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MODULI STARS IN THE NON RELATIVISTIC LIMIT

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When you come to a fork in the road, take it.
— Yogi Berra

Abstract

We study the possibility of formation of moduli oscillons analitically in the non relativistic regime. We start by introducing moduli as the scalar fields that arise in string theory and see how they can be stabilized in the KKLT scenario. We then consider the possibility of moduli coming together to form compact objects known as moduli stars and place them in the broader landscape of boson stars, taking into consideration also their formation mechanisms and possible experimental signals. The first step to do so is the construction of the non relativistic effective field theory, which we perform through the means of a non local operator that enables us to compute non relativistic corrections in a systematic way. This result is not present in the literature, so we check it by computing the NREFT via the traditional method of diagram matching: the two results turn out to be related by a field redefinition and are therefore equivalent low energy descriptions. Next, we turn to the corresponding effective Hamiltonian, looking for minima and maxima that would represent stable and unstable configurations. By studying it in different regimes we conclude that, in the absence of gravity, the only extremal point is a maximum non compatible with a bound state.

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Chapter 1

Introduction

The search for a unified theory of gravity and the interactions of the Standard Model is still ongoing. For many physicists, the perhaps most promising candidate remains string theory, a quantum theory of one dimensional objects that move in a N dimensional space-time whose vibrations make up the spectrum of all the matter particles and gauge bosons that we observe. Although elegant and capable of reproducing many aspects of our Universe, this theory also predicts many features of which we have not seen any experimental clue so far. The most striking and well known of them is the existence of six extra-dimensions, needed for internal consistency of the theory, that can be curled up on a specific geometry called Calabi-Yau manifold. Another feature, less familiar outside of the circle of string theorists, is the presence of hundreds of gravitationally coupled scalar fields called moduli that have a central role in defining many fundamental parameters of the theory through their vacuum expectation values. They, however, arise without a potential and the question of how to give them one is a significant open problem. One usually considers the effects of fluxes when compactifying the extra dimensions, but this is often not enough and further corrections need to be taken into considerations to stabilize all of the moduli of the theory. After being endowed with a potential they acquire a mass, a fixed VEV and self-interactions and could therefore in principle come together to form bound states. The compact objects that result are know as moduli stars and would significantly affect the history of the Early Universe, as well as many of the features of our current one, by being a source of baryon asymmetry, of a stochastic background of gravitational waves, by delaying thermalization or catalysing phase transitions.
Despite their importance in the theory, moduli stars are kept in the shadow by axion stars in the literature. The latter, compact objects made out of pseudo-scalar fields, have received much more attention in recent years because they could be a plausible candidate for cold dark matter [1] [4] and have been studied using both analytical and numerical methods. Moduli stars, on the other hand, have only been investigated through numerical methods and lattice simulations, so we take on the task of building an analytical approach to the study of their formation. In doing so, we focus on stars held together

by self-interactions (known as oscillons), as is the case for the objects formed during pre-heating in the Early Universe, and consider the non relativistic limit as it's usually done in the literature.
We start off by giving an overview of moduli in the context of type II string theory as scalar particles arising in the massless spectrum and from string compactification and see how their vacuum expectation values parametrize the coupling coefficient of the strings and the shape and size of the extra-dimensions. We then consider the KKLT scenario to stabilize the moduli of the theory and the Kähler moduli in particular: this step involves performing a flux compactification and considering non-perturbative corrections to the superpotential. The resulting minimum is anti deSitter, so it needs to be uplifted to values consistent with the cosmological constant by considering the effects of anti D3-branes at the end of warped throats that extend from the Calabi-Yau manifold of the extra-dimensions. The parameters of the theory can be fine-tuned to obtain the desired value for the minimum of the potential, however the corrections considered could back react on the geometry of the internal compact space and potentially disrupt the minimum itself. Nevertheless, the KKLT scenario remains a valid toy model to explore the implications of string theory. We conclude chapter 2 by expanding the KKLT potential in a power series around the minimum. Next, we consider the possibility of moduli stars as pseudo-solitonic solutions of the coupled Einsten-Klein-Gordon equations and review the broader landscape of boson stars. We also discuss the mechanisms of amplification of the fluctuations of the moduli field that could lead to the formation of moduli oscillons in the Early Universe and consider the possible experimental signals that moduli stars could give as long lived objects, sources of gravitational waves and origin of black holes of unusual small mass. In chapter 4 we dwell into the analytical study by constructing an effective field theory for the Kähler modulus in the KKLT scenario in the non relativistic regime. To do so we first outline the methods used in the literature for axion stars and then choose the non-local operator redefinition developed in [8] and extend it to our case with odd terms in the potential. Computing the equation of motion perturbatively and then expanding the non local operator in the non relativistic limit allows us to calculate relativistic corrections in a systematic way and up to an arbitrary order of approximation. The resulting effective Lagrangian is not found in the literature, so we also compute effective vertices through diagram matching to check our outcome. The two results are at first sight quite different, but we show that they are related by a field redefinition and therefore describe the same low energy theory at the given order of approximation. In chapter 5, to conclude, we consider the effective Hamiltonian and look for stable and unstable configurations of moduli objects as minima and maxima. As there is no know analytical solution to the equation of motion, we assume a reasonable ansatz to compute the Hamiltonian explicitly in terms of the radius and number of particles of the star. In order to understand the behaviour of the Hamiltonian and the nature of its extremal points we study it in different regimes and conclude that only a maximum exists, which is not compatible with a bound state. Several numerical studies in the literature, on

the other hand, agree in founding the formation of meta-stable moduli stars possible in the context of the KKLT scenario [2] [12]. This discrepancy could be attributed to the two significant simplifications we made: neglecting gravity and considering only the non relativistic limit. The inclusion of the first could in principle lead to the observation of a minimum of the Hamiltonian compatible with a bound stable state that could be identified and studied using the analytical approach described in this thesis. To witness the formation of oscillons, however, one needs to turn to the full relativistic regime in order to possibly recover the results in [12].
Throughout all the work we assume the convention $c = \hbar = 1$, which gives us the conversion rule $1GeV = 1.8 \cdot 10^{-24} g = 5 \cdot 10^{13} cm^{-1} = 1.5 \cdot 10^{24} Hz$. The Planck mass m_P is defined in terms of the Newton constant G $m_P = \sqrt{\frac{\hbar c}{G}} = 1.2 \cdot 10^{19} Gev = 2 \cdot 10^{-5} g$, while the Planck length is $l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \cdot 10^{-33}$. The value of the solar mass is $M_\odot = 2 \cdot 10^{33} g = 10^{57} GeV$. Capital latin letters such as N and M are used to indicate indices spanning over the D-dimensional space time of string theory and range from 0 to (D-1). When assuming a lower dimensional point of view, they are divided into greek indices for the external non compact space-time (typically $\mu, \nu = 0, 1, 2, 3$) and lower case latin indices for the internal compact space ($m, n = D - 3, ..., D$). Finally, greek indices like α and β are reserved for the coordinates of the worldsheet. All of the graphs, except where otherwise specified, are obtained using the computational software *Wolfram Mathematica*.

Chapter 2

Moduli

String theory, although elusive, is still one of the most promising theories that attempts to reconcile the Standard Model with General Relativity and to do so with a great deal of elegance. One of its prediction is the existence of hundreds of gravitationally coupled scalar fields that arise from string excitations and from compactification of the six extra dimensions required for internal coherence. These particles are collectively called moduli and play a central role in the theory: knowing their vacuum expectation values (VEVs) would set many values, including coupling constants and the shape and size of the extra dimension, leaving only the typical scale of strings as a free parameter. Moduli, however, do no acquire naturally a potential[1] and the question of how to construct it, thus endowing string theory with predictive power, represents one of the main open problems of string theory, known as moduli stabilization. The principal mechanism understood so far involves the inclusion of fluxes during compactification. This is however usually not enough to stabilize all of the moduli of the theory and further corrections (both perturbative and non-perturbative in nature) need to be taken into account. We are going to dwell deeper into the scenario proposed by Kachru, Kallosh, Linde and Trivedi in [11] that also considers the influence of a few number of anti-D3 branes positioned at the end of warped throats that extend from the Calabi-Yau manifolds of extra dimensions. The main effect of this addition is to uplift the minimum of the potential generated to positive values. The parameters of the scenario can be fine-tuned to obtain the proper deSitter vacuum for our Universe at current times, in accordance with the supposed value of the cosmological constant.

[1] This missing potential presents another issue: having massless moduli would result in another long-range force of which, so far, we have no experimental evidence.

2.1 String theory

Let us dive straight into string theory and get acquainted to it by calculating its massless spectrum. Strings are one dimensional objects moving in a D-dimensional space-time[2]. Doing so they sweep a two dimensional surface know as "world sheet", which we parametrize with the coordinates (σ, τ) with σ running along the string itself. The evolution of the worldsheet in described by its area embedded in the D-dimensional space-time, encoded by the so-called Poljankov action

$$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{\gamma} \gamma^{\alpha\beta} \eta_{MN} \partial_\alpha X^M \partial_\beta X^N \tag{2.1}$$

where capital latin indices span the D-dimensional manifold $(M, N = 0, ..., D-1)$ while greek indices are the coordinates of the worldsheet $\alpha = \tau, \sigma$. The X^M are functions that define the embedding of the worldsheet in space-time, $\gamma^{\alpha\beta}$ is the metric of the worldsheet while $\eta_{\alpha\beta}$ is the space-time Minkowski metric. α' is a parameter linked to tension of the string T and the typical energy scale of the string $1/l_s$ as $T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s}$. Let us introduce left- and right- moving coordinates for the worldsheet:

$$\sigma^\pm = \tau \pm \sigma \tag{2.2}$$

Varying the action 2.1 with respect to X^M gives us the equation of motion, which with our new coordinates reads

$$\frac{\partial}{\partial\sigma^+} \frac{\partial}{\partial\sigma^-} X^M(\tau, \sigma) \tag{2.3}$$

The straight forward conclusion from this equation is that the X^M is composed by left- and right- moving degrees of freedom. Now, σ can either range from 0 to π if it's describing an open string or from 0 to 2π in the case of the closed string. Let us choose the latter and impose as boundary conditions

$$X^M(\tau, 0) = X^M(\tau, 2\pi) \qquad\qquad X'^M(\tau, 0) = X'^M(\tau, 2\pi) \tag{2.4}$$

Then we can write X^M in a mode expansion as a pair of independently left- and right-moving travelling waves:

$$X^M(\tau, \sigma) = X_R^M(\sigma^-) + X_L^M(\sigma^+) \tag{2.5}$$

$$X_R^M(\sigma^-) = \frac{1}{2}x^M + \alpha' p^M \sigma^- + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n} \alpha_n^M e^{-2in\sigma^-} \tag{2.6}$$

$$X_R^M(\sigma^+) = \frac{1}{2}x^M + \alpha' p^M \sigma^+ + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^M e^{-2in\sigma^+} \tag{2.7}$$

[2] As is widely known this space is 10-dimensional or 11-dimensional, but as we will see D for now is just a parameter of the theory.

where x^M and p^M are the position and the momentum of the center of mass and asking for the solution to be real requires $\alpha_{-n}^M = (\alpha_n^M)^*$ and $\tilde{\alpha}_{-n}^M = (\tilde{\alpha}_{-n}^M)^*$. We can quantize the theory in a canonical way imposing the commutators

$$[\alpha_n^P, \alpha_m^Q] = [\tilde{\alpha}_n^P, \tilde{\alpha}_m^Q] = \delta_{n+m}\eta^{PQ} \qquad\qquad [x^P, p^Q] = i\eta^{PQ} \tag{2.8}$$

and interpret the alphas as creation and annihilation operators: α_n^M and $\tilde{\alpha}_n^M$ create respectively a left- and a right- moving excitation at level n while α_{-n}^M and $\tilde{\alpha}_{-n}^M$ annihilate them. The mass of a state is given by the operator[3]:

$$M^2 = \frac{2}{\alpha'}\Big(\sum_{n=1}^{\infty}\alpha_n\alpha_{-n} + \tilde{\alpha}_n\tilde{\alpha}_{-n} - 2\Big) \tag{2.9}$$

Now, varying the action with respect to X^M gave us the equation of motion, but we can also vary it with respect to $\gamma_{\alpha\beta}$ to obtain a constraint for physical states. Upon quantization this requirement turns into the vanishing of the Virasoro operators. Among the restrictions that this implies is setting a level matching condition for physical states $N = \tilde{N}$, where N and $\tilde{N}$ are the number operators for left- and right- moving modes. The least energetic state that we can construct will therefore need both a left- and a right- moving creation operator:

$$|\xi_{MN}\rangle = \xi_M\tilde{\xi}_N\alpha_1^M\tilde{\alpha}_1^N|0\rangle \tag{2.10}$$

is the massless state. k^M being the momentum of the center of mass, $k \cdot k = 0$; moreover the polarization vectors ξ_M, $\tilde{\xi}_M$ need to be space-like for the norm of the state to be positive and orthogonal to k^M for the conditions coming from the variation of 2.1 with respect to $\gamma_{\alpha\beta}$ to be realized. This means that with an appropriate choice of coordinates k can be written as $\mathbf{k} = (k, k, 0, ..., 0)$ and hence ξ_M, $\tilde{\xi}_M$ live in a space parametrized by $D-2$ coordinates: we can therefore conclude that the states are classified by their SO(D-2) representation.
Let us define the tensor $\xi_{MN} := \xi_M\tilde{\xi}_N$. We can decompose it into a scalar, a symmetric tensor and an anti-symmetric tensor as

$$\xi_{MN} = \xi_t\eta_{MN} + \xi^s_{MN} + \xi^a_{MN} \tag{2.11}$$

with

$$\xi_t := \frac{1}{D}\eta^{MN}\xi_{MN} \qquad \xi^s_{MN} := \frac{1}{2}(\xi_{MN}+\xi_{NM}-2\xi_t\eta_{MN}) \qquad \xi^a_{MN} := \frac{1}{2}(\xi_{MN}-\xi_{NM}) \tag{2.12}$$

So when we build a state 2.10 that's actually equivalent to producing a scalar, and two tensor fields. We call the scalar dilaton Φ: that's our first modulus. As the tensor fields

[3]The -2 term comes from normal ordering, i.e. is the zero point energy.

we recognize a massless spin 2 state, i.e. the graviton, while we call the field associated to the antisymmetric polarization B-field. There is another point to be made about having 2.10 as a massless state. When applying operator 2.9 to the ground state $|0\rangle$ we find it to have squared mass $-1/\alpha'$: we have a tachyonic instability that makes the bosonic string roll down to the true minimum of the theory. To solve this problem (among others) we will later introduce supersymmetric fermionic string partners. Let us forget about it for now and upgrade the Poljankov action to fit it in a σ-model:

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{\gamma}\gamma^{\alpha\beta} g_{MN}(X)\partial_\alpha X^M \partial_\beta X^N \tag{2.13}$$

with $g_{MN}(X)$ metric of the background space-time. By writing the path integral for action 2.13 and expanding it, we can see it as describing coherent states of gravitons, B-fields and dilatons. Keeping this in mind we can re-write the Poljankov action as

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{\gamma}\Big[\Big(\gamma^{\alpha\beta} g_{MN}(X) + i\epsilon^{\alpha\beta} B_{MN}(X)\Big)\partial_\alpha X^M \partial_\beta X^N + \alpha'\Phi R\Big] \tag{2.14}$$

with R being the Euler density compatible with g_{MN}. Imposing Weyl invariance $\gamma_{\alpha\beta} \to e^{\omega}\gamma_{\alpha\beta}$ gives us the equations of motion for the fields of the massless string spectrum. At one loop[4]

$$\alpha'\Big(R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{4}H_{MPQ}H_N^{PQ}\Big) + \mathcal{O}(\alpha'^2) = 0 \tag{2.15}$$

$$\alpha'\Big(-\frac{1}{2}\nabla^P H_{PMN} + \nabla^P\Phi H_{PMN}\Big) + \mathcal{O}(\alpha'^2) = 0 \tag{2.16}$$

$$\alpha'\Big(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_M\Phi\nabla^M\Phi - \frac{1}{24}H_{MNP}H^{MNP}\Big) + \mathcal{O}(\alpha'^2) = 0 \tag{2.17}$$

with $H_{MPQ} := \partial_{[M}B_{PQ]}$[5] being the field strength of B_{PQ}. The first equation resembles Einstein's equation with $\nabla\Phi$ and ∇B acting as sources, the second is the generalization of Maxwell's equation for the anti-symmetric tensor B_{PQ} and the third implies $D = 26$ in order to avoid the need for Φ and the B-field to have large gradients of order of the string energy scale[6]. The equation of motion for the dilaton set the number of space-time dimensions! Let us focus on this modulus for another comment. The action 2.14 does not have a potential for it. It does however present it alongside R, meaning that Φ couples to the Euler number for the worldsheet

$$\chi = \frac{1}{4}\int d^2\sigma\sqrt{-\gamma}R \tag{2.18}$$

[4]Notice from 2.14 how α' can be interpreted as a parameter akin to $\hbar$, around which to expand.

[5]Where the indices are anti-symmetrized.

[6]This solution is actually not mandatory and there are variations of the theory with $D \neq 26$, like the linear dilaton one.

which for a 2-dimensional surface can also be written as $\chi = 2 - 2h - b - c$ with h being the number of handles, b of boundaries and c cross-caps. The emission and adsorption of a close string onto the worldsheets ads an extra handle, so $\delta\chi = -2$. When considering the same event from the point of view of a tree level Feynman diagram we have two vertices, so a factor g_s^2 (with g_s being the coupling constant for the string). As from the path integral the amplitudes are weighted as $e^{-\chi\Phi}$ we have $g_s = e^{\langle\Phi\rangle}$, meaning that the vacuum expectation value of the dilaton also sets the g_s[7]. So far we focused on the closed string $0 \leq \sigma \leq 2\pi$, but we have another option, the open one $0 \leq \sigma \leq \pi$. We can either impose the Neumann boundary conditions $\partial_\sigma X^M(\tau, 0) = \partial_\sigma X^M(\tau, \pi) = 0$ or the Dirichlet boundary conditions and let the endpoint of the string be on a (p+1)-dimensional surface, the Dp-brane. The first case leads to the mode decomposition

$$X^M(\tau,\sigma) = x^M + 2\alpha' p^M \tau + i(2\alpha')^{1/2} \sum_{n\neq 0} \frac{1}{n} \alpha_n^M e^{-in\tau} cos(n\sigma) \tag{2.19}$$

Upon quantization we get the mass operator

$$M^2 = \frac{1}{\alpha'} (\sum_{n=1}^{\infty} \alpha_n \cdot \alpha_{-n} - 1) \tag{2.20}$$

We can therefore construct the massless state as:

$$|\xi_M\rangle := \xi_M \alpha_1^M |0\rangle \tag{2.21}$$

We no longer have a polarization tensor, but a vector. The corresponding state can therefore be interpreted as a gauge field A_M. Similarly to what we did before we can find the action for it, which turns out to be the Yang-Mills action

$$S = -\frac{1}{4} \int d^D X e^{-\Phi} F_{MN} F^{MN} + \mathcal{O}(\alpha') \tag{2.22}$$

The Dirichlet boundary conditions, on the other hand, makes the action depend on $D - p - 1$ scalars on the boundary of the string. The VEVs of these scalars actually set the position of the Dp-brane, making it a dynamical object and further proving the importance of moduli.

Now, starting from strings, we recovered a massless spectrum made out of scalars (the dilaton in particular), the graviton, the B-field and a gauge field. To complete the picture with fermions we can introduce the supersymmetric partners of X^M: the Majorana-Weyl spinors Ψ^M and $\tilde{\Psi^M}$. After a Wick rotation, the combined Poljankov action 2.1 becomes

$$S = \frac{1}{4\pi} \int d^2\sigma \eta_{MN} [\frac{1}{\alpha'} \partial X^M \bar{\partial} X^N + \Psi^M \bar{\partial} \Psi^N + \bar{\Psi}^M \bar{\partial} \bar{\Psi}^N] \tag{2.23}$$

[7]The only free parameter of the theory is in fact the string tension T, or equivalently the string energy scale.

where the ∂ are to be intended to be taken with respect with $z := e^{\tau - i\sigma}$. As done before we need to set the boundary conditions in order to proceed with a mode expansion. Being Ψ^M and $\tilde{\Psi}^M$ spinors, however, leaves us two different choices:

$$R : \Psi^M(\tau, 0) = \Psi^M(\tau, 2\pi) \qquad NS : \Psi^M(\tau, 0) = -\Psi^M(\tau, 2\pi) \tag{2.24}$$

The one we labelled with R is known as the Ramond condition, while NS stands for Neveu-Schwarz. Canonical quantization can be achieved by imposing the anticommutators

$$\{\psi_r^M, \psi_s^N\} = \{\tilde{\psi}_r^M, \tilde{\psi}_s^N\} = \eta^{MN}\delta_{r+s} \tag{2.25}$$

on which we once again have a level-matching condition for physical states. Imposing Weyl invariance gives us instead the condition $D = 10$[8]. We can construct the massless states combining the boundary conditions to get different sectors of the theory

$$\begin{aligned} R-R: &\quad \xi_{MN}\psi_0^M\tilde{\psi}_0^N|0\rangle & NS-NS: &\quad \xi_{MN}\psi_{1/2}^M\tilde{\psi}_{1/2}^N|0\rangle \\ NS-R: &\quad \xi_{MN}\psi_{1/2}^M\tilde{\psi}_0^N|0\rangle & R-NS: &\quad \xi_{MN}\psi_0^M\tilde{\psi}_{1/2}^N|0\rangle \end{aligned} \tag{2.26}$$

where the NS condition is associated to semi-integer modes and R is linked to integer modes. The same considerations made on ξ and k before make the states classified by their SO(8) representation. Let us consider a single left-moving R ground state to understand how to interpret this spectrum. The $\psi_0^M|0\rangle$ form a representation of the Clifford algebra obeyed by Ψ_0 and we can label it with the chiralities possible in four planes of two dimensions as $|s\rangle = |\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle$. This means that $\psi_0^M|0\rangle$ is a space-time fermion, although in representation **16**. To obtain the appropriate Standard Model fermions we need to eliminate half of them, as $\mathbf{16} = \mathbf{8} \oplus \mathbf{8}$. To do so we can project the state to the Majorana-Weyl spinor subspace of definite chirality. Mathematically we can do this by requiring $(\sum_{i=1}^4 s_i = 0)$ with the Gliozzi-Scherk-Olive projectors. This redundancy in the degrees of freedom gives us the possibility to project out the tachyonic vacuum that worried us when dealing with the bosonic string alone, and thus to make the theory stable.

The states 2.26 are therefore to be interpreted as $|s\rangle \otimes |s\rangle$, which after GSO-projection becomes $\mathbf{8} \otimes \mathbf{8}$. As left- and right-moving modes are projected indipendently, we can also choose the same chirality for both or opposite ones. This choice distinguishes between type IIB superstring theory (which is therefore chiral) or tipe IIA superstring theory[9]. Noting that, in the 8-dimensional space-time in which Ψ and $\tilde{\Psi}$ live, spinors s=1/2 and vectors s=1 both have eight degrees of freedom, we can treat them all as **8** representations that can be fit for our spectrum. Let us label the vector representation $\mathbf{8}_v$, the spinor of positive chirality representation $\mathbf{8}_s$ and the spinor of negative chirality representation $\mathbf{8}_c$[10]. We get:

[8] Is actually possible to construct a string theory also with $D = 11$, like M theory.

[9] As we will see in a moment the II refers to the number of gravitini in the massless spectrum.

[10] Triality is a symmetry of the theory that swaps the $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ among each other.

- R-R type IIB: $\mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ = C_0 \oplus C_2 \oplus C_4$
  R-R type IIA: $\mathbf{8}_s \otimes \mathbf{8}_c = \mathbf{8}_v \oplus \mathbf{56}_t = C_1 \oplus C_3$

- NS-NS type IIB: $\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \Phi \oplus B_2 \oplus G_{MN}$
  NS-NS type IIA: $\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = \Phi \oplus B_2 \oplus G_{MN}$

- R-NS type IIB: $\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s = \lambda^2 \oplus \Psi^{2M}$
  R-NS type IIA: $\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s = \lambda^2 \oplus \Psi^{2M}$

- NS-R type IIB: $\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_s \oplus \mathbf{56}_s = \lambda^1 \oplus \Psi^{1M}$
  NS-R type IIA: $\mathbf{8}_v \otimes \mathbf{8}_c = \mathbf{8}_c \oplus \mathbf{56}_c = \lambda^1 \oplus \Psi^{1M}$

We have quite a lot on our hands. The C_n are n-forms and play the part of potentials. The fermions λ^1 and λ^2 are the dilatini, while the fermions Ψ^{1M} and Ψ^{2M} are the gravitini. The Φ, the B_2 and the G_{MN} are, respectively, the dilaton, the B-field and the graviton that we already found in the massless closed bosonic string spectrum. Including the open superstring gives us the gauge fields and the fermions of the Standard Model that we are missing. The massless spectrum is actually enough for many application, as the massive states have masses of order $1/\sqrt{\alpha'}$ and can therefore be neglected at significantly lower energies. Type IIA and type IIB can be exchanged under a transformation that maps representations $\mathbf{8}_s$ into $\mathbf{8}_c$ and vice versa, known as T-duality. One can make up different types of string theories[11], each connected to one other via dualities. Moduli show their importance once again: changing their VEVs[12] lets you navigate among these options, which could therefore be different low energy realizations of a single string theory.

2.2 Moduli from compactification

In the previous section we outlined the NSR formulation of superstrings and showed how starting from one dimensional fundamental objects we can obtain the particles we are used to, along with many others. There is, however, still a significant point left to address: the internal consistency of the theory as we described it requires ten space-time dimensions. To reconcile this feature with our experience we could interpret the ten dimensional space-time as composed by our usual four dimensional space-time and a six dimensional compact internal space ($\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6$). The process of curling up these extra-dimensions to recover our four dimensional phenomenology is know as compactification. As is often the case in physics, this idea is actually much older than its application in string theory. The first to use a fifth dimension in the attempt to reconcile gravity and electromagnetism was Nordström in 1914 [42], followed by Kaluza

[11] Examples include M-theory, type I string theory, E_8xE_8 heterotic string theory and SO(32) heterotic string theory

[12] The space of all possible moduli VEVs is known as moduli space.

in 1921. Klein came in 1926 to suggest that this extra dimension should be finite and describing a circle of radius R. This kind of scenario in useful when dealing with unification in the sense that we can see the five-dimensional metric g_{MN} (with indices spanning from 0 to 4) as including the fourth-dimensional metric $g_{\mu\nu}$, a vector field $g_{\mu 4}$ and a scalar g_{44} (where μ, ν=0,1,2,3). Writing the vector field as $g_{\mu 4} = e^{2\sigma} A_\mu$[13], the invariance of the metric under a transformation $x^4 \to x^4 + \lambda(x^\mu)$ implies for A_μ the U(1) gauge transformation $A \to A - d\lambda$: standard gauge symmetries can therefore be derived from general coordinate transformations in extra-dimensions. Kaluza-Klein theories did not succeed in their unification programme, but they came back in the spotlight during the '70s when, with the development of string theory, compactification became a need.

Accounting for phenomenology imposes very strict topological constraints on the choice of manifold for $\mathcal{M}_6$. The $\mathcal{N} = 8$ supersymmetry that we introduced with fermionic strings needs to break down to $\mathcal{N} = 2$ in the resulting four-dimensional effective theory[14]. We would also like for this to happen at scales which are low compared to the compactification scale $1/R_c$ (where R_C is the typical length of the compact internal space, e.g. the radius of $\mathcal{M}_1 = S^1$ in Kaluza-Klein theories), as supersymmetry is not observed at low energies but is indeed very useful in our description. Mathematically this requirement implies the existence of a covariantly constant spinor on the manifold

$$\nabla_m \eta = 0. \tag{2.27}$$

The existence of a non-vanishing spinor on the whole manifold implies that in a point where more than one chart is defined their result must be compatible, which translates as SU(3) as the group of transition functions. Its integrability on the other hand asks the manifold to have SU(3) as the group of transformations that the spinors undergo when they are parallel transported with Levi-Civita connection around a closed loop (aka its holonomy group). Manifolds with these characteristics are called Calabi-Yau manifolds[15] and, as it was demonstrated by Calabi and Yau, they always admit a metric that is Ricci flat. This is compatible with the equation of motion 2.15 (and its equivalent in the full superstring NSR formulation), that in the absence of fluxes[16] requires $R_{mn} = 0$ $(m, n = 4, 5, 6, 7, 8, 9)$. In six dimensions there are many known such manifolds, possibly infinite.

Let us use a lower dimensional example to illustrate some features of the compactification procedure. In $D = 2$ an example of a Calabi-Yau manifold is the torus $\mathbb{T}^2$, so let us pick $\mathcal{M}_{10} = \mathcal{M}_8 \times \mathbb{T}^2$. The torus requires periodicity $x^m = x^m + 2\pi R^m$ for coordinates

[13]The field σ is actually another modulus, called radion. It is linked to the size of the fifth dimension through the determinant of g_{MN}, but having no potential nor its mass nor its VEV are specified.

[14]Including fluxes further breaks down the symmetry to $\mathcal{N} = 1$.

[15]Calabi-Yau manifolds can also be defined as being Kähler manifolds, so complex and symplectic manifolds, of trivial first Chern class ($c_1 = 0$).

[16]As we will see later, fluxes are introduced to give moduli a potential and they actually back-react on the topology of the manifold.

$m = 8, 9$. As the actor of our theory is a one-dimensional object, however, it can also wind around the torus: the periodicity condition for the (bosonic) string is therefore $X^m(\tau, \sigma + 2\pi) = X^m(\tau, \sigma) + 2\pi R_m w^m$ with w^m being integer called winding numbers. This periodicity modifies the mode expansion for left- and right- moving X^m

$$\begin{aligned} X^m_L(\sigma^+) &= (\frac{\alpha' n_m}{R_m} + w^m R_m)\sigma^+ + oscillations \\ X^m_R(\sigma^-) &= (\frac{\alpha' n_m}{R_m} - w^m R_m)\sigma^- + oscillations. \end{aligned} \tag{2.28}$$

assigning to the states two additional labels: the winding number w^m and the Kaluza-Klein number n_m. The massless states of the theory correspond to both of them vanishing; considering also the excitations along the torus dimensions gives us additional states compared to what we would get just on the flat eight-dimensional $\mathcal{M}_8$. We get a metric g_{MN}, that can be tough of as a metric $g_{\mu\nu}$, two vectors $g_{\mu 8}$ and $g_{\mu 9}$ and three scalars g_{88}, g_{89} and g_{99}, the two anti-symmetric form B_{MN}, so a tensor $B_{\mu\nu}$, two vectors $B_{\mu 8}$ and $B_{\mu 9}$ and a scalar B_{89}, the scalar Φ. Alongside with the dilaton, we can use the four scalars that are left to form two complex scalars: the Kähler modulus ρ

$$\rho = B_{89} + \mathcal{V}(\mathbb{T}^2) \tag{2.29}$$

with $\mathcal{V}(\mathbb{T}^2)$ being the volume of the torus, and the complex structure modulus τ, which contributes to the metric of $\mathbb{T}^2$

$$ds^2 = \frac{Im(\rho)}{Im(\tau)} |dx^8 + \tau dx^9|^2. \tag{2.30}$$

Compactification on proper Calabi-Yau manifolds shares an unwanted feature with Kaluza-Klein theories: an overall scaling symmetry that makes the volume of the extra dimension completely unfixed. As we have seen, some moduli (like the complex structure moduli for example) are linked to the size and shape of the extra dimensions, so working out a mechanism that assigns them a potential would fix this issue too.

2.3 Moduli stabilization: the KKLT scenario

We have seen moduli arising from superstrings and their compactification, we have seen how they set paramount parameters of the theory such as the coupling constant of the string g_s (and in general four dimensional masses and interactions) or the shape and size of the extra dimensions, but we do not have a potential for them yet. All these massless scalars would furthermore result in an additional long-range force in four-dimensions which has not been observed so far. We need to come up with a mechanism that assigns a potential to these particles, thereby fixing their VEV and mass: we call this procedure moduli stabilization. The only known way to do so is by turning on fluxes and account for them in the compactification procedure. Fluxes actually break the condition 2.27, so including them modifies the topology of the manifold which is no longer a Calabi-Yau. Another effect of this procedure is the breaking of supersymmetry to $\mathcal{N} = 1$, which is fit for phenomenology[17]. Flux compactification requires the formalism of generalized geometry, which was first developed by Hitchin in [43] to construct a unified description of complex and symplectic geometry[18]. Just including fluxes, however, does not generally stabilize all of the moduli. One can add further corrections, both perturbative and non-perturbative in nature, like the ones to the action in α' and g_s to leading order or the ones to the superpotential coming from D-brane instantons. The latter one is the approach devised by Kachru, Kallosh, Linde and Trivedi in [11] to stabilize all moduli in type IIB string theory compactified on Calabi-Yau orientifolds[19]. In the so-called KKLT scenario fluxes stabilize the complex structure moduli but still conserve a rescaling symmetry in the Lagrangian to leading order in α' and g_s, thus leaving the overall volume of the extra dimensions and the Kähler modulus ρ unfixed. The inclusion of non-perturbative corrections to the superpotential takes care of the Kähler modulus too and generates an exponential term for the superpotential

$$W = W_0 + Ae^{-a\rho} \tag{2.31}$$

where complex structure moduli τ and the dilaton Φ have been fixed to their VEVs, W_0 being the tree level contribution coming from the Gukov-Vafa-Witten flux that stabilizes them. A and a are coefficient whose values depend on the details of the non-perturbative corrections. The Kähler potential at tree level is

$$K = -3ln[-i(\rho - \bar{\rho})]. \tag{2.32}$$

Let us set the axion making up the real part of our complex valued Kähler modulus to zero to have $\rho = i\sigma$. The system has a supersymmetric minimum at σ_0 set by $D_\rho W = 0$

$$W_0 = -Ae^{-a\sigma_0}\Big(1 + \frac{2}{3}a\sigma_0\Big). \tag{2.33}$$

We can calculate the corresponding potential

$$V = \frac{aAe^{-a\sigma}}{2\sigma^2}\Big(\frac{1}{3}\sigma aAe^{-a\sigma} + W_0 + Ae^{-a\sigma}\Big). \tag{2.34}$$

[17]The inclusion of fluxes could also provide a solution for the hierarchy between the Planck and the electroweak scale via a non-trivial warp factor in the 10-dimensional metric.

[18]It's interesting how [43], while describing a new mathematical theory, makes large use of the corresponding physical lexicon for the application in string theory.

[19]The other scenario that is mostly used alongside KKLT is the so-called Large Volume scenario (LVS).

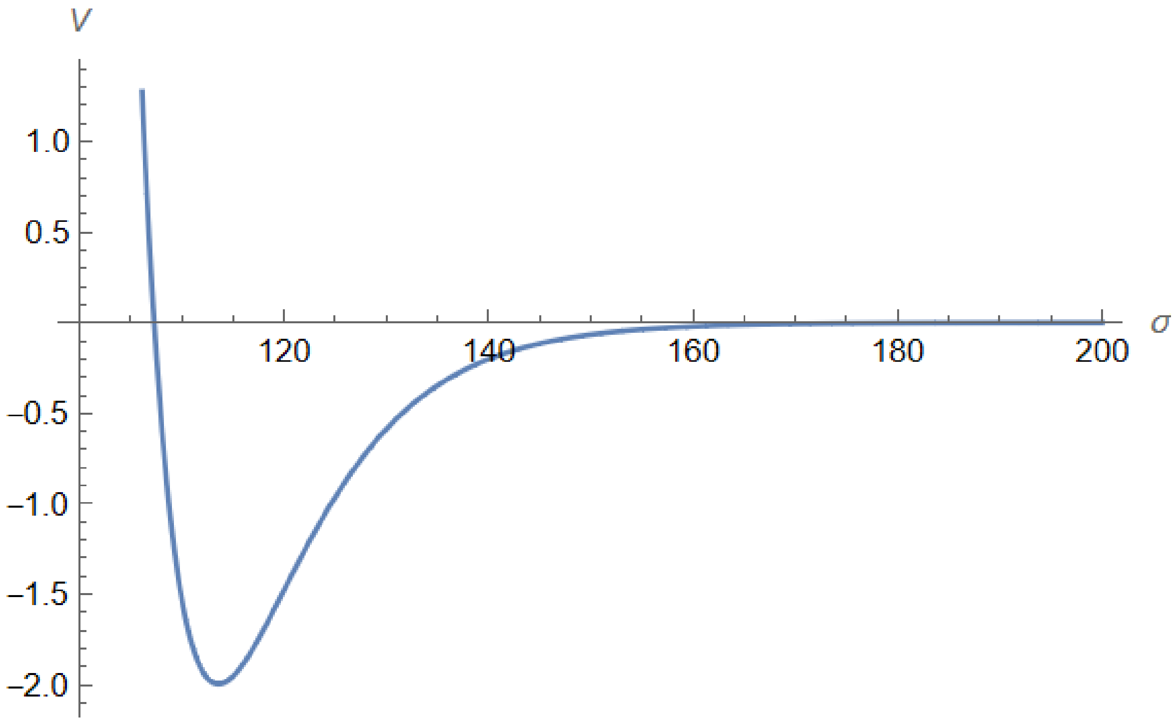


Figure 2.1: The KKLT potential for coefficients $W_0 = -10^{-4}$, $A = 1$ and $a = 0.1$, multiplied by a factor 10^{15} for readability. It presents an Anti deSitter minimum in $\sigma_0 \sim 113$ and a minimum at $\sigma \to \infty$, typical of string theories. If we want the theory to describe our Universe we need to uplift the AdS minimum to the value of $\sim 10^{-120} l_p^{-2}$.

We want its minimum to be an appropriate description of the system at low energy, for this to be the case we require $\sigma \gg 1$ and $a\sigma > 1$. A turns out to be of order $\mathcal{O}(1)$ and tuning fluxes we can make it so $W_0 \ll 1$ (more precisely $W_0 \simeq 10^{-4} - 10^{-12}$). a then has to be positive but smaller than 1. The minimum σ_0 is therefore an Anti deSitter one, as can be seen in Fig. 2.1, where we plotted potential 2.34 with reasonable values for the coefficients. For this minimum to be an appropriate description of the Universe in its current days state, we need to break its supersymmetry and to uplift it to the deSitter value of $V_{min} \sim 10^{-120} l_p^{-2}$ to have a positive cosmological constant. We can obtain this result by including a small number of anti D3-branes ($\overline{D3}$) at the end of warping throats that lounge out of the Calabi-Yau manifold. Their effects generate a term

$$\delta V \propto \frac{1}{\mathcal{V}^n} \sim \frac{1}{(\rho^{3/2})^n} \tag{2.35}$$

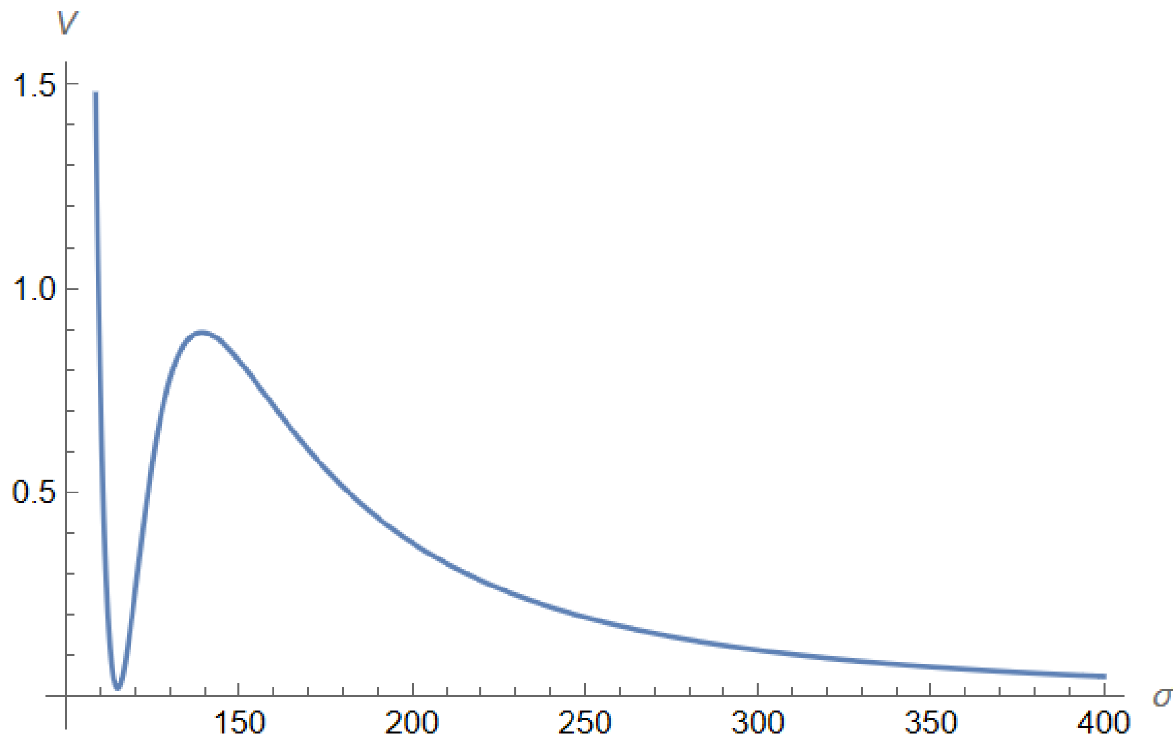


Figure 2.2: The KKLT potential 2.36 for coefficients $W_0 = -10^{-4}$, $A = 1$, $a = 0.1$ and $D = 3 \times 10^{-9}$, multiplied by a factor 10^{15}. We get a dS minimum very close to zero, the usual Dine-Seinberg run-away vacuum for string theories and a maximum that separates them. The scale on the σ axes is different from the one in Fig. 2.1 to improve readability of this graph, however a closer look shows how the position of the minimum has changed only slightly and so has the steepness of V around it.

where $\mathcal{V}$ is the volume of the extra dimensions[20]. Only a narrow range for the value of n is allowed; typically one takes $n = 2$ to have a term $1/\sigma^3$ as in the original paper [11] or $n = 4/3$ as in [2] for a term $1/\sigma^2$. With the former choice potential 2.34 becomes

$$V = \frac{aAe^{-a\sigma}}{2\sigma^2}\left(\frac{1}{3}\sigma aAe^{-a\sigma} + W_0 + Ae^{-a\sigma}\right) + \frac{D}{\sigma^3} \tag{2.36}$$

where fine tuning D can gives us the V_{min} we are looking for. The resulting potential is plotted in Fig.2.2. We get a positive minimum very close to zero, whose position σ_0 is shifted only slightly from the one of potential 2.34. More importantly, the behaviour of

[20]Up until recently this term was added a posteriori, but it can also be induced by a nilpotent superfield S, $S^2 = 0$.

V around it is still very similar to before, so in a first approximation one could neglect the D/σ^3 correction and work with the anti deSitter minimum. The term 2.35 has also generated a maximum, which was not present before, that separates the dS minimum from the vacuum at $\sigma \to \infty$. This latter additional minimum is known as Dine-Seinberg run-away vacuum and is a standard feature of string theories. Because of it, any dS minimum is a false vacuum which can be destabilized by tunnelling effects; however its lifetime is large enough to account for the age of the Universe [11].
Let us focus on the region of interest by expanding the potential 2.36 around its minimum. The lagrangian resulting from the Kähler potential and the superpotential has a non canonical kinetic term, so first we want to enforce a substitution from σ to the canonically normalised field ϕ:

$$\mathcal{L} \in \frac{3}{4}\left(\frac{\partial\sigma}{\sigma}\right)^2 = \frac{1}{2}(\partial\phi)^2. \tag{2.37}$$

The expansion in a power series of potential 2.36 around the minimum ϕ_0 gets us

$$\begin{aligned} V(\phi) =& \frac{a^2A^2e^{-2a\sigma_0}}{\sigma_0(-1+a\sigma_0)}\Big[\frac{1}{18}(4-3a\sigma_0+2a^3\sigma_0^3)(\phi-\phi_0)^2+ \\ &-\frac{1}{9\sqrt{6}}(8-a\sigma_0-3a^2\sigma_0^2+2a^3\sigma_0^3+2a^4\sigma_0^4)(\phi-\phi_0)^3+ \\ &+\frac{1}{324}(100+15a\sigma_0-18a^2\sigma_0^2+14a^3\sigma_0^3+8a^4\sigma_0^4)(\phi-\phi_0)^4+...\Big]. \end{aligned}, \tag{2.38}$$

To better understand the behaviour of V, let us consider the case

$$\delta V = \frac{D}{\sigma^2}: \tag{2.39}$$

the expansion around the minimum then looks like

$$\begin{aligned} V(\phi) =& \frac{a^3A^2e^{-2a\sigma_0}}{9}\Big[\frac{1}{2}(3+2a\sigma_0)(\phi-\phi_0)^2-\frac{1}{3\sqrt{6}}(9+13a\sigma_0+6a^2\sigma_0^2) \\ &(\phi-\phi_0)^3+\frac{1}{36}(21+28a\sigma_0+23a^2\sigma_0^2+14a^3\sigma_0^3)(\phi-\phi_0)^4+...\Big]. \end{aligned} \tag{2.40}$$

As all the coefficient of each term share the same sign, it is clear that the potential represented is asymmetric around ϕ_0, being steeper than quadratic to its left and shallower than quadratic to its right. Given that the focus of this work is on the formation of compact objects out of moduli, let us notice that such an asymmetry has interesting consequences on their stability, as it results in a self-interaction that alternates between being overall attractive and being overall repulsive during an oscillation.
There are two main issues to the KKLT procedure. One is that the potential was obtained by first fixing the complex structure moduli and the dilaton, which are assumed

to be heavy at the classical level. The correct method would instead ask to minimize the full potential, but such calculations become highly involved. The other point is that no corrections to the Kähler potential were considered, however at large volumes the α' corrections to it start to become dominant. Moreover, the inclusion of $\overline{D3}$ branes back-reacts on the geometry in such a way that may not be compatible with the existence of the dS minimum found. For these reasons the KKLT scenario should only be taken as a toy model, though its results remain effective.

Chapter 3

Moduli stars

String theory predicts a large number of scalar particles coupled gravitationally, specifically moduli and axions, and it's not the only theory to do so, as spin-0 fields have gained a central and pervasive role in many models that aim at describing Nature. The first example that comes to mind is that of the Higgs field, the key for a mass inducing mechanism in the Standard Model, which was first proposed in 1962 [28] [27] and found at CERN in 2012 after a long search [29]. From the infinitesimally small to cosmology, the inflaton is a scalar particle proposed in the context of the Big Bang theory to drive the exponential expansion of the very Early Universe [1]. Examples of other relevant scalar fields are found in solid milestone theories such us Supersymmetry or the Peccei-Quinn mechanism that addresses the strong CP problem of QCD [25], as well as in very recent attempts to extend the Standard Model via asymptotic safety [24][2].
This renewed interest in the topic of spin-0 fields has resulted in many papers in last few years investigating a particular consequence of the supposed existence of such fields: scalar stars. The scalar particles could in fact clump up together under the influence of their gravitational pull, their attractive self-interaction or an interplay of both to form an exotic object sustained by hydrostatic equilibrium (aka a star). To be more precise we are talking about a Bose-Einstein condensate, but one in the form of a localized clump instead of the usual long range one, as suggested in [45] and [26].
Moduli could therefore aggregate to form moduli stars, whose formation, stability and dynamics can be studied exploiting the literature on generic scalar stars by adapting it to the specific potential at hand, to form another feature of our Universe. These objects could form and evolve in the Early Universe or even survive until our present day, but either way they would leave a trace (e.g. in the stochastic background of gravita-

[1] Actually at least one scalar field is required in inflation models.

[2] The method is a UV completion mechanism which employs a non trivial UV fixed point in the renormalization group to solve problems such as the presence of Landau poles in the theory. Several fundamental scalars come into play to allow the presence of such a fixed point, as suggested by perturbative analysis in the Veneziano limit.

tional waves) which could even be within the reach of our current detectors like Gaia [30] or LIGO [31]. Such signals turn out to be extremely model dependent and hence carry significant information on the corresponding string theory and moduli stabilization mechanism which is realized. Their detection would give us an invaluable insight into a possible fundamental theory.

3.1 Scalar stars

The scalars predicted in the context of Supersymmetry give rise to a very natural parallelism: as fermion stars, like neutron stars, are such a common feature of our Universe, what about a bosonic counterpart? Is it theoretically possible for these objects to exist, even if we have not yet found them? A key part in the existence of a fermion star is played by the Pauli's principle. The Fermi-Dirac statistics forbids two identical particles from occupying the same energy level at the same time, providing a degeneracy pressure that counteracts the gravitational pull of the clump and allows hydrodynamical equilibrium of the fermion gas. The energy profile will therefore exhibit two terms with opposite signs, and for a spherically symmetric configuration of radius R of N fermions of mass m of a single field takes the form

$$E(R) = -\frac{GMn}{R} + \left(\frac{9\pi}{4}\right)^{1/3}\frac{N^{1/3}}{R} \tag{3.1}$$

where $M = Nm$ is the total mass of the star. The first term is of course the attractive gravitational potential, while the second is the relativistic kinetic energy of a fermion on the surface of the clump. The star will expand and the fermion density decrease until the kinetic energy is of order $\sim m$ and balanced by the self-gravitational pull. Supposing a maximum number N_{max} that such a compact object could sustain, the two energetic terms would be of the same order in that situation. We can therefore have an estimate of the associated maximum mass for a fermion star[3]:

$$M_{max} \sim \frac{M_P^3}{m^2} \tag{3.2}$$

equal to the Chandrasekhar mass. Taking $k \sim m$ gives us the associated minimum radius R_{min}, as the momentum k is related to the number density in Fermi-Dirac statistics through $\frac{k^3}{3\pi^2} = \frac{N}{(4/3)\pi R^3}$. We obtain

$$R_{min} \sim \frac{M_P}{m^2}. \tag{3.3}$$

For a neutron star ($m_N \approx 1\text{GeV}$) this means $M_{max} \sim M_\odot$ and $R_{min} \sim 2\text{km}$, as expected. When it comes to bosons we can no longer rely on the Pauli principle, as they obey the

[3] M_P is the reduced Planck mass, defined as $M_P^2 = \frac{1}{8\pi}G$.

Bose-Einstein statistics. They however still have to abide the Heisenberg principle. As $\Delta x \Delta p \geq \hbar$, we can interpret Δx as the radius of the star R and $\Delta p = mc$ as the energy of the boson of mass m and readily make a rough estimate of the minimum radius:

$$R_{min} \sim \frac{1}{m}. \tag{3.4}$$

To get the maximum mass we set the radius to the Schwarzschild value $R_S = 2GM$ to get

$$M_{mass} \sim \frac{M_P^2}{m}. \tag{3.5}$$

Supposing again a particle of mass $m = 1$ GeV we are now presented with a compact object of dimension $R_{min} \sim 10^{-15}$cm and mass $M_{max} \sim 10^{36}$ GeV$\sim 10^{-21} M_{\odot}$, many orders of magnitude away from our usual stars and even smaller than an atomic nucleus. Considering an attractive interaction however increases significantly the capability of the star to sustain itself and allows it to reach macroscopic dimensions and masses in the GeV scale. As the analytical study in [32] shows, considering the conglomerate in a general relativity framework with a quartic interaction term gives a mass akin to the Chandraseakar limit, considering a coupling $\lambda \sim \mathcal{O}(1)$.
Bosonic stars are actually realized as pseudo-solitonic solutions of the equation of motion of the corresponding field (aka the Einstein-Klein-Gordon equations or the Schrödinger-Poisson equations in the non relativistic limit), that is meta-stable solutions which are localized, in the sense that their boundary condition at infinity is the same as the one for the vacuum state, that represent long lived compact objects. Their (meta-)stability is protected by either the conservation of a topological charge, the conservation of a Nöther charge or an approximate symmetry[4]. The first case is that of the class of topological solitons, which includes monopoles, skyrmions, vortices and kinks, just as a few examples. While the existence of some has been ruled out, at least in the observable Universe, by the strict phenomenology that they would impose, others still provide interesting features[5]. In the absence of a topological charge we talk about non-topological solitons. Let us restrict for simplicity to the case of an object made out of a single scalar field. Derrick's theorem states that stationary localized solutions to the non-linear Klein-Gordon equation are unstable [46], however one can safeguard stability by requiring a periodic time-dependence [47]. If the field ϕ in question is complex and its Lagrangian presents a global U(1) symmetry, than the transformation $\phi \to e^{i\alpha}\phi$ can compensate the

[4]"I think it's fair to say that these two kinds of objects are in the same family but not the same genus" [14].

[5]Topological solitons are topics of interest also in the context of condensed matter systems, where they are often called topological defects. Skryrmions, in particular, have been shown to exist in magnetic materials at room temperature [34].Their topological charge could be interpreted as a bit state (i.e. the presence or absence of a skyrmion), placing them at the core of the next possible generation of data storage devices[33].

time translations for ϕ and allows for a stationary spherically symmetric solution to the Einstein-Klein-Gordon equation in the form

$$\phi(r,t) = \phi_R(r)e^{i\omega t} \tag{3.6}$$

with a static metric $ds^2 = -A(r)^2dt^2 + B(r)^2dt^2 + r^2(d\theta^2 + sin^2\theta d\phi)$. This symmetry also provides a Nöther charge, whose conservation further ensures the stability of the solution. When self-interacting terms other then gravity are negligible or weak we call the resulting objects mini-boson stars, as their dimensions are in the ballpark of those already calculated in equation 3.4 and equation 3.5 with our simple reflection on the Heisenberg principle. Turning on strong self-interaction terms allows the solution to grow up to fermion star scales, earning them the full fledged name of Boson stars. In the regimes in which we can neglect gravity, the lumps can still aggregate under the influence of attractive self-interactions of ϕ. This class of objects was first described by Coleman in 1985 in his paper [14], where he gave them the name of Q-balls referring to their conserved charge. By minimizing their Hamiltonian he showed that, for large Q, their energy and volume grow linearly with Q, or in other words they behave as homogeneous balls of ordinary matter with Q in the role of a particle number. In fact starting from the U(1) symmetric Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi^* - V(|\phi|) \tag{3.7}$$

we get an associated Nöther charge

$$Q = \int d^3xJ^0 = \int d^3x\frac{1}{2i}(\phi^*\dot{\phi} - \dot{\phi}^*\phi). \tag{3.8}$$

To find a stable solution we look for minima of the Hamiltonian, enforcing the requirement of fixed and non-vanishing Q through a Lagrangian multiplier ω as in

$$E_\omega = \int d^3\big[\frac{1}{2}|\dot{\phi}|^2 + \frac{1}{2}|\nabla\phi|^2 + V(|\phi|)\big] + \omega\big[Q - \frac{1}{2i}\int d^3(\phi^*\dot{\phi} - \dot{\phi}^*\phi)\big]. \tag{3.9}$$

A field of the familiar form

$$\phi(x,t) = \phi_R(x)e^{i\omega t} \tag{3.10}$$

cancels the kinetic term and shifts the problem onto finding a field $\phi_R(x)$ that extremizes a potential $\tilde{V}_\omega(|\phi|) = V(|\phi|) - \frac{1}{2}\omega^2|\phi|^2$. As we are looking for a localized solution, let us assume that the field vanishes outside of a defined region $\mathcal{V}$ and call its value inside it ϕ_0. Under the thin walls approximation (or large Q) we can neglect the gradients and eliminate ω from the theory by varying E_ω with respect to it. We get $\omega = Q/(\mathcal{V}\phi_0^2)$, that put back in energy gives

$$E = V(\phi_0)\mathcal{V} + \frac{Q^2}{2\mathcal{V}\phi_0^2}. \tag{3.11}$$

Varying the energy with respect to $\mathcal{V}$ gives us the volume of the star $\mathcal{V} = Q/\sqrt{2\phi_0^2 V(\phi_0)}$ and consequently the energy $E = Q\sqrt{\frac{2V(\phi_0)}{\phi_0^2}}$. Both, as promised, grow linearly with Q. The last thing to do is to find the actual value of ϕ_0, as the one to minimize equation 3.11 and therefore V/ϕ_R^2. Remembering that $\omega = Q/(\mathcal{V}\phi_0^2) = \sqrt{2V/\phi_o^2}$ shows that the ϕ_0 we are looking for is the same that non only minimizes $\tilde{V}$ but makes it vanish. As that's the same value that we postulated for the energy outside the Q-ball (by taking $\phi_R = 0$, the value $\tilde{V} = 0$ remains a minimum as long as $\omega^2 > V''(0)$), this minimum is degenerate and the Q-ball exists as long as $V/|\phi|^2$ has a non vanishing minimum. Outside the thin walls approximation, solutions can be found numerically and show that there is no classical lower limit on Q for the formation of a stable compact object, going as low as $Q \geq 1$ (when they are called Q-beads[6])[35].
The choice of a complex field is not required, but if we instead consider a real scalar ϕ field we can no longer rely on a conservation law to protect the stability of our star. We can however consider cases where there is an approximate symmetry that ensures the long-life of our compact object, which in this case will be a pseudo-soliton. Notable real pseudo scalar fields include axions (both the one related to the PQ mechanism for the strong CP problem of QCD and the ones that arise in string theory as the phase of an open string moduli or the imaginary part of a closed string moduli) and of course our dearest, the moduli. Choosing a real field also implies the fact the we can no longer write it as equation 3.6 and avoid Derrick's theorem, so the metric needs to become time dependent and the solution will involve expanding both the field and the metric components in Fourier series. This kind of solutions are called oscillatons, while if we can neglect gravity we are talking about oscillons. In general, fields are believed to support oscillons formation whenever their potential is shallower than quadratic, meaning that there is an attractive interaction (we will come back to this claim later). Moduli that present a potential which is asymmetric around the minimum, such as in the KKLT case, are therefore particularly interesting to study, as the attractive and repulsive overall potential alternate during an oscillation. While it is natural to assume that this erratic behaviour renders the corresponding star more unstable, it is actually found it not to be the case, with stability being very model dependent [2]. This dependence renders the traces that the moduli star would leave in Universe deeply linked to the specific model that generates the potential and therefore the underlying string theory, rendering the topic definitely worth studying (we will come back to this topic in a later section).
Axion stars have actually been more at the center of attention in the literature for two main reasons: they exhibit a potential which is even around the minimum, thus making their study easier, and they are of interest being a plausible candidate for dark matter. Ultra light axions of mass $m \sim 1 - 10 \cdot 10^{-22}$ eV, for example, could form axion solitons compatible with the so called fuzzy dark matter or ultra-light dark matter (ULDM). Let

[6]These kind of variation could actually be more significant for the evolution of our Early Universe, as Q-ball with small charges could be more easily generated at high temperatures.

us take a look at one of the most studied cases, the QCD axion clump with spherical symmetry. The QCD axion is actually a pseudo-Goldstone boson associated with the spontaneous symmetry breaking of the $U(1)_{PQ}$ and can be described as a scalar field ϕ with potential

$$V(\phi) = \Lambda[1 - cos(\phi/f_a)] \tag{3.12}$$

where f_a is the breaking scale of the $U(1)_{PQ}$ symmetry[7] and Λ, of order of the QCD scale, sets the scale of the potential. We can expand V around its minimum to get

$$V(\phi) = \frac{1}{2}m^2\phi^2 - m^2 f_a^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\phi}{f_a}\Big)^{2n}. \tag{3.13}$$

As it will be discussed more in depth in the next chapter, looking for a solution in the non relativistic regime it is useful to express ϕ in terms of complex field ψ

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}}\left(\psi(\mathbf{x}, t)e^{-imt} + \psi^*(\mathbf{x}, t)e^{imt}\right) \tag{3.14}$$

$$\pi(\mathbf{x}, t) = -i\sqrt{\frac{m}{2}}\left(\psi(\mathbf{x}, t)e^{-imt} - \psi^*(\mathbf{x}, t)e^{imt}\right). \tag{3.15}$$

Substituting in the Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big] \tag{3.16}$$

in the weak Newtonian gravity approximation of metric

$$ds^2 = -(1 + 2\Phi_N)dt^2 + (1 - \Phi_N)(dx^2 + dy^2 + dz^2) \tag{3.17}$$

with Φ_N being the Poisson potential gives us a naive effective Lagrangian for the non relativistic regime

$$\mathcal{L}_{eff} = \frac{i}{2}(\dot{\psi} - \psi\dot{\psi}^*) - \frac{1}{2m}\nabla\psi^*\nabla\psi - m\psi^*\psi\Phi_N + \frac{1}{16f_a^2}|\psi|^4 \tag{3.18}$$

where we dropped rapidly oscillating terms, considered only the leading non-linearity in the potential and took $|\dot{\psi}|/m \ll |\psi|$ in the kinetic term. A Legendre trasform gives us the effective Hamiltonian for the system, which we can compute explicitly by assuming a spherically symmetric configuration for the ground state. This is reasonable, as the theory respects rotational invariance and there is no mechanism that would spontaneously break it. We therefore assume the usual

$$\psi_0(r, t) = \Psi(r)e^{-i\mu t} \tag{3.19}$$

[7]The specific value of f_a depends on the details of the model, but typically $f_a \leq 10^{12}$ GeV.

Now we can study the system in two different regions. For $r \to \infty$ $\Psi \to 0$, as we are looking for a localized solution, so for large r we can ignore the non-linear self-interactions. This leaves as the equation of motion

$$\mu\Psi \simeq -\frac{1}{2m}\left(\Psi'' + \frac{2}{r}\right) - \frac{Gm^2 N}{r}\Psi \tag{3.20}$$

where we introduce $N = 4\pi \int_0^\infty dr' r'^2 \Psi(r')^2$ as the total number of particles. This equation is akin to the Schrödinger time independent equation for an hydrogen atom (just with $e^2 \to Gm^2 N$). The solutions are therefore of the form

$$\Psi(r) = Poly_n(r) e^{-Gm^3 Nr/n} \tag{3.21}$$

where $Poly_n(r)$ stands for a polynomial function in r of degree n. Getting in the near field region, however, self-interactions can no longer be neglected and there is no known analytical solution to the equation of motion. One way to gain an understanding of the system is to impose an ansatz for the whole space, and with that perform explicit calculation. A natural choice is

$$\Psi_R(r) = \sqrt{\frac{N}{\pi R^3}} e^{-r/R}, \tag{3.22}$$

where R is a variation parameter that sets the decay length scale and the prefactor ensures N to be the total number of particles. A straightforward evaluation of the Hamiltonian yields

$$H_{eff}(R) = \frac{N}{2mR^2} - \frac{5Gm^2 N^2}{16R} - \frac{N^2}{128\pi f_a^2 R^3}. \tag{3.23}$$

It is useful to perform a rescaling to deal with dimensionless quantities:

$$\tilde{R} := m f_a \sqrt{G} R \tag{3.24}$$

$$\tilde{N} := \frac{m^2 \sqrt{G}}{f_a} N \tag{3.25}$$

$$\tilde{H} := \frac{m}{f_a^3 \sqrt{G}} H_{eff}. \tag{3.26}$$

We have

$$\tilde{H}(\tilde{R}) = \frac{\tilde{N}}{2\tilde{R}^2} - \frac{5\tilde{N}^2}{16\tilde{R}} - \frac{\tilde{N}^2}{128\pi \tilde{R}^3}. \tag{3.27}$$

The solitons that we are looking for will be the maxima and minima for this $\tilde{H}$. Performing the first derivative of the last equation and setting it to zero gives a simple second order equation of solutions

$$\tilde{R} = \frac{8}{5\tilde{N}} \pm \frac{\sqrt{512\pi - 15\tilde{N}^2}}{10\sqrt{2\pi}\tilde{N}} \tag{3.28}$$

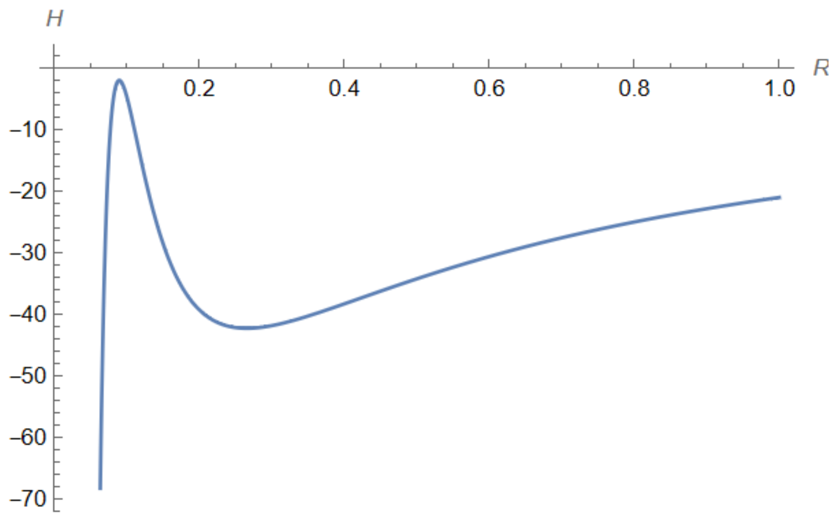


Figure 3.1: Plot of the dimensionless effective non relativistic Hamiltonian for the PQ axion with respect to the dimensionless Radius of a localized solution, chosen in the exponential form. H correctly goes to 0 as the parameter R approaches infinity, as required by the assumption that we are looking for localized solutions, and presents two critical points with negative energy that are compatible with bound configurations: a maximum (aka an unstable oscillaton) and a minimum (the stable oscillaton).

where the + solution corresponds to the minimum, while the - solution is the maximum, as can be seen in 3.1, where the dimensionless Hamiltonian is plotted for $\tilde{N} = 9$.

The two families of solutions are plotted in Fig. 3.2. As can be seen by both the graphs and eq. 3.28, there is a limit on $\tilde{N}$ to keep the solution physical (i.e. $\tilde{R}$ as a real, positive number)

$$\tilde{N}_{max} = \sqrt{\frac{512\pi}{15}} \simeq 10.36 \tag{3.29}$$

As $\tilde{N}$ increases the gravitational and self-interacting terms become comparable, while for small $\tilde{N}$ there is a significant difference in the configuration, depending on whether the dominant interaction is gravitation (for the stable solutions) or the quartic self-interaction (for the unstable solutions), resulting in more compact stars in the latter scenario. An additional stable branch related to a denser solution is sometimes found in the literature, but that seems to be an artefact of an inadequate use of the effective field theory, at least for axions and up to order $\mathcal{O}(g_4^3, g_6^2, g_4^2 g_6)$[8], as shown in [13]. The actual solution of the system that's been outlined requires numerical methods. Numerical

[8] g_4 and g_6 being the coefficient respectively of the quartic and sixth interacting terms in the Lagrangian for the axion expanded around its minimum, which has a $\mathbf{Z}_2$ symmetry

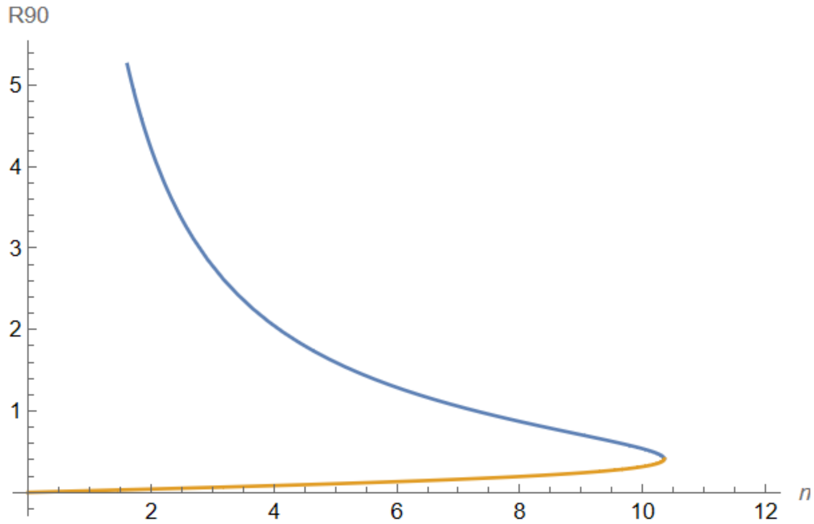


Figure 3.2: Plot of the dimensionless Radius $R90$ vs the dimensionless total number of particles for a PQ axion star in the non relativistic regime, where we introduce $R90$ as the radius inside which 90% of the matter is found (for the exponential ansatz one can calculate $R90 \simeq 2.661\tilde{R}$). The minimum and maximum of $\tilde{H}(\tilde{R})$ correspond to two families of solutions: the unstable branch (in yellow) and the stable branch (in blue). The first case is realized when self-interactions dominate, while in the second the star is stabilized by the effects of gravity. As N increases the two effects become of comparable magnitude and the configuration can migrate from one branch to another. The physicality of the solution requires a maximum value for $N \sim 10$. The choice of the ansatz makes the solution not reliable in the region $R90 \to 0$, where the non relativistic approximation is no longer valid.

methods and simulations like Floquet analysis or lattice simulations are the tools of choice found in most of the literature, however the solutions obtained analytically with our ansatz are in a nice agreement with the actual results and can be made better with a more careful choice of ansatz [3].
We can in general identify two regimes for oscillatons, the dense and the dilute one, and quantify them by labelling Λ as the typical field range of the canonically normalized field ϕ. Λ sets the natural scale for the mass of the star and the radius of the star as

$$\frac{M}{\tilde{M}} \simeq \frac{\Lambda^2}{m} \qquad\qquad \frac{R}{\tilde{R}} \simeq \frac{1}{m} \tag{3.30}$$

where $\tilde{M}$ and $\tilde{R}$ are dimensionless parameters to be computed numerically. The density of the star will therefore be linked to the quantity $C = M/R = (\tilde{M}/\tilde{R})\Lambda$, with the most compact objects being those for which $\Lambda = M_P$ as, recalling equation 3.5 and equation 3.4, for a fixed $\tilde{M}/\tilde{R}$ the quantity C is suppressed by a factor Λ^2/M_P^2. In which regime we are is set by how the core amplitude (i.e. the amplitude of the background field oscillations $max\{\phi(0,t)\}$) compares to this scale Λ, as the highest the first the denser the star. If the first is small compared to the second we are therefore in a diluted regime, where weak field Newtonian approximation for gravity are applicable and self-interactions can be neglected. The equations of motion to be solved will then be the Poisson equation coupled with either the Klein-Gordon equation or the Schrödinger equation, in the non relativistic limit. If instead the core amplitude is comparable to the scale Λ, we are in the dense regime and eventual self-interactions can no longer be ignored. For $\Lambda = M_P$ the system in the absence of self-interactions presents a stable branch of solutions and an unstable branch, whose perturbed configurations can either collapse to black hole or migrate to the stable branch, depending on the sign of the perturbation. When present, however, self-interactions prevail and it becomes possible to self-consistently neglect gravitational effects, entering in the domain of oscillons.
For both axions and moduli stars the answer to the questions of whether they can form, how self-gravity affects them and what is their fate (that is if under perturbation they are stable, disperse or collapse to black hole) is highly model dependent. Moduli stars can at most be meta-stable because the moduli will eventually decay, however they could survive until today provided $m \leq 10^{-2}$ GeV. In [2] it is shown how including effects of gravity on a moduli star that would otherwise disperse can either stabilize it or compress it so much that it eventually collapses to a black hole. Even this effect can have sizeable consequences in our Universe, as it would imply the formation of more stars than those predicted by lattice simulations for oscillons in pre-heating scenarios and hence more of the related signals. For a stable oscillon instead its transition to a oscillaton could signify an enhancement of the amplitude of ϕ oscillations (affecting the production of gravitational waves) or dynamically drive it to collapse into a black hole (which would affect the reheating history of the Universe). It is found that for the KKLT potential configurations tend to be meta-stable, even for particularly heavy initial conditions where

other models,like the α-attractor T model or the α-attractor E model, evolve into a black hole even for quite large radii (i.e. much larger than the Schwarzschild radius).

3.2 Mechanisms of formation

As stated, the outcomes of moduli vary greatly depending on the specific potential at hand and the actual formation needs to be checked explicitly via lattice simulations. Let us focus on oscillons, as that's the appropriate regime when discussing pre-heating formation. We need a mechanism that triggers the growth of perturbations. More than one are possible, they are successful for moduli in different models and they are classified depending on the values of the background field ϕ during the growth of perturbations. Let us assume that our moduli field ϕ is initially displaced beyond the inflection point, near the minimum. When ϕ rolls down towards the minimum it passes a tachyonic region ($V''(\phi(t)) < 0$) and there the infrared modes $\delta\phi_k$ grow exponentially, for $k^2/a^2 + V''(\phi) < 0$[9]. This is the fist mechanism, which is called tachyonic preheating. There are additional non linear effects from the interplay of the $\delta\phi_k$, but the results are subleading. The effects on the growth of the perturbations depend of course on the magnitude of the initial displacement, but to obtain oscillons we always need a follow up, more efficient mechanism for the growth of the oscillations. After ϕ reaches the minimum it starts to oscillate around it, passing periodically the extremant $V''(\phi) = 0$ and alternating its passage in the regions $V''(\phi) < 0$ and $V''(\phi) > 0$: we have tachyonic oscillations. Each oscillation provides a growth phase for the perturbations, when ϕ accelerates rolling down, and a decreasing phase, as ϕ decelerates climb up the minimum valley. Taking into account the expansion of the Universe, these oscillations lead to an overall growth of the fluctuations, whose spectrum is of course is closely related to the frequency of oscillations. As we have two processes at work in this mechanism, its efficiency will depend on how they compare. If the expansion rate of the Universe $a(t)$ is large compared to the frequency of the oscillations, ϕ will quickly relax to the minimum and we won't have a sufficient enhancement of the fluctuations. If instead the ratio is reversed, the expansion of the Universe can allow for a large enough number of oscillations so that non linear interplay among fluctuations with different wave number become relevant and we enter a non-perturbative regime. A third mechanism, parametric resonance, can dramatically amplify the perturbations, and it occurs when ϕ oscillates around the minimum with a time dependent frequency

$$\omega_k^2(t) = \frac{k^2}{a^2} + V''. \tag{3.31}$$

[9] a here is the time dependent coefficient of the FLRW metric which we are considering as background: $ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$

When the adiabatic condition $|\dot{\omega}_k(t)| \ll |\omega_k(t)|$ is violated,a range of fluctuation modes can grow exponentially, eventually leading to the fragmentation of ϕ.
The formation of the field perturbations and their dynamics during the evolution of the Universe can be tracked with lattice simulations. As the Universe expands, the dynamics between fluctuations eventually becomes less violent and can lead to the formation of stable,highly energetic, localized regions, which we recognize as oscillons finally being formed. In the KKLT model parametric resonance is fit to lead to the formation of oscillons in a gradual way, leading to particularly stable solutions. On the other hand, for example, the blow-up moduli in the LVS scenario can support oscillons formation via tachyonic oscillations, but the dynamic is significantly more violent and the stable configurations are achieved indirectly, through the fragmentation of previous, unstable inhomogeneities. Other moduli, like the volume modulus in LVS or the fibration moduli in Swiss cheese Calabi-Yau manifolds just don't support oscillon formation at all[biblio:Antush].

3.3 Possible experimentally relevant signals

The possible presence of moduli stars and their prevalence and dynamics would have significant effects on the cosmological evolution of our Early Universe. They could dominate the energy density before decaying and delay thermalization, catalyze second and first order phase transitions, be a source of baryon asymmetry, just to name a few. Adherence to phenomenology can only rule out some scenarios, but cannot provide a proof of existence. Although moduli stars could theoretically still be present in our current space landscape and direct encounter could be possible, that's not the preferred method to find experimental clues about them, especially as the majority of the interest focuses on how moduli star would affect eras of the Early Universe. We need something able to probe both the present and the past, to asses the existence of moduli stars at some point in the history of our Universe, and relatively young in their experimental use, as no data with a direct link to them has yet to be found: gravitational waves (GWs) represents the best candidate. GWs carry a signature of the event that originated them, and in some cases that signature for moduli stars turns out to be sharply distinctive to those with respect to other objects we are already acquainted to. Moreover since their experimental discovery in 2015, GWs have been the protagonists of much attention from the scientific community, leading to significant funding and upgrades of experimental facilities, making them the best area on which to bet at this time. Moduli stars could contribute both to new events, provided they are long enough lived, or to the stochastic background of primordial gravitational waves. In moduli star scenarios, GWs are generated during preheating in the process of the formation of the star itself. They carry a signature of their origin in the form of a distinctive multiple peaks structure, whose power spectrum

is defined by the size of the oscillon[10] [37].
If the moduli stars are sufficiently long lived to undergo dynamical interaction, they could form binary systems and eventually collide. Depending on the compactness $C = M/R$ the result is different: for increasing values of C of the colliding stars we have as a result [36]
i- an excited stable oscillaton,
ii-the formation of a black hole after collision,
iii-a pre-merger collapse of the oscillatons to separate black holes due to tidal forces.
The first case provides a possibly long lived source of GWs, with multiple post merge pulses which represents a distinct signature from the analogue black hole-black hole case. The second case presents a similar waveform to that of a black hole-black hole collision, but with a significantly bigger amplitude. These events could therefore be disguised as mergers of more massive black hole, but could have other identifying features. The last case is extremely akin to a black hole-black hole collision. The profile of the GWs pertaining to the cases *i*,*ii* and *iii*, compared to corresponding black holes events, obtained by Helfner, Lim, Garcia and Amin [36] are reported in Fig. 3.3. LIGO has the right frequency range to be sensitive to GW resulting from this collisions for a mass of the moduli 10^{-12} eV$\leq m \leq 10^{-10}$ eV. Another way in which we could have GWs from scalar stars is from the dynamics of a single object, provided it doesn't have a spherical symmetry. Although a spherically symmetric configuration is a common choice to look for a solution, there is nothing that imposes it other than usefulness in simplifying the calculations. For $m \sim 10^9$ GeV the peak of the spectrum $f \sim \mathcal{O}(m)$ falls within the LIGO frequency range and the energy density of GWs goes as the fourth power of the oscillon field amplitude, therefore increasing with compactness C [2].
Another array of possibilities is presented when considering moduli stars that have collapsed to black holes, either as their independent evolution or as a consequence of clustering with other compact objects. First of all, this contingency would make the object significantly more long lived: while typically real scalar stars have a lifetime $T_{star} \sim 10^3 - 10^4 \cdot m^{-1}$, black holes with the same order of mass $M \sim M_P^2/m$ (which is a reasonable outcome [2]) have a lifetime $T_{BH} \sim \frac{M^3}{M_P^4} \sim \left(\frac{M_P}{m}\right)^2 \cdot m^{-1}$, which is larger than T_{star} for $m \leq 10^{-2} M_P$. This extended lifetime would contribute to enhance their dynamics scenarios. Moreover, the black holes that arise as a result of the collapse of a moduli star would have very small mass compared to usual ones. They would therefore evaporate quickly into both the Standard Model sector and hidden sectors to be a plausible candidate per dark matter generation [23]. This contingency would provide strong constraints which so far have non been in contrast with phenomenological findings, e.g. CMB measurements [22].

[10]Oscillons, kept together by self interactions instead of gravity, are in fact the right limit to consider during pre-heating

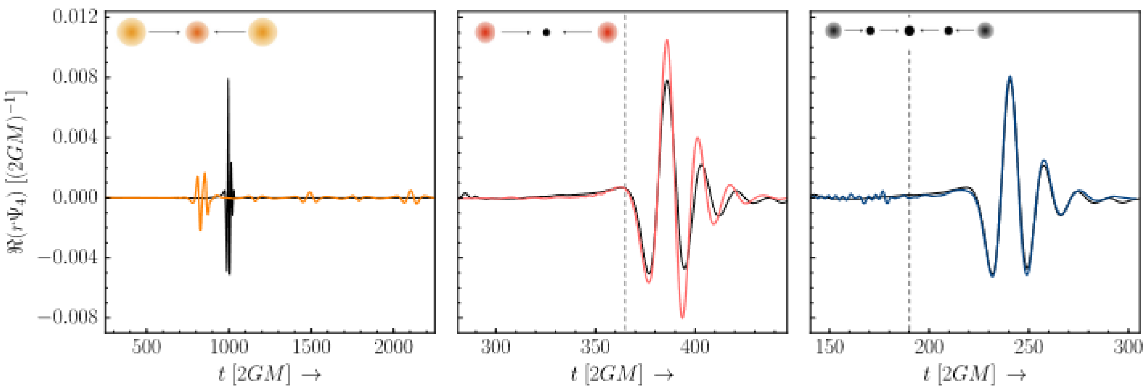


Figure 3.3: Graph from [36] showing their results for GWs generated by a scalar star-scalar star collision obtained with numerical simulations in full GR. Each panel also reports in a solid black line the GWs generated by the same events involving instead black holes of the same mass as the stars considered for comparison. From left to right the value of C is increasing (specifically we have $C = 0.03$, $C = 0.10$, $C = 0.15$). Panel i: the collision generates an excited oscillaton that keeps on emitting. Panel ii: the collision results in a black hole, whose formation is marked by the dashed line, and the GW profile is akin to that of an event involving more massive black holes. Panel iii: the stars turn into separate black holes under the effect of tidal waves and the collide, at t marked by the dashed line. They show a signal almost indistinguishable from the black hole-black hole case of the same mass.

Chapter 4

Non relativistic effective field theory

In the previous chapters we have seen how oscillons made out of moduli could be a feature of our Universe and what that would entail. Given the implication of this possibility, we would like to understand from the analytical point of view whether the KKLT potential can sustain such formations in the non relativistic regime and compare the result with the meta-stability found via numerical simulations [2] [12]. As a first step we first need to construct a non relativistic effective field theory for the moduli. Let us rewrite the expansion around the minimum for the KKLT potential as

$$V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \tag{4.1}$$

where the coefficients are defined accordingly to result 2.40 of chapter 2 and we took out the signs to highlight the asymmetric behaviour. The are many papers in the literature were the authors focus on obtaining a non-relativistic effective Lagrangian for oscillon formation [17]. The attention however is all reserved for the axion, and the reasons are mainly two. First of all, axion stars have recently sparked a good deal of interest for being plausible dark matter candidates; as a second point, their potential presents a discrete $\mathbb{Z}_2$ symmetry which simplifies the calculations. The moduli potential is not even, but we can still apply the methods developed to our case. To do so let us start with a review of the procedure for the axion used by Hertzberg and Schiappacasse in [3] and of the one devised by Namjoo, Guth and Kaiser [8]. Having developed an insight on the proper construction of the effective Lagrangian we can move onto the moduli case. We make use of the Namjoo-Guth-Kaiser method to account for relativistic corrections in a systematic way. The ensuing effective Lagrangian is a result which is not found in the literature. To check it we calculate effective vertices via Feynman diagrams and compare the outcome, which turns out to be related to ours by a redefinition of the field.

4.1 Non relativistic effective Lagrangian for the axion

We can start from the work that has been done on the axion, specifically on the QCD axion (which is usually chosen for its sound theoretical base), to construct the effective Lagrangian in the non-relativistic limit. Let us recall its potential, arising from non-perturbative effects:

$$V(\phi) = \Lambda[1 - cos(\phi/f_a)] \tag{4.2}$$

where f_a is the breaking scale of the $U(1)_{PQ}$ Peccei-Quinn symmetry, of typical value $f_a \leq 10^{12}$ GeV and Λ is of order of the QCD scale. We can expand V around its minimum to get

$$V(\phi) = \frac{1}{2}m^2\phi^2 - m^2 f_a^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!}\Big(\frac{\phi}{f_a}\Big)^{2n} \tag{4.3}$$

with $m = \Lambda^2/f_a$. We can actually interpret f_a as a loop counter: by taking $m/f_a \ll 1$, as is the case of the QCD axion[1], the diagrams are suppressed by a factor $(m/f_a)^2$ for each loop, so constructing a classical effective field theory is tantamount to keeping only the leading terms when expanding in m/f_a. So as a Lagrangian for oscillons formation we have

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \frac{m^2}{4!f_a^2}\phi^4. \tag{4.4}$$

Now, $\phi(x)$ is the (pseudo-)scalar field describing the axion. Its particles are identical spin-0 bosons, so we can think about an axion star as a localized, short-range order, Bose-Einstein condensate [45]. Because of its high occupacy number it can be well described by a classical complex field theory, even in the presence of strong self-interactions [44], where the complex mean field in question can be though of as the vacuum expectation value of a complex quantum field $\psi(x)$. For this reason, it's plausible to assume that in the non-relativistic regime the field theory for real axion field $\phi(x)$ reduces to a theory for a complex valued non-relativistic scalar field $\psi(x)$. We therefore perform the substitution

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}}\left(\psi(\mathbf{x}, t)e^{-imt} + \psi^*(\mathbf{x}, t)e^{imt}\right). \tag{4.5}$$

In may look like we are doubling the number of degrees of freedom, as we are trading a real field for two (the real and imaginary part of the complex field). Actually with the substitution $\phi \to \psi$ one simultaneously changes the structure of the Lagrangian too: the relativistic Lagrangian is second order in the time derivative of ϕ, while the non relativistic Lagrangian is first order in the time derivative of ψ. As in in the first case the number of actual propagating degrees of freedom is equal to the number of real fields,

[1]For the QCD axion $m/f_a \sim 10^{-48}$.

while in the latter case it's equal to just half of them; the reduction 4.5 is perfectly fine. One last consideration is in order before we can proceed with the substitution of relation 4.5 in the Lagrangian as a first step towards a relativistic reduction. Even though the two fields have the same number of propagating degrees of freedom, $\phi(x)$ is real valued while $\psi(x)$ is complex valued. In order to have a one-to-one mapping between the two at a fixed time t we also need to specify the conjugate momentum to ϕ, $\pi = \dot{\phi}$:

$$\pi(t, \mathbf{x}) = -i\sqrt{\frac{m}{2}} \left(\psi(t, \mathbf{x})e^{-imt} - \psi^*(t, \mathbf{x})e^{imt}\right). \tag{4.6}$$

Now that everything is settled, we can substitute the non relativistic reduction 4.5 into the Lagrangian 4.4. Considering the non relativistic limit we can drop the rapidly oscillating terms, as they would average to zero over long times, and consider $|\dot{\psi}/m| \ll |\psi|$ in the kinetic term. We obtain

$$\mathcal{L} = \frac{i}{2}(\dot{\psi}\psi^* - \dot{\psi}^*\psi) - \frac{1}{2m}\nabla\psi^*\nabla\psi - \frac{\lambda}{16f_a^2}(\psi^*\psi)^2. \tag{4.7}$$

This result, although used in the literature (e.g. in [3]), is indeed very rough. We need to take into account more terms and to do so a promising solution is offered in [8]. There Namjoo, Guth and Kaiser develop a construction method based on a non local operator that enables us to compute the relativistic corrections systematically. Let us consider a more generic Lagrangian then the one for the QCD axion and only keep the $\mathbb{Z}_2$ from it:

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{4.8}$$

The energy for the free relativistic particle is $E = \sqrt{p^2 + m^2}$, so by defining the non local operator $\mathcal{P}$ we can express it as $E = m\mathcal{P}$ with

$$\mathcal{P} := \sqrt{1 - \frac{\nabla^2}{m^2}}. \tag{4.9}$$

We can modify the non-relativistic reductions 4.5 and 4.6 to include $\mathcal{P}$ as

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}}\mathcal{P}^{-1/2} \left(\psi(\mathbf{x}, t)e^{-imt} + \psi^*(\mathbf{x}, t)e^{imt}\right) \tag{4.10}$$

$$\pi(\mathbf{x}, t) = -i\sqrt{\frac{m}{2}}\mathcal{P}^{1/2} \left(\psi(\mathbf{x}, t)e^{-imt} - \psi^*(\mathbf{x}, t)e^{imt}\right) \tag{4.11}$$

so that we recover the old relations at leading order when expanding in powers of ∇^2/m^2 (i.e. the non-relativistic limit). We can think of equations 4.10 and 4.11 as the relativistic equivalent of equations 4.5 and 4.6: they are in fact exact relations and not relativistic reductions as their counterpart. The expansion in a power series of $\mathcal{P}$ gives us a way to systematically compute terms in the non relativistic limit up to arbitrary order. Let us get acquainted with these redefinitions by considering the free field case. The Lagrangian 4.8 with $\lambda = 0$ gives us the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{4.12}$$

Applying Hamilton's equations we can compute the equations of motion

$$\begin{aligned} \dot{\phi} &= \frac{\delta H}{\delta \pi} = \pi \\ \dot{\pi} &= -\frac{\delta H}{\delta \phi} = (\nabla^2 - m^2)\phi. \end{aligned} \tag{4.13}$$

With the inverse relation of 4.10 and 4.6,

$$\psi(\mathbf{x}, t) = \sqrt{\frac{m}{2}} e^{imt} \mathcal{P}^{1/2} \left[\phi(\mathbf{x}, t) + \frac{i}{m}\mathcal{P}^{-1}\pi(\mathbf{x}, t) \right], \tag{4.14}$$

we can work out the Lagrangian that gives these equations of motion:

$$\mathcal{L} = \frac{i}{2}(\dot{\psi}\psi^* - \dot{\psi}^*\psi) - m\psi^*(\mathcal{P} - 1)\psi. \tag{4.15}$$

Although introducing a non-local operator at first sight seems to be a complication, we actually ended up with a Lagrangian naturally free from rapidly oscillating terms $e^{\pm inmt}$, without the need to neglect any term. The other aspect that emerges without the need to enforce it is a $U(1)$ explicit symmetry that reflects the conservation of the number of particles $N = \int d^3x \psi^* \psi$, which in the non relativistic theory is exact.
Now that we got an understanding of the usefulness of the non local description, let us put it to the test on the interacting theory of Lagrangian 4.8. Before performing the substitution 4.10 let us introduce an auxiliary field χ in place of the time derivative of $\dot{\phi}$, enforcing the relation through a lagrangian multiplier α

$$\mathcal{L} = \frac{1}{2}\chi^2 - \frac{1}{2}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \alpha(\chi - \dot{\phi}) \tag{4.16}$$

with $\chi - \dot{\phi} = 0$. There is no kinetic term for χ, which is therefore not a dynamical variable. This means that we can remove it from the theory by varying $\mathcal{L}$ with respect to it to obtain its value:

$$\frac{\delta \mathcal{L}}{\delta \chi} = \chi + \alpha = 0 \quad \rightarrow \quad \chi = -\alpha. \tag{4.17}$$

Putting it back in we get a Lagrangian in ϕ and α

$$\mathcal{L} = -\frac{1}{2}\alpha^2 - \frac{1}{2}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \alpha\dot{\phi}. \tag{4.18}$$

Given result 4.17, we can construct the equivalent to relations 4.10 and 4.11 as

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}}\mathcal{P}^{-1/2}\left[\psi(\mathbf{x}, t)e^{-imt} + \psi^*(\mathbf{x}, t)e^{imt}\right] \tag{4.19}$$

$$\alpha(\mathbf{x}, t) = i\sqrt{\frac{m}{2}}\mathcal{P}^{1/2}\left[\psi(\mathbf{x}, t)e^{-imt} - \psi^*(\mathbf{x}, t)e^{imt}\right] \tag{4.20}$$

and finally substitute them in the Lagrangian to have an expression in terms of the non-relativistic field ψ (and its complex conjugate ψ^*). After some algebra, which involves moving around the ∇ operator via an integration by parts[2] and recognising $\left(\frac{i}{2}\dot{\psi}\psi e^{-2imt} + \frac{m}{2}\psi^2 e^{-2imt}\right)$ as the total derivative $\frac{d}{dt}\left[\frac{i}{4}e^{-2imt}\psi^2\right]$, we can write the Lagrangian as:

$$\mathcal{L} = \frac{i}{2}(\dot{\psi}^*\psi - \dot{\psi}\psi^*) - m\psi^*(\mathcal{P} - 1)\psi - \frac{\lambda}{4\cdot 4!m^2}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{1/2}\psi^*\right)^4. \tag{4.21}$$

This result is still exact as we have not made any approximation to Lagrangian 4.8 yet. We can get the equation of motion by applying the usual Euler-Lagrange equation

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} = \frac{\partial\mathcal{L}}{\partial\psi^*}, \tag{4.22}$$

which results in

$$i\dot{\psi} = m(\mathcal{P} - 1)\psi + \frac{\lambda}{4!m^2}e^{imt}\mathcal{P}^{-1/2}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi^*\right)^3. \tag{4.23}$$

We have everything to start moving towards the non-relativistic limit. There the non local operator $\mathcal{P}$ can be expanded in a power series of ∇^2/m^2. The equation of motion becomes

$$i\dot{\psi} = -\frac{1}{2m}\nabla^2\psi + \frac{\lambda}{8m^2}|\psi|^2\psi + \frac{\lambda}{4!m^2}\left(e^{-2imt}\psi^3 + e^{4imt}\psi^{*3} + 3e^{2imt}|\psi|^2\psi^*\right) + \mathcal{O}(\nabla^4/m^4). \tag{4.24}$$

In the first three terms we recognize the usual Schrödinger equation, while the last term is usually neglected with the justification that the fast oscillating terms would average to 0 over times $\Delta t \gg m^{-1}$, just as we assumed before. It is actually worthwhile to be more careful in treating these terms, as they could back-react on the slowly varying part

[2] The integration by parts results in an overall - sign for the terms involving ∇ and an overall + sign when moving $\mathcal{P}$, as $\mathcal{P}$ expands in even powers of ∇.

of ψ and affect the non relativistic effective theory. This is indeed the case, as we can check by dividing ψ into the slowly varying contribution and a small term that oscillates fast:

$$\psi = \psi_s + \delta\psi e^{2imt}. \tag{4.25}$$

Even if in the Lagrangian we only keep the slowly varying terms, there is still a contribution by $\delta\psi$:

$$i\dot{\psi}_s = -\frac{1}{2m}\nabla^2\psi_s + \frac{\lambda}{8m^2}|\psi_s^2|\psi_s + \frac{\lambda}{8m^2}\psi_s(\psi_s\delta\psi + 2\psi_s^*\delta\psi^*) + ... \tag{4.26}$$

The correct way to factor in the contribution of the fast oscillating terms is through a perturbative approach, as Namjoo, Guth and Kaiser do in [8]. To do so, let us take a closer look at the exact equation of motion 4.23. There we can parametrize the variations of the field as

$$\frac{\nabla^2\psi}{m^2} \sim \epsilon_x\psi \qquad , \qquad \frac{\dot{\psi}}{m} \sim \epsilon_t\psi \tag{4.27}$$

with ϵ_x, ϵ_t, $\lambda \ll 1$ and take these coefficients as the small quantities on which to construct our perturbative analysis. The proper way to isolate the slowly varying portion of ψ is with the mode decomposition

$$\psi(\mathbf{x}, t) = \sum_{\nu=-\infty}^{\infty} \psi_\nu(\mathbf{x}, t)e^{i\nu mt}, \tag{4.28}$$

where the ψ_ν are slowly varying. Writing the equation of motion 4.23 as

$$i\dot{\psi}(\mathbf{x}, t) = m(\mathcal{P} - 1)\psi(\mathbf{x}, t) + \frac{\lambda}{4!m^2}\tilde{L}(\mathbf{x}, t) \tag{4.29}$$

we can expand the interaction term

$$\tilde{L}(\mathbf{x}, t) := e^{imt}\mathcal{P}^{-1/2}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi^*\right)^3 \tag{4.30}$$

collectively as

$$\tilde{L}(\mathbf{x}, t) = \sum_{\nu=-\infty}^{\infty} \tilde{L}_\nu(\mathbf{x}, t)e^{i\nu mt}. \tag{4.31}$$

The non relativistic effective field theory that we are looking for is then given by the equation of motion for the mode $\nu = 0$, given that we assume that in the non relativistic limit the full ψ varies only slightly from $\psi_s = \psi_0$. With the mode expansion in equation 4.23 we can obtain the equation of motion for a fixed mode ψ_ν

$$i\dot{\psi}_\nu - \nu m\psi_\nu = m(\mathcal{P} - 1)\psi_\nu + \frac{\lambda}{4!m^2}\tilde{L}_\nu, \tag{4.32}$$

where to obtain the expression

$$\tilde{L}_\nu(\mathbf{x},t) = \mathcal{P}^{-1/2}\sum_{\mu,\mu'}\{\Psi_\mu\Psi_{\mu'}\Psi_{2+\nu-\mu-\mu'} + \Psi^*_\mu\Psi^*_{\mu'}\Psi^*_{4-\nu-\mu-\mu'} + \\ 3\Psi_\mu\Psi_{\mu'}\Psi^*_{-\nu+\mu+\mu'} + 3\Psi^*_\mu\Psi^*_{\mu'}\Psi_{-2+\nu+\mu+\mu'}\} \tag{4.33}$$

we recognized the exponential representation of the delta function and used the notation $\Psi_\nu := \mathcal{P}^{-1/2}\psi_\nu$, which will prove to be very useful in the following calculations. The back-reaction from the higher energy modes that we saw in 4.26 shows here in equation 4.33: to evaluate $\tilde{L}_{\nu=0}$ exactly we need the contribution of all the other modes Ψ_μ, with μ ranging from 0 all the way up to ∞. To see more clearly how to proceed let us multiply both sides of the equation of motion 4.32 and rearrange it to have

$$\Psi_\nu = -\frac{i}{m}\Gamma_\nu\dot{\Psi}_\nu + \lambda L_\nu \tag{4.34}$$

where we defined

$$\Gamma_\nu := (1-\nu-\mathcal{P})^{-1} \qquad , \qquad L_\nu(\mathbf{x},t) = \frac{\Gamma_\nu}{4!m^3}\mathcal{P}^{-1/2}\tilde{L}_\nu(\mathbf{x},t). \tag{4.35}$$

The two terms on the right hand side are suppressed relatively to Ψ_ν and Ł_ν respectively, as per the parametrization 4.27. We can therefore threat them as a perturbative source for Ψ_ν and solve the equation iteratively at increasing order of approximation. Let us write such orders explicitly

$$\Psi_\nu(\mathbf{x},t) = \begin{cases} \sum_{n=0}^\infty \Psi_\nu^{(n)}(\mathbf{x},t) & \nu \neq 0 \\ \Psi_s(\mathbf{x},t) & \nu = 0 \end{cases} \tag{4.36}$$

and

$$L_\nu(\mathbf{x},t) = \sum_{n=0}^\infty L_\nu^{(n)}(\mathbf{x},t). \tag{4.37}$$

As per equation 4.36, we defined the lowest energy mode as being exactly Ψ_s, while we consider the other modes relevant only at higher orders of approximation. So, at the zeroth-order approximation we have

$$\Psi_\nu^{(0)}(\mathbf{x},t) = \begin{cases} \Psi_s(\mathbf{x},t) & \nu = 0 \\ 0 & \nu \neq 0. \end{cases} \tag{4.38}$$

Given these expansions, equation 4.32 for the modes $\nu \neq 0$ is

$$\Psi_\nu^{(1)} = \lambda L_\nu^{(0)} \tag{4.39}$$

at first order and

$$\Psi_\nu^{(n)} = -\frac{i}{m}\Gamma_\nu \dot{\Psi}_\nu^{(n-1)} + \lambda L_\nu^{(n-1)} \qquad\qquad n > 1 \tag{4.40}$$

at higher order. Now we can finally focus on the mode Ψ_s: from equation 4.32 at $\nu = 0$ we get its equation of motion

$$i\dot{\psi}_s = m(\mathcal{P} - 1)\psi_s + m\lambda\Gamma_0^{-1}\mathcal{P}^{1/2}L_0. \tag{4.41}$$

The value of the field is exact, while we have to compute L_0 at some perturbative order, thereby approximating it. At order $n = 1$ we have

$$L_\nu^{(0)} = \frac{\Gamma_\nu \mathcal{P}^{-1}}{4!m^3}\left[\Psi_s^3\delta_{\nu,-2} + \Psi_s^{*3}\delta_{\nu,4} + 3|\Psi_s|^2\Psi_s^*\delta_{\nu,2} + 3|\Psi_s|^2\Psi_s\delta_{\nu,0}\right], \tag{4.42}$$

$$L_\nu^{(1)} = \frac{3\Gamma_\nu \mathcal{P}^{-1}}{4!m^3}\left[\Psi_s^2\Psi_{\nu+2}^{(1)} + \Psi_s^{*2}\Psi_{-\nu+4}^{(1)*} + \Psi_s^2\Psi_{-\nu}^{(1)*} + 2|\Psi_s|^2\Psi_\nu^{(1)} + \Psi_s^{*2}\Psi_{-2+\nu}^{(1)} + 2|\Psi_s|^2\Psi_{-\nu+2}^{(1)*}\right].$$

We can evaluate the $\Psi_{\nu\neq 0}^{(1)}$ trough their equation of motion at first order 4.39. By substituting everything back in the equation of motion we get the first relation which is the result of some approximation. After some simple algebra we get

$$\begin{aligned} i\dot{\psi}_s =& m(\mathcal{P} - 1)\psi_s + \frac{\lambda\mathcal{P}^{-1/2}}{8m^2}|\Psi_s|^2\Psi_s + \frac{3\lambda^2\mathcal{P}^{-1/2}}{(4!)^2m^5}\big\{3\Psi_s^2\Gamma_2\mathcal{P}^{-1}(|\Psi_s|^2\Psi_s^*) + \\ & \Psi_s^2\Gamma_4\mathcal{P}^{-1}(\Psi_s^3) + \Psi_s^{*2}\Gamma_{-2}\mathcal{P}^{-1}(\Psi_s^3) + 6|\Psi_s|^2\Gamma_2\mathcal{P}^{-1}(|\Psi_s|^2\Psi_s)\big\} + \mathcal{O}(\lambda^3, \epsilon_t^3, \lambda^2\epsilon_t, \lambda\epsilon_t^2). \end{aligned} \tag{4.43}$$

At this point we are working in the realm of the non relativistic limit, where $|\nabla^2\psi| \ll m^2$. We can therefore expand the non local operator $\mathcal{P}$ in a power series, and we do so up to order $\epsilon_x^2 \sim \nabla^4/m^4$. We have

$$\begin{aligned} i\dot{\psi}_s =& -\frac{1}{2m}\nabla^2\psi_s + \frac{\lambda}{8m^2}|\psi_s|^2\psi_s - \frac{1}{8m^3}\nabla^4\psi_s + \frac{\lambda}{32m^4}[\psi_s^2\nabla^2\psi_s^* + 2|\psi_s|^2\nabla^2\psi_s + \\ & + \nabla^2(|\psi_s|^2\psi_s)] - \frac{17\lambda^2}{768m^5}|\psi_s|^4\psi_s + \mathcal{O}[\lambda^3, \epsilon_t^3, \epsilon_x^3, \lambda^2\epsilon_t, \lambda^2\epsilon_x, \lambda\epsilon_t^2, \lambda\epsilon_x^2, \lambda\epsilon_t\epsilon_x, \epsilon_t\epsilon_x^2, \epsilon_t^2\epsilon_x] \end{aligned} \tag{4.44}$$

which is finally the equation of motion for ψ in the non relativistic limit. There we can recognize the usual Schrödinger equation for a field with a quartic self-interaction in the first three terms, but we also have relativistic corrections to it arising from two sources: the terms involving the ∇ operator come from the expansion of the non-local operator (and are therefore associated to the relativistic energy of the particles of ϕ), while the λ^2 term is originated by the effect of the higher energy modes $\psi_{\nu\neq 0}^{(1)}$. A Lagrangian that produces such an equation of motion, and that is therefore suitable to be the non relativistic effective Lagrangian we were looking for, is

$$\begin{aligned} \mathcal{L}_{eff} =& \frac{i}{2}(\dot{\psi}_s\psi_s^* - \psi_s\dot{\psi}_s^*) - \frac{1}{2m}\nabla\psi_s\nabla\psi_s^* - \frac{\lambda}{16m^2}|\psi_s|^4 + \frac{1}{8m^3}\nabla^2\psi_s\nabla^2\psi_s^* + \\ & - \frac{\lambda}{32m^4}|\psi_s|^2(\psi_s^*\nabla^2\psi_s + \psi_s\nabla^2\psi_s^*) + \frac{17\lambda^2}{2304m^5}|\psi_s|^6. \end{aligned} \tag{4.45}$$

Let us compare this result with the original Lagrangian 4.8 and with the non relativistic "effective" Lagrangian of our first attempt 4.7. We have a few more terms here, but the most interesting one is the $|\psi|^6$ interaction that wasn't present in the original theory. Going to higher order in considering the iterative approximations for ψ would give us more $|\psi|^{2n}$ terms associated with $n \to n$ scattering, all obeying an explicit $U(1)$ symmetry consistent with the conservation of the number of particles. Processes that violate it in fact, although permitted in the relativistic theory, require energies that are out of the range of applicability of the non relativistic field theory that we constructed.

4.2 Non relativistic effective field theory for the moduli

Now that we got acquainted with the non local operator method of [8], we can put it to work to construct a non relativistic effective theory for moduli in the KKLT scenario. The Lagrangian we are working on is

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3 - \frac{1}{4!}\lambda\phi^4 \tag{4.46}$$

and can easily be generalized to different moduli potentials. The Hamiltonian then has the form

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\phi^3 + \frac{1}{4!}\lambda\phi^4 \tag{4.47}$$

where the canonical momentum is $\pi = \dot{\psi}$ the related equations of motion are

$$\begin{aligned}\dot{\phi} &= \frac{\delta H}{\delta\pi} = \pi \\ \dot{\pi} &= -\frac{\delta H}{\delta\phi} = (\nabla^2 - m^2)\phi + \frac{1}{2}g\phi^2 - \frac{1}{3!}\lambda\phi^3.\end{aligned} \tag{4.48}$$

Let us suppose that it's still appropriate to look at the moduli star as a Bose-Einstein condensate. The discussion about it being properly described by a complex scalar field ψ in the non relativistic limit then carries over from the axion case and so do the relations

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}}\mathcal{P}^{-1/2}\left(\psi(\mathbf{x}, t)e^{-imt} + \psi^*(\mathbf{x}, t)e^{imt}\right) \tag{4.49}$$

$$\pi(t, \mathbf{x}) = -i\sqrt{\frac{m}{2}}\mathcal{P}^{1/2}\left(\psi(t, \mathbf{x})e^{-imt} - \psi^*(t, \mathbf{x})e^{imt}\right) \tag{4.50}$$

along with the observation that the number of propagating degrees of freedom is conserved. We therefore want to write the Lagrangian for $\phi(x)$ 4.46 as a Lagrangian for $\psi(x)$. To do so properly we introduce an auxiliary field χ of value $\chi = \dot{\phi}$ and enforce this relation in the Lagrangian with a lagrangian multiplier α. Having no kinetic term, χ is a non dynamical variable: it can be removed from the theory by varying $\mathcal{L}$ with respect to it and substituting back in the Lagrangian the resulting expression, which turns out to be $\chi = -\alpha$. This relation is the same as in the axion case, so we can use relations that look exactly like the 4.19 and 4.20 to perform the substitution $(\phi, \alpha) \to (\psi, \psi^*)$ in the Lagrangian 4.46. After some algebra, which involves integrating by parts to rearrange terms with ∇ and $\mathcal{P}$ and taking out total time derivatives, we get to

$$\begin{aligned}\mathcal{L} =& \frac{i}{2}(\dot{\psi}^*\psi - \dot{\psi}\psi^*) - m\psi^*(\mathcal{P}-1)\psi + \frac{g}{12\sqrt{2}m^{3/2}}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{1/2}\psi^*\right)^3 \\ &- \frac{\lambda}{4\cdot 4!m^2}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{1/2}\psi^*\right)^4 .\end{aligned} \tag{4.51}$$

With Euler-Lagrange equation we can calculate the equation of motion for ψ:

$$i\dot{\psi} = m(\mathcal{P}-1) - \frac{g}{4\sqrt{2}m^{3/2}}\tilde{G} + \frac{\lambda}{4!m^2}\tilde{L} \tag{4.52}$$

where we introduced the following notation for the interaction terms

$$\tilde{G}(\mathbf{x},t) := e^{imt}\mathcal{P}^{-1/2}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi^*\right)^2 , \tag{4.53}$$

$$\tilde{L}(\mathbf{x},t) := e^{imt}\mathcal{P}^{-1/2}\left(e^{-imt}\mathcal{P}^{-1/2}\psi + e^{imt}\mathcal{P}^{-1/2}\psi^*\right)^3 . \tag{4.54}$$

This equation is our starting point to repeat the perturbative approach outlined in the previous section. As a first step let us expand into modes the field ψ

$$\psi(\mathbf{x},t) = \sum_{\nu=-\infty}^{\infty} \psi_\nu(\mathbf{x},t)e^{i\nu mt} \tag{4.55}$$

and the interaction terms as

$$\tilde{G}(\mathbf{x},t) = \sum_{\nu=-\infty}^{\infty} \tilde{G}_\nu(\mathbf{x},t)e^{i\nu mt} \qquad \tilde{L}(\mathbf{x},t) = \sum_{\nu=-\infty}^{\infty} \tilde{L}_\nu(\mathbf{x},t)e^{i\nu mt}. \tag{4.56}$$

With these expansions into equation 4.52 we can get to the equation of motion for a single mode ψ_ν

$$i\dot{\psi}_\nu - \nu m\psi_\nu = m(\mathcal{P}-1)\psi_\nu - \frac{g}{4\sqrt{2}m^{3/2}}\tilde{G}_\nu + \frac{\lambda}{4!m^2}\tilde{L}_\nu \tag{4.57}$$

with

$$\tilde{G}_\nu(\mathbf{x},t) = \mathcal{P}^{-1/2}\sum_\mu\{\Psi_\mu\Psi_{1+\nu-\mu} + \Psi^*_\mu\Psi^*_{3-\nu-\mu} + 2\Psi_\mu\Psi^*_{1-\nu+\mu}\} \tag{4.58}$$

$$\tilde{L}_\nu(\mathbf{x},t) = \mathcal{P}^{-1/2} \sum_{\mu,\mu'} \{\Psi_\mu \Psi_{\mu'} \Psi_{2+\nu-\mu-\mu'} + \Psi^*_\mu \Psi^*_{\mu'} \Psi^*_{4-\nu-\mu-\mu'} + \\ + 3\Psi_\mu \Psi_{\mu'} \Psi^*_{-\nu+\mu+\mu'} + 3\Psi^*_\mu \Psi^*_{\mu'} \Psi_{-2+\nu+\mu+\mu'}\} \tag{4.59}$$

where we used again the useful notation $\Psi_\nu := \mathcal{P}^{-1/2}\psi_\nu$ and the indices come from summing over delta functions in the exponential representation. It is reasonable to assume that in the non relativistic limit the field will not deviate in a significant way from the lowest energy, slowly varying mode $\psi_0 := \psi_s$, $|\psi - \psi_s| \ll |\psi|$. Our goal of constructing the non relativistic effective field theory can therefore be achieved by calculating the equation of motion for the mode corresponding to $\nu = 0$. This cannot be done by simply ignoring any contribution from other terms, as higher energy modes back-react on the effective behaviour of ψ_s through the interaction terms. An exact evaluation of $\tilde{G}_\nu$ and $\tilde{L}_\nu$ would require the calculation of all the higher energy modes, which is obviously not viable. Let us now rewrite equation 4.57 by multiplying both sides by $\mathcal{P}^{-1/2}$

$$\Psi_\nu = -\frac{i}{m}\Gamma_\nu \dot{\Psi}_\nu - gG_\nu + \lambda L_\nu \tag{4.60}$$

where we defined the operator

$$\Gamma_\nu := (1 - \nu - \mathcal{P})^{-1} \tag{4.61}$$

and the interaction terms as

$$G_\nu(\mathbf{x},t) = \frac{\Gamma_\nu}{4\sqrt{2}m^2\sqrt{m}}\mathcal{P}^{-1/2}\tilde{G}_\nu(\mathbf{x},t) \qquad , \qquad L_\nu(\mathbf{x},t) = \frac{\Gamma_\nu}{4!m^3}\mathcal{P}^{-1/2}\tilde{L}_\nu(\mathbf{x},t). \tag{4.62}$$

We constructed the mode decomposition 4.55 defining the ψ_ν as being slowly varying, so we can parametrize their variation in space and time in terms of small coefficients

$$\frac{\nabla^2\psi_\nu}{m^2} \sim \epsilon_x \psi_\nu \qquad\qquad \frac{\dot{\psi}_\nu}{m} \sim \epsilon_t \psi_\nu \tag{4.63}$$

with $\epsilon_x, \epsilon_t \ll 1$. Considering that the coupling constants g and λ are small too, all the terms on the right hand side of the equation 4.60 are suppressed, compared respectively to Ψ_ν, G_ν and L_ν. We can therefore treat the equation perturbatively, with the left hand side acting as a source for Ψ_ν. At the zeroth-order of approximation the field is completely defined by the slow varying lower energy mode

$$\Psi^{(0)}_\nu(\mathbf{x},t) = \begin{cases} \Psi_s(\mathbf{x},t) & \nu = 0 \\ 0 & \nu \neq 0 \end{cases} \tag{4.64}$$

while at higher order we have the contribution of the other modes

$$\Psi_\nu(\mathbf{x},t) = \begin{cases} \sum_{n=0}^\infty \Psi^{(n)}_\nu(\mathbf{x},t) & \nu \neq 0 \\ \Psi_s(\mathbf{x},t) & \nu = 0. \end{cases} \tag{4.65}$$

We also write the interaction terms is a series of increasing order of approximation

$$G_\nu(\mathbf{x},t) = \sum_{n=0}^{\infty} G_\nu^{(n)}(\mathbf{x},t) \qquad ; \qquad L_\nu(\mathbf{x},t) = \sum_{n=0}^{\infty} L_\nu^{(n)}(\mathbf{x},t). \tag{4.66}$$

Equation 4.60 then gives us the expression for Ψ_ν at the first order of approximation

$$\Psi_\nu^{(1)} = -gG_\nu^0 + \lambda L_\nu^0 \tag{4.67}$$

and at the higher orders

$$\Psi_\nu^{(n)} = -\frac{i}{m}\Gamma_\nu \dot{\Psi}_\nu^{(n-1)} - gG_\nu^{(n-1)} + \lambda L_\nu^{(n-1)}. \tag{4.68}$$

Now that we have a plan to approach equation 4.57 we can write it down for our mode of interest, ψ_s

$$i\dot{\psi}_s = m(\mathcal{P}-1)\psi_s - mg\Gamma_0^{-1}\mathcal{P}^{1/2}G_0 + m\lambda\Gamma_0^{-1}\mathcal{P}^{1/2}L_0. \tag{4.69}$$

For this equation to be exact G_0 and L_0 have to be computed up to an infinite number of terms. We can however be satisfied with the correction of order $n = 1$. Then the cubic interaction term is the sum of

$$\begin{aligned} G_\nu^{(0)} &= \frac{\Gamma_\nu \mathcal{P}^{-1}}{4\sqrt{2}m^2\sqrt{m}}\left[\Psi_s^2\delta_{\nu,-1} + \Psi_s^{*2}\delta_{\nu,3} + 2|\Psi_s|^2\delta_{\nu,1}\right] \\ G_\nu^{(1)} &= \frac{2\Gamma_\nu \mathcal{P}^{-1}}{4\sqrt{2}m^2\sqrt{m}}\left[\Psi_s\Psi_{\nu+1}^{(1)} + \Psi_s^*\Psi_{-\nu+3}^{(1)*} + \Psi_s\Psi_{-\nu+1}^{(1)*} + \Psi_s^*\Psi_{-1+\nu}^{(1)}\right] \end{aligned} \tag{4.70}$$

and the quartic interaction term is the sum of

$$\begin{aligned} L_\nu^{(0)} &= \frac{\Gamma_\nu \mathcal{P}^{-1}}{4!m^3}\left[\Psi_s^3\delta_{\nu,-2} + \Psi_s^{*3}\delta_{\nu,4} + 3|\Psi_s|^2\Psi_s^*\delta_{\nu,2} + 3|\Psi_s|^2\Psi_s\delta_{\nu,0}\right] \\ L_\nu^{(1)} &= \frac{3\Gamma_\nu \mathcal{P}^{-1}}{4!m^3}\left[\Psi_s^2\Psi_{\nu+2}^{(1)} + \Psi_s^{*2}\Psi_{-\nu+4}^{(1)*} + \Psi_s^2\Psi_{-\nu}^{(1)*} + 2|\Psi_s|^2\Psi_\nu^{(1)} + \Psi_s^{*2}\Psi_{-2+\nu}^{(1)} + 2|\Psi_s|^2\Psi_{-\nu+2}^{(1)*}\right]. \end{aligned} \tag{4.71}$$

Substituting relations 4.70, 4.71 and 4.67 into the equation of motion we get to the equation

$$\begin{aligned} i\dot{\psi}_s =& m(\mathcal{P}-1)\psi_s + \frac{\lambda\mathcal{P}^{-1/2}}{8m^2}|\Psi_s|^2\Psi_s + \\ &+ \frac{g^2\mathcal{P}^{-1/2}}{16m^4}\left\{2\Psi_s\Gamma_1\mathcal{P}^{-1}|\Psi_s|^2 + \Psi_s^*\Gamma_3\mathcal{P}^{-1}\Psi_s^2 + 2\Psi_s\Gamma_1\mathcal{P}^{-1}|\Psi_s|^2 + \Psi_s^*\Gamma_{-1}\mathcal{P}^{-1}\Psi_s^2\right\} + \\ &+ \frac{3\lambda^2\mathcal{P}^{-1/2}}{(4!)^2m^5}\left\{3\Psi_s^2\Gamma_2\mathcal{P}^{-1}(|\Psi_s|^2\Psi_s^*) + \Psi_s^2\Gamma_4\mathcal{P}^{-1}(\Psi_s^3) + \Psi_s^{*2}\Gamma_{-2}\mathcal{P}^{-1}(\Psi_s^3) + \right. \\ &\left. 6|\Psi_s|^2\Gamma_2\mathcal{P}^{-1}(|\Psi_s|^2\Psi_s)\right\} + \mathcal{O}(g^3, \lambda^3, \epsilon_t^3, g^2\epsilon_t, \lambda^2\epsilon_t, g\lambda\epsilon_t, g\epsilon_t^2, \lambda\epsilon_t^2). \end{aligned} \tag{4.72}$$

At this point all that's left is to expand the non local operator $\mathcal{P}$ in a power series in the non relativistic limit up to $\epsilon_x^2 \sim \nabla^4/m^4$:

$$
\begin{aligned}
i\dot{\psi}_s =& -\frac{1}{2m}\nabla^2\psi_s + \frac{\lambda}{8m^2}|\psi_s|^2\psi_s - \frac{1}{8m^3}\nabla^4\psi_s + \\
&+ \frac{\lambda}{32m^4}\left[\psi_s^2\nabla^2\psi_s^* + 2|\psi_s|^2\nabla^2\psi_s + \nabla^2(|\psi_s|^2\psi_s)\right] - \frac{5g^2}{24m^2}\psi_s|\psi_s|^2 - \frac{17\lambda^2}{768m^5}|\psi_s|^4\psi_s + \\
&+ \mathcal{O}[g^3, \lambda^3, \epsilon_t^3, \epsilon_x^3, g^2\epsilon_t, g^2\epsilon_x, \lambda^2\epsilon_t, \lambda^2\epsilon_x, g\lambda\epsilon_t, g\lambda\epsilon_x, g\epsilon_t^2, g\epsilon_x^2, \lambda\epsilon_t^2, \lambda\epsilon_x^2, \lambda\epsilon_t\epsilon_x, g\epsilon_t\epsilon_x, \epsilon_t\epsilon_x^2, \epsilon_t^2\epsilon_x],
\end{aligned}
\tag{4.73}
$$

which is finally an equation of motion for ψ_s only in terms of the field itself. A Lagrangian associated to it can be constructed as

$$
\begin{aligned}
\mathcal{L}_{eff} =& \frac{i}{2}(\dot{\psi}_s\psi_s^* - \psi_s\dot{\psi}_s^*) - \frac{1}{2m}\nabla\psi_s\nabla\psi_s^* + \frac{1}{8m^3}\nabla^2\psi_s\nabla^2\psi_s^* + \\
&+ \left(\frac{5g^2}{48m^4} - \frac{\lambda}{16m^2}\right)|\psi_s|^4 - \frac{\lambda}{32m^4}|\psi_s|^2(\psi_s^*\nabla^2\psi_s + \psi_s\nabla^2\psi_s^*) + \frac{17\lambda^2}{2304m^5}|\psi_s|^6.
\end{aligned}
\tag{4.74}
$$

This is the effective Lagrangian for the non relativistic limit that we were looking for. The relativistic corrections changed equation 4.46 quite a bit: we no longer have a cubic term (it got cancelled due to the value of the indices in equation 4.70), the quartic term instead survived but its coupling got corrected with a g^2 term and a brand new term $|\psi|^6$ emerged. Had we gone further in considering orders of approximation more terms $|\psi|^{2n}$ would have appeared, representing more $n \to n$ effective vertices and adding corrections to the ones we already have. The overall Lagrangian presents a U(1) symmetry that's related to the fact that the processes allowed by the energies of the non relativistic limit ($E \ll m$) conserve the total number of particles. Considering the effect of the higher energy modes produced terms proportional to the square of the couplings g and λ so, no matter the original sign with which they appeared in Lagrangian, they will be attractive in nature.
Our last result is not found in the literature and we obtained it exploiting a quite unusual and recent method. Let us check it by comparing it with the effective theory calculated in the traditional way. We follow the approach outlined in [7] by Mukaida, Takimoto and Yamada to have a sound starting point for the comparison. They use the Lagrangian

$$
\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g_3}{3}\phi^3 + \frac{g_4}{4}\phi^4. \tag{4.75}
$$

We want to compute effective vertices by integrating out the relativistic fluctuations. To do so we first separate the latter from the non relativistic part of the field

$$
\phi(\mathbf{x}, t) = \phi_{NR}(\mathbf{x}, t) + \delta\phi(\mathbf{x}, t) \tag{4.76}
$$

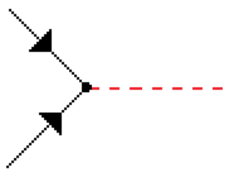
and write the two components in momentum space with a Fourier transform

$$
\phi_{NR}(x) := \int_{k\in NR} dk e^{-ik\cdot x}\tilde{\phi}(k) \qquad \delta\phi(x) := \int_{k\in\overline{NR}} dk e^{-ik\cdot x}\tilde{\phi}(k), \tag{4.77}
$$

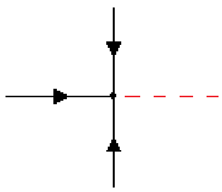
where x and k are four vectors and the dot $\cdot$ in the exponential stands for the scalar product. $\tilde{\phi}$ is of course the Fourier transform of the scalar field and, the latter being real, it satisfies $\tilde{\phi}(k) = \tilde{\phi}^*(-k)$. The momentum is integrated over NR, which is the region of momentum space nearby the on-shell poles for non-relativistic excitations ($NR := \{k = (k_0, \mathbf{k}) : \pm k_0 \sim m + \mathcal{O}(mv^2), \mathbf{k} \sim \mathcal{O}(mv)$ for $v := |\mathbf{v}| \ll 1\}$) or over its complementary set $\overline{NR}$. The non relativistic part can also be expressed in terms of a complex slowly varying field χ as

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{2m}} \left(\chi(\mathbf{x}, t) e^{-imt} + \chi^*(\mathbf{x}, t) e^{imt} \right) . \tag{4.78}$$

Performing these substitutions into the Lagrangian 4.75 we can read out the value for the vertices of the theory. Our aim is to integrate out the relativistic modes, encapsulating their contribution into an effective vertex, so the kind of diagrams that we want to build are of the kind $n \to n$ with high energy modes in the inner part. The suitable vertices to take into consideration are therefore only

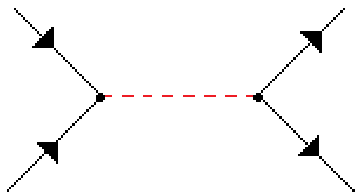


$$= \frac{g_3}{2} |\chi|^2 \delta\phi \tag{4.79}$$

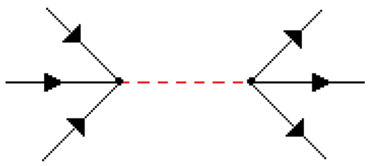


$$= -\frac{g_4}{8} \chi^3 e^{-3imt} \delta\phi \tag{4.80}$$

The full incoming line represents a non relativistic mode χe^{-imt} while the red dashed line is a relativistic mode $\delta\phi$. The operators χ by definition cannot carry relativistic energies nor comparable to the mass scale m, so energy conservation doesn't allow terms $\delta\phi |\chi|^{2n}$ and only diagrams $n \to n$ are suitable. The corresponding diagrams are



$$= \frac{g_3}{16} \chi^2 e^{-2imt} \frac{-1}{m^2 + \Box} \chi^{\dagger 2} e^{2imt} \tag{4.81}$$



$$= \frac{g_4^2}{64} \chi^3 e^{-3imt} \frac{-1}{m^2 + \Box} \chi^{*3} e^{3imt} \tag{4.82}$$

To compute both let us take out m^2 in the propagator[3]. We then have a term $-m^2(1+\frac{\Box}{m^2})$, that in the non relativistic limit $\Box \ll m^2$ we can expand in a power series of $\frac{\Box}{m^2}$. As the field χ is by definition slowly varying, the leading contribution will come from the action of ∂_t on e^{3imt}. The corresponding corrections from diagram 4.81 and 4.82 are $\mathcal{L} \supset \frac{5g_3^2}{48m^2}|\chi|^4 - \frac{g_4^2}{512m^2}|\chi|^6$.
In [7] the authors consider the effective action

$$S_{eff}[\chi] = \int_x \frac{1}{4}\left[\chi^\dagger(2im\partial_t - \partial_t^2 + \nabla^2)\chi - V_{eff}(|\chi|)\right] - i\Gamma[\chi] \tag{4.83}$$

that also has an imaginary part that holds relativistic fluctuations and the breaking of the $U(1)$ symmetry in that regime. Taking into account the 1/4 factor that appears in the action is, at leading order

$$V_{eff}(|\chi|) = -\left(\frac{5g_3^2}{12m^2} + \frac{3g_4}{8}\right)|\chi|^4 + \frac{g_4^2}{128m^2}|\chi|^6. \tag{4.84}$$

This effective potential looks quite different than what is in the Lagrangian 4.74, but the two effective field theories could be equivalent if they were related by a field redefinition. Following the suggestion of Namjoo, Guth and Kaiser in Appendix C of [8] for the even potential, let us compare how we introduced the complex fields redefinition in the two situations:

$$\frac{1}{2}\left[\chi(\mathbf{x},t)e^{-imt} + \chi(\mathbf{x},t)^* e^{imt}\right] = \frac{1}{\sqrt{2m}}\mathcal{P}^{-1/2}\left[(\psi_s + \psi_2^*)e^{-imt} + (\psi_s^* + \psi_2)e^{imt}\right]. \tag{4.85}$$

χ, ψ_s and ψ_2 were all constructed as being slowly oscillating by definition. So the last equation entails

$$\sqrt{\frac{m}{2}}\chi = \mathcal{P}^{-1/2}(\psi_s + \psi_2^*). \tag{4.86}$$

There we can work on the right hand side by using 4.67 to compute $\psi_2^* = \psi_2^{*(1)} + ...$ and expand $\mathcal{P}$. Then

$$\sqrt{\frac{m}{2}}\chi = \left(1 + \frac{1}{4m^2}\nabla^2\right)\psi_s - \frac{\lambda}{16m^3}|\psi_s|^2\psi_s + \mathcal{O}(\lambda^2, \epsilon_t^2, \epsilon_x^2, \lambda\epsilon_t, \lambda\epsilon_x, \epsilon_t\epsilon_x). \tag{4.87}$$

Let us move to the equation of motion in order to compare the two effective theories at low energies. Considering only the real part of the Lagrangian in the action 4.83, we can derive the equation of motion by varying it:

$$i\dot{\chi} = \frac{1}{2m}\ddot{\chi} - \frac{1}{2m}\nabla^2\chi + \frac{1}{2m}\frac{\partial V_{eff}}{\partial \chi^*}. \tag{4.88}$$

[3] In the propagator the $\Box$ actually stands for $\Box - i\epsilon$ to include the correct boundary condition.

Using relation 4.87 to substitute χ with an expression in ψ we get an equation of motion for ψ in the diagram matching construction[4]

$$\begin{aligned} i\dot{\psi}_s =& -\frac{1}{2m}\nabla^2\psi_s + \frac{\lambda}{8m^2}|\psi_s|^2\psi_s - \frac{5g^2}{24m^4}|\psi_s|^2\psi_s + \\ &+ \frac{1}{2m}\ddot{\psi}_s - \frac{i}{4m^2}\nabla^2\dot{\psi}_s + \frac{i\lambda}{16m^3}\psi_s(2\dot{\psi}_s\psi_s^* + \psi_s\dot{\psi}_s^*) + \\ &- \frac{1}{8m^3}\nabla^4\psi_s + \frac{\lambda}{32m^4}\left[\psi_s^2\nabla^2\psi_s^* + 2|\psi_s|^2\nabla^2\psi_s + \nabla^2(|\psi_s|^2\psi_s)\right] - \frac{17\lambda^2}{768m^5}|\psi_s|^4\psi_s + \\ &+ \mathcal{O}(\epsilon^3). \end{aligned} \tag{4.89}$$

We already accounted for the different definition of coupling constants in the two Lagrangians[5], and yet this equation looks quite different than the one at 4.73. Let us take a look at the second time derivative of ψ_s, which we can calculate by looking at 4.89 as an expression for $\dot{\psi}$. At order ϵ^3

$$i\ddot{\psi}_s = -\frac{1}{2m}\nabla\dot{\psi}_s + \frac{\lambda}{8m^2}\psi_s(2\dot{\psi}_s\psi_s^* + \psi_s\dot{\psi}_s^*) + \mathcal{O}(\epsilon^3) \tag{4.90}$$

which exactly cancels out the second line of equation 4.89. We then have

$$\begin{aligned} i\dot{\psi}_s =& -\frac{1}{2m}\nabla^2\psi_s + \frac{\lambda}{8m^2}|\psi_s|^2\psi_s - \frac{1}{8m^3}\nabla^4\psi_s + \\ &+ \frac{\lambda}{32m^4}\left[\psi_s^2\nabla^2\psi_s^* + 2|\psi_s|^2\nabla^2\psi_s + \nabla^2(|\psi_s|^2\psi_s)\right] - \frac{5g^2}{24m^2}\psi_s|\psi_s|^2 - \frac{17\lambda^2}{768m^5}|\psi_s|^4\psi_s + \\ &+ \mathcal{O}[g^3, \lambda^3, \epsilon_t^3, \epsilon_x^3, g^2\epsilon_t, g^2\epsilon_x, \lambda^2\epsilon_t, \lambda^2\epsilon_x, g\lambda\epsilon_t, g\lambda\epsilon_x, g\epsilon_t^2, g\epsilon_x^2, \lambda\epsilon_t^2, \lambda\epsilon_x^2, \lambda\epsilon_t\epsilon_x, g\epsilon_t\epsilon_x, \epsilon_t\epsilon_x^2, \epsilon_t^2\epsilon_x]. \end{aligned} \tag{4.91}$$

This is finally in agreement with our result 4.73, proving that at this order of approximation and for low energies our effective Lagrangian describes the same theory as the one established in the literature for the non relativistic limit.

[4] We used the convenient notation $\mathcal{O}(\epsilon^3)$ to mean $\mathcal{O}(g^3, \lambda^3, \epsilon_t^3, \epsilon_x^3, g^2\epsilon_t, g^2\epsilon_x, \lambda^2\epsilon_t, \lambda^2\epsilon_x, g\lambda\epsilon_t, g\lambda\epsilon_x, g\epsilon_t^2, g\epsilon_x^2, \lambda\epsilon_t^2, \lambda\epsilon_x^2, \lambda\epsilon_t\epsilon_x, g\epsilon_t\epsilon_x, \epsilon_t\epsilon_x^2, \epsilon_t^2\epsilon_x)$.

[5] Specifically $g_3 = -g/2$ and $g_4 = -\lambda/6$.

Chapter 5

Hamiltonian study

In the previous chapter we found an effective Lagrangian in the non relativistic limit for the moduli in the KKLT scenario. Now we want to understand whether it allows for the formation of oscillons: to do so we first compute explicitly the Hamiltonian assuming spherical symmetry and a reasonable ansatz for the ground state, as the equation of motion 4.73 doesn't have any known exact analytical solutions. Stable and unstable configurations will then correspond to minima and maxima of the resulting function with respect to the radius. Their expression, however, turns out to be quite complicated, so to gain an intuition on the behaviour of the Hamiltonian we study it in different regimes, first focusing on just one coupling coefficient at a time and then in different regions of the values of the radius.

5.1 The Hamiltonian

We have the effective field theory in the non relativistic limit, as described by the Lagrangian

$$\begin{aligned}\mathcal{L}_{eff} =& \frac{i}{2}(\dot{\psi}_s\psi_s^* - \psi_s\dot{\psi}_s^*) - \frac{1}{2m}\nabla\psi_s\nabla\psi_s^* + \frac{1}{8m^3}\nabla^2\psi_s\nabla^2\psi_s^* + \\ &+ \left(\frac{5g^2}{48m^4} - \frac{\lambda}{16m^2}\right)|\psi_s|^4 - \frac{\lambda}{32m^4}|\psi_s|^2(\psi_s^*\nabla^2\psi_s + \psi_s\nabla^2\psi_s^*) + \frac{17\lambda^2}{2304m^5}|\psi_s|^6.\end{aligned} \tag{5.1}$$

that we build and whose validity we checked in the previous chapter. The number of particles is a conserved quantity and equal to $N = \int d^3\psi^*\psi$, so in the non relativistic limit we can see $n(\mathbf{x}) = \psi^*(\mathbf{x})\psi(\mathbf{x})$ as the local density of particles and $\rho(\mathbf{x}) = m\psi^*(\mathbf{x})\psi(\mathbf{x})$.

Performing a Legendre transform gives us the Hamiltonian density

$$\begin{aligned}\mathcal{H}_{eff} =& \frac{1}{2m}\nabla\psi_s\nabla\psi_s^* - \frac{1}{8m^3}\nabla^2\psi_s\nabla^2\psi_s^* + \left(-\frac{5g^2}{48m^4} + \frac{\lambda}{16m^2}\right)|\psi_s|^4 + \\ &+ \frac{\lambda}{32m^4}|\psi_s|^2(\psi_s^*\nabla^2\psi_s + \psi_s\nabla^2\psi_s^*) - \frac{17\lambda^2}{2304m^5}|\psi_s|^6.\end{aligned} \tag{5.2}$$

The corresponding equation of motion that we wrote explicitly in the previous chapter in 4.73 has no known exact analytical solution. We can however construct a reasonable ansatz that qualitatively mimics the true field and study the Hamiltonian for it. The first assumption that we can make is for the solution to have a spherical symmetry, as there are no mechanisms for the Lagrangian that can spontaneously break its rotational symmetry. We also take out the time dependence of the field to write

$$\psi_s(r,t) = \Psi(r)e^{-i\mu t} \tag{5.3}$$

where μ is the chemical potential and we can take $\Psi(r)$ to be real. Now to explicitly calculate the Hamiltonian we need an ansatz for the spacial part $\Psi(r)$. A simple but effective choice is a damping exponential parametrized by R (which can then be interpreted as the radius of the moduli star). To ensure $N = \int d^3x|\psi|^2 = 4\pi\int drr^2n(r)$ we impose an appropriate normalization factor, so the expression reads

$$\Psi_R(r) = \sqrt{\frac{N}{\pi R^3}}e^{-r/R}. \tag{5.4}$$

This ansatz enables us to compute each term of Hamiltonian by writing 5.2 in spherical coordinates and the using the ansatz to perform the integration in r. All the terms are of the form $r^n exp(-ar/R)$ and can be solved with a number of integration by parts. The result is

$$H_{eff} = \frac{N}{2mR^2} - \frac{5N}{8m^3R^4} + \frac{(-5g^2 + 3m^3\lambda)\,N^2}{384m^4\pi R^3} - \frac{3\lambda N^2}{128m^4\pi R^5} - \frac{17\lambda^2N^3}{62208m^5\pi^2R^6}. \tag{5.5}$$

Let us now perform a rescaling of the quantities involved in the Hamiltonian to turn to dimensionless variables and simplify the analysis. As is common in theoretical physics we assumed $\hbar$ to be equal to one and dimensionless; it follows that the action must be dimensionless too, so we can work out the dimensionality of the coupling constants from any of the Lagrangians. The condition $\hbar = 1$, alongside with $c = G = 1$, allows us to interpret everything as having mass-like dimension m^n. In particular [λ]=0 and [g]=m, so we can define the dimensionless "tilde" quantities

$$\begin{cases}\lambda =: \tilde{\lambda} \\ g =: m\tilde{g} \\ R =: \tilde{R}/m \\ N =: \tilde{N} \\ H_{eff} =: m\tilde{H}.\end{cases} \tag{5.6}$$

Substituting these expressions into the Lagrangian we can get rid of all the masses and have

$$\tilde{H} = \frac{\tilde{N}}{2\tilde{R}^2} + \frac{\left(-5\tilde{g}^2 + 3\tilde{\lambda}\right)\tilde{N}^2}{384\pi\tilde{R}^3} - \frac{5\tilde{N}}{8\tilde{R}^4} - \frac{3\tilde{\lambda}\tilde{N}^2}{128\pi\tilde{R}^5} - \frac{17\tilde{\lambda}^2\tilde{N}^3}{62208\pi^2\tilde{R}^6}. \tag{5.7}$$

We are looking for stable enough configurations compatible with the characteristics of a moduli star. The compactness is ensured by the form of our ansatz, with the field being different from zero only inside the radius R. Stationary solutions correspond to the minima and maxima of the Hamiltonian, the former being stable and the latter unstable. To find them we have to solve the following equation in $\tilde{R}$

$$\tilde{R}^4 + (-5\tilde{g}^2 + 3\tilde{\lambda})\frac{\tilde{N}}{128\pi}\tilde{R}^3 - \frac{5}{2}\tilde{R}^2 - \frac{15\tilde{\lambda}N}{128\pi}\tilde{R} - \frac{17\tilde{\lambda}^2\tilde{N}^2}{10368\pi^2} = 0. \tag{5.8}$$

The four solutions to this quartic equation are quite complicated and long, which makes it hard to interpret them at first sight. To gain some intuition we can start by selectively turning off terms in the Hamiltonian 5.7. Let us first disregard the original quartic interaction by taking $\tilde{\lambda} = 0$: the corresponding equation for stationary configurations is then quadratic and much easier to solve and interpret. It presents two maxima, one in a positive value $\tilde{R}_1$ and the other in the negative value $\tilde{R}_2$. To understand to which of the full solutions they match we can expand those in the limit $\tilde{\lambda} \to 0$. The comparison leaves only two options as possible physical solution. The same procedure done for $\tilde{g}$ further restricts the pool, enabling us to understand that out of the four solutions for equation 5.8 only one is physical and it represents a maximum. This interpretation is indeed confirmed by the graph in figure 5.1, where we plotted $\tilde{H}(\tilde{R})$ for fixed values of $\tilde{g}$, $\tilde{\lambda}$ and $\tilde{N}$.

5.2 Understanding the Hamiltonian

Let us analyse our Hamiltonian more rigorously and ground our interpretation by comparing our result to what's present in the literature. In the third chapter, "Moduli stars", we saw how Schiappacasse and Hertzberg studied the stable and unstable configurations allowed by the Hamiltonian for the PQ axion in the presence of gravity

$$H_{eff}(R) = \frac{N}{2mR^2} - \frac{5Gm^2N^2}{16R} - \frac{N^2}{128\pi f_a^2 R^3} \tag{5.9}$$

in [3]. As previously seen, this Hamiltonian presents a minimum and a maximum for negative values of the energy that represent a stable and an unstable bound state respectively. Let us now turn off the gravitational interaction to compare their results with

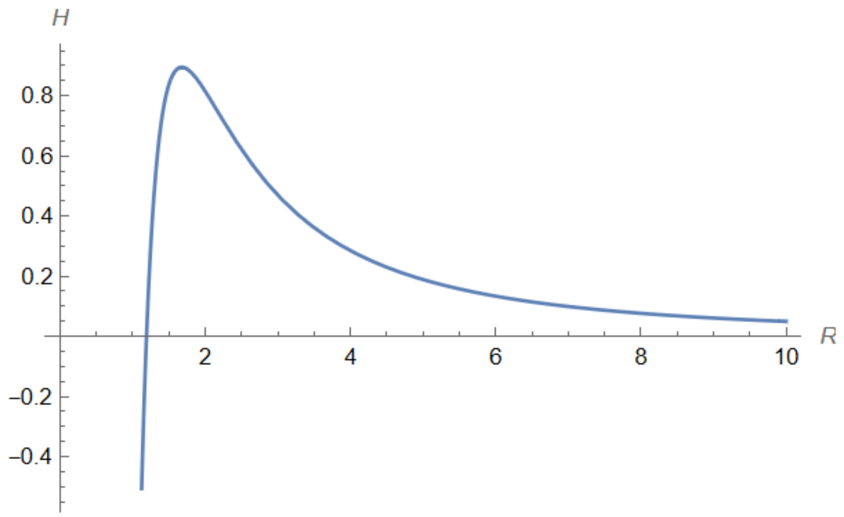


Figure 5.1: Plot of the dimensionless effective Hamiltonian for the KKLT potential vs the dimensionless radius of the configuration, in the absence of gravity and in the non-relativistic limit, for $\tilde{g}, \tilde{\lambda} = 1$ and $\tilde{N} = 10$. For a very small dimensionless radius R the value of the Hamiltonian drops to minus infinity; for very large dimensionless radius R the value of the Hamiltonian approaches zero, as is required by a localized solution. In between those two regimes we have a maximum: its value as a function of the dimensionless number of particles is plotted in Fig. 5.3. The function representing the Hamiltonian also has values for negative R, showing a similar although mirrored behaviour to the positive axis counterpart, but the corresponding maximum is not a physical solution of equation 5.8.

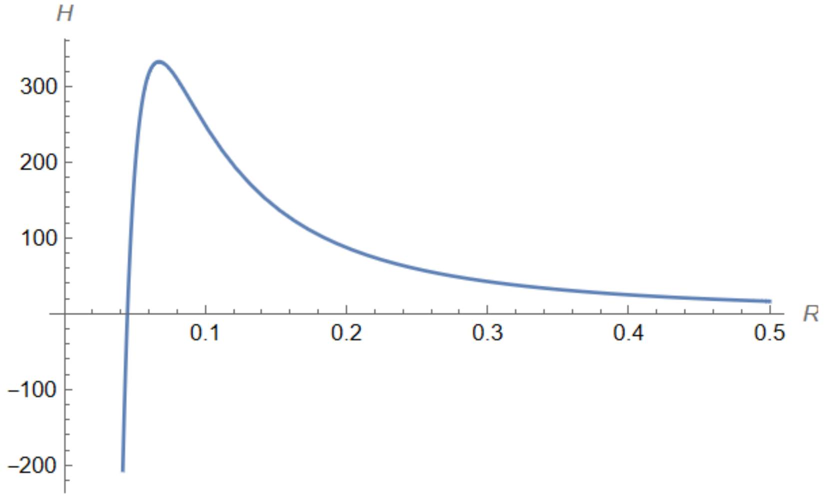


Figure 5.2: Plot of the dimensionless Hamiltonian for the PQ axion in the absence of gravity vs the dimensionless radius for the value of $\tilde{N} = 9$. Comparing it to the plot of the full Hamiltonian in Fig. 3.1, where we used the same value of $\tilde{N}$, we can see how discarding the gravitational term modifies the asymptotic behaviour for large $\tilde{R}$, that used to go to zero as $\propto -1/\tilde{R}$ and now does so as $\propto 1/\tilde{R}^2$. This change results in having just one extremal point, a maximum associated to positive values of the energy, which is no longer compatible with a bound state.

our situation. Then the Hamiltonian 5.9, after the rescaling 5.6, reads

$$\tilde{H}(\tilde{R}) = \frac{\tilde{N}}{\tilde{R}^2} - \frac{\tilde{N}^2}{128\pi\tilde{R}^3}. \tag{5.10}$$

We can find the extremals of this function by taking its first derivative with respect to $\tilde{R}$ and requiring it to be equal to zero. We then find just one maximum in

$$\langle R \rangle = \frac{3\tilde{N}}{256\pi} \tag{5.11}$$

and evaluating the energy in that point we get the positive value $\tilde{H} = 65536\pi^2/27\tilde{N}$. Neglecting the gravitational term modified significantly the behaviour of the Hamiltonian, resulting not only in the disappearance of the minimum but also in the uplifting of the maximum, which is no longer compatible with a bound state. This happened because gravity exerts a stronger impact at large values of $\tilde{R}$, so not considering the term $-\frac{5Gm^2N^2}{16R}$ changed the asymptotic behaviour of $\tilde{H}$ from $-1/\tilde{R}$ to $1/\tilde{R}^2$, as one can see by comparing Fig. 5.2 with the plot of the full Hamiltonian 5.9 in chapter 2 (Fig. 3.1). With this understanding of how gravity affects the existence of bound configurations, let us now return to our effective Hamiltonian 5.7 and rewrite it to highlight the dependence from the radius as

$$\tilde{H} = \frac{C_2}{\tilde{R}^2} + \frac{C_3}{\tilde{R}^3} - \frac{C_4}{\tilde{R}^4} - \frac{C_5}{\tilde{R}^5} - \frac{C_6}{\tilde{R}^6}, \tag{5.12}$$

where all coefficients but C_3 and C_5 have a defined positive sign that comes from the structure of the Lagrangian. The signs of two mentioned coefficient depend on the couplings $\tilde{g}$ and $\tilde{\lambda}$: C_3 is positive when $\tilde{\lambda} > \frac{5}{3}\tilde{g}^2$, while in the KKLT potential we consider a repulsive quartic self-interaction that makes C_5 positive. Now we can study the behaviour of the Hamiltonian in different regimes by considering only three terms at a time. Then we will have equations for the extremal points which are quadratic in $\tilde{R}$ and therefore easily solvable and interpretable. For large radius the terms that shape the Hamiltonian the most are those with bigger exponents for $\tilde{R}$, therefore we can approximate

$$\tilde{H} = \frac{C_2}{\tilde{R}^2} + \frac{C_3}{\tilde{R}^3} - \frac{C_4}{\tilde{R}^4}. \tag{5.13}$$

Asymptotically, $\tilde{H}$ will behave like $1/\tilde{R}^2$, reaching zero from above, while for small radius it will head down to $-\infty$ is a way that is at least as steep as $-1/\tilde{R}^4$. Taking the first derivative of 5.13 with respect to $\tilde{R}$ and requiring it to vanish gives us the equation for the extremal points

$$2\tilde{R}^2 C_2 + 3\tilde{R}C_3 - 4C_4 = 0 \tag{5.14}$$

which is solved by

$$\langle\tilde{R}\rangle = \frac{3|C_3|}{4C_2}\Big(- sgn(C_3) \pm \sqrt{1 + \frac{32C_2C_3}{9C_3^2}}\Big), \tag{5.15}$$

where the sign of the coefficient C_3 depends on how the original couplings of the theory $\tilde{g}$ and $\tilde{\lambda}$ compare to each other. They, in fact, define C_3 as $\frac{\left(-5\tilde{g}^2+3\tilde{\lambda}\right)\tilde{N}^2}{384\pi\tilde{R}^3}$ and appear in the Lagrangian with opposite sign; the value of the coefficient then signals whether attractive or repulsive self-interactions dominate in the quartic term $|\psi|^4$ and is of particular importance when studying the possibility of stable configurations. For both the case of C_3 being positive and the one of it being negative, however, only the + solution of expression 5.15 is positive and therefore represents a physical option. Continuity, alongside our previous considerations on the behaviour of $\tilde{H}$ in the limit $\tilde{R} \to \infty$ and in the region of smaller radius, requires this solution to be a maximum in the first quadrant, as it has to connect the branch of the Hamiltonian coming up from negative values from the left to the one that, on the right, goes to zero from above.
In the region of small radius, instead, the Hamiltonian behaves like $-1/\tilde{R}^6$ for $\tilde{R} \to 0$ before becoming shallower ($\propto -1/\tilde{R}^4$) as $\tilde{R}$ grows larger. In our Hamiltonian 5.12 all the terms that contribute the most in this zone have the same sign, but if instead $\tilde{\lambda}$ were negative we would have some competition among them which could result in more

extremal points and, in principle, even a minimum for negative values of $\tilde{H}$, which could be associated to the bound states we are looking for. To investigate this interesting case, and to see how the sign of the original quartic self-interaction term affects the formation of oscillons, let us consider the Hamiltonian approximated by

$$\tilde{H} = -\frac{C_4}{\tilde{R}^4} + \frac{C_5}{\tilde{R}^5} - \frac{C_6}{\tilde{R}^6}, \tag{5.16}$$

where the coefficients C_4 and C_6 are still positive definite, while we leave the sign of C_5 unspecified. We once again have a quadratic equation for the extremal points, of solution

$$\langle \tilde{R}_\pm \rangle = \frac{5|C_5|}{8C_4}\Big(sgn(C_5) \pm \sqrt{1 - \frac{96C_4C_6}{25C_5^2}} \Big). \tag{5.17}$$

Now, for $sgn(C_5) = -1$, as is the case for the KKLT potential, both solutions are negative and the only extremal point we are left with is the maximum found at larger radius. For C_5 positive, however, both solutions are physical and represent two more possible configurations. Evaluating the second derivative of $\tilde{H}$ in $\langle \tilde{R}_- \rangle$ and in $\langle \tilde{R}_+ \rangle$ identifies the previous as a maximum and the latter as a minimum. We are particularly interested in the latter, as it could be a stable oscillon. To understand whether it represents a bound state let us check the sign of the Hamiltonian 5.16 in $\langle \tilde{R}_+ \rangle$, only for the case of interest $C_5 > 0$. We have

$$sgn(\tilde{H}) = -sgn\Big(1 - \sqrt{1 - \frac{96C_4C_6}{25C_5^2}} - \frac{16C_4C_6}{5C_5^2} \Big), \tag{5.18}$$

where we can see that the result depends on the value of the ratio C_4C_6/C_5^2. For $C_4C_6/C_5^2 \ll 1$ we can approximate the square root as $1 - \frac{96C_4C_6}{50C_5^2}$, which gives an overall plus sign for the Hamiltonian. For $\frac{96C_4C_6}{25C_5^2} \sim 1$, instead, the square root vanishes and we are left with a positive quantity inside the brackets that corresponds to a negative value of the Hamiltonian in $\langle \tilde{R}_+ \rangle$, which can therefore be a stable bound state. Let us recall that the coefficients C_i are defined by the Hamiltonian 5.7 with a change of sign for $\tilde{\lambda}$, so we can make a more precise statement about the possibility of $\langle \tilde{R}_+ \rangle$ corresponding to configurations of negative energy. Substituting the expression for C_4, C_5 and C_6 back in the evaluation of the Hamiltonian gives us in fact a positive value, which means that not even a completely attractive self-interaction is enough for a stable bound state to form. To conclude we can consider the region for intermediate values of $\tilde{R}$ when $C_5 > 0$, where

$$\tilde{H} = \frac{C_3}{\tilde{R}^3} - \frac{C_4}{\tilde{R}^4} + \frac{C_5}{\tilde{R}^5}. \tag{5.19}$$

As we still have freedom over the choice of sign of C_3, we can have two separate cases. For both the Hamiltonian behaves like $1/\tilde{R}^5$ from the right and then flattens to the

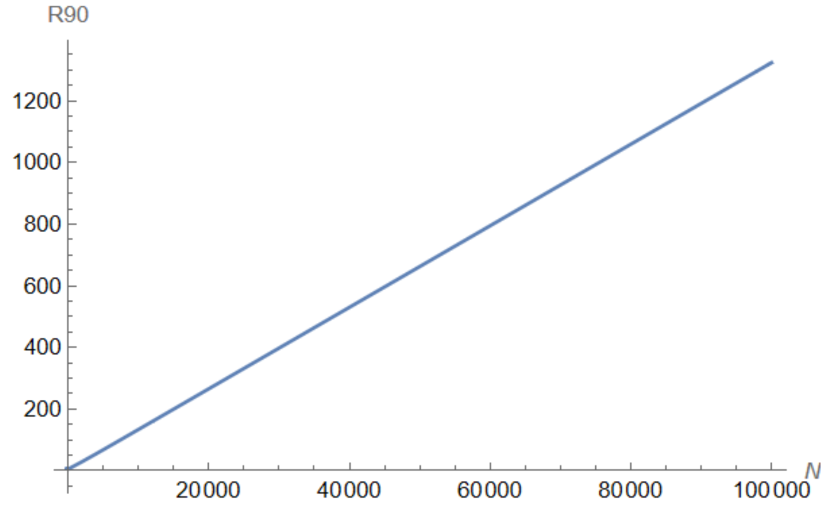


Figure 5.3: Plot of the rescaled dimensionless radius $R90$ as a function of the dimensionless number of particles $\tilde{N}$. $R90$ is the (dimensionless) radius encompassing 90% of the total mass of the star, as introduced in definition 5.21, and in our case it is equal to $\sim 2.661\tilde{R}$. The dimensions of the star grow linearly with $\tilde{N}$ and do so indefinitely.

ordinate axis as the inverse of the cube of $\tilde{R}$, but it does so coming from either above or below, depending on the sign of the coefficient. For positive C_3 we therefore have room for a minimum followed by a maximum, whose expressions are

$$\langle \tilde{R}_{\pm} \rangle = \frac{2C_4}{3C_3}\Big(1 \pm \sqrt{1 - \frac{15C_3C_5}{4C_4^2}}\Big). \tag{5.20}$$

In the case of negative C_3, instead, the trend of the Hamiltonian allows for only a minimum, which corresponds to the $\langle \tilde{R}_{-} \rangle$ of the previous equation.

For our KKLT case we therefore found just one extremal point, a maximum, which is furthermore associated to a positive value of the energy and cannot represent a bound configuration, not even an unstable one. Still, let us consider it for a moment as if it where a star configuration in order to compare it with the result found in the literature [3]. In Fig. 5.3 we plotted the value of the dimensionless radius, rescaled to encompass ninety percent of the total mass of the star, as a function of the dimensionless number of particles. To find the value of the rescaled value for $\tilde{R}$, which we call $R90$, one has to solve numerically

$$0.9N = 4\pi \int_0^{R90} dr' r'^2 \Psi(r')^2. \tag{5.21}$$

For our exponential ansatz 5.3 the solution is $R90 \sim 2.661R$. The radius of the "star" then grows linearly and indefinitely with the number of particles, as does the solution 5.11 that we found for [3] when turning off gravity.

Chapter 6

Conclusions

We first introduced moduli as scalar fields arising in string theory in the massless spectrum and in the process of compactification of the extra dimensions. We saw how they hold a central role in the theory, parametrizing many fundamental aspects such as the coupling of the strings or the shape and size of the extra-dimensions. In order to assign them a potential and in the process fix their vacuum expectation value, however, one needs to consider the effects of fluxes during compactification and corrections, both perturbative and non perturbative in nature, to the superpotential. We did so in the context of the KKLT scenario, where we focused on the stabilization of the Kähler modulus. A peculiar aspect of this particular procedure is the uplifting of the resulting minimum to a proper deSitter value, which is done by considering the effects of a few anti D3-branes at the end of warped throats lounging from the Calabi-Yau manifold. We finished chapter two by expanding the KKLT potential around its minimum: the resulting expression highlights in a clear way the asymmetric nature of the potential, which translates to an overall self-interaction that alternates between being attractive and being repulsive during oscillations of a bound state. Having understood what moduli are, we then turned to examining the possibility of compact objects made out of them: moduli stars. This eventuality actually positions itself in a broader landscape of boson stars, which we outlined in chapter 3. Moduli stars could form as oscillatons, i.e. under the influence of gravity, or as oscillons, kept together by self-interactions. In this last case the fluctuations of the field start from an initial displacement with respect to the minimum and can grow via three different mechanisms: tachyonic pre-heating, tachyonic oscillations and parametric resonance. If moduli stars do form, they present an interesting array of possible experimental signals, particularly as gravitational waves within the range of LIGO and as black holes with an unusual small mass that would be interesting scenarios in which to study Hawking radiation.

The literature on scalar stars mainly focuses on axion stars, given their importance as a plausible candidate for dark matter, so we chose to start from the methods developed in that context in order to construct our analytical treatise for moduli stars (on which only numerical studies exist). The first step consisted in building an effective field theory for the moduli in the KKLT scenario, chosen because its asymmetric nature mentioned above provides ground for possible interesting effects on the stability of the star. We started by reviewing the approaches used on axion stars in [3] and [8]. Having recognized the effectiveness of the non local definition used in the latter, we applied it to our case. Treating the equation of motion perturbatively and then expanding the non local operator in the non relativistic limit, we were able to find a non relativistic effective Lagrangian whose relativistic corrections can be computed in a systematic way. This result is not found in the literature, so to check it we also computed the effective vertices with the traditional diagram matching method. The two final Lagrangians can be related by a field redefinition, which proves that they describe the same theory at low energies and at the considered level of approximation. We then turned to the effective Hamiltonian and looked for minima and maxima that could represent stable and unstable oscillons. To evaluate the Hamiltonian we assumed the ground state to have a spherical symmetry and considered a reasonable exponential ansatz to perform an explicit computation, as the equation of motion does not have an exact known analytical solution. We computed the extremal points as the values of the radius for which the first derivative of the Hamiltonian vanishes; the results, however, turned out to be quite lengthy and complicated solutions of an equation of the fourth order. Although we can form an intuition numerically on the nature of these solutions, to perform a proper study we need to consider the Hamiltonian in different regimes: we did so for small, intermediate and larger values of the radius. The KKLT scenario presents only one maxima which is not compatible with a bound configuration, as it is associated with positive values of the energy. Several numerical studies in the literature, on the other hand, predict the existence of metastable configurations for the moduli of the KKLT potential [2], even in the absence of gravity [12]. The reason for this discrepancy needs to be searched in the two significant approximations that we made: considering the non relativistic limit and neglecting the effects of gravity. A fully analytic relativistic treatment that could in principle recover the results found for oscillons in [12] is currently out of reach. The inclusion of gravity, instead, could lead to the presence of a minimum associated to a bound stable state that could be studied using the method outlined in this thesis. The non-relativistic analysis requires the consideration of the weak Newtonian limit and the introduction of a metric expressed in terms of the Poisson gravitational potential $\Phi_N(\phi)$. The system is then described by the coupled Schrödinger-Poisson equation, that admits a solitonic solution. We have initiated such an analysis using the non-local field redefinition employed in this thesis, but found its perturbative treatment technically challenging and so we leave this extension for future work.

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