# MAKING SENSE OF ZERO TO MAKE SENSE OF NEGATIVE NUMBERS

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*Few studies so far have examined the meaning attributed to zero by students and how this can affect the handling of negative numbers. The objective of this paper is to analyse the relationship between the understanding of zero and the ability to perform integer additions. A paper-and-pencil test was submitted to 166 grade 6 students who had not been taught about negative numbers, in order to analyse the meaning they attributed to zero in relation to their ability to perform integer additions correctly. The results show that these students had two main conceptions of zero: zero as "nothing" and zero as a "point on a number line". We also found that students with this latter conception of zero were significantly more likely to succeed in integer additions.*

## THEORETICAL BACKGROUND

Few studies have considered the question of zero in learning, and in particular the relationship between understanding zero and the ability to perform operations with negative numbers. However, to perform operations with these numbers it is important to be able to envisage the existence of numbers below zero. All too often, zero is regarded as "nothing" (Wheeler & Feghali, 1983), the "nil" below which no number is conceivable; this reflects the idea of "absolute zero" (Glaeser, 1981).

Like many mathematical symbols, zero has various different meanings. According to Volken (2000) and Toma (2008), there are two main uses of zero, which are both essential, but also slightly different. The first is as a way of marking an empty place in our positional notation system for numbers. Volken (2000) refers to this as a "meta sign", in that it indicates the absence of other signs. The zero in this context therefore seems to be of a different nature from the digits 1 to 9. The second use is as a number in its own right, in the form that we currently attribute to it, 0. Historically, this use took a long time to become established: zero was not a natural candidate to be a number, as numbers initially designated sets of objects (Toma, 2008). If there were no items to count, there was therefore no need to mention this "nothing", and even less need to give it a symbol (Ruttenberg-Rozen, 2018). The polysemy of zero makes it complex to understand and use for today's students at all stages of schooling (Levenson et al., 2007). It can also create obstacles in the extension of the natural number domain to the integers (Glaeser, 1981).

The purpose of this paper is to examine the different meanings attributed to zero by grade 6 students who have not yet learned about negative numbers, and to relate these meanings to their ability to solve integer addition problems.



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## Zero in history

Historically, according to Toma (2008), the first trace of a zero was not found until around 400 BC, among the Babylonians, who used a special symbol to indicate empty space in their positional notation system. This sign can be considered the precursor of zero notation. However, Ruttenberg-Rozen (2018) notes that zero as we know it today in our base-10 positional system was developed in India from the 5th century AD onwards, when mathematicians began to use a small circle to represent the empty space in a number. The invention of this symbol by the Hindus made conceptual progress possible with regard to zero, whose status changed from that of a mere placeholder to that of a number less than one – a number in its own right (Ruttenberg-Rozen, 2018). Thus, in a treatise on astronomy in 628, Brahmagupta was able to define zero as the subtraction of a number from itself ($a - a = 0$). However, it was not until the 1600s that zero finally held an uncontested place and mathematics could further progress with the inclusion of this important number (Toma, 2008).

From the new understanding of zero as a number, other mathematical understandings were able to develop. Volken (2000) emphasises that the “zero” symbol was also the starting point for remarkable developments in arithmetic and algebra, resulting in the appearance of a new conception of numbers that was more abstract and more unified. Numbers gradually acquired autonomy from the pre-existing objects they had hitherto been deemed to represent. This abstraction made it possible to consider the numbers below zero, the negatives, which are not properties of sets of objects (Toma, 2008). Zero thus became “origin zero” (Glaeser, 1981). However, the introduction of negative quantities in the West was slow and difficult, due in particular to this ambiguity of zero, and many mathematicians throughout history have found it difficult to distinguish “origin zero” from “absolute zero” (Glaeser, 1981).

## Zero in school education

What is zero? When asked, many students will reply that zero is “nothing” (Russell & Chernoff, 2011). Primary school teachers in pre-service education also appear frequently to use the words “zero” and “nothing” interchangeably or synonymously (Levenson et al., 2007).

Among other misconceptions about zero, it has also been observed that students believe that zero is not a number, or that it is only part of the symbol for the number ten, or that it “doesn’t do anything” and can therefore be ignored (Russell & Chernoff, 2011). Some of this confusion can be attributed to the fact that many students think of zero as being “nothing”. According to Levenson et al. (2007), this conception hinders effective teaching of the deep and complex structure of zero.

From the point of view of the relationship between zero and integer operations, Peled et al. (1989) carried out a study on addition and subtraction of integers among primary school students before they had learned about negative numbers. They examined the intuitive models used by students to perform these operations. The authors showed that the main mental model used by students was that of the number line, which is in fact relatively abstract. However, some students, while referring to the number line model, did not necessarily make effective use of it. They displayed the misconception known as the "divided number line" (DNL): the numbers were seen as two symmetrical sets on either side of zero, which was often thought of as a barrier rather than a number. By contrast, other students were already showing the ability to take a unified view of the positive and negative numbers and zero as integers, in other words the conception of the "continuous number line" (CNL). These students performed calculations smoothly by going "to the right" for addition and "to the left" for subtraction, without having to create special partitioning rules to move past zero.

## METHOD

The objective of this paper is to answer the following two research questions.

1. What meaning do grade 6 students, who have not been taught about negative numbers, attribute to zero? (RQ1)
2. To what extent does the meaning attributed to zero go hand in hand with the ability to perform integer additions correctly? (RQ2)

The analyses presented in this paper come from a larger study presented previously (Vlassis & Demonty, 2022). A total of 166 grade 6 students in 13 classes at eight primary schools in the Grand Duchy of Luxembourg took part in the study by completing a paper-and-pencil test. The students in our sample had not been taught about negative numbers, as this topic, as well as the meaning of zero, is not included in the Luxembourg primary curriculum (MENFP, 2011). The paper-and-pencil test was designed to be taken individually, and took approximately one hour to complete. The test had two parts: the first part consisted of decontextualised questions relating to integer additions and subtractions, the role of zero, the order of negative numbers, opposite numbers, and mental computations in addition and subtraction operations with more than two terms, and in subtractions with two terms where the compensation strategy would clearly be useful. The second part consisted of contextual problems, the solutions to which required operations similar to those in the first part. For the purposes of this study, only decontextualised questions relating to (1) integer additions and subtractions, and (2) the role of zero were analysed.

The two questions used for the analyses were as follows:

**Question 1** (Q1.a and Q1.b)

**a) What answer would you give to the problem 0 – 14 = ?**
**b) In 0 – 14 = what does the zero represent?**

More than one answer may be correct. Tick the **one** answer that seems most appropriate to you.

1. The zero has no real value and could be removed
2. It represents a position on the number line
3. It represents a nil quantity

4. The zero is used to separate the positive and negative numbers
5. It represents emptiness

The five options presented in Question 1.b derive from students' misconceptions as described in the theoretical background:

- Option 1 refers to the "doesn't do anything" conception (Levenson et al. 2007);
- Options 2 and 4 both concern the "origin zero" (Glaeser, 1981), and are based either on the CNL (2) or on the DNL (4) (Peled et al., 1989);
- Options 3 and 5 correspond to "zero = nothing" which refers to the "absolute zero" conception (Glaeser, 1981).

**Question 2** (Q2)

| | | |
|---|---|---|
| $4 - 6 = \square$ | $-3 - 5 = \square$ | $-5 + 8 = \square$ |
| $8 + \square = 5$ | $\square - 3 = -8$ | $-12 - 8 = \square$ |
| $\square + 9 = 6$ | $\square + 4 = -6$ | $4 + (-9) = \square$ |
| $5 - 10 = \square$ | $-7 + 4 = \square$ | $7 + \square = -3$ |

In Question 2, some additional items with natural numbers were added to prevent students from deducing that all the answers to this question were negative numbers. These items show good internal consistency (Cronbach's alpha: 0.90) (Vlassis & Demonty, 2022).

## RESULTS

### The role of zero

In order to identify students' conception of zero (RQ1), we have analysed in Table 1 below the options chosen by students in Question 1.b.

| Conception of zero | Choice | % of students |
|---|---|---|
| Nothing or emptiness ("nothing") | 1 | 8 |
| | 3 | 11 |
| | 5 | 9 |
| | more than one option | 12 |
| | Total | 40 |
| Number line (NL) | 2 | 8 |
| | 4 | 28 |
| | more than one option | 7 |
| | Total | 43 |

| | | |
|---|---|---|
| Mixed conception | | 9 |
| No answer given | | 8 |
| Total (N=166) | | 100 |

Table 1: Percentage of choices made by students regarding the conception of zero (Q1.b)

In Table 1, the five options have been divided into two main categories: the "nothing" category consisting of Options 1, 3 and 5 and related to the "absolute zero" conception, and the "number line" category (NL), referring to the "origin zero" conception (Options 2 and 4). First, as can be seen from the results presented in Table 1, despite the instruction asking students to select a single option among the five, some of them ticked more than one. It should be noted that most of the students who ticked several options did so within the same category, thereby demonstrating some degree of consistency. Only 9% of students selected options in both categories ("Mixed conception"). Next, it is notable that the students' choices were evenly distributed between the two categories (40% for "nothing" versus 43% for "NL"). It is striking that even before any learning about negative numbers, 43% of the students already chose the NL conception (exclusively), implying an acceptance of numbers under zero. We use the term "implying" deliberately, because the students were not interviewed and we do not know what they actually thought on this subject. It is possible that the general context of the test, in which most of the questions concerned negative numbers, led to this choice being favoured by some students – a choice that they perhaps might not have made in a neutral context. Within this category, most of the students selected Option 4, referring to the DNL, indicating that their understanding of the integers and zero was not yet unified.

Finally, in the "nothing" category, the majority of choices refer to the "zero = nothing" conception, with 19% of the students choosing either Option 3 (zero = nil quantity) (11%) or Option 5 (zero = emptiness) (8%). A mere 8% of students ticked Option 1 only. It might be concluded that the idea of a zero that "doesn't do anything" was not particularly widespread among the students in the sample. However, closer analysis of the data (not presented in Table 1) shows that this conception was indeed present among the students. Among the 12% of students who selected "more than one option", 10% chose Option 1 in combination with Option 3 and/or 5. These students apparently thought that the idea of emptiness or nil quantity could be combined with the idea of "doesn't do anything".

**Relations between conception of zero and operations with negatives**

In order to answer RQ 2, we examined the results of the students relating to operations with negatives according to the conception of zero that they revealed. We thus related the results for Question 1.b with those for Questions 1.a and 2, hypothesising that students presenting an NL conception would succeed better in the operations with negatives than those presenting a conception labelled as “nothing+”, this latter group including this time not only the main conception of “nothing” (40%), but also the “mixed conceptions” (9%) and students who failed to answer the question (8%). Table 2 below presents the answers given to Q1.a (0 – 14 =) according to the conception of zero (Q1.b).

| Conception of zero (Q1.b) | Correct answers (%) (Q1.a) | Incorrect answers (%) (Q1.a) | | | | No answer given (%) (Q1.a) | Total (%) |
|---|---|---|---|---|---|---|---|
| | *–14* | *14* | *-0.86/ –16/–6* | *0.86/86/ 6* | *0* | | |
| NL conception | 73 | 1 | 10 | - | - | 16 | 100 (N= 70) |
| “Nothing+” conception | 49 | 13 | 4 | 2 | 2 | 28 | 100 (N= 96) |

Table 2: The relationship between conception of zero (Q1.b) and correct solution to the question 0 – 14 = (Q1.a)

The results in Table 2 show that students with an NL conception of zero were significantly more likely (73%) to give the correct solution, –14, than students with a “nothing+” conception. The correlation of 0.24 between the results obtained in Questions 1.a and 2 confirms this link between the two variables. It is also worth noting that all these students, even when they were wrong, gave a negative answer (with the exception of one student who answered 14), and were much less likely to fail to answer (16%, against 28% of “nothing+” students). Forty-nine percent of the students with a “nothing+” conception were able to find the correct solution, –14. However, 17% of them did not consider a negative solution: they answered 14 (13%) or 0.86/86/6 (2%) or even 0 (2%). A final and somewhat unexpected observation concerns the students who, regardless of their conception of zero, put forward solutions such as ±0.86, ±6, 86 or even –16, as if they were attributing a numerical value to zero such as 1 or even 100. 10% of students who displayed NL conceptions and 6% of those displaying “nothing+” conceptions fell into this category. These students seem to have considered zero as a unit or a hundred and thus to have confused operations under zero with operations in the decimal system (Stacey & Steinle, 2001).

While the results of Table 2 made it possible in particular to examine the type of solution given by the students, those in Table 3 below present the students’ success rate with a set of integer additions (Q2) according to their conception of zero (Q1.b).

| Conception of zero (Q1.b) | % success with integer additions (Q2) |
|---|---|
| NL conception | 60 |
| “Nothing+” conception | 46 |

Table 3: The relationship between conception of zero (Q1.b) and success with integer additions (Q2)

In table 2, we observe again that students with an NL conception of zero are also more successful at adding integers. The difference is significant (ANOVA: $F=9.92$; $p<0.005$).

**CONCLUSION**

The “nothing” conception seems to be regarded in scholarship on zero in learning as predominant among students (Levenson et al., 2007; Russell & Chernoff, 2011; Wheeler & Feghali, 1983). Our results reveal a more nuanced reality. Although some of the grade 6 students in our sample – who, it should be remembered, had not yet learned about negative numbers – did demonstrate a “nothing” conception, we saw that an almost equivalent proportion had a conception that related to the number line (NL). The general context of the test about the negative numbers and the MCQ format may have favoured this tendency, but the results also suggest that the “nothing” conception may be less entrenched among students than might appear to be the case. Within the NL conception, the notion of the DNL was especially favoured by the students. While not yet reflecting a unified vision of integers, this conception makes it possible to accept numbers below zero. Our results showed that an NL conception of zero, essentially therefore of the DNL type (in our student sample), was associated with significantly higher success than that associated with a “nothing” conception.

It should be emphasised that a “nothing” conception did not necessarily mean an inability to consider numbers below zero and to perform integer additions; however, the rate of success in performing such calculations was lower. It is also worth noting that, in Q1.a, the correct solution, –14, could also have been arrived at by students who thought that zero “doesn’t do anything”, and that therefore 0 – 14 = –14, because zero has no effect. A somewhat distinctive conception of zero was apparent in the case of students who gave solutions to 0 – 14 = such as ±0.86, 86, etc., as if they thought zero was equal to a unit or a hundred. Stacey and Steinle (2001) pointed to a similar problem with students who became confused between decimal numbers and negative numbers. According to these authors, these students were confusing the classical model of the number line with the place value columns and treating the spatial arrangement of the usual place value numeration as a kind of “number line” along which the numbers were distributed. We wonder if this confusion could have arisen from students confusing a conception of zero as an empty place indicator with a conception of zero as a number. Ultimately, it would seem that among students, as in the history of mathematics, working with integers is accompanied by an evolution of the conception of zero towards origin zero (NL), so that zero can be regarded as an integer in its own right. It is therefore vital in school for the extension of natural numbers to integers to go hand in hand with a broadening of the use of zero through a clear explanation of its different functions in different contexts. However, learning about the different meanings of zero does not seem to be part of the curriculum at either primary or secondary level (MENFP, 2011; MENFP, 2008).

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