume is that the two agents are behaving in accordance with the deductive rules of second-order arithmetic. To see this, recall that the naturalistic constraint requires that determinacy must be understood as arising through characteristics of publicly articulable and communicable practices. Recall also that the naturalistic constraint was a prime motivation for rejecting the external categoricity of second-order arithmetic (under the full semantics) as a guarantor of determinacy. As noted above, nothing in our explicitly (publicly articulable and communicable) practice of second-order logic manifests the concept of the full powerset concept, on which the full semantics depends. Our explicit practice of second-order logic is captured by the deductive system for second-order logic. This motivated the move by Button and Walsh to base determinacy on internal categoricity, which relies only on the explicit deductive rules of second-order applied to the axioms of PA_{int}. Accordingly it is only on those rules that the No Divergence Argument can rely.

6.2.3 The No Divergence Argument requires an additional premise

The No Divergence Argument assumes that the two agents are behaving in accordance with the deductive rules of second-order arithmetic and seeks to establish that they cannot consistently diverge on any arithmetical claim.

Now, there is prima facie reason to doubt that the No Divergence Argument can succeed. After all, if we consider any arithmetical claim φ which is not decided by PA_{int} and *all we know is that the two agents are behaving in conformity with the deductive rules of second-order arithmetic*, then it seems

[43]Lavine, *Understanding the Infinite*: 236.

clear that they *can* consistently diverge: one can adopt φ and the other $\neg\varphi$, (suitably subscripted) without any contradiction.

I claim that this prima facie impression is correct and that the No Divergence Argument can only succeed with an additional premise, beyond the assumption that the two agents are acting in conformity with second-order arithmetic. The additional premise may be understood in a number of ways. Essentially it is an assumption that the two agents' practices are tied together in certain way.

We are interested in two (potentially diverging) arithmetical practices, each (perhaps partially) represented by a second-order theory. The No Divergence Argument assumes that two such potentially diverging arithmetical practices can be represented by two "internal" copies of the respective theories. The situation may be illustrated informally as follows. The two (potentially diverging) practices of Kurt and Michael correspond to (are partially characterised by) two deductive systems:

$$PA(N_k 0_k \mathrm{s}_k) \vdash \qquad\qquad PA(N_m 0_m \mathrm{s}_m) \vdash$$

The No Divergence Argument represents the two practices by a single deductive system:

$$PA(N_k 0_k \mathrm{s}_k) \wedge PA(N_m 0_m s_m) \vdash$$

Call this representation of the two practices the *Common System Representation* and the assumption that it is a faithful representation of the two practices the *Common System Assumption*. The Common System Representation introduces a constraint on the deductive possibilities of the two practices which does not follow from the mere assumption that the two agents behave in conformity with the deductive rules of second-order arithmetic. We can see this with reference to the Henkin semantics as follows. In considering the possible deductive consequences of the two "internal" copies of the theories, the Common System Representation restricts attention to deductive consequences of those theories in *shared* Henkin models. However, if *all we know is that the two agents are behaving in conformity with the deductive rules of second-order arithmetic* then the deductive practices of the two agents may range (independently) over *all* Henkin models and not be tied together in this way.

To further understand the implications of the Common System Assumption, observe that each of the following follows from confining attention to shared Henkin models:

1. each of the agents' arithmetical models is drawn from a common Henkin model;
2. each agent's second-order variables have the same range (drawn from a common Henkin model); and
3. each agent is capable of referring to the same predicates and relations as the other (via their second-order variables and using Comprehension).

As is the case for the Common System Assumption itself, none of these is implied by the mere fact that the two agents act in accordance with the deductive rules of second-order arithmetic.[44]

It follows that No Divergence does not follow from the mere commitment of the agents to deductive second-order arithmetic, as the No Divergence Argument claims. It requires an additional premise, which constrains the deductive consequences of the two practices.

Note that the mere fact that both agents can prove the same internal categoricity result does not preclude divergence. If Kurt and Michael were to come to "intend" incompatible classes of Henkin models of PA_{int}, with Kurt affirming, say, φ and Michael affirming $\neg\varphi$ (suitably subscripted), then each of them can prove an identical internal categoricity theorem. That theorem would show that all internal arithmetical structures within each of their own Henkin models are isomorphic. However, the agents would not have any Henkin models in common and neither would be able to refer to the internal arithmetical structures of the other agent (φ holds in all of Kurt's internal arithmetical structures, and $\neg\varphi$ in all of Michael's).[45]

[44]This diagnosis of the No Divergence Argument is related to Hartry Field's criticism of Parsons' version of the No Divergence Argument, see Hartry Field, *Truth and the Absence of Fact* (Oxford: Clarendon Press, 2001): 358-360. See also Parsons' discussion of Field's argument in Parsons, *Mathematical Thought and Its Objects*: 284. In essence, Field denies the assumption that one of the agents, say Kurt, can refer (through the predicates and relations of his language) to Michael's natural numbers (N_m) and consequently denies that induction on predicates defined in Kurt's language can be used to prove the existence of an isomorphism between N_m and N_k. From the point of view advanced here, this amounts to denying the assumption that Kurt's arithmetical practice can properly be represented by an "internal copy" of Kurt's theory within Michael's theory, that is, denying the Common System Assumption.

[45]More precisiely, since Kurt's and Michael's theories have different vocabularies, due to the subscripts, their Henkin models are actually models of different vocabularies. In the situation envisaged, the crucial point is that their Henkin models have no consistent

6.2.4 The Common System Assumption begs the question

The original concern, which the No Divergence Argument was designed to address, was that the two agents, despite both adhering to PA_{int}, may somehow "intend" (or come to intend) different ranges of quantification for their predicates and relations. If they did, then, compatibly with PA_{int}, they might be led to adopt divergent arithmetical claims. The No Divergence Argument purports to show that this is impossible, but by relying on the Common System Assumption, attention is restricted to circumstances in which the two agents have the same range of quantification for their relations and predicates (drawn from a common Henkin model). The result is that the No Divergence Argument effectively assumes the original problem away because attention is restricted to "internal" models which have the same range of quantification for relations and predicates.

To be clear, this does not detract from the mathematical significance of internal categoricity. That theorem shows that the axioms of second-order arithmetic are sufficiently strong (assuming Comprehension) to ensure that all arithmetical structures within a given Henkin model are isomorphic. That is a significant mathematical property but it is not sufficient to show No Divergence.

6.2.5 Can the Common System Assumption be justified?

The naturalistic constraint requires that determinacy be understood as arising through characteristics of publicly articulable and communicable practices. In Button and Walsh's approach, the publicly articulable and communicable practices of the two agents are defined by the conformity of the practices with the deductive rules of second-order arithmetic. And, as we have seen above, the Common System Assumption does not follow from that conformity. Thus the Common System Assumption cannot be justified in accordance with the naturalistic constraint as that constraint is construed in Button and Walsh's approach.

It might be pressed that, if the two agents are behaving in conformity with PA_{int}, then it is reasonable to assume that they intend the same ranges for their predicates and relations, at least until something they say or do

common expansion to a model of the union of the vocabularies. See Jouko Väänänen, "Second-Order Logic or Set Theory?" *The Bulletin of Symbolic Logic* Vol. 18, No. 1 (2012): 98-99.

indicates otherwise. I agree. However the No Divergence Argument seeks to prove something much stronger: that the two agents' conformity with PA_{int} *precludes* that they can consistently diverge, that is, that if they do so, then "on pain of contradiction" one of them is right and the other wrong. The Common System Assumption *builds in* the assumption that their predicates and relations have the same range. It is not simply a defeasible presumption.

6.3 Objections and responses

It might be said that Kurt and Michael's practices can only diverge if the axioms of second-order arithmetic and the deductive rules of second-order logic are given the Henkin semantics. If instead they are given the full semantics then their practices cannot diverge. This is indeed true. But of course, the whole point of the internal categoricity approach to determinacy is to avoid reliance on the full semantics. According to the internal categoricity approach, divergence is supposed to be ruled out based only on the explicit rules of second-order logic (applied to the axioms of PA_{int}), that is, proof-theoretically. It is the Henkin semantics which faithfully reflects the proof theory (due to the completeness result for second-order logic with respect to the Henkin semantics).

Button and Walsh, in the context of responding to Field's criticisms, concede certain limitations as to what internal categoricity can show but dismiss these limitations as irrelevant to the "internalist" use of internal categoricity:

> Suppose that a model theoretical sceptic has suggested that Michael and Kurt are discussing nonisomorphic Henkin models of second-order arithmetic. We cannot answer that sceptic by pointing out that Kurt and Michael have produced literally the same proof [of internal categoricity], line by line. For, if the sceptical scenario obtained, then that same proof would mean different things in their respective mouths, for it would concern nonisomorphic models. This point is worth emphasising. But it is also worth emphasising that it is irrelevant to our internalist's imagined use of internal categoricity. We stressed ... that internal categoricity results cannot be used to pin down [semantic models], or to rule out Henkin models, or whatever.[46]

[46] Button and Walsh. *Philosophy and Model Theory*: 242.

This response is not sufficient to address the argument presented here. That argument is that internal categoricity can only guarantee No Divergence if the Common System Assumption is made. While the effect of the Common System Assumption can profitably be understood in terms of Henkin models, it can also be understood purely proof-theoretically – the Common System Assumption constrains the deductive possibilities of the two practices in a way which does not follow from their conformity with deductive second-order arithmetic. This is a proof theoretic fact about the effect of the Common System Assumption: there is no necessary reference to semantic models. Thus, even if Button and Walsh are uninterested in semantic models, they cannot dismiss the fact that the Common System Assumption is required to show No Divergence, and that this assumption does not follow merely from conformity of arithmetical practice with deductive second-order arithmetic. To say that the issue is irrelevant to the proposed internalist use of internal categoricity amounts simply to a reassertion of the Common System Assumption, without argument for that assumption.

Tim Button concedes that he cannot prove that Michael and Kurt "share a logical language" and, moreover that if they do not share a logical language, then in principle Michael might affirm φ and Kurt might affirm $\neg\varphi$, and each could be "right in their own languages." However he goes on to discount this possibility:

> Having raised this abstract possibility, though, I should immediately point out that it is hard to see how it could actually come about. Indeed, it is not obvious that this abstract possibility is even *intelligible* to internalists. After all, the logical language in question is to be understood *deductively* rather than *semantically*, and we can take it for granted that [Michael and Kurt] accept exactly the same rules of inference. But, given this, it is hard to see what it could even *mean*, to say that they *do not* share a logical language."[47]

There is an ambiguity here in the idea of "sharing a logical language." In one sense, which Button appears to invoke, it could mean simply accepting the same rules of inference (for example those of second-order logic applied to PA_{int}). But we have seen above that this is perfectly compatible with the indeterminacy of some arithmetic propositions, and indeed is compatible

[47] Button, "Mathematical Internal Realism": 20.

with potential divergence (one agent affirming φ and the other affirming $\neg\varphi$) which can happen if the practice over time of the two agents, while conforming to second-order logic applied to PA_{int} is not exhaustively characterised by second-order logic applied to PA_{int}. This first sense of "sharing a logical language" does not rule out arithmetical indeterminacy. To secure determinacy Button must take "sharing a logical language" to mean something strictly stronger – invoking the Common System Assumption. However, we have seen that this additional assumption does not follow from the fact that the agents accept the same rules of inference.

Button and Walsh also concede that internal categoricity cannot "guarantee that Kurt might not one day do something which makes Michael do a double-take, and exclaim 'but then your induction axiom was restricted after all, for you have rejected this instance of induction!'" (This alludes to the situation in which the range of the agents' second-order variables may differ). However they go on to claim that this "has nothing much to do with the induction axiom" or specifically to do with mathematics or logic, taking it to be an instance of rule-following scepticism:

> Kurt and Michael might have both used the word 'green' to apply to similar things for a very long time, until one day one of them starts using the word 'green' where the other uses 'red', causing Michael to exclaim 'but then you did not mean greenness by "green" after all!' No theorem can block these sorts of concerns, which are particular instances of a much more general worry: scepticism about meaning, or rule-following scepticism.[48]

Again, however, this response misses the point in relation to the argument presented here. Far from depending on rule-following scepticism, the argument that internal categoricity does not guarantee arithmetical determinacy presented here assumes that the relevant arithmetical practices fully conform with the deductive system of second-order logic applied to the axioms of PA_{int}, and assumes that such conformity is a fully determinate matter. The argument shows that such determinate conformity with the axioms of PA_{int} and second-order logic is insufficient on its own to guarantee arithmetical determinacy.

[48] Button and Walsh. *Philosophy and Model Theory*: 242.

7 Conclusion

I conclude that internal categoricity of second-order arithmetic does not entail the determinacy of arithmetic in a way which satisfies the naturalistic constraint, as that constraint is construed by Button and Walsh. To satisfy the naturalistic constraint, Button and Walsh rely on characterising arithmetical practice by conformity with deduction in second-order arithmetic. However, their argument that arithmetical determinacy follows from internal categoricity depends on an additional assumption (the Common System Assumption) which does not itself follow from that characterisation of arithmetical practice.[49] Of course, I have not set out to show that arithmetical practice *is* indeterminate. Other arguments may be available to show that arithmetical practice is determinate, perhaps even arguments compatible with the naturalistic constraint.

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