https://doi.org/10.15407/dopovidi2023.02.010
UDC 517.5

V.Ya. Gutlyanskiĭ[1,2], https://orcid.org/0000-0002-8691-4617
V.I. Ryazanov[1, 2], https://orcid.org/0000-0002-4503-4939
E.A. Sevost'yanov[1,3], https://orcid.org/0000-0001-7892-6186
E. Yakubov[4], https://orcid.org/0000-0002-2744-1338

[1] Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slov'yansk
[2] Institute of Mathematics of the NAS of Ukraine, Kyiv
[3] Zhytomyr Ivan Fanko State University, Zhytomyr
[4] Holon Institute of Technology, Israel
E-mail: vgutlyanskii@gmail.com, vl.ryazanov1@gmail.com,
esevostyanov2009@gmail.com, eduardyakubov@gmail.com

Hydrodynamic normalization conditions in the theory of degenerate Beltrami equations

Presented by Corresponding Member of the NAS of Ukraine V.Ya. Gutlyanskiĭ

We study the existence of normalized homeomorphic solutions for the degenerate Beltrami equation $f_{\bar{z}} = \mu(z) f_z$ *in the whole complex plane* $\mathbb{C}$, *assuming that its measurable coefficient* $\mu(z)$, $|\mu(z)|<1$ *a. e., has compact support and the degeneration of the equation is controlled by the tangential dilatation quotient* $K_\mu^T(z, z_0)$. *We show that if* $K_\mu^T(z, z_0)$ *has bounded or finite mean oscillation dominants, or satisfies the Lehto type integral divergence condition, then the Beltrami equation admits a regular homeomorphic* $W_{\mathrm{loc}}^{1,1}$ *solution f with the hydrodynamic normalization at infinity. We also give integral criteria of Calderon-Zygmund or Orlicz types for the existence of the normalized solutions in terms of* $K_\mu^T(z, z_0)$ *and the maximal dilatation* $K_\mu(z)$.

Keywords: *BMO, bounded mean oscillation, FMO, finite mean oscillation, degenerate Beltrami equations, hydrodynamic normalization.*

1. Introduction. It is well known that quasiconformal mappings and functions and their generalizations, the mathematical basis for the study of which is the analytic and geometric theory of linear and quasilinear partial differential equations of elliptic type, are a powerful tool in the theory of two-dimensional subsonic compressible flows (see, e. g., [1, Ch. 2]). The Beltrami PDE, that generates quasiconformal mappings, plays here a crucial role. Among the variety of approaches related to the study of such flows, special attention is paid to the proof of existence theorems for homeomorphisms of the whole complex plane that satisfy the degenerate Beltrami equation, i. e. when the condition of

Citation: Gutlyanskiĭ V.Ya., Ryazanov V.I., Sevost'yanov E.A., Yakubov E. Hydrodynamic normalization conditions in the theory of degenerate Beltrami equations. *Dopov. Nac. akad. nauk Ukr.* 2023. No 2. P. 10—17. https://doi.org/10.15407/dopovidi2023.02.010

uniform ellipticity for the equation is violated. Some effective criteria for the existence of such homeomorphic solutions can be found, e. g., in [2-9], see also the references therein. Taking into account the behaviour of a subsonic flow at infinity (see [1, Ch. 3]), the existence of theorems for homeomorphic solutions of the degenerate Beltrami equation with hydrodynamic normalization at the infinity acquires a special role. In this paper, we just give some integral criteria for the existence of such solutions both in terms of the tangential dilatation quotient and the maximal dilatation coefficient.

Let D be a domain in the complex plane $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a. e. in D. A *Beltrami equation* is an equation of the form

$$f_{\bar{z}} = \mu(z) f_z \tag{1}$$

with the formal complex derivatives $f_{\bar{z}} = (f_x + if_y)/2$, $f_z = (f_x - if_y)/2$, $z = x + iy$, where f_x and f_y are usual partial derivatives of f in x and y, correspondingly. The function μ is said to be the *complex coefficient* for the Beltrami equation. The measurable function

$$K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{2}$$

is called the *maximal dilatation* of equation (1) at point z. The Beltrami equation is called *degenerate* if *ess* sup $K_\mu(z) = \infty$.

It is known that if K_μ is bounded, then the Beltrami equation has homeomorphic solutions (see, e. g., historic comments with relevant references in the monographs [2] and [3]). The corresponding criteria on the existence of homeomorphic solutions for the degenerate Beltrami equations were formulated both in terms of K_μ and the more refined quantity

$$K_\mu^T(z, z_0) := \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \tag{3}$$

that takes into account not only the modulus of the complex coefficient μ but also its argument.

This quantity is called the *tangent dilatation quotient* of the Beltrami equation (1) with respect to a point $z_0 \in \mathbb{C}$. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z, z_0) \leqslant K_\mu(z) \quad \forall\, z \in D, \qquad z_0 \in \mathbb{C}. \tag{4}$$

2. The main lemma. Assuming that the complex coefficient $\mu(z)$, $|\mu(z)| < 1$ a. e. in $\mathbb{C}$, has compact support, we study the existence of homeomorphic solutions for the degenerate Beltrami equation (1) in the whole complex plane $\mathbb{C}$ with *hydrodynamic normalization*: $f(z) = z + o(1)$ as $z \to \infty$. Recall also that a function $\mathbb{C} \to \mathbb{C}$ in Sobolev's class $W^{1,1}_{\mathrm{loc}}$ is called a *regular solution* of the Beltrami equation (1) if f satisfies (1) a. e. and its Jacobian $J_f(z) > 0$ a. e. in $\mathbb{C}$.

Lemma 1. *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with compact support S, $|\mu(z)| < 1$ a. e. and $K_\mu \in L^1(S)$. Suppose that, for every $z_0 \in S$, there is a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \varepsilon_0) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 = \varepsilon(z_0) > 0$, such that*

$$I_{z_0}(\varepsilon) := \int_\varepsilon^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0) \tag{5}$$

and

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\cdot\psi_{z_0,\varepsilon}^2(|z-z_0|)dm(z)=o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon\to 0 \quad \forall\, z_0\in S\,. \tag{6}$$

Then equation (1) *has a regular homeomorphic solution* f *with* $f(z)=z+o(1)$ *as* $z\to\infty$.

Here and further $dm(z)$ stands for the Lebesgue measure in $\mathbb{C}$.

Proof. By Lemma 3 and Remark 2 in [6] the Beltrami equation (1) has a regular homeomorphic solution f in $\mathbb{C}$ under the hypotheses on μ given above. Note that f is holomorphic and univalent (one-to-one), i. e. conformal, and with no zeros outside of a closed disk $|z|\leqslant R$ because the support S of μ is compact.

Let us consider the function $F(\varsigma)=f(1/\varsigma)$, $\varsigma\in\mathbb{C}_0:=\overline{\mathbb{C}}\setminus\{0\}$, $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, that is conformal in a punctured disk $\mathbb{D}_r\setminus\{0\}$, where $\mathbb{D}_r=\{\varsigma\in\mathbb{C}:|\varsigma|<r\}$, $r=1/R$, and 0 is its isolated singular point. In view of the Casorati-Weierstrass theorem (see, e. g., Proposition II.6.3 in [10]), 0 cannot be an essential singular point because of the mapping F is homeomorphic.

Moreover, 0 cannot be a removable singular point of F. Indeed, let us assume that F has a finite limit $\lim\limits_{\varsigma\to 0}F(\varsigma)=c$. Then the extended mapping $\tilde{F}$ is a homeomorphism of $\overline{\mathbb{C}}$ into $\mathbb{C}$. However, by a stereographic projection of $\overline{\mathbb{C}}$ is homeomorphic to the sphere $\mathbb{S}^2$ and, consequently, by the Brouwer theorem on the invariance of domain the set $C:=\tilde{F}(\overline{\mathbb{C}})$ is open in $\overline{\mathbb{C}}$ (see, e. g., Theorem 4.8.16 in [11]). In addition, the set C is compact as a continuous image of the compact space $\overline{\mathbb{C}}$. Hence the set $\overline{\mathbb{C}}\setminus C\neq\varnothing$ is also open in $\overline{\mathbb{C}}$. The latter contradicts the connectivity of $\overline{\mathbb{C}}$ (see, e. g., Proposition I.1.1 in [10]).

Thus, 0 is a (unique) pole of the function F in the disk. Hence the function $\Phi(\zeta):=1/F(\zeta)$ has a removable singularity at 0 and $\Phi(0)=0$. By the Riemann extension theorem (see, e. g., Proposition II.3.7 in [10]), the extended function $\tilde{\Phi}$ is conformal in $\mathbb{D}_r$. By the Rouche theorem $\tilde{\Phi}'(0)\neq 0$ (see, e. g., Theorem 63 in [12]), and, consequently, the function $\tilde{\Phi}$ has the expansion of the form $c_1\zeta+c_2\zeta^2+\ldots$ in the disk $\mathbb{D}_r$ with $c_1\neq 0$. Hence

$$f(z)=\frac{1}{\Phi(1/z)}=\frac{1}{c_1z^{-1}+c_2z^{-2}+\ldots}=\frac{z}{c_1}\left(1+\frac{c_2}{c_1}z^{-1}+\ldots\right)^{-1}=c_1^{-1}z-c_1^{-2}c_2+o(1)$$

along the set $\{z\in\mathbb{C}:|z|>R\}$, i. e. the function $c_1f(z)+c_2/c_1$ gives the desired regular homeomorphic solution of the Beltrami equation with the hydrodynamic normalization at infinity.

In particular, by relations (4) we obtain from Lemma 1 the following.

Corollary 1. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be with compact support* S, $|\mu(z)|<1$ *a. e.,* $K_\mu\in L^1(S)$ *and* $\psi:(0,\varepsilon_0)\to(0,\infty)$, $\varepsilon_0>0$, *be a measurable function with*

$$\int\limits_0^{\varepsilon_0}\psi(t)dt=\infty, \qquad \int\limits_\varepsilon^{\varepsilon_0}\psi(t)dt<\infty \qquad \forall\varepsilon\in(0,\varepsilon_0). \tag{7}$$

Suppose that

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z)\cdot\psi^2(|z-z_0|)dm(z)\leqslant O\left(\int\limits_\varepsilon^{\varepsilon_0}\psi(t)dt\right) \quad as\, \varepsilon\to 0 \quad \forall z_0\in S\,. \tag{8}$$

Then the Beltrami equation (1) *has a regular homeomorphic solution f with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

3. BMO and FMO. Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of *bounded mean oscillation* in D, abbr. $u\in \mathrm{BMO}(D)$, if $u\in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|}\int_B |u(z)-u_B|\,dm(z)<\infty, \tag{9}$$

where the supremum is taken over all discs in D and

$$u_B=\frac{1}{|B|}\int_B u(z)\,dm(z).$$

The class BMO was introduced by John and Nirenberg (1961) in the paper [13] and soon became an important concept in harmonic analysis, partial differential equations and related areas (see, e. g., [14]).

Following [15], given a domain D in $\mathbb{C}$, we say that a function $\varphi: D\to R$ has *finite mean oscillation* at a point $z_0\in D$, abbr. $\varphi\in \mathrm{FMO}(z_0)$, if

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{\pi\varepsilon^2}\int_{B(z_0,\varepsilon)}|\varphi(z)-\tilde{\varphi}_\varepsilon(z_0)|\,dm(z)<\infty, \tag{10}$$

where $\tilde{\varphi}_\varepsilon(z_0)$ is the mean value of $\varphi(z)$ over $B(z_0,\varepsilon):=\{z\in\mathbb{C}: |z-z_0|<\varepsilon\}$.

The following statement follows by the triangle inequality.

Proposition 1. *If, for a collection of numbers* $\varphi_\varepsilon\in R$, $\varepsilon\in(0,\varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{\pi\varepsilon^2}\int_{B(z_0,\varepsilon)}|\varphi(z)-\varphi_\varepsilon|\,dm(z)<\infty, \tag{11}$$

then φ *is of finite mean oscillation at* z_0.

In particular, choosing here $\varphi_\varepsilon\equiv 0$, $\varepsilon\in(0,\varepsilon_0]$ in Proposition 1, we obtain the following.

Corollary 2. *If, for a point* $z_0\in D$,

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{\pi\varepsilon^2}\int_{B(z_0,\varepsilon)}|\varphi(z)|\,dm(z)<\infty, \tag{12}$$

then φ *has finite mean oscillation at* z_0.

Versions of the next lemma have been first proved for the class BMO in [5]. For the FMO case, see the paper [15] and the monographs [8] and [9].

Lemma 2. *Let* D *be a domain in* $\mathbb{C}$ *and let* $\varphi: D\to\mathbb{R}$ *be a non-negative function of the class* $\mathrm{FMO}(z_0)$ *for some* $z_0\in D$. *Then*

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0}\frac{\varphi(z)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2}=O\left(\log\log\frac{1}{\varepsilon}\right)\quad \text{as}\quad \varepsilon\to 0 \tag{13}$$

for some $\varepsilon_0\in(0,\delta_0)$ *where* $\delta_0=\min(e^{-e},d_0)$, $d_0=\sup_{z\in D}|z-z_0|$.

4. The main results. Choosing $\psi(t)=1/(t\log(1/t))$ in Lemma 1, we obtain by Lemma 2 :

Theorem 1. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be a measurable function with a compact support* S *and* $|\mu(z)|<1$ *a. e. Suppose that* $K_\mu^T(z,z_0)\leqslant Q_{z_0}(z)$ *a. e. in* U_{z_0} *for every point* $z_0\in S$, *a neighbourhood* U_{z_0} *of* z_0 *and a function* $Q_{z_0}:U_{z_0}\to[0,\infty]$ *in the class* FMO(z_0). *Then the Beltrami equation* (1) *has regular homeomorphic solutions* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

Since $K_\mu^T(z,z_0)\leqslant K_\mu(z)$ for all z and $z_0\in C$, we obtain the following consequence.

Corollary 3. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S *and* $|\mu(z)|<1$ *a. e. Suppose that* $K_\mu(z)\leqslant Q(z)$ *a. e. in* $\mathbb{C}$ *with* $Q:\mathbb{C}\to[1,\infty]$ *in the class* BMO. *Then the Beltrami equation* (1) *has regular homeomorphic solutions* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

By Corollary 2, we obtain the next consequence of Theorem 1.

Corollary 4. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S, $|\mu(z)|<1$ *a. e.,* $K_\mu\in L^1(D)$ *and*

$$\overline{\lim_{\varepsilon\to 0}}\frac{1}{\pi\varepsilon^2}\int\limits_{B(z_0,\varepsilon)}K_\mu^T(z,z_0)\,dm(z)<\infty\qquad\forall\, z_0\in S. \tag{14}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

Next, if we take in Lemma 1 $\psi(t)=1/t$, we come to Calderon-Zygmund type conditions.

Theorem 2. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support S,* $|\mu(z)|<1$ *a. e.,* $K_\mu\in L^1(S)$ *and, for some* $\varepsilon_0>0$,

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0}K_\mu^T(z,z_0)\frac{dm(z)}{|z-z_0|^2}=o([\log 1/\varepsilon]^2)\quad as\ \varepsilon\to 0\quad\forall\, z_0\in S. \tag{15}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions f with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

Remark 1. Choosing in Lemma 1 the function $\psi(t)=1/(t\log 1/t)$ instead of $\psi(t)=1/t$, we are able to replace (15) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0}\frac{K_\mu^T(z,z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2}=o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right)\quad as\ \varepsilon\to 0. \tag{16}$$

In general, we are able to give here the whole scale of the corresponding conditions, using the function $\psi(t)$ in terms of the iterated logarithms: $1/(t\log 1/t\cdot\log\log 1/t\cdot\ldots\cdot\log\ldots\log 1/t)$.

If we take in Lemma 1 $\psi_{z_0,\varepsilon}(t)\equiv\psi_{z_0}(t):=1/[tk_\mu^T(z_0,t)]$, where $k_\mu^T(z_0,r)$ is the integral mean of $K_\mu^T(z,z_0)$ over the circle $S(z_0,r):=\{z\in\mathbb{C}:|z-z_0|=r\}$, we arrive at the Lehto type criterion.

Theorem 3. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S, $|\mu(z)|<1$ *a. e.,* $K\ \in L\ (S)$ *and, for some* $\varepsilon_0>0$,

$$\int\limits_0^{\varepsilon_0}\frac{dr}{rk_\mu^T(z_0,r)}=\infty\qquad\forall\, z_0\in S. \tag{17}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

Corollary 5. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S, $|\mu(z)|<1$ *a. e.,* $K_\mu\in L^1(S)$ *and*

$$k_\mu^T(z_0,\varepsilon)=O(\log 1/\varepsilon) \qquad as\ \varepsilon\to 0 \qquad \forall\, z_0\in S. \tag{18}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

Remark 2. In particular, the conclusion of Corollary 5 holds if

$$K_\mu^T(z,z_0)=O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z\to z_0 \quad \forall\, z_0\in S. \tag{19}$$

Moreover, (18) can be replaced by the weaker conditions

$$k_\mu^T(z_0,\varepsilon)=O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\, z_0\in S\,. \tag{20}$$

Combining Theorems 2.5 and 3.2 in [7] and Theorems 3, we come to the Orlicz type conditions.

Theorem 4. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S, $|\mu(z)|<1$ *a. e.,* $K_\mu\in L^1(S)$ *and, for a neighborhood* U_{z_0} *of* z_0,

$$\int\limits_{U_{z_0}}\Phi_{z_0}(K_\mu^T(z,z_0))dm(z)<\infty \qquad \forall\, z_0\in S\,, \tag{21}$$

where $\Phi_{z_0}:[0,\infty]\to[0,\infty]$ *is a convex non-decreasing function with*

$$\int\limits_{\Delta(z_0)}^{\infty}\log\Phi_{z_0}(t)\frac{dt}{t^2}=+\infty, \quad \Delta(z_0)>0, \qquad \forall\, z_0\in S. \tag{22}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions* f *with* $f(z)=z+o(1)$ *as* $z\to\infty$.

Corollary 6. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S, $|\mu(z)|<1$ *a. e. and*

$$\int\limits_S\Phi(K_\mu(z))dm(z)<\infty \tag{23}$$

for a convex non-decreasing function $\Phi:[0,\infty]\to[0,\infty]$ *with*

$$\int\limits_\delta^\infty\log\Phi(t)\frac{dt}{t^2}=+\infty \tag{24}$$

for some $\delta>0$. *Then the Beltrami equation* (1) *has regular homeomorphic solutions f with hydrodynamic normalization at the infinity.*

Remark 3. By Theorem 5.1 in [7] the condition (24) is not only sufficient but also necessary to have a regular solution in $\mathbb{C}$ for arbitrary Beltrami equations (1) with the integral constraints (23).

Corollary 7. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support* S, $|\mu(z)|<1$ *a.e.,* $K_\mu\in L^1(S)$ *and, for a neighborhood* U_{z_0} *of* z_0 *and* $\alpha(z_0)>0$,

$$\int\limits_{U_{z_0}} e^{\alpha(z_0)K_\mu^T(z,z_0)}\,dm(z)<\infty \qquad \forall\, z_0\in S. \tag{25}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions f, normalized by* $f(z)=z+o(1)$ *as* $z\to\infty$.

See the paper [4] and the monograph [8], Ch. A1, for similar results.

In particular, the following consequence can be found as Theorem 20.4.9 in the monograph [2], where the corresponding solutions are called *principal solutions* of the Beltrami equations.

Corollary 8. *Let* $\mu:\mathbb{C}\to\mathbb{C}$ *be measurable with a compact support S,* $|\mu(z)|<1$ *a. e. and, for some* $\alpha>0$,

$$\int\limits_S e^{\alpha K_\mu(z)}\,dm(z)<\infty. \tag{26}$$

Then the Beltrami equation (1) *has regular homeomorphic solutions* f *with the hydrodynamic normalization* $f(z)=z+o(1)$ *as* $z\to\infty$.

It is known that if μ has a compact support, then there exists a number $\alpha_0>1$ such that the Beltrami equation (1) for μ satisfying (26) with $\alpha\geqslant\alpha_0$ admits a unique principal solution f with $f(z)-z\in W^{1,2}(\mathbb{C})$ (see, e. g., [2, Ch. 20]).

The first two authors are partially supported by the project "Mathematical modelling of complex dynamical systems and processes caused by the state security", No. 0123U100853, of National Academy of Sciences of Ukraine and by the Grant EFDS-FL2-08 of the fund of the European Federation of Academies of Sciences and Humanities (ALLEA).

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Received 03.12.2022

В.Я. Гутлянський[1, 2], https://orcid.org/0000-0002-8691-4617
В.І. Рязанов[1, 2], https://orcid.org/0000-0002-4503-4939
Є. О. Севостьянов[1, 3], https://orcid.org/0000-0001-7892-6186
Е. Якубов [4], https://orcid.org/0000-0002-2744-1338

[1] Інститут прикладної математики і механіки НАН України, Слов'янськ

[2] Інститут математики НАН України, Київ

[3] Житомирський національний університет ім. Івана Франка, Житомир

[4] Інститут технологій Холона, Ізраїль

E-mail: vgutlyanskii@gmail.com, vl.ryazanov1@gmail.com,
esevostyanov2009@gmail.com, eduardyakubov@gmail.com

ГІДРОДИНАМІЧНІ УМОВИ НОРМУВАННЯ В ТЕОРІЇ ВИРОДЖЕНИХ РІВНЯНЬ БЕЛЬТРАМІ

Досліджено існування нормалізованих гомеоморфних розв'язків для виродженого рівняння Бельтрамі у всій комплексній площині з припущенням, що його вимірний коефіцієнт має компактний носій, а виродження рівняння контролюється коефіцієнтом тангенціальної дилатації. Доведено, що якщо коефіцієнт тангенціальної дилатації має обмежені чи скінченні середні осциляційні домінанти або задовольняє умову інтегральної розбіжності типу Лехто, то рівняння Бельтрамі допускає регулярний гомеоморфний розв'язок із гідродинамічною нормалізацією на нескінченності. Також розглянуто деякі інші інтегральні критерії типу Кальдерона-Зігмунда і Орліча для існування нормалізованих регулярних розв'язків як у термінах коефіцієнта тангенціальної дилатації, так і в термінах коефіцієнта максимальної дилатації. Зокрема, наведено низку критеріїв існування регулярних гомеоморфних розв'язків для виродженого рівняння Бельтрамі із гідродинамічною нормалізацією на нескінченності в термінах ітеративних логарифмів. Отримані результати можуть бути використані для дослідження крайових задач гідромеханіки в сильно анізотропних і неоднорідних середовищах.

Ключові слова: *ВМО, обмежене середнє коливання, FMO, скінченне середнє коливання, вироджені рівняння Бельтрамі, гідродинамічні умови нормування.*