Explanation in Descriptive Set Theory

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September 5, 2023

Abstract

In this paper we investigate explanatory proofs in mathematics. We do this via an extended case study from descriptive set theory. The aim is to shed light on what makes explanatory proofs explanatory. Building on earlier work, we argue that there may be more than one notion of explanation in operation in mathematics: there does not seem to be a single account that ties together the different explanatory proofs we find in the various areas of mathematics. We then attempt to give a characterisation of the different notions of explanation in play and how these sit with accounts of explanation found elsewhere.

1 Introduction

We are interested in explanations in mathematics. These are sometimes called *intra-mathematical explanations* and involve one mathematical result being explained in terms of further mathematics. For example, some proofs are explanatory: they do more than merely show *that* a given theorem is true; they demonstrate *why* the theorem is true. It is an interesting, open question whether explanation in mathematics is always connected with a proof of a theorem. While there is good reason to suspect that proof may not be the only locus of explanation in mathematics,[1] it is, at least, *one* such locus. For present purposes, we set aside the issue of non-proof-based explanation and concentrate on explanations arising from proofs.

Explanation in mathematics is important for a number of reasons. For a start, such explanation is clearly not causal so is not accommodated by causal accounts of explanation, such as those advanced by Lewis (1986). At least as traditionally construed, mathematical facts are necessarily true, so counterfactual accounts of explanation run into

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[1]See, for example, Colyvan (2012); Lange (2018); D'Alessandro (2020).

trivialism issues when applied to mathematics.[2] In light of all this, mathematical explanation is an interesting test case for theories of explanation and presents problems for any ambitions for a single, unified theory of explanation (see Reutlinger et al. (2022). Of course, mathematical explanation is interesting in its own right. In this paper, we are less interested in the broader philosophical issues[3] and more concerned with understanding mathematical explanation in its own terms.

In pursuit of this goal, it is instructive to look at theorems that have different styles of proofs. In particular, it is good to look at explanatory proofs and non-explanatory proofs of a particular result. The fact that such pairs of proofs exist for some theorems helps establish that it is not the theorem itself that is explanatory or not. The explanation seems to reside in at least some proofs. Moreover, looking at such pairs of proofs allows us to identify the explanatorily-relevant differences between the proofs and thus help identify what makes a proof explanatory. This, in turn, helps us get a grip on what a theory of explanation in mathematics might look like. Also of interest are pairs of proofs of a particular theorem that each has some claim to being explanatory but in different ways (or perhaps at different levels of generality).

Thus far, the philosophical literature on mathematical explanation has mostly focussed on examples of proofs from elementary number theory, Euclidian geometry, and the like. Focussing on such basic mathematics is understandable. Indeed, it is usually advisable to use as simple an example or case study as is needed for the task at hand. And, of course, examples from elementary mathematics will be accessible to a broader range of readers. The problem with this, however, is that we run a risk of developing an account of explanation that is based on too limited a stock of examples and does not do justice to mathematics as a whole. We think it is important to draw examples from different areas of mathematics. We hold this view for a couple of reasons. First, if we focus too narrowly on elementary examples, we might be misled about the nature of mathematical explanation in higher mathematics. We need at least some examples from advanced mathematics. Second, there may well be different explanatory goals and even different standards of proof in different areas of mathematics. We thus must consider examples from at least some of the many different branches of mathematics. Ideally, we would draw examples from across all areas of mathematics. This is impractical in a paper such as this. Instead we see this paper as a contribution to this larger task: the diversification of examples needed for informed philosophical

[2]The core idea of a counterfactual account of explanation is that A explains B just in case the following counterfactual holds: had A not been the case, then B would not be the case. But in mathematics, both A and B are necessary, so the counterfactual in question has an impossible antecedent so is trivially true (at least, according to the usual semantics for counterfactuals). There has been some work on extending such counterfactual accounts to deal with mathematical explanation by invoking counterpossibles (see Baron et al. (2020)).

[3]See Mancosu (2008); Colyvan (2018) for more on this.

discussion about mathematical explanation.[4] Our main focus will be on proofs in one advanced area of modern mathematics — descriptive set theory — where there has been some very interesting debate over mathematical explanation in dichotomy theorems.

Finally, we note that we need to draw on the judgements of mathematicians about which proofs are explanatory, if we are to respect mathematical practice here. It is all too easy for philosopher's judgements about which proofs are explanatory to be clouded by other philosophical commitments (e.g. in metaphysics, in epistemology, and in the philosophy of explanation). In a naturalist spirit, we see our task to be that of taking the judgements of mathematicians and trying to make philosophical sense of these.

2 Fermat's Little Theorem

Before we get to our main case study in descriptive set theory, it will be useful to warm up with an elementary example. This will help to get a feel for the issues in question. The example of this section is from number theory and is known as *Fermat's Little Theorem*.[5]

Theorem 1 (Fermat's Little Theorem)**.** *If p is prime and a is a positive integer such that $p \nmid a$ (p does not divide a), then* $a^{p-1} \equiv 1 \pmod{p}$.

There are many different proofs of this theorem. Arguably many of these proofs give different insights into the theorem and forge connections with different branches of mathematics. Here we're content to provide sketches of three different proofs and make some suggestions about their relative explanatoriness.

2.1 A Number Theory Proof

Consider the set of $p-1$ integers $S = \{a, 2a, 3a, \ldots (p-1)a\}$. None of these integers is divisible by p, for if $p \mid ja$ (i.e. p divides ja) for $1 \leq j \leq (p-1)$, then, since $p \nmid a$, we'd have the impossibility: $p \mid j$. Moreover, no two of the integers in S are congruent modulo p. If they were, we'd have j and k less than $(p-1)$ such that $ja \equiv ka \pmod{p}$. But since $p \nmid a$, this means that $j \equiv k \pmod{p}$ but this is impossible since both j and k are less than $p-1$. This means that the least positive residues (modulo p) of the members of S are the integers $1, 2, 3, \ldots (p-1)$. So $a \cdot 2a \cdot 3a, \cdots (p-1)a \equiv 1 \cdot 2 \cdot 3 \cdots (p-1) \pmod{p}$. Thus $a^{p-1}(p-1)! \equiv (p-1)! \pmod{p}$. Since $(p-1)!$ and p are relatively prime, we can divide both sides of the last equivalence by $(p-1)!$ to give us the required result: $a^{p-1} \equiv 1 \pmod{p}$.[6]

This proof uses only number-theoretic resources and has some claim to being explanatory. It shows that the result holds because of facts

[4] There has already been some work in this direction, e.g. Lange (2017); Colyvan et al. (2018).

[5] Thanks to Hannes Leitgeb for suggesting this example and for his mathematically-informed intuitions about which of the proofs is more explanatory.

[6] A version of this proof can be found in Rosen (1988).

about prime numbers, divisibility, and the like. In essence, we have a number-theory result spelled out in terms of the properties of numbers. Such proofs are valued in number theory and are called "elementary proofs" and are contrasted with some proofs that proceed via methods from complex analysis. It is not clear that elementary proofs in number theory are valued for their explanatoriness but this is a fair assumption. After all, if we have a theorem about prime numbers, we could reasonably expect that an explanation of this would be in terms of properties of prime numbers — not rely on facts about analytic functions on the complex domain.[7] This proof fits the bill and seems to give us insights into why the theorem holds. But can we do better?

2.2 A Group Theory Proof

It is straightforward to show that $G = \{1, 2, 3, \ldots p-1\}$, with the operation of multiplication (mod p), is a group. Next we reduce a modulo p so we can assume that $1 \leq a \leq p-1$. That is, $a \in G$. Let k be the smallest positive integer such that $a^k \equiv 1 \pmod{p}$. Then the set containing the numbers $1, a, a^2, \ldots a^{k-1}$, reduced modulo p, forms a subgroup of G with order k. We then invoke Lagrange's Theorem[8] to show that k divides $p-1$ (which is the order of G). So we have $p-1 = kn$, for some positive integer n. Thus $a^{p-1} \equiv a^{kn} \equiv 1^n \equiv 1 \pmod{p}$, as required.[9]

This proof shows that Fermat's Little Theorem is an instance of a more general group-theoretic result. At least, the proof places this number-theoretic result in a broader context of group theory. Indeed, Lagrange's theorem is the key to this particular proof. It is worth noting that there are group-theoretic proofs that do not invoke the full generality of Lagrange's theorem but, instead, prove the crucial step directly by proving, in effect, a special case of Lagrange's theorem.[10] It is the generality delivered by this proof that gives it its claim to explanatoriness. While Fermat's Little Theorem is a number-theoretic result, this group theory proof, we think, is more explanatory. But this does raise an interesting question about whether it is generality that matters most or proving a result in a particular area by appealing to details of the area in question.[11] The number theory proof in the previous section had the latter virtue. We might think of this earlier proof as delivering a local or intrinsic notion of explanation, while the group theory proof offers a more unifying or global notion of explanation. Indeed, these might be thought of as distinct axes of evaluation of a proof, both relevant in their own right, but not always offering

[7]The preference for elementary proofs in number theory resonates with Hartry Field's Field (2016) argument preferring *intrinsic explanation* in science.

[8]This theorem states that the order of any subgroup of a finite group G, divides the order of G.

[9]A version of this proof can be found in Weil and Rosenlicht (1979).

[10]Euler provided such a proof Euler (1761).

[11]This raises the interesting question of whether levels of generality might correspond to levels of explanation.

a best balance between the two. We will return to such issues in our discussion of the major case study presented in the next section.

2.3 A Combinatorial Proof

As before, assume that p is prime and a is a positive integer such that $p \nmid a$. Suppose we have a different coloured beads and we wish to make necklaces with p beads in each. First we place p beads on a string and we note that there are a^p such possible strings. We discard the strings consisting of beads of only one colour. This leaves $a^p - a$ strings. Now we join the ends of the strings to form the necklaces. We note that some of the necklaces are cyclic permutations of others. While the cyclic permutations are distinguishable as untied strings, they are indistinguishable as necklaces. Since there are p cyclic permutations of the beads on the string, and p is prime, the number of distinguishable necklaces is $(a^p - a)/p$ and this must be an integer. From this the result follows.[12]

This is an interesting proof. It uses the least sophisticated mathematics: there's no group theory or even much by way of number theory here. Moreover, the appeal to necklaces helps with visualisation. Indeed, the proof has the reader build a mental model in the service of delivering the result in question. For these reasons, this proof is very useful pedagogically. It is explanatory in the sense that it helps newcomers to number theory get a grip on Fermat's Little Theorem. But it also has some claim to being explanatory in the sense of revealing the real reason that the result holds. After all, the construction of necklaces and discarding duplicates is, in a sense, just some do-it-yourself group theory. Or rather, what we have here is a particular instance of the group-theoretic approach but without invoking the full generality of group theory. There is no appeal to groups or Lagrange's theorem to be seen in this proof, yet it is a particular instance of Lagrange's theorem, applied to the case at hand, that lies at the heart of this proof. So this proof might be thought to have many of the virtues of the group theory proof but without the full generality afforded by group theory. This proof thus does not (explicitly, at least) forge a connection between number theory and group theory. For this reason, it might be argued that this proof is, indeed, explanatory but perhaps not as explanatory as the more unifying group theory proof.

Nothing hangs on our tentative suggestions about the relative explanatoriness of the above three proofs. We simply note that if these proofs all have some claim to explanatoriness, arguably, it is for different reasons. Moreover, it seems that explanatoriness comes in degrees; we are not dealing with an all-or-nothing concept here.

[12]This proof can be found in Goloumb (1956).

3 Dichotomy Theorems in Descriptive Set Theory

This case study is from descriptive set theory, a sub-area (or perhaps even a neighbouring area) of set theory that is more strongly connected to standard mathematics than the more abstract, logical areas of common set theory. Here we will outline an example from recent descriptive set theory that showcases different aspects of explanatoriness in the proofs of a class of theorems: *the dichotomy theorems*. This class encompasses a number of theorems which are based on a classical result by Cantor and then generalised to ever more abstract levels. Our focus here lies on the existence of two main proof types for these theorems, where the introduction of the second proof type signified a strong discontinuity in proving such theorems. In the following we will give the argument that both proof types present us with elements that make them explanatory, albeit in very different ways.

The main aim of giving this case study is to present an example about explanatoriness from very recent research, something that is missing in the literature on mathematical explanation. We think that such an example can shed further light on the complexities of mathematical explanation and its impact on recent research. In particular, our example will show that the search for explanatoriness is a major motivating factor for producing new and fruitful research, leading to fundamental discussions in the expert community and influencing the direction of research. Furthermore, we will show how a type of pluralism in explanatoriness can occur that is related to different sub-areas in mathematics and their respective communities.

Studying such an advanced example brings some peculiarities in presentation as well as content. We will usually not be able to give the whole proofs under consideration or even a detailed outline of them, as the mathematical background theory is too technical and would involve more setting up than we can accomplish here. Instead, we will present the main mathematical intuitions behind the results in question, leaving the details to textbooks and articles on the subject. However, we think that these limits in exposition are compensated by some unique insights with which examples from recent research provides us.

One advantage is the possibility to observe current discussions about explanatoriness and related questions by mathematicians themselves. We can see this more clearly in recent research because we have access not only to the formalised content as presented in textbooks or papers but also to informal material such as slides from talks, programmatic research statements and discussions with the mathematicians themselves.[13]

[13]For historical case studies, similar things can sometimes be found in correspondences or in cases the theorems are especially surprising (for such a case study see Hafner and Mancosu (2005)).

Further, when considering recent research we are often presented with a far more complex and advanced mathematical setting, making it hard for an average investigator into explanatoriness to develop an intuition of her own that goes beyond reconstructing the reasoning of the experts. Here we have to solely rely on the intuition about explanatoriness of the mathematicians in the relevant field thus making our approach more independent from our own views on the matter. The complexity of recent mathematical research can also highlight problems with accounts of explanatoriness that are not so clearly seen when considering examples from more elementary mathematics. One instance for this is that of the explanatoriness of parts of proofs. For example, one could ask whether for a proof to be explanatory, do all the proofs of all the lemmata, theorems or basic facts that are used in it have to be explanatory as well? Typical examples from contemporary mathematics will make issues such as these more pressing, as they usually rely on a wide-spread network of existing mathematical results.

Descriptive set theory is a part of set theory that studies definable subsets of the real numbers in certain topological spaces. Although it is considered to be a sub-discipline of set theory, it is also connected to more mainstream mathematics — areas such as topology and functional analysis.

Dichotomy theorems are a class of theorems that go back to the beginnings of (descriptive) set theory. Indeed, as with so many things in set theory, the earliest version of such a dichotomy theorem arose in the works of Cantor when searching for a solution to the Continuum Hypothesis (CH), the hypothesis that there are no infinite cardinals strictly between the size of the natural and the real numbers. One partial result by Cantor (1884) implies that the CH holds for closed sets, i.e. sets that contain all of their limit points. This was the starting point for a line of theorems, the set-theoretic dichotomy theorem (they state either-or results), that generalise Cantor's result step by step by considering every more abstract definable subsets of the real line. Together they provide detailed insight into the mathematical structure of the continuum and constitute one of the core areas of descriptive set theory.

Following the exposition of the history of set-theoretic dichotomy theorems provided in Miller (2012), we can see that the continuity in ever more general versions of set-theoretic dichotomy results did not transfer to the proof structure of these theorems. Instead, Miller (2012) points to a strong discontinuity between the proof of Cantor's theorem and early generalisations to Borel and analytic sets[14], on the one hand, and, on the other hand, the proofs of a later generalisation by Silver (1980) and subsequent work. There is a proof type for the earlier theorems that has a mathematical construction at heart that is considered to be especially informative (we will call this *the classical proof type*). For the later theorems such a type of proof was not available for some decades. Instead, these proof relied on very advanced

[14]The definitions will be provided in the next sections, when the relevant theorems are considered in more detail.

techniques from other areas of mathematical logic, in particular from recursion theory and general set theory (we will call this *the advanced logic proof type*). Only very recently B. Miller was able to find a proof that relies on comparable principles as the one for the older theorems (see for example Miller (2011)). For a partial timeline of the theorems and proofs, see the chart below:

Year	Dichotomy Theorems	Proof Type
1884	Cantor-Bendixon	Classical
1916	Hausdorff/Alexandroff	Classical
1917	Souslin	Classical
$\vdots$	$\vdots$	$\vdots$
1980	Silver	Advanced logic
$\vdots$	$\vdots$	$\vdots$
1990	Harrington-Kechris-Louveau	Advanced logic
1999	Kechris-Solecki-Todorcevic (KST)	Advanced logic
to the present	$\vdots$	$\vdots$
2010	KST, Silver etc.	(new) classical
to the present	$\vdots$	$\vdots$

As we will see below, that classical proof type relies on the construction of so-called Cantor-Bendixon derivatives (see Definition 2). The new classical proof scheme uses a similar construction while at the same time forgoing the use of advanced logical techniques that where introduced for the original proof of Silver's Theorem. In the literature, the classical and new classical proof scheme are therefore considered as one type of proof and the advanced logic proof as another.

In the following we will analyse these two types of proofs with respect to their explanatory value. As the main arguments for the explanatoriness of the proof types often involves several of the dichotomy theorems or the interrelations between them, we will mostly consider (parts of) the class of dichotomy theorems instead of one single theorem.[15]

3.1 The Classical Proof Schema

3.1.1 Early Dichotomy Theorems

As the classical proof schema goes back to the first version of a dichotomy theorem related to Cantor's result in Cantor (1884). We will start by considering this example in greater detail, as it provides the basic construction that is used in the classical proof schema: "We can think of the Cantor-Bendixon Theorem as a construction principle, since it gives us a method of building up the closed sets from the ap-

[15] However, there are a few that come up more often, because of their general significance. Amongst these are the Cantor-Bendixon theorem Cantor (1884), Souslin's theorem for analytic sets Souslin (1917), and Silver's theorem Silver (1980)).

parently simpler perfect sets and countable sets."(Moschovakis, 2009, 51)

Definition 1. [16]

- *A space is* perfect *if all its points are limit points. If P is a subset of a topological space X, we call P* perfect in X *if P is closed and perfect in its relative topology.*
- *A point x in a topological space X is a condensation point if every open neighbourhood of x is uncountable.*

Theorem 2 (Cantor-Bendixon)**.** *Let X be a Polish space (i.e. a separable completely metrisable space). Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C is countable and open.*[17]

This theorem can be proven in a quite simple manner, where we provide a construction of the partition of X:
$P = \{x \in X : x \text{ is a condensation point of } X\}$ and $C = X \setminus P$.[18]

However there is also a more general construction mechanism for the perfect set. The idea is that the perfect set we are looking for is a specific set in a decreasing transfinite sequence of closed subsets of the space X. The definition goes as follows:

Definition 2. *For any topological space X, let*

$$X' = \{x \in X : x \textit{ is a limit point of } X\}.$$

We call X' the Cantor-Bendixon derivative of X. Then X' is closed, X is perfect if and only if $X = X'$. Repeating this definition transfinitely many times gives rise to the following construction: Let X^α be the iterated Cantor-Bendixon derivatives for all ordinals α, defined as follows:

$$\begin{aligned} X^0 &= X, \\ X^{\alpha+1} &= (X^\alpha)', \\ X^\lambda &= \bigcap_{\alpha<\lambda} X^\alpha, \ \textit{if } \lambda \textit{ is limit.} \end{aligned}$$

Then $(X^\alpha)_{\alpha \in ORD}$ is a decreasing transfinite sequence of closed subsets of X.

It can now be shown that the perfect kernel P of the Cantor-Bendixon Theorem is X^{α_0}, where α_0 is a countable cardinal for which $X^\alpha = X\alpha_0$ for all $\alpha \geq \alpha_0$ (that such an α_0 exists follows from a more general fact about specific descending sequences).[19]

[16] Unless marked otherwise, the following definitions, theorems, and proofs are taken from Kechris (1995). See there for more details and background.

[17] To better see the connection with the later theorems, consider a different version of this theorem: Suppose that X is a Polish space and $C \subset X$ is closed. Then exactly one of the following holds: Either C is countable or there is a perfect subset of C.

[18] It remains to show the desired properties of P and C and prove the uniqueness of the partition; see (Kechris, 1995, 32).

[19] See (Kechris, 1995, 33–34) for the full proof.

3.1.2 The Explanatory Value of the Classical Proof Type

It is interesting to note how these two proof for the Cantor-Bendixon theorem are evaluated: Although the proof via condensation points is simpler that the proof via the derivative, the derivative proof is of greater importance: Two of the most standard textbooks of descriptive set theory, Kechris (1995) and Moschovakis (2009), point out that it is important for generalisations of the theorem, for example for analytic sets.[20]. But they also consider this proof to be "more informative" than the simpler proof via condensation points.[21]

We understand this use of "more informative" at least partly to mean "more explanatory": the more informative construction of the Cantor-Bendixon derivative provides us with greater insight into the general nature of these perfect sets. In other words, the easier construction (P as the set of condensation points) shows us what the perfect set looks like, but the construction via derivatives additionally shows us why the perfect set looks like that and this holds not only for one theorem, but all the dichotomy theorems before Silver's:

> One was therefore led naturally to the belief that the abundance of such derivatives is the driving force underlying the great variety of dichotomy theorems in descriptive set theory. (Miller, 2009, Introduction/A brief history)

What is this "driving force" in terms of explanatoriness? Considering the proof using the iterated Cantor-Bendixon derivatives, its explanatoriness arises from the way in which the perfect set is built. We construct a set that consists solely of limit points by "sorting out" more and more of the non-limit points and at the same time closing under limit points in transfinitely many steps. By construction, there is a point in these steps, α_0, where this process stabilises and naturally produces the desired set: the set that contains all and only its limit points. Following this construction we are able to "observe" how the perfect set grows out of the inherent properties of the construction (i.e. the definition of the derivative operation, the property that such an α_0 exists etc.).

This construction fits a type of explanatoriness given by Steiner (1978).[22] Steiner identifies two components explanatory proofs should have, namely they should refer to a characterising property of an entity in the theorem such that "from the proof it is evident that the result depends on the property" (Steiner, 1978, 143); and they should

[20] See for example (Moschovakis, 2009, 59–60) for Souslin's theorem

[21] See (Kechris, 1995, 33): "a more informative construction of the kernel", the kernel being the perfect set; Moschovakis (2009, 59) calls the whole argument "more informative".

[22] Here we do not claim, that Steiner's account of explanatoriness is the best or "right" one; we see it as one way of giving an account for the explanatoriness of a proof. In our case, mathematicians see the proof as explanatory so we use Steiner's theory to try to ascertain what features make this proof explanatory. See Colyvan and Resnik (forthcoming) for discussion of Steiner's account and its influence in contemporary philosophy of mathematics.

provide the possibility of generalising the feature(s) connected to this characterising property to produce other, related theorems and proofs. Both features are present in the classical proof type: The characterising property occurring in the theorem is the property of being perfect.[23] The classical proof refers to this property by showing how it can be produced through the derivative construction. So one fundamental feature of the property of being perfect is its construction via derivatives and this feature fulfils Steiner's second criterion of generalisability. In the case of the early dichotomy theorems, the classical proof fulfils this in a very strong way, as varying this feature produces proofs for ever more general dichotomy theorems.

Colyvan et al. (2018) discuss the local dependence-based explanation, a more general form of explanation stemming from considerations in the philosophy of science.[24] Local dependence-based explanations involve constructions of mathematical objects that then exhibit the desired property in a deep way, meaning that the property "[does not only follow] logically [...] from the construction in question, rather, we mean that the [...] property naturally arises from the core properties of the construction in question." (Colyvan et al., 2018, 14). This fits the classical proof quite well; we already used terms like "naturally produces" and "grows out of" above to describe the construction of the perfect set.

We conclude that the classical proof for the early dichotomy theorems is explanatory, based primarily on the intuition of mathematicians such as Miller, Kechris and Moschovakis. But it is interesting that this proof also fits well with a couple of philosophical accounts of mathematical explanation.

3.1.3 A "New" Classical Proof

As we already noted, the classical proof type was not (and could not be) used any longer for Silver's theorem and later generalisations. However, around 2010 Ben Miller developed a new proof for the Kechris-Solecki-Todorcevic theorem,[25] a very advanced general dichotomy result, that exhibits features that are very similar to the classical proof type. Miller introduces what he calls the "graph-theoretic approach to dichotomy theorems".[26] This approach developed out of the modern work on dichotomy theorems such as Silver's that generalise the older theorems by focusing on definable equivalence relations, i.e. subsets of $X \times X$ for a Polish space X that are definable in a certain manner (e.g. analytic, Borel etc.). Most notable are the new dichotomies introduced in (Harrington et al., 1990), (Hjorth and Kechris, 1997) and (Kechris et al.,

[23] Some of these theorems are also called "perfect set theorems".
[24] See (Colyvan et al., 2018, 13–16) for more information.
[25] See Kechris et al. (1999).
[26] See for example (Miller, 2012).

1999), where the latter one specifically produces dichotomy theorems for graphs.[27]

Miller based his work specifically on the notions used in (Kechris et al., 1999) using graphs and colourings of graphs to build up a new proof for the Kechris-Solecki-Todoercevic (KST) dichotomy theorem that had until then only a proof of the advanced logic type. This proof is much too complex to present here; an excellent survey of this approach is presented in (Miller, 2012). We therefore refer to this exposition for more details and only discuss its details against the backdrop of the classical Cantor-Bendixon derivative proof given above.

Most importantly, the heart of Miller's proof consists of a transfinite construction similar to the iterative Cantor-Bendixon derivative construction, only that here we transfinitely construct Borel sets on which a graph G has a Borel $\aleph_0$-colouring. Miller himself considers this construction to be the more complex analogue to the classical proof of the early dichotomy theorems: "[One] obtains a classical proof [...] resembling that of Cantor's perfect set theorem via the Cantor-Bendixson derivative." (Miller, 2012, 6)

Although presenting us with a much more complex situation than in the contexts of the earlier classical proofs, Miller's proof inherits its explanatory features from them. Miller himself remarks upon this fact in a lecture given on his work on the subject, when his proof was still work in progress (emphasis added):

> The new ideas described here appear to be leading towards a *classical explanation* of descriptive set-theoretic dichotomy theorems. [...] the new proofs restore the intuition that the abundance of derivatives is at the heart of the matter. (Miller, 2009, Conclusions/Advantages of the new technique)

Returning to Steiner's account, we see a very similar picture to the classical proof of the early theorems: we have a construction that builds up the characterising feature we are after, the $\aleph_0$-colouring of a graph. The generalisability is a strong point in favour of this proof. Most of (Miller, 2012) is concerned with presenting various ways in which the graph-theoretic approach can be varied to produce a cornucopia of variations, specifications, and generalisations of the previously known dichotomy results. The generalisability of the graph-theoretic proof is therefore much greater than that of the standard classical proof,which ceased to be generalisable for Silver's theorem and later ones.

When looking at the local dependence-based model of explanation the situation is not that clear, as it will be nearly impossible for the average investigator into mathematical explanation to judge the "naturalness" of the graph-theoretic approach to dichotomy theorems. The mathematical details are too complex and the mathematical theory on which it rests is too expansive to be clearly and deeply understood by more than a handful of experts in this area. We therefore have to

[27] A graph is an irreflexive symmetric subset of the product of the underlying Polish space.

rely on the intuition of the experts to find evidence for explanatoriness. In our case, this intuition points towards a strong analogy in the general set-up of the classical early proofs and the proof in the graph-theoretic setting, going so far as to see the different types of transfinite constructions used in the proof to produce the same type of entity, namely derivatives. Furthermore, both constructions are judged to be explanatory because of *the same way* in which they produce these derivatives (see (Miller, 2009) and (Miller, 2012)). So based on our judgement that the classical proof is explanatory in the way in which it naturally gives rise to the property in question, the graph-theoretic approach does the same in its respective construction.

3.2 The Advanced Logic Proof

We now turn to the advanced logic proof type that was initially developed by Silver to prove a further generalisation of the early dichotomy theorems:

Theorem 3 (Silver). *If X is a Polish space and $E \subseteq X^2$ a $\mathbf{\Phi^1_1}$ equivalence relation, then either E has only countable many equivalence classes or there exists a perfect set of pairwise inequivalent elements.*[28]

The proof given by Silver (1980) and the improvement thereof by Harrington (1976) and Louveau (1979) are of a totally different kind than proofs by derivative-style constructions. The Silver proof makes use of an advanced logical set-up, relating to the other, perhaps more "meta-mathematical" fields of set theory and mathematical logic in general.

Silver himself used three logical tools in the proof of his theorem, namely the technique of forcing — developed to show undecidability results in set theory — methods from effective descriptive set theory — a recursion theoretic analog to descriptive set theory — and iterates of the Power Set axiom. Harrington simplified this by getting rid of the last technique (therefore he called it a "powerless" proof in Harrington (1976)), but he still relied on forcing and methods from recursion theory (for a different version see also Harrington and Shelah (1982)). Finally, Louveau (1979) produced a proof that formulates the forcing part of Harrington's proof in a topological manner using the so-called Gandy-Harrington topology.[29] It therefore does not use forcing,

[28]The expressions $\Sigma^n_n, \Phi^n_n, \Delta^n_n$ marks the complexity of a set (or formula that defines a set) with respects to some mathematical concept. According to this complexity hierarchies of sets can be given. One example is the Borel hierarchy, where the complexity is measured with respect to taking countable unions and complements; a different example is the Lévy hierarchy where formulas are more complex the more often their unbounded quantifiers change. Here Φ^1_1 refers to a co-analytic set, meaning that it is a complement of an analytic (Σ^1_1) set; it is part of the Projective hierarchy.

[29]For a definition see (Harrington et al., 1990, 917)

but still relies on techniques from topology and effective descriptive set theory.[30]

Both the proof of Harrington[31] and the proof type of Louveau are still in use, however the topological (and not forcing-related) approach of the latter seems to be used more often in the proof of recent dichotomy results. In particular, the use of effective methods is more essential than the use of forcing because it is used in both proof types and because the relationship between effective descriptive set theory (EDST) and the non-effective, classical descriptive set theory (CDST) has more wide-ranging applications, unrelated to dichotomy theorems. We will therefore focus our account of explanatoriness on this advanced logic part of the proof type.

Effective descriptive set theory goes back to the work of S. Kleene.[32] Initially he developed it outside of descriptive set theory by using methods from recursion theory to study sets. Instead of considering sets that are definable in a certain manner, one studies their recursion-theoretic analogues (that still are definable in a certain manner). As a basic example, a set is recursive if it is computable in the sense that there exists an algorithm that always decides in a finite amount of time whether something is an element of the set or not. Likewise, a recursively-enumerable set is one in which the algorithm decides in the above way whether something is an element, but can sometimes return no answer. These notions can be spelled out mathematically via functions and are basic notions in the field of recursion theory.

It turns out, that there is a whole field of research analogous to CDST that makes use of these recursion-theoretic notions. As an example let us consider basic sets studied in CDST: Here a $\boldsymbol{\Delta^0_1}$ pointclass (in bold font) is one where the elements are closed and open. It corresponds to the (non-bold font) Δ^0_1 pointclass that consists of the recursive pointsets; likewise the $\boldsymbol{\Sigma^0_1}$ pointclass (open sets) correspond to the Σ^0_1 pointclass (recursively enumerable sets) and so on (the effective versions are the parameter free versions of the classical definitions). Based on these relations, whole hierarchies of sets can be build up in CDST that have an analogous version in EDST, giving rise to theorems that have classical versions (the CDST version) and effective versions in the formulation of EDST.

Based on Kleene's work, Addison developed the exact analogies between CDST and EDST. Since then, mathematical work has shown that this is not a local phenomenon but holds on a fundamental level in many areas of DST.[33]

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[30] Such a topological version of Harrington's proof is also given in (Martin and Kechris, 1980), however, according to (Harrington et al., 1990, 907) it is based on seminar notes of Louveau.

[31] See for example (Miller, 1995, 113–115).

[32] For short introductions into EDST with reference to dichotomy theorems see (Harrington et al., 1990, chap. 3) or (Martin and Kechris, 1980). For a more general account see for example (Moschovakis, 2009).

[33] For more on the historical development see Kanamori (1995) and (Moschovakis, 2009, Introduction).

It therefore turns out that CDST and EDST are very tightly interconnected up to a point where both can be seen to be refinements of the other (emphasis in the original):

> Over the years and with the work of many people, what was first conceived as "analogies" developed into a general theory which yields in a unified manner both the classical results and the theorems of the recursion theorists; more precisely, this effective theory yields *refinements* of the classical results. (Moschovakis, 2009, 5)[34]

We will now study what role this relation plays in the proof of dichotomy theorems; in particular we will study the example of the KST theorem (Theorem 6.3 in (Kechris et al., 1999)) as this was also the theorem for which the first "new" classical proof was developed.

Theorem 4 (Kechris-Solecki-Todorcevic)**.** *Let X be a Polish space and $\mathcal{G}(X,R)$ an analytic graph (i.e. $R \subseteq X^2$ is analytic). Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$ *or*
2. $\mathcal{G}_0 \leq_c \mathcal{G}$.

Here $\mathcal{G}_0$ is a certain minimal graph and $\chi_B(\mathcal{G})$ the Borel chromatic number of the graph $\mathcal{G}$ (all of the definitions can be found in (Kechris et al., 1999)). We don't need to understand the exact definitions of the notions involved here to study this as an example of the set-up of the advanced logic proof type. For that let us look at how the authors of the theorems begin its proof. Directly after stating the theorem, they continue in the following manner:

> This result is proved using methods of effective descriptive set theory, in particular the Gandy-Harrington topology. In fact one has the following effective version (which by standard arguments implies the above theorem) (Kechris et al., 1999, 21)

They proceed to give the effective version:

Theorem 5. *Let $\mathcal{G} = (\mathbb{N}^{\mathbb{N}}, R)$ be a Σ^1_1-graph (i.e. $R \subseteq (\mathbb{N}^{\mathbb{N}})^2$ is Σ^1_1). Then exactly one of the following holds:*

1. *There is a Δ^1_1 colouring $c : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ for $\mathcal{G}$;*
2. $\mathcal{G}_0 \leq_c \mathcal{G}$.

Again, we don't need to grasp all the relevant definitions to see the main point of this approach: we have a theorem that can be presented in two versions, one using classical notions and one using effective notions from descriptive set theory (e.g. in the first, the graph is given as analytic, which is boldface $\boldsymbol{\Sigma}^1_1$; in the second, it is lightface Σ^1_1, the effective version). This procedure goes back to proofs of dichotomy

[34] For the other direction of refinements see Moschovakis (2010)

theorems in (Harrington et al., 1990), where the classical results are obtained by relativising the effective version to a parameter.[35]

To summarise the above: The advanced logic proof does not show the connections with derivatives; in fact, it represents a clear break with the proofs of the earlier dichotomy theorems. Instead, it links the dichotomy theorems to EDST as well as providing a further example for the connection of CDST and EDST. Thereby it both uses and strengthens the interconnection between CDST and its effective counterpart. Through this interconnection, the proof situates the dichotomy theorems in the larger context of this general feature of descriptive set theory and therefore unifies dichotomy theorems with other results that make use of this feature as well.

This interconnection also lies at the heart of the explanatory value that advanced logic proof provides us with. We mentioned above that the classical proof type fits the local dependency-based account of explanation as given in (Colyvan et al., 2018). There the authors also outline another kind of explanation, the global unification-based explanation:[36]

> [A] theorem is explained by deriving the theorem using a proof that unifies many diverse theorems, and thereby showing that the theorem is part of a very general, perhaps utterly pervasive, pattern of theorems in mathematics. (Colyvan et al., 2018, 15)

As we are already looking at a certain class of theorems — the dichotomy theorems in DST — that is defined by the common characteristics of its members, let us rephrase this for this situation: a class of theorems is explained by deriving its members using a proof type that shows that the class of theorems is part of a very general, perhaps utterly pervasive, pattern that is characteristic for the area of mathematics it is a part of.

The advanced logic proof type provides us with this kind of explanatory value: Looking at the pervasive pattern of the close analogies between CDST and EDST that have shown themselves in various areas before the advanced logic proof, shows how dichotomy theorems are a very fruitful part of this pattern. As one example, consider how this back and forth between DST and its effective version is used in the proofs of dichotomy theorems such as Theorem 6.3 and Theorem 6.4 in (Kechris et al., 1999, 21).

This back and forth can also be applied to earlier dichotomy theorems that are usually proven via the classical proof and give rise to

[35]See (Harrington et al., 1990, 916) for an introduction to such relativisations; the complete proof on which the proof of the KST theorem depends on in a crucial manner can be found in (Harrington et al., 1990, 919–927).

[36]This account can also be related to Kitcher's unificatory account for explanations in mathematics and the sciences, see for example (Kitcher, 1989). Indeed we think a case can be made that the advanced logic type is a very good candidate for what Kitcher calls the explanatory store for a system of beliefs.

concrete examples of explanation. For an example that provides a concrete way in which the relation to EDST can be explanatory consider the following:[37]

Theorem 6 (Souslin, 1917)**.** *Every uncountable analytic set has a non-empty perfect subset.*

Moschovakis points out[38] that an effective version of this result provides an explanation of the theorem "in terms of definability rather than size: an analytic set P has a perfect subset if it has at least one member which is *more difficult to define* than P itself." Such an effective version is due to a result by Harrison from 1967 (see Harrison (1967)). After Silver's theorem provided a proof of the advanced logic type for the first time, it thereby established the connections to effective DST. So although Souslin's theorem can be proven directly by an easier classical-type proof, the connection to effective DST provide a different kind of explanation.

This is also emphasised by Ramez Sami who points out the added value the advanced logic proofs provide:

> The present note greatly antedates the more recent "back-to-classical" movement developed *con maestria* [by Ben Miller]. We still hold that effective methods will often yield simpler proofs of stronger and finer results. (Sami, 2019, 4039)

A similar sentiment with regards to explanatoriness was expressed by Sami in an online event with one of the authors on 25 November 2020. We therefore have a similar situation for the advanced logic proof type as with the classical proof type: intuitions by mathematicians tell us that the proof is explanatory and we can back this up by showing how it fits with at least some of the philosophical accounts of mathematical explanation.

4 Conclusions

The dichotomy theorems in DST are good examples of how questions of explanatoriness not only play a role in but also can direct mathematical research.[39] The lack of a classical proof for Silver's theorem and further generalisations was the main motivation for searching, and finally producing, a different proof that supplies us with a deeper understanding of the theorem, showing not only that it holds but also *why* it holds.[40] This new proof led to a reconsideration of the field of dichotomy results in descriptive set theory, not only by re-proving already known theorems, but also by introducing a new approach to this area that produces new theorems and new generalisations.

[37]The authors would like to thank Yiannis Moschovakis for pointing out this example.

[38]In private communication with the authors; e-mail from 12 September 2020.

[39]This has been called into question for example by Zelcer (2013).

[40]B. Miller confirmed that this was his main motivation in reproving Silver's theorem and generalisations thereof. (Personal communication with one of the authors; virtual meeting on 8 July 2020)

These arguments and others given in Section 3.1 show that the classical proof can be considered to be explanatory. However, we have also seen that there are arguments for the explanatoriness of the advanced logic proof. This points towards a pluralist picture of what explanation is in mathematics. Taking the intuitions of mathematicians as our evidence, it might be argued that one type of explanatoriness is as valid as the other, precisely because mathematicians' intuitions do not converge.

If we accept such a pluralist picture of explanatoriness, we might ask where this pluralism comes from. For the case of dichotomy theorems, let us outline one possibility.[41] Here, we can relate the different intuitions about explanatoriness to the communities of different subareas of DST. So, for researchers who primarily consider dichotomy theorems from the classical point of view, focusing on inherent similarities of the dichotomy theorems "from within" (i.e. the way in which they are built up via derivatives and the inherent properties of objects like perfect sets), the classical proof is more explanatory because it provides one with a vivid picture of how the inherent properties of the objects in question give rise to the various theorems.

If one approaches the dichotomy theorems from the viewpoint of general descriptive set theory, where a lot of research has gone into the interconnections between CDST and EDST, we consider the theorems "from outside" and see them as one example for a more general pattern in the theory. The advanced logic proof is thus seen as more explanatory because it ties in with these general patterns.

Indeed, such sentiments were expressed by the mathematicians themselves. B. Miller, for example, mentioned that people who deeply work with dichotomy theorems prefer the classical proof, while it might be different for people whose work is more closely connected to effective DST.[42] So what mathematicians see as explanatory seems to be motivated by the epistemic interest they pursue in their practice. These epistemic interests can be the search for basic entities that lie at the heart of a collection of theorems — the search for overarching patterns in mathematical reasoning etc.

We conclude that our main case study — the dichotomy theorems in descriptive set theory — as well as the elementary example we started with — Fermat's Little Theorem from elementary number theory — suggest a kind of explanatory pluralism. It would appear that whether a proof is seen as more explanatory than another depends on the epistemic interests and goals of the practitioners. These epistemic interests can be connected to (unofficial) research agendas or other practices of

[41] We do not claim that this is the only or even the main reason for every case of pluralism in explanations. There is also the difficult task of distinguishing explanatoriness from other virtues found in good mathematics and, indeed, determining what makes for good mathematical proofs — such as those that Paul Erdös referred to as coming from God's book of the best proofs or simply "from the book" (Aigner and Ziegler (2010)). The virtues of good mathematics include beauty, simplicity, elegance explanatory power, and so on (see Tao (2007); Inglis and Aberdein (2015)).

[42] Personal communication with one of the authors; virtual meeting on 8 July 2020.

sub-communities of a discipline. In this sense, the explanatoriness or otherwise of a proof must be assessed in relation to the wider mathematical context in which it sits. Perhaps this is not surprising. But accepting such a context-sensitive notion of explanation in mathematics would be a serious blow for those of us with monist leanings — those who seek a single, elegant, and unified account of mathematical explanation. For better or worse, we have good reason to believe that explanation in mathematics is more complex and more interesting than the monist would wish.

This raises interesting questions about the nature of any potential pluralism. For instance, if there is explanatory pluralism in mathematics, does this arise because there are different levels of explanation in operation in mathematics? As we have seen, there are different levels of abstraction in mathematics and explanations arising at these different levels. It seems a good working hypothesis that these different explanations correspond to different levels of explanation. But an alternative might be that the different explanations are in fact answering different why questions—perhaps why questions pitched at different levels of abstraction. What might these different why questions be? In our experience, mathematicians tend to be interested in explanation in an apparently unitary sense: *why does the theorem in question holds*. On the face of it, at least, this looks like a single why question but appearances may be deceptive here. Indeed, the lack of precision in the question "why does the theorem hold?" may be hiding ambiguity about exactly what is being asked. Is it an ambiguous question or is it inviting different levels of explanation?

These interesting issues require much further work. Any attempt to settle them now would be misguided. As things currently stand, philosophical work on mathematical explanation is in its infancy. In our view, work on mathematical explanation requires more case studies to draw upon before we tackle some of the questions just raised about the nature of any potential explanatory pluralism in mathematics.[43] The purpose of this paper is to provide a couple more case studies that we trust will be helpful in addressing some of the many puzzles about mathematical explanation.[44]

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[43] For obvious reasons, we do not want to develop a philosophical account of mathematical explanation based on a few examples from one or two areas on mathematics. Just as an account of scientific explanation needs to work across all areas of empirical science, an account of mathematical explanation should work across all areas of mathematics.

[44] We'd like to thank Neil Barton, Hazel Brickhill, Erik Curiel, Clio Cresswell, Ben Eva, Stephan Hartmann, Anthony Henderson, Leon Horsten, Silvia Jonas, Deborah Kant, Daniel Kuby, Randall McCutcheon, Ben Miller, Yiannis Moschovakis, Hannes Leitgeb, Miklós, Rédei, Máté Wierdl, Geordie Williamson, and Alistair Windsor for very helpful discussions on the topic of this paper. We are especially indebted to Alistair WIlson's for detailed comments on an earlier draft of this paper and for his many helpful suggestions. Carolin Antos's work on this paper was supported by the Volkswagen Foundation (Freigeist Grant) and the Zukunftskolleg (University of Konstanz, Germany). Mark Colyvan's work on this paper was supported by an Australian Research Council Future Fellowship (grant number: FT110100909) and a Carl Friedrich von Siemens Research Award of the Alexander von Humboldt Foundation.

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