Stochastic Operator Variance: an observable to diagnose noise and scrambling Supplementary Material

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We here provide more details on the evolution of an operator driven by a stochastic Hamiltonian and the equation of motion for the stochastic operator variance (c.f. Sec. I), dissipative out-of-time-order correlators (c.f. Sec. II), the bipartite interpretation of matrix products (c.f. Sec. III), the quantum (c.f. Sec. IV) and classical (c.f. Sec. V) stochastic Lipkin-Meshkov-Glick models, and the corresponding numerical computations (c.f. Sec. VI).

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I. STOCHASTIC EVOLUTION: DETAILS ON THE MAIN DERIVATION

A. On the Heisenberg picture of an explicitly time-dependent Hamiltonian

We consider an explicitly time-dependent Hamiltonian $\hat{H}_t$, which in general does not commute with itself at different times, $[\hat{H}_t, \hat{H}_{t'}] \neq 0$. It governs the dynamics of a state through the Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t} |\psi_t\rangle = -i\hat{H}_t |\psi_t\rangle , \tag{S1}$$

whose solution $|\psi_t\rangle = \hat{U}_t |\psi_0\rangle = \mathcal{T}_{\leftarrow} e^{-i\int_0^t \hat{H}_\tau \mathrm{d}\tau} |\psi_0\rangle$, involves the propagator from time 0 to t, $\hat{U}_t$, where $\mathcal{T}_{\leftarrow}$ is the chronological time-ordering operator [1]. An (explicitly time-independent) operator $\hat{A}$ in the Heisenberg picture then evolves as

$$\hat{A}_t^{(\mathrm{H})} = \hat{U}_t^\dagger \hat{A} \hat{U}_t. \tag{S2}$$

Differentiating this evolution gives Heisenberg's equation for the evolution of $\hat{A}_t$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{A}_t^{(\mathrm{H})} = \dot{\hat{U}}_t^\dagger \hat{A}\hat{U}_t + \hat{U}_t^\dagger \hat{A}\dot{\hat{U}}_t = i\hat{U}_t^\dagger \hat{H}_t \hat{A}\hat{U}_t - i\hat{U}_t^\dagger \hat{A}\hat{H}_t\hat{U}_t = i[\hat{H}_t^{(\mathrm{H})}, \hat{A}_t^{(\mathrm{H})}], \tag{S3}$$

where $\hat{H}_t^{(\mathrm{H})} = \hat{U}_t^\dagger \hat{H}_t \hat{U}_t \neq \hat{H}_t$ is the Hamiltonian in the Heisenberg picture with respect to itself, which is not equal to $\hat{H}_t$ since it does not commute with itself at the different times $t' < t$ included in $\hat{U}_t$. Note that the Heisenberg equation (S3) is recovered when we introduce the Hamiltonian in the Heisenberg picture with respect to itself $\hat{H}_t^{(\mathrm{H})}$, but that its solution (S2) only requires the bare Hamiltonian $\hat{H}_t$ in the propagator $\hat{U}_t$.

We bring attention to the fact that Heisenberg operators evolve *"backwards in time"*. This can be illustrated by splitting the evolution over two different times $0 \leq t_1 \leq t_2$. The expectation value of any general operator $\hat{A}$ then reads

$$\langle\psi|\mathcal{T}_{\rightarrow} e^{+i\int_0^{t_1} \hat{H}_\tau \mathrm{d}\tau} \mathcal{T}_{\rightarrow} e^{+i\int_{t_1}^{t_2} \hat{H}_\tau \mathrm{d}\tau} \hat{A} \mathcal{T}_{\leftarrow} e^{-i\int_{t_1}^{t_2} \hat{H}_\tau \mathrm{d}\tau} \mathcal{T}_{\leftarrow} e^{-i\int_0^{t_1} \hat{H}_\tau \mathrm{d}\tau} |\psi\rangle\,, \tag{S4}$$

where $\mathcal{T}_{\rightarrow}$ denotes anti-chronological time-ordering. While states evolve forwards in time, $0 \rightarrow t_1 \rightarrow t_2$, the equivalent evolution at the level of operators is for them to evolve backwards in time, $t_2 \rightarrow t_1 \rightarrow 0$. Of course this is just a result of the Heisenberg representation and does not correspond to a backwards time evolution.

B. Heisenberg evolution with a stochastic Hamiltonian

Adjoint stochastic master equations (SMEs) have been studied in [2, 3] for continuous homodyne detection and read

$$\mathrm{d}\hat{A}_t = i[\hat{H}, \hat{A}]\mathrm{d}t + \sqrt{\eta}(\hat{c}^\dagger \hat{A}_t + \hat{A}_t \hat{c})\mathrm{d}Y_{t-\mathrm{d}t} + \sum_m \left(\hat{L}_m \hat{A}_t \hat{L}_m - \frac{1}{2}\{\hat{L}_m^\dagger \hat{L}_m, \hat{A}_t\}\right)\mathrm{d}t, \tag{S5}$$

where we use the backwards differential $\mathrm{d}\hat{A}_t \equiv \hat{A}_{t-\mathrm{d}t} - \hat{A}_t$. η is the efficiency of the detector, $\hat{c}$ is the measurement observable, $\hat{L}_m$ are the Lindblad operators describing coupling to a general bath, and $\mathrm{d}Y_t$ is the measurement record. $\mathrm{d}Y_t$ is a stochastic process. Note that this equation does not preserve trace. When $\mathrm{d}Y_t$ represents white noise without drift, i.e. $\langle \mathrm{d}Y_t\rangle = 0$, the average over the noise recovers the standard adjoint master equation for the average operator $\langle \hat{A}_t\rangle$.

We draw attention to the fact that the measurement record appears as $\mathrm{d}Y_{t-\mathrm{d}t}$. To apply Itō's rules, we need to evaluate the noise at the beginning of the time interval. In the Schrödinger picture, this corresponds to evaluating the noise at time t in the interval $[t, t+\mathrm{d}t]$. But since we work in the Heisenberg picture we propagate the operator in the interval $[t-\mathrm{d}t, t]$ and we have to evaluate the noise at $t - \mathrm{d}t$. The propagator over dt thus reads

$$\hat{U}_{dt} = e^{-i\hat{H}\mathrm{d}t - i\sqrt{2\gamma}\hat{L}\mathrm{d}W_{t-\mathrm{d}t}}, \tag{S6}$$

which gives the SME (3) in the main text. Note that this equation is trace preserving. So although in the case of driftless white noise $\langle \mathrm{d}Y_t\rangle = 0$, the SME's (3) and (S5) give the same results at the average level, they describe different dynamics at the level of single trajectories.

C. Finding the equation of motion for the SOV

The equation of motion for the second stochastic moment simply reads $\frac{d}{dt}\langle \hat{A}_t^2\rangle = \mathcal{L}^\dagger[\langle \hat{A}_t^2\rangle]$, and that for the first moment squared reads

$$\frac{d}{dt}\langle \hat{A}_t\rangle^2 = \mathcal{L}^\dagger[\langle \hat{A}_t\rangle]\langle \hat{A}_t\rangle + \langle \hat{A}_t\rangle \mathcal{L}^\dagger[\langle \hat{A}_t\rangle]. \tag{S7}$$

In the latter, we want to introduce the term $\mathcal{L}^\dagger[\langle \hat{A}_t\rangle^2]$ in order to recover $\mathcal{L}^\dagger[\Delta \hat{A}_t^2]$ through linearity of the Lindbladian superoperator. The difference reads

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{A}_t\rangle^2 - \mathcal{L}^\dagger[\langle \hat{A}_t\rangle^2] &= -\gamma\left(\left\{[\hat{L}, [\hat{L}, \langle \hat{A}_t\rangle]], \langle \hat{A}_t\rangle\right\} - [\hat{L}, [\hat{L}, \langle \hat{A}_t\rangle^2]]\right) \\
&= -2\gamma\left(\hat{L}\langle \hat{A}_t\rangle^2 \hat{L} + \langle \hat{A}_t\rangle \hat{L}^2 \langle \hat{A}_t\rangle - \hat{L}\langle \hat{A}_t\rangle \hat{L}\langle \hat{A}_t\rangle - \langle \hat{A}_t\rangle \hat{L}\langle \hat{A}_t\rangle \hat{L}\right) \\
&= -2\gamma[\hat{L}, \langle \hat{A}_t\rangle]^2,
\end{aligned} \tag{S8}$$

which yields the equation of motion (4) for the SOV given in the main text

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta\hat{A}_t^2 = \mathcal{L}^\dagger[\Delta\hat{A}_t^2] - 2\gamma[\hat{L},\langle\hat{A}_t\rangle]^2. \tag{S9}$$

D. The SOV uncertainty principle

The theory of positive matrices (see pg. 75 [4]) tells us that given a completely positive unital map, like $e^{\mathcal{L}^\dagger t}[\bullet]$, for all operators $\hat{A},\ \hat{B}$, the following matrix is positive semidefinite

$$\begin{pmatrix} \Delta\hat{A}_t^2 & \Delta\widehat{AB}_t \\ \Delta\widehat{AB}_t^\dagger & \Delta\hat{B}_t^2 \end{pmatrix} \geq 0, \tag{S10}$$

i.e. the $2N \times 2N$ matrix has to be positive semidefinite. While this statement is powerful, it does not give any straightforward inequality. For this reason, we take the expectation value of each of the blocks over a certain state ρ, denoted $\langle\bullet\rangle_\rho = \mathrm{Tr}(\bullet\rho)$. This allows recasting the inequality for a 2x2 matrix,

$$\begin{pmatrix} \langle\Delta\hat{A}_t^2\rangle_\rho & \langle\Delta\widehat{AB}_t\rangle_\rho \\ \langle\Delta\widehat{AB}_t^\dagger\rangle_\rho & \langle\Delta\hat{B}_t^2\rangle_\rho \end{pmatrix} \geq 0. \tag{S11}$$

Let us now apply Sylvester's criterion: a matrix is positive semi-definite if and only if all its principal minors, i.e. the determinants of the matrix in which we delete the columns and rows with the same index, are non-negative. Doing so, we can recast the complete positivity condition into

$$\langle\Delta\hat{A}_t^2\rangle_\rho \geq 0, \quad \langle\Delta\hat{B}_t^2\rangle_\rho \geq 0, \quad \langle\Delta\hat{A}_t^2\rangle_\rho\langle\Delta\hat{B}_t^2\rangle_\rho - |\langle\Delta\widehat{AB}_t\rangle_\rho|^2 \geq 0. \tag{S12}$$

The two first inequalities above are fulfilled because the SOV is positive semidefinite, the last condition yields an analog of the Schwarz inequality

$$\langle\Delta\hat{A}_t^2\rangle_\rho\langle\Delta\hat{B}_t^2\rangle_\rho \geq |\langle\Delta\widehat{AB}_t\rangle_\rho|^2. \tag{S13}$$

Let us now expand the r.h.s. of the inequality. Writing $\hat{A}\hat{B} = \frac{1}{2}(\{\hat{A},\hat{B}\} + [\hat{A},\hat{B}])$ and $\hat{B}\hat{A} = \frac{1}{2}(\{\hat{A},\hat{B}\} - [\hat{A},\hat{B}])$, we obtain

$$\begin{aligned}
|\langle\Delta\widehat{AB}_t\rangle_\rho|^2 &= (\langle e^{\mathcal{L}^\dagger t}[\hat{A}\hat{B}]\rangle_\rho - \langle\hat{A}_t\hat{B}_t\rangle_\rho)(\langle e^{\mathcal{L}^\dagger t}[\hat{B}\hat{A}]\rangle_\rho - \langle\hat{B}_t\hat{A}_t\rangle_\rho) \\
&= \frac{1}{4}\Big(\langle e^{\mathcal{L}^\dagger t}[\{\hat{A},\hat{B}\}]\rangle_\rho^2 - \langle e^{\mathcal{L}^\dagger t}[[\hat{A},\hat{B}]]\rangle_\rho^2 - 2\langle e^{\mathcal{L}^\dagger t}[\{\hat{A},\hat{B}\}]\rangle_\rho\langle\{\hat{A}_t,\hat{B}_t\}\rangle_\rho + 2\langle e^{\mathcal{L}^\dagger t}[[\hat{A},\hat{B}]]\rangle_\rho\langle[\hat{A}_t,\hat{B}_t]\rangle_\rho \\
&\quad + \langle\{\hat{A}_t,\hat{B}_t\}\rangle_\rho - \langle[\hat{A}_t,\hat{B}_t]\rangle_\rho^2\Big) \\
&= \frac{1}{4}\left(\langle e^{\mathcal{L}^\dagger t}[\{\hat{A},\hat{B}\}]\rangle_\rho - \langle\{\hat{A}_t,\hat{B}_t\}\rangle_\rho\right)^2 - \frac{1}{4}\left(\langle e^{\mathcal{L}^\dagger t}[[\hat{A},\hat{B}]]\rangle_\rho - \langle[\hat{A}_t,\hat{B}_t]\rangle_\rho\right)^2 \\
&= \frac{1}{4}\left(D_+^2(\hat{A},\hat{B}) - D_-^2(\hat{A},\hat{B})\right),
\end{aligned}$$

In the last line, we introduced $D_\eta(\hat{A},\hat{B}) = \langle e^{\mathcal{L}^\dagger t}([\hat{A},\hat{B}]_\eta)\rangle_\rho - \langle[\hat{A}_t,\hat{B}_t]_\eta\rangle_\rho$, where $[\hat{X},\hat{Y}]_\eta = \hat{X}\hat{Y} + \eta\hat{Y}\hat{X}$ with $\eta = \pm 1$ is the generalized commutator. Therefore the SOV uncertainty principle reads

$$\mathrm{Tr}(\Delta\hat{A}_t^2\rho)\mathrm{Tr}(\Delta\hat{B}_t^2\rho) \geq \frac{1}{4}\left(D_+^2(\hat{A},\hat{B}) - D_-^2(\hat{A},\hat{B})\right). \tag{S14}$$

II. DISSIPATIVE OUT-OF-TIME-ORDER CORRELATORS

A. Short-time expansion of OTOC

It is insightful to consider the short-time expansion of the OTOC

$$\mathrm{d}_t\mathrm{Tr}(\Delta\hat{A}_t^2) = -2\gamma\mathrm{Tr}\left([\hat{L},\hat{A} + t\mathcal{L}^\dagger[\hat{A}] + \mathcal{O}(t^2)]^2\right) = -2\gamma\mathrm{Tr}\left([\hat{L},\hat{A}]^2 + 2t[\hat{L},\hat{A}][\hat{L},\mathcal{L}^\dagger[\hat{A}]]\right) + \mathcal{O}(t^2). \tag{S15}$$

The linear term in t has an oscillatory contribution $-4i\gamma t\mathrm{Tr}([\hat{L},\hat{A}][\hat{L},[\hat{H}_0,\hat{A}]])$ and a dissipative contribution

$$-C_0\frac{t}{\tau_D}=\frac{2\gamma}{N}t\mathrm{Tr}([\hat{L},\hat{A}][\hat{L},[\hat{L},[\hat{L},\hat{A}]]])=-\frac{2\gamma}{N}t\mathrm{Tr}([\hat{L},[\hat{L},\hat{A}]]^2), \tag{S16}$$

where we used the cyclic property of the trace to write $\mathrm{Tr}(\hat{X}[\hat{Y},\hat{Z}])=\mathrm{Tr}([\hat{X},\hat{Y}]\hat{Z})$ with $\hat{X}=[\hat{L},\hat{A}]$, $\hat{Y}=\hat{L}$, and $\hat{Z}=[\hat{L},[\hat{L},\hat{A}]]$. Then the inverse dissipation time, $\tau_D^{-1}\propto 2\gamma\mathrm{Tr}(\mathcal{D}[\hat{A}]^2)=4(\mathcal{D}[\hat{A}],\mathcal{D}[\hat{A}])\geq 0$ is related to the Hilbert-Schmidt norm of the dissipator $\mathcal{D}[\bullet]=-[\hat{L},[\hat{L},\bullet]]$ acting on the initial operator $\hat{A}$. The short-time behavior is then determined by the exponential

$$C_t\approx C_0e^{-t/\tau_D}. \tag{S17}$$

This approximation is compared to the full OTOC in Fig. 1, where the crossover between the different exponential regimes of the dissipative OTOC is apparent.

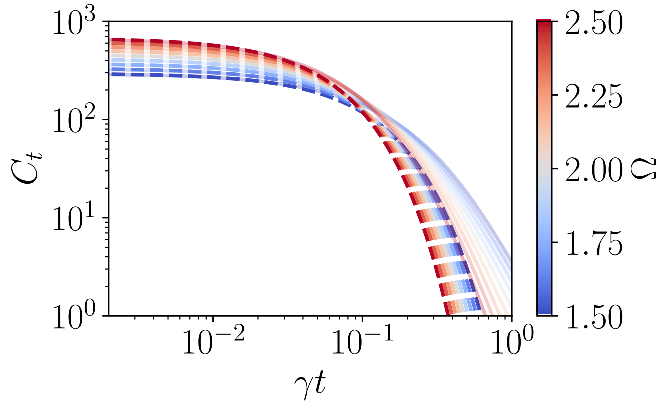

FIG. 1. **Short-time behavior of the OTOC as computed from the SOV-OTOC relation**. Full OTOC (solid line) and short-time expansion (S17) (dashed line). The crossover between the two exponentially decaying regimes is seen around $\gamma t\approx 0.1$.

B. The dissipative OTOC for $[\hat{H}_0,\hat{L}]=0$

In the simple case that $[\hat{H}_0,\hat{L}]=0$, the operators $\hat{H}_0$ and $\hat{L}$ share a common eigenbasis, such that $\hat{H}_0=\sum_n E_n|n\rangle\langle n|$ and $\hat{L}=\sum_n l_n|n\rangle\langle n|$. The solution of the adjoint master equation then reads

$$\hat{A}_t=\sum_{m,n}A_{mn}e^{i(E_m-E_n)t-\gamma(l_m-l_n)^2t}|m\rangle\langle n|. \tag{S18}$$

The behavior of the dissipative OTOC then is simply

$$\mathrm{Tr}([\hat{L},\hat{A}_t]^2)=\sum_{m,k}(l_m-l_k)^2|A_{km}(t)|^2=\sum_{m,k}(l_m-l_k)^2e^{-2\gamma(l_m-l_k)^2t}|A_{km}|^2. \tag{S19}$$

This shows that the short-time dynamics is governed by the two eigenvalues with largest difference $\max_{m,k}(l_m-l_k)^2$ over which the operator has support, i.e. $A_{mk}\neq 0$, and the long-time dynamics is governed by the smallest possible non-zero difference $\min_{m,k}(l_m-l_k)\neq 0$ over which the initial operator has support.

III. PRODUCT OF OPERATORS AS A DOUBLED HILBERT SPACE OPERATION

The standard product between operators in a Hilbert space $\mathscr{H}$ with closure relation $\hat{\mathbb{1}}_{\mathscr{H}}=\sum_n|n\rangle\langle n|$ reads

$$\hat{X}\hat{Y}=\sum_{n,m,k}\langle n|\hat{X}|k\rangle\langle k|\hat{Y}|m\rangle|n\rangle\langle m|. \tag{S20}$$

In turn, the tensor product of the operators over a doubled Hilbert space $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ with closure relation $\mathbb{1} = \mathbb{1}_{\mathscr{H}_1} \otimes \mathbb{1}_{\mathscr{H}_2} = \sum_{n_1,n_2} |n_1, n_2\rangle \langle n_1, n_2|$ where $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$ reads

$$\hat{X} \otimes \hat{Y} = \sum_{n_1,m_1,n_2,m_2} \langle n_1|\hat{X}|m_1\rangle \langle n_2|\hat{Y}|m_2\rangle |n_1, n_2\rangle \langle m_1, m_2| \,. \tag{S21}$$

Thus, applying the swap operator, defined from $\mathbb{S}\,|n_1\rangle \otimes |n_2\rangle = |n_2\rangle \otimes |n_1\rangle$, yields

$$(\hat{X} \otimes \hat{Y})\mathbb{S} = \sum_{n_1,m_1,n_2,m_2} \langle n_1|\hat{X}|m_1\rangle \langle n_2|\hat{Y}|m_2\rangle |n_1, n_2\rangle \langle m_2, m_1| \,. \tag{S22}$$

The partial trace over the second Hilbert space $\mathrm{Tr}_{\mathscr{H}_2}(\bullet) = \sum_{n_2} \langle n_2| \bullet |n_2\rangle$ gives

$$\mathrm{Tr}_{\mathscr{H}_2}\Big((\hat{X} \otimes \hat{Y})\mathbb{S}\Big) = \sum_{n_1,m_1,m_2} \langle n_1|\hat{X}|m_1\rangle \langle m_1|\hat{Y}|m_2\rangle |n_1\rangle \langle m_2| = \hat{X}\hat{Y}.$$

IV. QUANTUM STOCHASTIC LIPKIN MESHKOV GLICK MODEL

A. Transport in quantum sLMG

In the main text, we presented the eigenvalues of the SOV and their transport analog in the sLMG for $[\hat{H}_0, \hat{L}] = 0$, see Fig. 2(a). The case in which $[\hat{H}_0, \hat{L}] \neq 0$ shows a richer phenomenology, even for the simple sLMG model. The eigenvalues of the SOV are shown in Fig. 2. The smallest eigenvalue first evolves ballistically, $\Lambda_0(t) \sim t^2$, then turns into superdiffusive $\Lambda_0(t) \sim t^{3/2}$ before changing further to a slower growth t^α with $\alpha < 1$, to ballistic again $\Lambda_0(t) \sim t^2$ and then saturates. Another interesting feature is that at long times all eigenvalues saturate to the same value, a feature not present in the case $[\hat{H}_0, \hat{L}] = 0$.

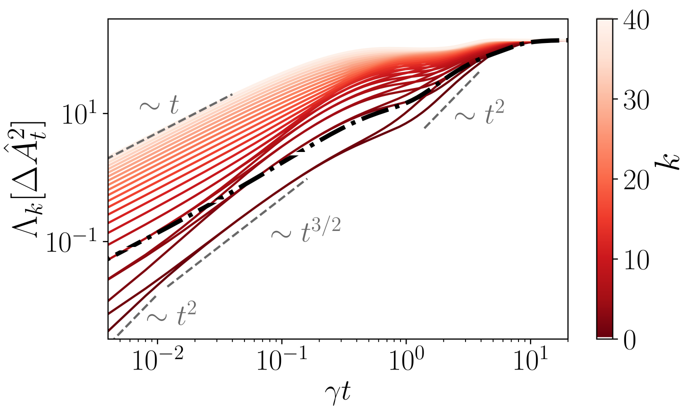

FIG. 2. Eigenvalues of the SOV as a function of time for noncommuting Hamiltonian and Jump operator $[\hat{H}_0, \hat{L}] \neq 0$ (red, colorscale shows order of the eigenvalues k) and the expectation value over the minimum SOV state $\langle \Psi|\Delta \hat{A}_t^2|\Psi\rangle$ (black dash-dotted). The parameters are $\gamma = 2, \Omega = 1, S = 20, \hat{A} = (\hat{S}_x + \hat{S}_y + \hat{S}_z)/\sqrt{3}$ and $\hat{L} = \hat{S}_x$.

B. Visualizing the quantum stochastic operator variance

The SOV $\Delta \hat{A}_t^2$ that we introduced is an operator over the Hilbert space. We illustrate this quantity for the case of the quantum sLMG model. In general, this operator is d dimensional, and it has non-zero components over all the elements of the considered operator basis. However, since the sLMG is a mean-field description, we are mainly interested in its projection over the subspace spanned by $\{\mathbb{1}, \hat{S}_x, \hat{S}_y, \hat{S}_z\}$. We would like to have these elements be orthonormal with each other. For this reason, we choose the Hilbert-Schmidt inner product

$$(\hat{A}, \hat{B}) = \frac{1}{S(S+1)(2S+1)/3} \mathrm{Tr}(\hat{A}^\dagger \hat{B}), \tag{S23}$$

where the normalization has been chosen such that $(\hat{S}_i, \hat{S}_j) = \delta_{ij}$ and comes from $\mathrm{Tr}(\hat{S}_z^2) = \sum_{j=0}^{2S}(S-j)^2 = S(S+1)(2S+1)/3$. Note that this normalization leads to $(\mathbb{1}, \mathbb{1}) = \frac{3}{S(S+1)} \neq 1$.

We further introduce a notion of Stochastic Operator Standard Deviation (SOSD) as the matrix square root of the SOV, $\Delta\hat{A}_t = \sqrt{\Delta\hat{A}_t^2}$. The SOV and its deviation are thus illustrated in Fig. 3.

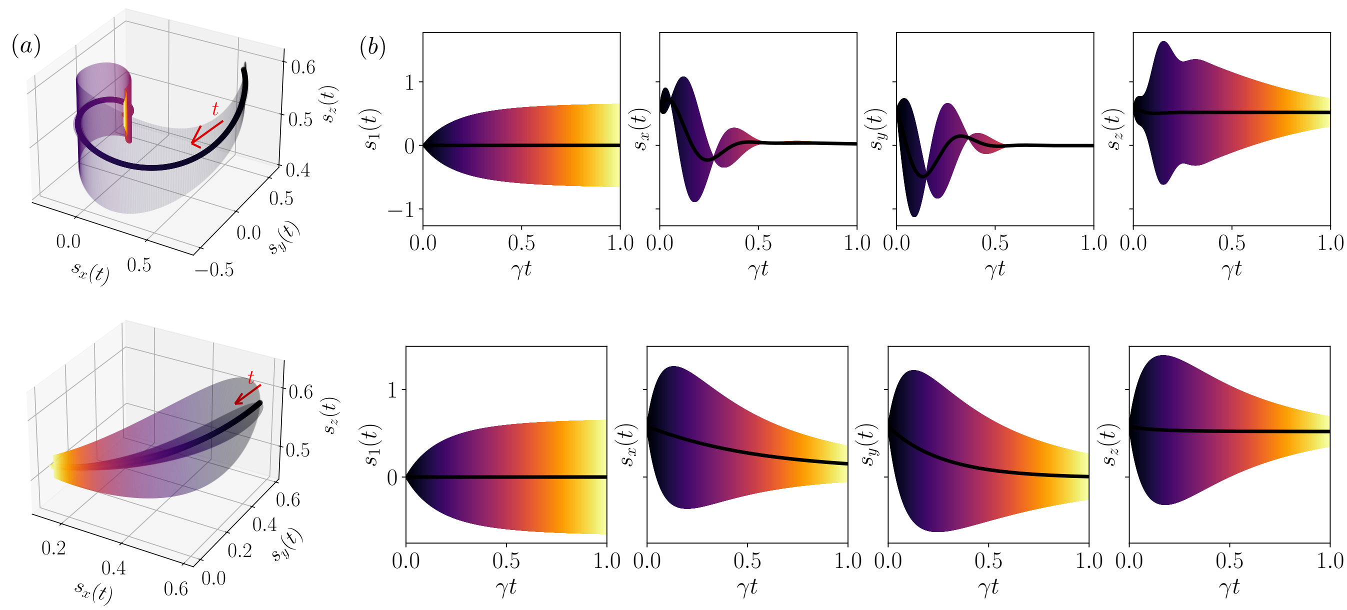

FIG. 3. **Visualization for the evolution of the stochastic operator variance and its standard deviation for the quantum sLMG model**. (a) Projection over the different spin operators of the noise averaged observable $(\langle\hat{A}_t\rangle, \hat{S}_j)$ (solid line) and the SOSD $(\Delta\hat{A}_t, \hat{S}_j)$ (error bar), with the flow of time indicated by the color scale and the red arrow. (b) Projections of the noise-averaged observable (black line) $(\langle\hat{A}_t\rangle, \hat{X})$ and the SOV $(\Delta\hat{A}_t^2, \hat{X})$ (error bar) over $\hat{X} \in \{\hat{\mathbb{1}}, \hat{\mathbf{S}}\}$. The initial operator is $\hat{A} = (\hat{S}_x + \hat{S}_y + \hat{S}_z)/\sqrt{3}$. The parameters are $S = 20$ and $\Omega = 1.5$, with $\gamma = 0.1$ (upper) or $\gamma = 2$ (lower). The times for which the error bar vanishes correspond to the SOV being orthogonal to the projected operator.

V. CLASSICAL STOCHASTIC LMG MODEL

A. Classical limit of the LMG Hamiltonian

Let us introduce the SU(2) coherent states as [5]

$$|\zeta\rangle = \frac{e^{\zeta\hat{S}_+}}{(1+|\zeta|^2)^S}\,|S,-S\rangle\,, \tag{S24}$$

where $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$ is the raising operator and $|S,-S\rangle$ is the eigenstate of $\hat{S}^2$ and $\hat{S}_z$ with smallest z component of the spin, namely $\hat{S}^2\,|S,-S\rangle = S(S+1)\,|S,-S\rangle$ and $\hat{S}_z\,|S,-S\rangle = -S\,|S,-S\rangle$. The coherent states $|\zeta\rangle$ represent a wavepacket with the minimum width allowed by the uncertainty principle localized around $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ where $\zeta = -\tan\frac{\theta}{2}e^{-i\phi}$ and corresponds to a stereographic projection. Following [6], we define the classical Hamiltonian H_{LMG} as

$$H_{\text{LMG}} = \lim_{S\to\infty}\frac{1}{S}\,\langle\zeta|\hat{H}_{\text{LMG}}|\zeta\rangle\,, \tag{S25}$$

where $S = N/2$. The expectation value of the relevant spin operators between coherent SU(2) states is [5]

$$\langle\mathbf{n}|\hat{S}_z|\mathbf{n}\rangle = -S\cos\theta, \qquad \langle\mathbf{n}|\hat{S}_x^2|\mathbf{n}\rangle = S(S-\tfrac{1}{2})\sin^2\theta\cos^2\phi + \tfrac{S}{2}. \tag{S26}$$

The classical LMG Hamiltonian then reads

$$H_{\text{LMG}} = -\Omega\cos\theta - \sin^2\theta\cos^2\phi + \mathcal{O}(S^{-1}). \tag{S27}$$

We then introduce the canonical variables $Q,\ P$ as

$$\zeta = \frac{Q - iP}{\sqrt{4-(Q^2+P^2)}} = -\tan\frac{\theta}{2}e^{-i\phi}, \tag{S28}$$

that yield the relations

$$\frac{Q}{\sqrt{4-Q^2-P^2}} = -\tan\frac{\theta}{2}\cos\phi, \qquad \frac{P}{\sqrt{4-Q^2-P^2}} = -\tan\frac{\theta}{2}\sin\phi. \tag{S29}$$

Inverting theses equations gives

$$\tan\phi = \frac{P}{Q}, \quad \tan^2\frac{\theta}{2} = \frac{Q^2+P^2}{4-(Q^2+P^2)}, \tag{S30}$$

that can be substituted in (S27) and yield

$$H_{\text{LMG}} = \frac{\Omega}{2}(Q^2+P^2) - \Omega - \frac{1}{4}(4Q^2 - Q^2P^2 - Q^4). \tag{S31}$$

This corresponds to the classical Hamiltonian given in the main text, up to a constant shift in energy.

B. Heuristic reason for stabilization of sLMG

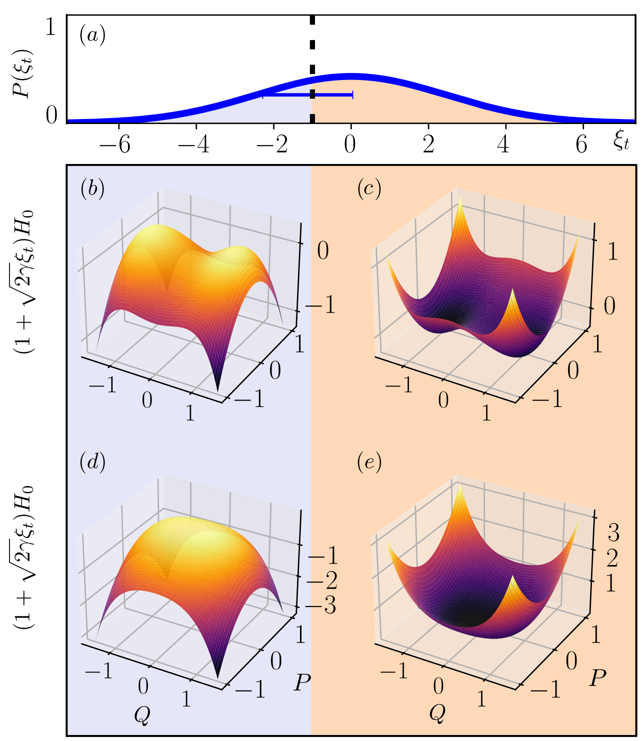

FIG. 4. **Visualization of the stochastic LMG model**. (a) Histogram of the Gaussian white noise. The standard deviation $\sqrt{2\gamma}$ is indicated by the horizontal blue line and the vertical dotted line delimits the sign flip of $(1+\sqrt{2\gamma}\xi_t)$. LMG Hamiltonian in the double well (b;c) and single well (d;e) phases multiplied by a negative (b;d) and positive (c;e) number corresponding to $(1+\sqrt{2\gamma}\xi_t)$.

The stabilization we discussed in the main text can be understood in terms of the energy landscape associated with the stochastic Hamiltonian, illustrated in Fig. 4. In the limit that $\sqrt{2\gamma} \ll 1$, most realizations are as in the noiseless LMG. The behavior is then close to that of the deterministic model, with the double well (DW) phase showing an unstable point and the single well (SW) phase being stable. In the opposite limit, $\sqrt{2\gamma} \gg 1$, almost half of the realizations flip the Hamiltonian since $(1+\sqrt{2\gamma}\xi_t) < 0$, as shown in Fig. 4 (a). This brings a major difference between the DW and SW phases. In the SW phase, all points in phase space (Q, P) either gain (d)→(e) or lose (e)→(d) energy. In the DW phase, however, when going from (b)→(c), the points inside the wells lose energy while the points outside the wells gain it, and viceversa for (c)→(b). This difference between the points of phase space provides a rationale for the stochastic stabilization seen for the DW phase—blue region in Fig. 3(b) of the main text.

C. Analytical determination of the Lyapunov exponent for the sLMG model

We consider a vector variable $\mathbf{u}_t$ subject to deterministic and fluctuating external perturbations, $\mathbb{A}_d$ and $\mathbb{A}_s(t)$, respectively. Its equation of motion is described by the stochastic differential equations

$$\dot{\mathbf{u}}_t = [\mathbb{A}_d + \sqrt{2\gamma}\mathbb{A}_s(t)]\mathbf{u}_t. \tag{S32}$$

Following van Kampen [7], we go in the interaction picture with respect to the deterministic evolution and consider $\mathbf{v}_t = e^{-\mathbb{A}_d t}\mathbf{u}_t$. In second-order of $\sqrt{\gamma}$, the average with fixed initial conditions evolves as

$$\langle \mathbf{v}_t \rangle = \mathbf{v}_0 + 2\gamma \int_0^t dt_1 \int_0^{t_1} d\tau e^{-t_1\mathbb{A}_d} \langle \mathbb{A}_s(t_1) e^{\tau \mathbb{A}_d} \mathbb{A}_s(t_1 - \tau)\rangle e^{(t_1-\tau)\mathbb{A}_d}\mathbf{v}_0, \tag{S33}$$

valid for $\sqrt{2\gamma}t \ll 1$. We recognize the solution to order γ of the linear differential equation, also known as Bourett's integral equation, written back in the original representation as

$$\partial_t \langle \mathbf{u}_t \rangle = \left[\mathbb{A}_d + 2\gamma \int_0^t \langle \mathbb{A}_s(t) e^{\mathbb{A}_d t} \mathbb{A}_s(t-\tau)\rangle e^{-\mathbb{A}_d \tau} d\tau\right] \langle \mathbf{u}_t \rangle. \tag{S34}$$

This equation is derived assuming the standard rules of calculus, and thus assumes Stratonovich formalism. The latter defines the stochastic integral as $\int_0^t \delta(t-\tau) f(\tau) d\tau = \frac{1}{2} f(t)$. So in the case that $\mathbb{A}_d$ and $\mathbb{A}_s$ commute, and for $\mathbb{A}_s(t) = \xi_t \mathbb{A}_s$ fluctuating with Gaussian white noise, Eq. (S34) simplifies to

$$\frac{\partial}{\partial t}\langle \mathbf{u}_t \rangle = \left(\mathbb{A}_d - 2\gamma \mathbb{A}_d^2 \int_0^t \langle \xi_t \xi_{\tau'} \rangle d\tau'\right) \langle \mathbf{u}_t \rangle = \left(\mathbb{A}_d - \gamma \mathbb{A}_d^2\right) \langle \mathbf{u}_t \rangle,$$

where the change of integration variable $\tau' = t - \tau$ brings the minus sign. In systems exhibiting chaos, the Lyapunov gives the exponential divergence of the trajectory. We interpret this as the maximum eigenvalue of $\mathbb{A}_d - \gamma \mathbb{A}_d^2$.

The LMG at the origin, $Q = P = 0$, can be linearized into the harmonic oscillator $H = \frac{1}{2}[\Omega P^2 + (\Omega - 2)Q^2]$. Hamilton's equation of motion gives $\dot{Q}_t$ and $\dot{P}_t$, from which we can compute the evolution of the quadratic terms as [8]

$$\frac{d}{dt}\begin{pmatrix} Q_t^2 \\ P_t^2 \\ Q_t P_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Omega \\ 0 & 0 & -(\Omega-2) \\ -\frac{\Omega-2}{2} & \frac{\Omega}{2} & 0 \end{pmatrix} \begin{pmatrix} Q_t^2 \\ P_t^2 \\ Q_t P_t \end{pmatrix} \equiv \mathbb{A}_d \mathbf{u}_t. \tag{S35}$$

The maximum eigenvalue of this matrix is $\lambda_{\text{LMG}} = \sqrt{2\Omega - \Omega^2}$, which recovers the Lyapunov exponent at the origin in the noiseless LMG model [6]. For the sLMG, we add noise in the energy scale and consider an evolution dictated by

$$\dot{\mathbf{u}}_t = \mathbb{A}_d (1 + \sqrt{2\gamma}\xi_t)\mathbf{u}_t, \tag{S36}$$

where ξ_t is Gaussian white noise. The maximum eigenvalue of $\mathbb{A}_d - \gamma \mathbb{A}_d^2$ thus gives the average Lyapunov exponent as

$$\lambda = \sqrt{2\Omega - \Omega^2} - \gamma(2\Omega - \Omega^2). \tag{S37}$$

VI. DETAILS ON THE NUMERICAL SOLUTIONS

A. Vectorization

The formal solution to the master equation reads $\langle \hat{A}_t \rangle = e^{\mathcal{L}^\dagger t}[\hat{A}]$, where $\mathcal{L}^\dagger[\bullet]$ is the adjoint Liouvillian superoperator. To numerically solve it, we apply vectorization by which operators $\hat{A}_t$ become vectors in a larger Hilbert space $|\hat{A}_t)$. Superoperators become operators over this larger Hilbert space through the mapping

$$\hat{X}\hat{A}_t\hat{Y} \to (\hat{X} \otimes \hat{Y}^T)|\hat{A}_t), \tag{S38}$$

therefore the vectorized Lindbladian reads

$$\mathcal{L}^\dagger[\bullet] \to i\hat{H}_0 \otimes \mathbb{1} - i\mathbb{1} \otimes \hat{H}_0^T + \gamma(2\hat{L} \otimes \hat{L}^T - \hat{L}^2 \otimes \mathbb{1} - \mathbb{1} \otimes (\hat{L}^2)^T). \tag{S39}$$

B. Solver of Stochastic Differential Equations

We solve the SDE using the *explicit order 1.0 strong scheme* [9], which briefly consists of the following. Consider a Itō SDE of the form

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + b(X_t)\mathrm{d}W_t, \tag{S40}$$

where X_0 is the initial condition. Let Y_n be the solution at time $t_n = n\delta$, where δ is the time-step. We first set the initial condition $Y_0 = X_0$ and compute recursively the solution as

$$Y_{n+1} = Y_n + a_n\delta + b_n\Delta W_n + \frac{1}{2\sqrt{\delta}}(b(\Upsilon_n) - b_n)((\Delta W_n)^2 - \delta), \tag{S41}$$

where $\Upsilon_n = Y_n + a_n\delta + b_n\sqrt{\delta}$, $a_n = a(Y_n)$, $b_n = b(Y_n)$, and $\Delta W_n = W_{n+1} - W_n$ are independent identically distributed normal random variables with zero average and variance δ.

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