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Homogenization of the Navier–Stokes equations in perforated domains in the inviscid limit

Richard M Höfer

Faculty of Mathematics, University of Regensburg, Regensburg, Germany

E-mail: richard.hoefer@ur.de

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Abstract

We study the solution u_ε to the Navier–Stokes equations in $\mathbb{R}^3$ perforated by small particles centered at $(\varepsilon\mathbb{Z})^3$ with no-slip boundary conditions at the particles. We study the behavior of u_ε for small ε, depending on the diameter ε^α, $\alpha > 1$, of the particles and the viscosity ε^γ, $\gamma > 0$, of the fluid. We prove quantitative convergence results for u_ε in all regimes when the local Reynolds number at the particles is negligible. Then, the particles approximately exert a linear friction force on the fluid. The obtained effective macroscopic equations depend on the order of magnitude of the collective friction. We obtain (a) the Euler–Brinkman equations in the critical regime, (b) the Euler equations in the subcritical regime and (c) Darcy's law in the supercritical regime.

Keywords: homogenization, perforated domain, Navier–Stokes equations, inviscid limit, Euler equations, Darcy's law, Euler–Brinkman equations

Mathematics Subject Classification numbers: 35Q30, 35Q31, 76D07, 76M50, 76S05,76T25

(Some figures may appear in colour only in the online journal)

1. Introduction

The homogenization of fluid flows in perforated domains has been intensively studied in the last decades. Various models for the fluids reaching from incompressible inviscid flows (see e.g. [HLW22, LLN18, LM16a, MP99]) to compressible viscous flows (see e.g. [BO23, HKS21, Mas02, Osc22]) and even non-Newtonian fluids (see e.g. [Mik18]) have been considered with different boundary conditions, including Navier slip conditions (see e.g. [All91]) and so-called sedimentation boundary conditions (see e.g. [DG21, GH21, NS20]).

From the application oriented point of view, interest in such homogenization problems arises from the study of flow through porous media and of suspension flows. In the case of such particulate flows, homogenization problems where the particle evolution is frozen or prescribed can be considered as a first step towards the derivation of fully coupled models between the fluid flow and the dispersed phase.

The limiting behavior of solutions to the incompressible (Navier-)Stokes equations with fixed viscosity in perforated domains with no-slip boundary conditions is by now quite well understood. On the microscopic lengthscale of the particles, the fluid inertia becomes negligible. Therefore, in the limit of many small particles, a linear friction relation (Stokes law) prevails, giving rise to an effective massive term, the so-called Brinkman term. Depending on the particle sizes and number density, the Brinkman term becomes negligible, dominant or of order one in the homogenization limit, leading to the (Navier-)Stokes equations, Darcy's law and the (Navier-)Stokes–Brinkman, respectively, see e.g. [All90a, All90b, CH20, DGR08, FNN16, GH19, Giu21, HJ20, HMS19, LY23, Mik91, Tar80].

For the case of the Navier–Stokes equations with vanishing viscosity, only very few results are available though. The problem of considering such fluids in perforated domains with very small viscosity (or more precisely large macroscopic Reynolds numbers) is a very relevant one in applications. Indeed, in the modeling of sprays, it is not unusual to couple kinetic equations for the dispersed phase to the Euler equations (see e.g. [BD06, CDM11]). On the other hand, regarding porous media, understanding flow at large Reynolds number is very important (see e.g. [BMW10]) and nonlinear extensions of Darcy's law, in particular the Darcy–Forchheimer equations, are proposed at very large Reynolds numbers. Although the rigorous derivation of such *nonlinear* effective models seems currently out of reach, the present work aims at identifying the effective behavior in all scaling limits where a linear friction law prevails. We emphasize that the effective models we obtain are completely different from the ones that result by starting from the Euler equations in perforated domains (see e.g. [HLW22, LLN18, LM16a, MP99] for such models). Instead, correspondingly to the (Navier-)Stokes equations with constant viscosity, we identify and prove homogenization limits in a critical, subcritical and supercritical regime yielding the Euler–Brinkman equations, the Euler equations and Darcy's law, respectively. To the author's knowledge, the Euler–Brinkman equations have not even been formally derived in the literature before. This can be viewed as a first step towards the rigorous justification of spray models like the one analyzed in [CDM11] that couples the incompressible Euler equations to a Vlasov equation through a linear friction force.

1.1. Setting and outline of the main results

Let $\mathcal{T} \Subset B_{1/4}(0)$, the reference particle, be a fixed closed set with smooth boundary, such that $B_1(0) \setminus \mathcal{T}$ is connected and $0 \in \mathring{\mathcal{T}}$. For $0 < \varepsilon < 1$, we consider particles centered at $x_i^\varepsilon := \varepsilon i$, $i \in \mathbb{Z}^3$. Moreover, precisely, for $\alpha \geqslant 1$, we define

$$\Omega_\varepsilon := \mathbb{R}^3 \setminus \cup_{i \in \mathbb{Z}^3} \mathcal{T}_i^\varepsilon, \qquad \mathcal{T}_i^\varepsilon := x_i^\varepsilon + \varepsilon^\alpha \mathcal{T}.$$

Then, for some $T > 0$, $\gamma > 0$ and $\mu_0 > 0$, we consider solutions u_ε to the Navier–Stokes equations

$$\begin{aligned} \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon - \mu_0 \varepsilon^\gamma \Delta u_\varepsilon + \nabla p_\varepsilon &= f_\varepsilon && \text{in } (0,T) \times \Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon &= 0 && \text{in } (0,T) \times \Omega_\varepsilon, \\ u_\varepsilon &= 0 && \text{on } (0,T) \times \partial\Omega_\varepsilon, \\ u_\varepsilon(0,\cdot) &= u_0^\varepsilon && \text{in } \Omega_\varepsilon \end{aligned} \tag{1.1}$$

for some given $f_\varepsilon \in L^2(0,T;L^2(\mathbb{R}^3))$ and $u_0^\varepsilon \in L^2_\sigma(\Omega_\varepsilon)$, where

$$L^2_\sigma(\Omega_\varepsilon) := \left\{ v \in L^2(\Omega_\varepsilon) : \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \partial\Omega_\varepsilon \right\}.$$

It is well known that then at least one Leray solution u_ε exist, i.e. a weak solution which satisfies the energy inequality

$$\begin{aligned} &\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \mu_0\varepsilon^\gamma \|\nabla u_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\quad \leqslant \frac{1}{2}\|u_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon \cdot u_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \quad \forall 0 \leqslant t \leqslant T. \end{aligned} \tag{1.2}$$

We focus on the case $\alpha > 1$ which characterizes the regime where the particle diameters ε^α are small compared to the inter-particle distance ε. In a nutshell, the effect of the particles on the fluid can then be described through a superposition of linear friction laws provided that the fluid inertia is negligible on the lengthscale of the particles. More precisely, we consider the particle Reynolds number

$$\mathrm{Re}^\varepsilon_{\mathrm{part}} := \frac{\text{particle diameter} \times \text{fluid velocity}}{\text{viscosity}} = U_\varepsilon \varepsilon^{\alpha-\gamma} \tag{1.3}$$

where U_ε, the order of magnitude of the fluid velocity, has yet to be determined. Then, if $\mathrm{Re}^\varepsilon_{\mathrm{part}} \ll 1$, the influence of each particle on the fluid can be approximated by a friction force determined from the unique solutions $(w_k, q_k) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ to the linear Stokes problem

$$\begin{aligned} -\Delta w_k + \nabla q_k &= 0 && \text{in } \mathbb{R}^3 \setminus \mathcal{T}, \\ \operatorname{div} w_k &= 0 && \text{in } \mathbb{R}^3 \setminus \mathcal{T}, \\ w_k &= e_k && \text{on } \partial\mathcal{T} \end{aligned} \tag{1.4}$$

through the associated resistance matrix $\mathcal{R} \in \mathbb{R}^{3\times3}$

$$\mathcal{R}_{jk} = \int_{\mathbb{R}^3\setminus\mathcal{T}} \nabla w_k : \nabla w_j, \tag{1.5}$$

which is a positive definite symmetric matrix. Neglecting fluid inertia and particle interaction, classical scaling considerations imply that each particle approximately contributes a friction force $F_i = -\mu_0\varepsilon^{\alpha+\gamma}\mathcal{R}(u_\varepsilon)_i$ where $(u_\varepsilon)_i$ should be understood as a suitable average of u_ε on some lenthscale $\varepsilon^\alpha \ll d_\varepsilon \leqslant \varepsilon$ around x_i^ε. Taking into account that the particle number density is ε^{-3} leads to approximating the fluid velocity u_ε by $\tilde{u}_\varepsilon$ which satisfies the Navier–Stokes

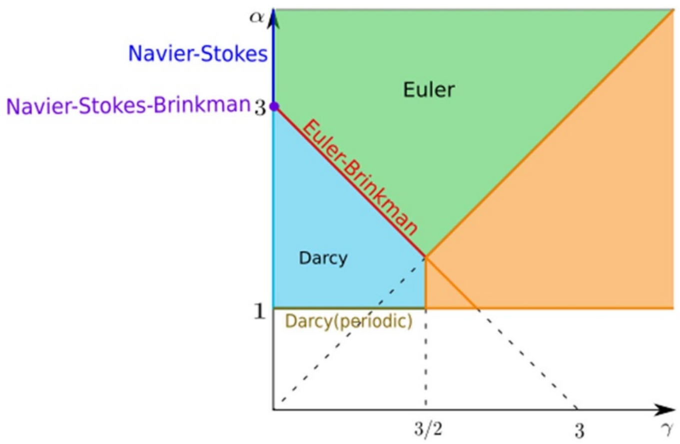

Figure 1. Scaling regimes of effective equations.

equations in the *whole space* with an additional linear friction term $\mu_0\varepsilon^{\alpha+\gamma-3}\mathcal{R}\tilde{u}_\varepsilon$, sometimes referred to as *Brinkman force*. More precisely, provided $\mathrm{Re}^\varepsilon_{\mathrm{part}} \ll 1$, we expect $u_\varepsilon \approx \tilde{u}_\varepsilon$ where

$$\begin{aligned} \partial_t\tilde{u}_\varepsilon + \tilde{u}_\varepsilon\cdot\nabla\tilde{u}_\varepsilon - mu_0\varepsilon^\gamma\Delta\tilde{u}_\varepsilon + \mu_0\varepsilon^{\alpha+\gamma-3}\mathcal{R}\tilde{u}_\varepsilon + \nabla\tilde{p}_\varepsilon &= f_\varepsilon \qquad \text{in } (0,T)\times\mathbb{R}^3, \\ \operatorname{div} u_\varepsilon &= 0 \qquad \text{in } (0,T)\times\mathbb{R}^3. \end{aligned} \tag{1.6}$$

From this approximation, we may easily identify the limiting behavior, where we distinguish the *critical* regime as $\gamma+\alpha = 3$, the *subcritical* regime as $\gamma+\alpha > 3$ and the supercritical regime as $\gamma+\alpha < 3$. Before writing down the limiting equations, we revisit the constraint $\mathrm{Re}^\varepsilon_{\mathrm{part}} \ll 1$. In the critical and subcritical regime, the Brinkman force is at most of order one, and therefore the solution $\tilde{u}_\varepsilon$, and thus u_ε and U_ε from (1.3), are expected to be of order 1, provided u_0^ε and f_ε are of order 1. Thus, in the critical and subcritical regime,

$$\mathrm{Re}^\varepsilon_{\mathrm{part}} = \frac{\varepsilon^{\alpha-\gamma}}{\mu_0},$$

which leads to the condition $\alpha > \gamma$.

On the other hand, in the supercritical regime, the Brinkman force dominates thus slows down the fluid velocity to $U_\varepsilon = \varepsilon^{3-\alpha-\gamma}$. Therefore, in the supercritical case,

$$\mathrm{Re}^\varepsilon_{\mathrm{part}} = \frac{\varepsilon^{3-2\gamma}}{\mu_0},$$

leading to the condition $\gamma < 3/2$.

Taking the formal limit in (1.6), assuming $f_\varepsilon \to f$ and $u_0^\varepsilon \to u_0$ leads to the following limit systems. The regimes are illustrated in figure 1.

- In the critical regime $\alpha+\gamma=3$ with $\alpha>1$ and $\alpha>\gamma>0$, we obtain (for $\mu_0=1$ for simplicity) the Euler–Brinkman equations[2]

$$\begin{aligned} \partial_t u + u\cdot\nabla u + \mathcal{R}u + \nabla p &= f && \text{in } (0,T)\times\mathbb{R}^3, \\ \operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^3, \\ u(0,\cdot) &= u_0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{1.7}$$

- In the subcritical regime for $\alpha+\gamma>3$ with $\alpha>1$ and $\alpha>\gamma>0$, we obtain the Euler equations

$$\begin{aligned} \partial_t u + u\cdot\nabla u + \nabla p &= f && \text{in } (0,T)\times\mathbb{R}^3, \\ \operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^3, \\ u(0,\cdot) &= u_0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{1.8}$$

  Since the particles do not create any effective perturbation on the limit system, the asymptotically linear friction law guaranteed by $\alpha>\gamma>0$ is actually not required to obtain this limit case but it instead suffices that $\mathrm{Re}^\varepsilon_{\mathrm{part}}\leqslant c_0$ for some $c_0>0$ independent of ε. This corresponds to the regime $\alpha=\gamma>3/2$ with $\mu_0\geqslant M$ for some M sufficiently large.
- In the supercritical regime, for $\alpha+\gamma<3$ with $\alpha>1$ and $\gamma<3/2$, $u_\varepsilon\to 0$. Thus, we rescale time and velocities to obtain a nontrivial limit. More precisely, if $\hat{u}_\varepsilon$ is a solution to (1.1) with $\mu_0=1$, we consider the function $u_\varepsilon(t,x)=\varepsilon^{\alpha+\gamma-3}\hat{u}_\varepsilon(\varepsilon^{\alpha+\gamma-3}t,x)$. This rescaled velocity solves (after rescaling accordingly f_ε, p_ε and u^0_ε without renaming them)

$$\begin{aligned} \varepsilon^{6-2\alpha-2\gamma}(\partial_t u_\varepsilon + u_\varepsilon\cdot\nabla u_\varepsilon) - \varepsilon^{3-\alpha}\Delta u_\varepsilon + \nabla p_\varepsilon &= f_\varepsilon && \text{in } (0,T)\times\Omega_\varepsilon, \\ \operatorname{div} u_\varepsilon &= 0 && \text{in } \Omega_\varepsilon, \\ u_\varepsilon &= 0 && \text{on } \partial(0,T)\times\Omega_\varepsilon, \\ u_\varepsilon(0,\cdot) &= u^\varepsilon_0 && \text{in } \Omega_\varepsilon. \end{aligned} \tag{1.9}$$

  Performing the same rescaling on the system (1.6), we formally obtain Darcy's law in the limit $\varepsilon\to 0$, namely

$$\begin{aligned} \mathcal{R}u + \nabla p &= f && \text{in } (0,T)\times\mathbb{R}^3, \\ \operatorname{div} u &= 0 && \text{in } (0,T)\times\mathbb{R}^3. \end{aligned} \tag{1.10}$$

1.2. Statement of the main results

The precise results are the following quantitative convergence results for u_ε in all three regimes under regularity assumption on the solution u to the respective limit system. Smooth solutions exist at least for short times. Moreover, in the supercritical regime, we obtain in addition a weak convergence result in $L^2(0,T;L^2(\mathbb{R}^3))$ assuming only a weak solution $u\in L^2(0,T;L^2(\mathbb{R}^3))$ to Darcy's law (1.10).

Theorem 1.1 (Critical regime). *Let $\alpha\in(3/2,3)$, $\gamma=3-\alpha$ and $\mu_0=1$. Let $T>0$, $u_0\in H^4(\mathbb{R}^3)$, $f\in C(0,T;H^2(\mathbb{R}^3))$ and $(u,p)\in C^1(0,T;H^4(\mathbb{R}^3))\times L^\infty(0,T;H^3_{\mathrm{loc}}(\mathbb{R}^3))$ be a solution to (1.7). Moreover, for $0<\varepsilon<1$ let $u^\varepsilon_0\in L^2_\sigma(\Omega_\varepsilon)$, $f_\varepsilon\in L^2(0,T;L^2(\Omega_\varepsilon))$ and let $u_\varepsilon\in L^2(H^1_0(\Omega_\varepsilon))\cap C(0,T;L^2(\Omega_\varepsilon))$ be a Leray solution to (1.1). Then, there exists $C>0$ which*

[2] One might argue that Euler–Darcy would be a more appropriate name for this system but this is already used for a different system that arises as homogenization limit of the 2-dimensional Euler equations in perforated domains, see e.g. [MP99].

depends only on the reference particle $\mathcal{T}$ and, monotonously, on T, $\|f\|_{L^\infty(0,T;H,^2(\mathbb{R}^3))}$, $\|u\|_{C^1(0,T;H^4(\mathbb{R}^3))}$ and $\|\nabla p\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$ such that for all $t \leqslant T$

$$\| (u_\varepsilon - u)(t) \|^2_{L^2(\Omega_\varepsilon)} \leqslant C \left(\|u^0_\varepsilon - u_0\|^2_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon - f\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left(\varepsilon^{2\alpha-3} + \varepsilon^{6-2\alpha}\right)\right).$$

Theorem 1.2 (Subcritical regime). *Let $\mu_0 > 0$, $\alpha > 3/2, \gamma > 0$ satisfy $3 - \alpha < \gamma \leqslant \alpha$. Let $T > 0$, $u_0 \in H^4(\mathbb{R}^3)$, $f \in C(0,T;H^2(\mathbb{R}^3))$ and $(u,p) \in C^1(0,T;H^4(\mathbb{R}^3)) \times L^\infty(0,T;H^3_{\mathrm{loc}}(\mathbb{R}^3))$ be a solution to (1.8). Moreover, for $0 < \varepsilon < 1$ let $u^\varepsilon_0 \in L^2_\sigma(\Omega_\varepsilon)$, $f_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon))$ and let $u_\varepsilon \in L^2(H^1_0(\Omega_\varepsilon)) \cap C(0,T;L^2(\Omega_\varepsilon))$ be a Leray solution to (1.1). Then, there exists $M > 0$ depends only on the reference particle $\mathcal{T}$ and, monotonously, on T, $\|f\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$, $\|u\|_{C^1(0,T;H^4(\mathbb{R}^3))}$ and $\|\nabla p\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$, and $C > 0$ which depends additionally on μ_0 such that, if either $\alpha > \gamma$ or $\mu_0 \geqslant M$, we have for all $t \leqslant T$*

$$\| (u_\varepsilon - u)(t) \|^2_{L^2(\Omega_\varepsilon)} \leqslant C \left(\|u^0_\varepsilon - u_0\|^2_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon - f\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left(\varepsilon^{2\alpha+2\gamma-6} + \varepsilon^{2\alpha-3} + \varepsilon^{2\gamma}\right)\right).$$

In the supercritical regime, we remind that we consider the rescaled system (1.9). The corresponding energy inequality reads

$$\begin{aligned}\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} &+ \varepsilon^{2\gamma+\alpha-3}\|\nabla u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \\ &\leqslant \frac{1}{2}\|u^\varepsilon_0\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{2\gamma+2\alpha-6}\int_0^t\int_{\Omega_\varepsilon} f_\varepsilon \cdot u_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{1.11}$$

for all $0 \leqslant t \leqslant T$.

Theorem 1.3 (Supercritical regime—quantitative result). *Let $\alpha \in (1,3)$ and $0 < \gamma < \min\{3/2, 3 - \alpha\}$. Let $T > 0$ and $f \in C^1(0,T;H^4(\mathbb{R}^3))$ and let $(u,p) \in C^1(0,T;H^4(\mathbb{R}^3)) \times C^1(0,T;H^5_{\mathrm{loc}}(\mathbb{R}^3))$ be the unique solution to (1.10) (up to constants for the pressure). For $\varepsilon > 0$ let $u^\varepsilon_0 \in L^2_\sigma(\Omega_\varepsilon)$ and let u_ε be a Leray solutions to (1.9). Then, there exists $C > 0$ which depends only on the reference particle $\mathcal{T}$ and, monotonously, on T, $\|f\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$, $\|u\|_{C^1(0,T;H^4(\mathbb{R}^3))}$, $\|\nabla p\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$ and $\|u^0_\varepsilon\|_{L^2(\Omega_\varepsilon)}$ such that for all $t \leqslant T$*

$$\begin{aligned}\|u_\varepsilon - u\|^2_{L^2((0,T)\times\Omega_\varepsilon)} \leqslant C \Big(&\varepsilon^{6-2\alpha-2\gamma}\|u^0_\varepsilon - u_0\|^2_{L^2(\Omega_\varepsilon)} + \|f_\varepsilon - f\|^2_{L^2((0,T)\times\Omega_\varepsilon)} \\ &+ \varepsilon^{\frac{6-4\gamma}{3}} + \varepsilon^{\alpha-1} + \varepsilon^{9-3\alpha} + \varepsilon^{12-4\alpha-4\gamma}\Big).\end{aligned}$$

Remark 1.4.

- The three theorems above imply in particular that for any sequence $\varepsilon \to 0$ with $\|u^0_\varepsilon - u_0\|_{L^2(\mathbb{R}^3)} \to 0$ (respectively $\varepsilon^{6-2\alpha-2\gamma}\|u^0_\varepsilon - u_0\|^2_{L^2(\Omega_\varepsilon)} \to 0$), and $f_\varepsilon \to f$ in $L^2(0,T;L^2(\mathbb{R}^3))$ we have $u_\varepsilon \to u$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ (respectively in $L^2(0,T;L^2(\mathbb{R}^3))$). Here, f_ε, u^0_ε and u_ε are to be understood as defined in $\mathbb{R}^3$ through extension by 0. Note that one may choose $f_\varepsilon = f$ in Ω_ε. Moreover, one may choose $u^0_\varepsilon = w^\varepsilon u_0$ with w^ε as in section 2. Then, estimate (2.2) guarantees $\|u^0_\varepsilon - u_0\|_{L^2(\mathbb{R}^3)} \to 0$ for any choice of the parameter $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$ that w^ε depends on. Optimizing η_ε yields $\|u^0_\varepsilon - u_0\|^2_{L^2(\mathbb{R}^3)} \leqslant C\varepsilon^{3\alpha-3}$.
- The regularity assumptions on u could probably be weakened but we do not pursue to optimize here.

- In the supercritical regime, we do not obtain pointwise estimates in time. Indeed, there are boundary layers in time which prevent pointwise estimates under the stated assumptions. These boundary layers are due to the initial datum u_ε but also due to possible jumps in time of the force f_ε.

Theorem 1.5 (Supercritical regime—qualitative result). *Assume $\alpha \in (1,3)$, $0<\gamma<\min\{3/2, 3-\alpha\}$. For $T>0$ and $\varepsilon>0$, assume $u_0^\varepsilon \in L^2(\Omega_\varepsilon)$ such that $\varepsilon^{3-\alpha-\gamma}\|u_0^\varepsilon\|_{L^2(\Omega_\varepsilon)}$ is uniformly bounded and $f_\varepsilon \in L^2(0,T;L^2(\mathbb{R}^3))$ converges weakly to some f in $L^2(0,T;L^2(\mathbb{R}^3))$. Let $u_\varepsilon \in L^2(H_0^1(\Omega_\varepsilon)) \cap C(0,T;L^2(\Omega_\varepsilon))$ be a Leray solution to (1.9). Then, $\tilde{u}_\varepsilon \rightharpoonup u$ in $L^2(0,T;L^2(\mathbb{R}^3))$, where u is the unique weak solution in $L^2(0,T,L^2(\mathbb{R}^3))$ to (1.10) and where $\tilde{u}_\varepsilon$ is the extensions of u_ε to $\mathbb{R}^3$ by $\tilde{u}_\varepsilon = 0$ in $\mathbb{R}^3 \setminus \Omega_\varepsilon$.*

1.3. *Previous results*

The vanishing viscosity limit is a classical problem in the study of incompressible fluids, we refer to [MM18] for a review on the topic. In bounded domains with no-slip boundary conditions, the limiting behavior is not well-understood due to the onset of boundary layers. This is the reason why we consider the whole space in this paper.

In dimensions two and three, the vanishing viscosity limit has been studied in [ILN09] in the presence of a single shrinking body. The convergence to the Euler equations has been established provided that the local Reynolds number is sufficiently small i.e. the same condition $a_\varepsilon \leqslant c\mu_\varepsilon \ll 1$, where a_ε and μ_ε denote the particle diameter and fluid viscosity, respectively, and c is a sufficiently small constant (depending on the initial data, time, and the reference particle).

There is a vast literature on homogenization in perforated domains. Modeling the fluid velocity u_ε by the stationary Stokes equations, Darcy's law has been obtained in [Tar80] in the case of particle of the same size as the inter-particle distance, i.e. $\alpha = 1$. Later, Allaire [All90a, All90b] proved homogenization results for the Stokes equations for all ranges of $\alpha > 1$, identifying Darcy's law for $\alpha \in (1,3)$, the Stokes–Brinkman equations for $\alpha = 3$ and the Stokes equations for $\alpha > 3$. Allaire's results cover all space dimensions $d \geqslant 2$ with appropriate adaptations of the ranges of α for $d \geqslant 4$. In the two-dimensional case, the critical regime corresponds to particle diameters a_ε such that $\varepsilon^{-2}\log a_\varepsilon \sim 1$. By compactness, Allaire's results also apply to the stationary Navier–Stokes equations (in dimensions $d \leqslant 4$).

The results of Allaire have been refined in a number of works, for example considering more general distributions of particles, non-homogeneous Dirichlet boundary conditions, the study of higher order approximations and fluctuations. We refer to the recent results [CH20, DGR08, GH19, Giu21, HJ20, HMS19] and the references therein.

The homogenization limits for the full instationary Navier–Stokes for *fixed viscosity* correspond to the one of the stationary Stokes equations and are displayed in figure 1. Formally they are obtained by setting $\gamma = 0$ in (1.6) and taking the limit $\varepsilon \to 0$. The critical regime, $\alpha = 3$, leading to the Navier–Stokes–Brinkman equations, has been considered by Feireisl, Nečasová and Namlyeyeva [FNN16], whereas the subcritical case $\alpha > 3$ and the supercritical case $\alpha \in (1,3)$ has been treated recently by Lu and Yang [LY23].

The case $\alpha = 1$, including the full range of vanishing viscosities $\gamma \in [0,3/2)$ has been treated by Mikelić [Mik91].

We emphasize that the Darcy's law in [LY23, All90b] is exactly the same as (1.10) whereas the Darcy's law in [Tar80, Mik91] differs quantitatively, in terms of a different resistance tensor $\mathcal{R}_{\text{per}}$ which is obtained analogously as $\mathcal{R}$ from (1.5) but by solving the Stokes equations in the torus instead of the whole space. The reason for this difference is that in the case $\alpha = 1$

the particle diameter is comparable to the interparticle distance. Therefore, the superposition of friction forces through single particle problems in the whole space (cf (1.4)) must be replaced by studying the collective forces through the problem with periodic boundary conditions. Mathematically, the analysis of the case $\alpha = 1$ is somewhat easier as it only involves two lengthscales, the microscopic lengthscale ε and the macroscopic lengthscale. Since the study of the case $\alpha = 1$ requires different corrector problems and is rather well understood, we restrict our attention to $\alpha > 1$ in the present paper.

Reflecting its importance for applications, there are several works concerning the derivation of non-linear Darcy's laws, especially the Darcy–Forchheimer equations. They seem to focus on the case $\alpha = 1$, where nonlinear effects are expected to become important for $\gamma \geqslant 3/2$. Most of these works do not contain rigorous proofs, we refer to [BMW10] for an overview of the literature. Concerning rigorous results, Mikelić [Mik95] and Marušić-Paloka and Mikelić [MM00] tackled the critical case $\alpha = 1$, $\gamma = 3/2$ in dimensions two and three starting from the stationary Navier–Stokes equations. The obtained limit system is a nonlinear nonlocal Darcy type equation. Moreover, in the subcritical case, $\alpha = 1$, $\gamma < 3/2$, Bourgeat, Marušić-Paloka and Mikelić [BMM95] justified nonlinear versions of Darcy's law as higher order corrections to the linear law.

We also mention that the homogenization of the instationary Stokes equations with vanishing viscosity has been studied by Allaire [All92] for $\alpha = 1$. In this case, the critical scaling (in any space dimension) is $\gamma = 2$ and a Darcy's law with memory effect is obtained as limit system.

The only previous result the author is aware of concerning the homogenization of the Navier–Stokes equations with vanishing viscosities when the particle diameters are much smaller than the interparticle distance ($\alpha > 1$) is due to Lacave and Mazzucato [LM16b]. In dimension two, they recover the unperturbed Euler equations under assumptions on the particle sizes, distances and the viscosity, which guarantee that the particle Reynolds number is sufficiently small and that the particles do not exert a significant collective force on the fluid (subcritical regime).

1.4. Elements of the proof

The proof of the (quantitative) main results is based on an energy argument to estimate $u_\varepsilon - u$ which is, at its core, classical in the study of vanishing viscosity limits. However, similarly as in [ILN09, LM16b], we face the problem, that the limit fluid velocity u does not vanish inside of the particles and thus u is not an admissible testfunction for the PDE of u_ε. As in [ILN09, LM16b], we therefore consider functions $\hat{u}_\varepsilon$ obtained from u by a suitable truncation. In [ILN09], the truncation is performed on the level of the stream function (respectively the vector potential in three dimensions). In [LM16b], the fluid velocity itself is truncated, i.e.

$$\hat{u}_\varepsilon = \phi_\varepsilon u + h_\varepsilon,$$

where h_ε is a suitable Bogovskii type correction such that $\hat{u}_\varepsilon$ is divergence free.

As in [LM16b], we perform the truncation on the level of the fluid velocity itself. However, we need to be more careful, since the truncation needs to contain information of the boundary layers at the particles that produce the Brinkman term in the limit. Thus, instead of the scalar function ϕ_ε in [LM16b] that truncate in a ε^α neighborhood around the particles, we choose a variant of the matrix-valued oscillating testfunction w^ε used by Allaire [All90a, All90b] that are build on the solutions to the resistance problem (1.4).

These functions w^ε from [All90a, All90b] (which go back to corresponding functions in [Tar80] and similar functions for the Poisson equations used by Cioranescu and Murat in [CM82]) have been used with some modifications in many related works, see e.g. [GH19, LY23]. However, w^ε truncates on an ε-neighborhood around the particles, and therefore we could only use them directly in the present context provided the Reynolds number on the ε-lengthscale is small. This is the case if $\gamma < 1$ in the (sub-)critical regime and $\gamma < 2 - \alpha/2$ in the supercritical regime. To overcome this restriction, we modify the testfunctions of Allaire, to truncate on a lengthscale η_ε, $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$. Aside from estimates analogous to their standard versions, we then use a Hardy-type estimate in order to control some error-terms arising from the nonlinear convection term.

1.5. Some possible generalizations and open problems

In this paper, we focus on periodic distributions of identical particles for the sake of the clarity of the presentation. The methods of proof do not rely on periodicity, though, and presumably apply to more general settings.

From the viewpoint of applications to suspensions, it would also be interesting to study non-homogeneous Dirichlet boundary conditions, i.e. $u_\varepsilon = V_i$ on ∂T_i^ε which have been treated for the corresponding model without vanishing viscosity in [DGR08, FNN16].

As in many related works, we focus here on the three-dimensional case. Extensions to two dimensions are possible with the necessary modifications similar as in [All90a, All90b]. As mentioned above, parts of the subcritical regime is treated in [LM16b]. There is one important difference between the two- and three-dimensional case, however, that seems to make it more difficult to analyze all the cases in dimensions two where the particle Reynolds number tends to zero. Namely, in three dimensions, the Stokes resistance of a particle of size a_ε in the whole space is well approximated by solving Stokes problems in an η_ε-neighborhood of the particle, for any lengthscale η_ε with $\eta_\varepsilon \gg a_\varepsilon$. This allows us to consider the intermediate scale η_ε as outlined in the previous subsection. In two dimensions, however, just like for capacities, only *relative* Stokes resistances are meaningful. As observed in [All90a, All90b], it turns out that the relative resistance in a cell of order of the inter-particle distance ε is the correct object to consider in order to study the collective effect of the particles[3]. Therefore, the use of an intermediate lengthscale η_ε does not seem suitable in 2 dimensions, at least not in the critical and supercritical regimes. As discussed above, this would restrict to assuming that the Reynolds number on the scale ε is of order one, in order that the (accordingly modified) proof given in this paper still works.

It would be of great interest to understand the regimes where the particle Reynolds number $\mathrm{Re}^\varepsilon_{\mathrm{part}}$ is not tending to zero, i.e. $\gamma \geqslant \max\{\alpha, 3/2\}$, displayed in orange in figure 1. However, as discussed above, the case when the particle Reynolds number is large is not even understood in the case of a single shrinking particle. In the case where the particle Reynolds number is small but fixed, we proved that one still obtains the Euler equations in the subcritical regime. One could still expect convergence to the Euler equations in the subcritical regime. In the critical and supercritical regimes, one could expect the onset of nonlinear behavior similar to the one obtained in [Mik95, MM00] at $\gamma = 3/2$.

[3] To be more precise, since the relative Stokes resistance scales like $|\log(\eta_\varepsilon/a_\varepsilon)|^{-1}$ in two dimensions, it does not matter whether one chooses $\eta_\varepsilon = \varepsilon$ or $\eta_\varepsilon = \varepsilon^\beta$. However, one should allow a_ε to be much smaller than powers of ε in order to include the critical case $-\varepsilon^2 \log a_\varepsilon \sim 1$.

1.6. Outline of the rest of the paper

The rest of the paper is organized as follows.

In section 2, we define the correctors w^ε and prove some useful estimates on them. Mostly, these are standard adaptions of previously established estimates.

Section 3 contains the proofs of the main results. In section 3.1 we give the proofs of theorems 1.1 and 1.2, which are largely analogous.

Section 3.2 contains the proof of theorems 1.5 and 1.3. The proof of theorem 1.3 is very similar to those of theorems 1.1 and 1.2. For the proof of theorem 1.5, we first use a well-known Poincaré inequality in the perforated domain (see proposition 2.4) to get a uniform a-priori estimate of u_ε in $L^2(0,T;L^2(\mathbb{R}^3))$. We use a classical duality argument that allows us to pass to the limit in the weak formulation of the PDE by applying the correctors w^ε to smooth testfunctions instead of the solution u of the limit problem as in the proof of the quantitative results.

2. Corrector estimates

Throughout this section, we write $A \lesssim B$ for $A, B \in \mathbb{R}$ when $A \leqslant CB$ for some constant C that depends only on the reference particle $\mathcal{T}$ and possibly the exponent p of some Sobolev space involved in the estimate.

Let $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$. We denote by Q_i^ε the open cubes of length ε centered at x_i^ε that (essentially) cover $\mathbb{R}^3$. We split each cube Q_i^ε into four areas, displayed in figure 2,

$$\begin{aligned}
&Q_i^\varepsilon = \mathcal{T}_i^\varepsilon \cup C_i^\varepsilon \cup D_i^\varepsilon \cup K_i^\varepsilon, \\
&C_i^\varepsilon := B_{\frac{\eta_\varepsilon}{4}}(x_i^\varepsilon) \setminus \mathcal{T}_i^\varepsilon, \\
&D_i^\varepsilon := B_{\frac{\eta_\varepsilon}{2}}(x_i^\varepsilon) \setminus B_{\frac{\eta_\varepsilon}{4}}(x_i^\varepsilon), \\
&K_i^\varepsilon := Q_i^\varepsilon \setminus B_{\frac{\eta_\varepsilon}{2}}(x_i^\varepsilon).
\end{aligned}$$

Then, recalling the definition of (w_k, q_k) from (1.4), we define $w_k^\varepsilon, q_k^\varepsilon$ as the ε-periodic functions that satisfy $(w_k^\varepsilon, q_k^\varepsilon) \in W_0^{1,\infty}(\Omega_\varepsilon) \times L^\infty(\Omega_\varepsilon)$, and, in Q_i^ε

$$\begin{aligned}
w_k^\varepsilon(x) = e_k - w_k\left(\tfrac{x - x_i^\varepsilon}{\varepsilon^\alpha}\right), \quad q_k^\varepsilon(x) = -\varepsilon^{-\alpha} q_k\left(\tfrac{x - x_i^\varepsilon}{\varepsilon^\alpha}\right) \qquad &\text{in } C_i^\varepsilon, \\
-\Delta w_k^\varepsilon(x) + \nabla q^\varepsilon = 0, \quad \operatorname{div} w_k^\varepsilon = 0 \qquad &\text{in } D_i^\varepsilon, \\
w_k^\varepsilon = e_k, \quad q_k^\varepsilon = 0 \qquad &\text{in } K_i^\varepsilon.
\end{aligned}$$

Here, e_k denotes the kth unit vector of the standard basis of $\mathbb{R}^3$. Note that the Stokes equations in D_i^ε are complemented with inhomogeneous no slip boundary conditions due to the requirement $w_k^\varepsilon \in W_0^{1,\infty}(\Omega_\varepsilon)$. We will write w^ε for the matrix-valued function with columns w_k^ε, and q^ε for the (row-)vector with entries q_k^ε. We summarize properties of w^ε in the following lemmas. Some of the estimates are very similar to the ones given in [All90a, All90b] and other works.

Lemma 2.1. *The functions w^ε, q^ε satisfy*

(i) $w^\varepsilon \in W_0^{1,\infty}(\Omega_\varepsilon)$, $q^\varepsilon \in L^\infty(\Omega_\varepsilon)$, $\operatorname{div} w_k^\varepsilon = 0$ *for* $k = 1, 2, 3$ *and*

$$\|w^\varepsilon\|_{L^\infty(\mathbb{R}^3)} + \varepsilon^\alpha \left(\|\nabla w^\varepsilon\|_{L^\infty(\mathbb{R}^3)} + \|q^\varepsilon\|_{L^\infty(\mathbb{R}^3)}\right) \lesssim 1. \tag{2.1}$$

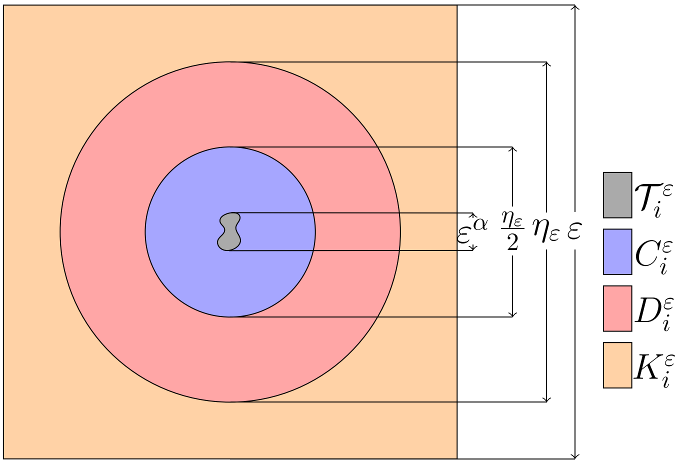

Figure 2. Decomposition of cell the Q_i^ε.

(ii) For all compact sets $K \subseteq \mathbb{R}^3$, we have $w^\varepsilon \to Id$ strongly in $L^2(K)$. Moreover, for all $3/2 < p < 3$ and all $\varphi \in W^{2,p}(\mathbb{R}^3)$

$$\|\varphi\,(Id - w^\varepsilon)\|_{L^p(\mathbb{R}^3)} \lesssim \eta_\varepsilon^{\frac{3}{p}-1} \varepsilon^{\alpha - \frac{3}{p}} \|\varphi\|_{W^{2,p}(\mathbb{R}^3)}. \tag{2.2}$$

Furthermore,

$$\|\varphi\,(Id - w^\varepsilon)\|_{L^3(\mathbb{R}^3)} \lesssim \varepsilon^{\alpha-1} |\log \varepsilon|^{\frac{1}{3}} \|\varphi\|_{W^{2,3}(\mathbb{R}^3)} \qquad \textit{for all } \varphi \in W^{2,3}\left(\mathbb{R}^3\right), \tag{2.3}$$

$$\|\varphi \nabla w^\varepsilon\|_{L^2(\mathbb{R}^3)} + \|\varphi q^\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{\alpha-3}{2}} \|\varphi\|_{H^2(\mathbb{R}^3)} \qquad \textit{for all } \varphi \in H^2\left(\mathbb{R}^3\right), \tag{2.4}$$

$$\||\nabla w^\varepsilon|^{\frac{1}{2}} \varphi\|_{L^2(\mathbb{R}^3)} + \||q^\varepsilon|^{\frac{1}{2}} \varphi\|_{L^2(\mathbb{R}^3)} \lesssim \eta_\varepsilon^{\frac{1}{2}} \varepsilon^{\frac{\alpha-3}{2}} \|\varphi\|_{H^2(\mathbb{R}^3)} \qquad \textit{for all } \varphi \in H^2\left(\mathbb{R}^3\right). \tag{2.5}$$

(iii) For all $\varphi \in H_0^1(\Omega_\varepsilon)$

$$\||\nabla w^\varepsilon|^{\frac{1}{2}} \varphi\|_{L^2(\Omega_\varepsilon)} + |q^\varepsilon|^{\frac{1}{2}} \varphi\|_{L^2(\Omega^\varepsilon)} \lesssim \eta_\varepsilon^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}. \tag{2.6}$$

Proof. *Step 1: Pointwise estimates and proof of (i).*

$$|\mathrm{Id} - w^\varepsilon|\,(x - x_i^\varepsilon) \lesssim \frac{\varepsilon^\alpha}{|x - x_i^\varepsilon|} \qquad \text{in } C_i^\varepsilon \cup D_i^\varepsilon, \tag{2.7}$$

$$|\nabla w^\varepsilon| + |q^\varepsilon| \lesssim \frac{\varepsilon^\alpha}{|x - x_i^\varepsilon|^2} \qquad \text{in } C_i^\varepsilon \cup D_i^\varepsilon. \tag{2.8}$$

The estimates on C_i^ε follow immediately from standard decay estimates for the Stokes equations in exterior domains (see [Gal11, theorem V.3.2]) applied to (w_k, q_k) from (1.4) and the definition of $w^\varepsilon, q^\varepsilon$ through rescaling on C_i^ε. Consequently, the estimates on D_i^ε are deduced from the estimates on ∂D_i^ε and standard regularity theory for the Stokes equations.

Clearly, (i) follows directly from these pointwise estimates.

Step 2: Proof of (ii). Using (2.7) and $w^\varepsilon = \mathrm{Id}$ in K_i^ε, we compute for one cell, for all $p < 3$,

$$\|\mathrm{Id} - w^\varepsilon\|^p_{L^p(Q_i^\varepsilon)} \lesssim \varepsilon^{\alpha p} \int_{B_{\eta_\varepsilon/2}(x_i^\varepsilon)} |x - x_i^\varepsilon|^{-p} \, \mathrm{d}x \lesssim \eta_\varepsilon^{3-p} \varepsilon^{\alpha p}.$$

For any compact $K \subseteq \mathbb{R}^3$, we can cover K by $C(K)\varepsilon^{-3}$ many cubes Q_i^ε. Hence, $\|\mathrm{Id} - w^\varepsilon\|^2_{L^2(K)} \lesssim C(K)(\eta_\varepsilon/\varepsilon)\varepsilon^{2(\alpha-1)} \to 0$ as $\varepsilon \to 0$ since $\eta_\varepsilon \leqslant \varepsilon$ and $\alpha > 1$.

Denoting $(\varphi)_i = \fint_{Q_i^\varepsilon} \varphi \, \mathrm{d}x$, we have for $p > 3/2$ by the Sobolev embedding $W^{2,p}(Q_i^\varepsilon) \subseteq L^\infty(Q_i^\varepsilon)$ and the Poincaré inequality that

$$\|\varphi - (\varphi)_i - (\nabla\varphi)_i (x - x_i^\varepsilon)\|_{L^\infty(Q_i^\varepsilon)} \leqslant C_\varepsilon \|\nabla^2 \varphi\|_{L^p(Q_i^\varepsilon)}.$$

Scaling considerations imply $C_\varepsilon = C\varepsilon^{2-3/p}$. Thus, using also that $|(\psi)_i| \leqslant \varepsilon^{-3/p}\|\psi\|_{L^p(Q_i^\varepsilon)}$,

$$\begin{aligned}\|\varphi\|_{L^\infty(Q_i^\varepsilon)} &\leqslant \|\varphi - (\varphi)_i - (\nabla\varphi)_i (x - x_i^\varepsilon)\|_{L^\infty(Q_i^\varepsilon)} \\ &\quad + |(\varphi)_i| + \varepsilon|(\nabla\varphi)_i| \lesssim \varepsilon^{-3/p}\|\varphi\|_{W^{2,p}(Q_i^\varepsilon)}.\end{aligned}$$

Hence, for $p \in (3/2, 3)$

$$\|\varphi(\mathrm{Id} - w^\varepsilon)\|^p_{L^p(\mathbb{R}^3)} \lesssim \sum_i \eta_\varepsilon^{3-p}\varepsilon^{\alpha p}\|\varphi\|^p_{L^\infty(Q_i^\varepsilon)} \lesssim \eta_\varepsilon^{3-p}\varepsilon^{\alpha p - 3}\|\varphi\|^p_{W^{2,p}(\mathbb{R}^3)}. \tag{2.9}$$

Estimates (2.3)–(2.5) are proved analogously. For (2.3) we use in addition that $B_{\delta\varepsilon^\alpha}(x_i^\varepsilon) \subseteq \mathcal{T}_i^\varepsilon$ for some $\delta > 0$ that depends only on the reference particle $\mathcal{T}$. Therefore $w^\varepsilon = 0$ in $B_{\delta\varepsilon^\alpha}(x_i^\varepsilon)$.

Step 3: Proof of (iii)*:* It suffices to prove that for all $\varphi \in C^\infty(Q_i^\varepsilon)$ with $\varphi = 0$ in $\mathcal{T}_i^\varepsilon$, we have

$$\||\nabla w^\varepsilon|\varphi^2\|_{L^1(Q_i^\varepsilon)} \lesssim \eta_\varepsilon \|\nabla\varphi\|^2_{L^2(Q_i^\varepsilon)}.$$

Without loss of generality, we assume $x_i^\varepsilon = 0$. By the pointwise estimate (2.8) and the fundamental theorem of calculus, we have for all $x \in C_i^\varepsilon \cup D_i^\varepsilon$ with $\delta > 0$ as above

$$|\nabla w^\varepsilon(x)||\varphi(x)|^2 \lesssim \frac{\varepsilon^\alpha}{|x|^2}|\varphi(x)|^2 \leqslant \frac{\varepsilon^\alpha}{|x|^2}\left(\int_{\delta\varepsilon^\alpha}^{|x|} \left|\nabla\varphi\left(\frac{tx}{|x|}\right)\right| \mathrm{d}t\right)^2.$$

This implies

$$\begin{aligned}\||\nabla w^\varepsilon|\varphi^2\|_{L^1(Q_i^\varepsilon)} &\lesssim \varepsilon^\alpha \int_{S^2}\int_{\delta\varepsilon^\alpha}^{\eta_\varepsilon/2} |\varphi(rn)|^2 \, \mathrm{d}r \, \mathrm{d}n \leqslant \varepsilon^\alpha \int_{S^2}\int_{\delta\varepsilon^\alpha}^{\eta_\varepsilon/2} \left(\int_{\delta\varepsilon^\alpha}^{\eta_\varepsilon/2} |\nabla\varphi(tn)| \, \mathrm{d}t\right)^2 \mathrm{d}r \, \mathrm{d}n \\ &\lesssim \eta_\varepsilon\varepsilon^\alpha \int_{S^2}\int_{\delta\varepsilon^\alpha}^{\eta_\varepsilon/2} r^2|\nabla\varphi(rn)|^2 \, \mathrm{d}r \, \mathrm{d}n \int_{\delta\varepsilon^\alpha}^{\eta_\varepsilon/2} \frac{1}{r^2} \, \mathrm{d}r \lesssim \eta_\varepsilon \|\nabla\varphi\|^2_{L^2(Q_i^\varepsilon)},\end{aligned}$$

as claimed. The proof of the estimate for the term involving q^ε is analogous. □

Lemma 2.2. *We can write*

$$-\Delta w^\varepsilon + \nabla q^\varepsilon = \varepsilon^{\alpha-3} M_\varepsilon - \gamma_\varepsilon \tag{2.10}$$

for some $M_\varepsilon, \gamma_\varepsilon \in W^{-1,\infty}(\mathbb{R}^3)$ *where* $\langle \gamma_\varepsilon, v\rangle = 0$ *for all* $v \in H^1_0(\Omega_\varepsilon)$ *and, for all* $\varphi \in H^3(\mathbb{R}^3)$ *and all* $\psi \in H^1(\mathbb{R}^3)$,

$$\langle (M_\varepsilon - \mathcal{R})\varphi, \psi\rangle \lesssim \left(\eta_\varepsilon^{-1}\varepsilon^\alpha \|\psi\|_{L^2(\mathbb{R}^3)} + \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\psi\|_{H^1(\mathbb{R}^3)}\right)\|\varphi\|_{H^3(\mathbb{R}^3)}, \tag{2.11}$$

where the matrix $\mathcal{R}$ *is defined in* (1.5).

Proof. We observe that $-\Delta w^\varepsilon + \nabla q^\varepsilon$ is supported on $\bigcup_i \partial C_i^\varepsilon \cup \partial D_i^\varepsilon = \bigcup_i \partial D_i^\varepsilon \cup \partial\Omega_\varepsilon$ and we define γ_ε to be the part supported on $\partial\Omega_\varepsilon$ which consequently satisfies $\langle \gamma_\varepsilon, v\rangle = 0$ for all $v \in H^1_0(\Omega_\varepsilon)$. Then (2.10) holds with M_k^ε, the columns of M^ε, being

$$M_k^\varepsilon = \varepsilon^{3-\alpha}\sum_i \left(m_{k,i}^\varepsilon + \operatorname{div}\left(1_{D_i^\varepsilon}\left(q_k^\varepsilon \mathrm{Id} - \nabla w_k^\varepsilon\right)\right)\right) \tag{2.12}$$

where

$$m_{k,i}^\varepsilon = \varepsilon^{-\alpha}\left(q_k \mathrm{Id} - \nabla w_k\right)\left(\varepsilon^{-\alpha}x\right) n |\partial B_{\eta_\varepsilon/4}| \delta^i_{\eta_\varepsilon/4}, \qquad \delta^i_{\eta_\varepsilon/4} = \frac{\mathcal{H}^2|_{\partial B_{\eta_\varepsilon/4}(x_i^\varepsilon)}}{|\partial B_{\eta_\varepsilon/4}(x_i^\varepsilon)|}, \tag{2.13}$$

and where w_k, q_k are as in (1.4) and n is the unit normal on $\partial B_{\eta_\varepsilon/4}(x_i^\varepsilon)$. By [All90a, lemma 2.3.5] (which follows from the fact that w_k, q_k asymptotically behave as the fundamental solution of the Stokes equations), we have

$$m_{k,i}^\varepsilon = \frac{\varepsilon^\alpha}{2}\left(\mathcal{R}_k + 3(\mathcal{R}_k\cdot n)n + \eta_\varepsilon^{-1}\varepsilon^\alpha r_{k,i}^\varepsilon\right)\delta^i_{\eta_\varepsilon/4}, \qquad \|r_{k,i}^\varepsilon\|_{W^{1,\infty}\left(\partial B_{\eta_\varepsilon/4}\right)} \lesssim 1.$$

To conclude the proof, it suffices to show that for all $\varphi \in H^3(\mathbb{R}^3)$ and all $\psi \in H^1(\mathbb{R}^3)$

$$\left\|\varphi\left(\mathcal{R}_k - \frac{\varepsilon^3}{2}\sum_i\left(\mathcal{R}_k + 3(\mathcal{R}_k\cdot n)n\right)\right)\delta^i_{\eta_\varepsilon/4}\right\|_{H^{-1}(\mathbb{R}^3)} \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\varphi\|_{H^3(\mathbb{R}^3)}, \tag{2.14}$$

$$\varepsilon^{3-\alpha}\left\|\varphi\sum_i \operatorname{div}\left(1_{D_i^\varepsilon}\left(q_k^\varepsilon\mathrm{Id} - \nabla w_k^\varepsilon\right)\right)\right\|_{H^{-1}(\mathbb{R}^3)} \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\varphi\|_{H^3(\mathbb{R}^3)}, \tag{2.15}$$

$$\left\langle \varphi\varepsilon^3\sum_i r_{k,i}^\varepsilon\delta^i_{\eta_\varepsilon/4}, \psi\right\rangle \lesssim \|\varphi\|_{H^3(\mathbb{R}^3)}\left(\|\psi\|_{L^2(\mathbb{R}^3)} + \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\psi\|_{H^1(\mathbb{R}^3)}\right). \tag{2.16}$$

Indeed, $\eta_\varepsilon^{-1}\varepsilon^\alpha \leqslant 1$ by assumption and thus (2.12)–(2.16) imply the assertion.

To prove (2.14), we begin by observing that for all $v \in H^1(Q_i^\varepsilon)$ we have due to Sobolev embedding

$$\|v - (v)_i\|_{L^6(Q_i^\varepsilon)} \leqslant C\|\nabla v\|_{L^2(Q_i^\varepsilon)}, \tag{2.17}$$

where we recall the notation $(v)_i = \fint_{Q_i^\varepsilon} v$ and where the constant C is universal due to scaling considerations. Similarly, we have the Poincaré-type inequality

$$\fint_{\partial B_{\eta_\varepsilon/4}(x_i^\varepsilon)}\left|v - \fint_{B_{\eta_\varepsilon/4}(x_i^\varepsilon)} v\,\mathrm{d}x\right|\mathrm{d}y \lesssim \eta_\varepsilon^{-\frac{1}{2}}\|\nabla v\|_{L^2\left(B_{\eta_\varepsilon/4}(x_i^\varepsilon)\right)}. \tag{2.18}$$

Since

$$\fint_{\partial B_{\eta_\varepsilon/4}} \frac{1}{2}\left(\mathcal{R}_k + 3\left(\mathcal{R}_k\cdot n\right)n\right)\mathrm{d}x = \mathcal{R}_k,$$

we deduce that for any $v \in H^1(Q_i^\varepsilon)$ that

$$\begin{aligned}
&\left|\int_{Q_i^\varepsilon}\left(v\cdot\mathcal{R}_k - \frac{1}{2}\fint_{\partial B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)} v\cdot\left(\mathcal{R}_k + 3\left(\mathcal{R}_k\cdot n\right)n\right)\mathrm{d}y\right)\mathrm{d}x\right| \\
&\quad = \frac{\varepsilon^3}{2}\left|\fint_{\partial B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)}\left(v-(v)_i\right)\cdot\left(\mathcal{R}_k + 3\left(\mathcal{R}_k\cdot n\right)n\right)\mathrm{d}x\right| \\
&\quad \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^3\|\nabla v\|_{L^2\left(B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)\right)} + \varepsilon^3\fint_{B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)}|v-(v)_i|\,\mathrm{d}x \\
&\quad \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^3\|\nabla v\|_{L^2\left(B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)\right)} + \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^3\||v-(v)_i\|_{L^6\left(B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)\right)} \\
&\quad \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^3\|\nabla v\|_{L^2\left(Q_i^\varepsilon\right)}.
\end{aligned}$$

Therefore, for $\varphi \in H^3(\mathbb{R}^3)$ and $\psi \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}
&\left\langle \varphi\left(\mathcal{R}_k - \frac{\varepsilon^3}{2}\sum_i\left(\mathcal{R}_k + 3\left(\mathcal{R}_k\cdot n\right)n\right)\delta^i_{\eta_\varepsilon/4}\right),\psi\right\rangle \\
&\quad \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^3\sum_i\|\nabla(\varphi\psi)\|_{L^2\left(Q_i^\varepsilon\right)} \\
&\quad \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\psi\|_{H^1(\mathbb{R}^3)}\left(\varepsilon^3\sum_i\|\varphi\|^2_{W^{1,\infty}\left(Q_i^\varepsilon\right)}\right)^{\frac{1}{2}} \\
&\quad \lesssim \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\psi\|_{H^1(\mathbb{R}^3)}\|\varphi\|_{H^3(\mathbb{R}^3)},
\end{aligned}$$

where the last inequality is shown as in (2.9)

We turn to (2.15). We use the pointwise estimates (2.8) to bound

$$\begin{aligned}
&\varepsilon^{3-\alpha}\left\langle\varphi\sum_i \operatorname{div}\left(1_{D_i^\varepsilon}\left(q_k^\varepsilon\mathrm{Id} - \nabla w_k^\varepsilon\right)\right),\psi\right\rangle \\
&\quad \lesssim \varepsilon^{3-\alpha}\eta_\varepsilon^{\frac{3}{2}}\sum_i\|q_k^\varepsilon\mathrm{Id} - \nabla w_k^\varepsilon\|_{L^\infty\left(D_i^\varepsilon\right)}\|\psi\|_{H^1\left(Q_i^\varepsilon\right)}\|\varphi\|_{W^{1,\infty}\left(Q_i^\varepsilon\right)} \\
&\quad \lesssim \varepsilon^{3-\alpha}\eta_\varepsilon^{\frac{3}{2}}\eta_\varepsilon^{-2}\varepsilon^{\alpha}\varepsilon^{-3/2}\|\psi\|_{H^1(\mathbb{R}^3)}\|\varphi\|_{H^3(\mathbb{R}^3)} \\
&\quad = \eta_\varepsilon^{-\frac{1}{2}}\varepsilon^{\frac{3}{2}}\|\psi\|_{H^1(\mathbb{R}^3)}\|\varphi\|_{H^3(\mathbb{R}^3)}.
\end{aligned}$$

It remains to show (2.16). Using again (2.17) and (2.18), we have for any $v \in H^1(Q_i^\varepsilon)$

$$\begin{aligned}
\left|\fint_{\partial B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)} v\,\mathrm{d}x\right| &\lesssim \fint_{\partial B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)}\left|v - \fint_{B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)} v\,\mathrm{d}y\right|\mathrm{d}x + \fint_{B_{\eta_\varepsilon/4}\left(x_i^\varepsilon\right)}|v-(v)_i|\,\mathrm{d}x + |(v)_i| \\
&\lesssim \eta_\varepsilon^{-\frac{1}{2}}\|\nabla v\|_{L^2\left(Q_i^\varepsilon\right)} + \varepsilon^{-3/2}\|v\|_{L^2\left(Q_i^\varepsilon\right)}.
\end{aligned}$$

Thus, for $\varphi \in H^3(\mathbb{R}^3)$ and $\psi \in H^1(\mathbb{R}^3)$, using (2.13),

$$\begin{aligned}\left\langle \varphi \varepsilon^3 \sum_i r^\varepsilon_{k,i} \delta^i_{\eta_\varepsilon/4}, \psi \right\rangle &\lesssim \eta_\varepsilon^{-\frac{1}{2}} \varepsilon^3 \sum_i \|\psi\|_{H^1(Q_i^\varepsilon)} \|\varphi\|_{W^{1,\infty}(Q_i^\varepsilon)} \\ &\quad + \varepsilon^{3/2} \sum_i \|\psi\|_{L^2(Q_i^\varepsilon)} \|\varphi\|_{L^\infty(Q_i^\varepsilon)} \\ &\lesssim \|\varphi\|_{H^3(\mathbb{R}^3)} \left(\eta_\varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{3}{2}} \|\psi\|_{H^1(\mathbb{R}^3)} + \|\psi\|_{L^2(\mathbb{R}^3)} \right).\end{aligned}$$

This finishes the proof. □

Lemma 2.3. *For all $1 < p < \infty$, there exists a linear operator $\mathcal{B}_\varepsilon : W^{1,p}(\mathbb{R}^3) \to W^{1,p}(\mathbb{R}^3)$ such that for all $\varphi \in W^{1,p}(\mathbb{R}^3)$ that are divergence free we have*

$$div \mathcal{B}_\varepsilon(\varphi) = w^\varepsilon : \nabla \varphi \tag{2.19}$$

and

$$\|\nabla \mathcal{B}_\varepsilon(\varphi)\|_{L^p} \lesssim \|(Id - w^\varepsilon) : \nabla\varphi\|_{L^p}, \qquad \|\mathcal{B}_\varepsilon(\varphi)\|_{L^p} \lesssim \eta_\varepsilon \|(Id - w^\varepsilon) : \nabla\varphi\|_{L^p}. \tag{2.20}$$

Proof. It suffices to construct the linear operator on the subspace of divergence free functions $\varphi \in W^{1,p}(\mathbb{R}^3)$. We observe that then $w^\varepsilon : \nabla\varphi = 0$ in $\mathbb{R}^3 \setminus A_i^\varepsilon$ where $A_i^\varepsilon := C_i^\varepsilon \cup D_i^\varepsilon$ and, since the functions w_k^ε are divergence free,

$$\int_{A_i^\varepsilon} w^\varepsilon : \nabla \varphi \, \mathrm{d}x = \int_{A_i^\varepsilon \cup \mathcal{T}_i^\varepsilon} w^\varepsilon : \nabla \varphi \, \mathrm{d}x = \int_{A_i^\varepsilon \cup \mathcal{T}_i^\varepsilon} \mathrm{div}\left((w^\varepsilon - \mathrm{Id})\varphi\right) \mathrm{d}x = 0$$

as $w^\varepsilon = \mathrm{Id}$ on ∂D_i^ε. Therefore we may employ a Bogovski operator in A_i^ε. More precisely, by [DFL17, lemma 3.1] (which is a consequence of [ADM06, DRS10]), there exist operators $\mathcal{B}_i^\varepsilon : L^p_0(A_i^\varepsilon) \to W^{1,p}_0(A_i^\varepsilon)$ ($L^p{}_0$ denotes the subspace of L^p functions with vanishing mean) such that for all $h \in L^p_0(A_i^\varepsilon)$

$$\mathrm{div} \mathcal{B}_i^\varepsilon(h) = h, \qquad \|\mathcal{B}_i^\varepsilon(h)\|_{W_0^{1,p}(A_i^\varepsilon)} \lesssim \|h\|_{L_0^p(A_i^\varepsilon)}.$$

We then deduce that $\mathcal{B}_\varepsilon(\varphi) := \sum_i \mathcal{B}_i^\varepsilon(w^\varepsilon : \nabla\varphi)$ satisfies (2.19) as well as the first inequality in (2.20). The second inequality in (2.20) follows from the first one and the Poincaré inequality in the domains $A_i^\varepsilon \subseteq B_{\eta_\varepsilon}(x_i)$. □

For the treatment of the subcritical case, we will rely on the following Poincaré inequality in Ω_ε. It is proved in [All90b, lemma 3.4.1] when Ω_ε is a bounded domain. Since the proof is based on a local Poincaré inequality in each of the cubes Q_i^ε, it still applies here.

Proposition 2.4 ([All90b, lemma 3.4.1]). *For all $\varphi \in H^1_0(\Omega_\varepsilon)$*

$$\|\varphi\|_{L^2(\Omega_\varepsilon)} \lesssim \varepsilon^{\frac{3-\alpha}{2}} \|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}. \tag{2.21}$$

3. Proof of the main results

As outlined in section 1.4, the strategy for the proof of the main results is based on energy estimates for the difference

$$v_\varepsilon = w^\varepsilon u - u_\varepsilon - \mathcal{B}_\varepsilon(u). \tag{3.1}$$

Here u_ε is the solution to (1.1) in the critical and subcritical case and to (1.9)in the supercritical case and u is the solution to (1.7), (1.8) and (1.10), respectively. Moreover, w^ε is the matrix valued function defined at the beginning of section 2 and depends on a parameter $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$ that we will choose later. Finally, $\mathcal{B}_\varepsilon$ is the operator from lemma 2.3.

We first observe that the difference $(w^\varepsilon - \mathrm{Id})u - \mathcal{B}_\varepsilon(u)$ between v_ε and $u - u_\varepsilon$ is very small, namely

$$\|v_\varepsilon - (u - u_\varepsilon)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leqslant C\eta_\varepsilon^{\frac{1}{2}}\varepsilon^{\alpha-\frac{3}{2}}, \tag{3.2}$$

where the constant C depends only on $\mathcal{T}$ and $\|u\|_{L^\infty(0,T;H^3(\mathbb{R}^3))}$. Indeed, this follows immediately from (2.2) and (2.20).

3.1. Proof of theorems 1.1 and 1.2

Throughout this subsection, we assume that the parameters α and γ are in the range of the critical or subcritical regime specified in theorems 1.1 and 1.2, respectively, that is $\alpha > 3/2$ and $\gamma > 0$, $\gamma \in [3-\alpha, \alpha)$ or $\gamma = \alpha$ and $\mu_0 \gg 1$. Moreover, v_ε is defined by (3.1) where u_ε is the solution to (1.1) and u is the solution to (1.8) or (1.7).

The main technical part of the proof of the main results is an energy estimate for v_ε stated in the following proposition. Thereafter, we show how theorems 1.1 and 1.2 follow from this proposition and Gronwall's inequality.

Proposition 3.1. *Let $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$. Then,*

(i) Then, under the assumptions of theorem 1.1 we have for all $t \leqslant T$

$$\begin{aligned}\|v_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} &+ (\varepsilon^\gamma - C\eta_\varepsilon)\|\nabla v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant \|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + C\|(f_\varepsilon - f)\|^2_{L^2((0,T)\times\Omega_\varepsilon)} + C\|v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\ &+ C\left(\eta_\varepsilon\varepsilon^{2\alpha-\gamma-3} + \eta_\varepsilon^{-1}\varepsilon^{3-\gamma} + \varepsilon^{2\gamma} + \eta_\varepsilon^2\right)\end{aligned} \tag{3.3}$$

for some constant C which depends only on $\mathcal{T}$, T $\|f\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$, $\|u\|_{C^1(0,T;H^4(\mathbb{R}^3))}$ and $\|\nabla p\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$.

(ii) Under the assumptions of theorem 1.2 we have for all $t \leqslant T$

$$\begin{aligned}\|v_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} &+ (\mu_0\varepsilon^\gamma - C\eta_\varepsilon)\|\nabla v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\ &\leqslant \|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + C\|(f_\varepsilon - f)\|^2_{L^2(0,T)\times\Omega_\varepsilon)} + C\|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \\ &+ C_{\mu_0}\left(\varepsilon^{2\alpha+2\gamma-6} + \eta_\varepsilon\varepsilon^{2\alpha-\gamma-3} + \eta_\varepsilon^{-1}\varepsilon^{2\alpha+\gamma-3} + \varepsilon^{2\gamma} + \eta_\varepsilon^2\right)\end{aligned} \tag{3.4}$$

for some C which depends only on $\mathcal{T}$, T, $\|f\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$, $\|u\|_{C^1(0,T;H^4(\mathbb{R}^3))}$, $\|\nabla p\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}$ and some C_{μ_0} which depends additionally on μ_0.

Proof of theorem 1.1. We choose $\eta_\varepsilon = \frac{1}{C}\varepsilon^\gamma$ such that we may drop the second term on the left-hand side of (3.3). Note that as $\gamma = 3-\alpha$ and $\alpha \in (3/2, 3)$, the assumption $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$ is satisfied for all ε sufficiently small (for ε of order 1, the assertion of the theorem is an immediate consequence of the energy inequality (1.2)).

Then, by Gronwall's inequality, proposition 3.1 yields

$$\|v_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \lesssim \|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \|(f_\varepsilon - f)\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left(\varepsilon^{2\alpha-3} + \varepsilon^{6-2\alpha}\right)$$

and we deduce with (3.2), which only gives a higher order error, that

$$\|(u_\varepsilon - u)(t)\|^2_{L^2(\Omega_\varepsilon)} \lesssim \|(u_\varepsilon - u)(0)\|^2_{L^2(\Omega_\varepsilon)} + \|(f_\varepsilon - f)\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + \left(\varepsilon^{2\alpha-3} + \varepsilon^{6-2\alpha}\right).$$

This finishes the proof. □

Proof of theorem 1.2. We choose $\eta_\varepsilon = \delta\varepsilon^\beta$ with $\beta = \min\{\gamma, 1\}$ and

$$\delta = \begin{cases} 1 & \text{if } \gamma = \alpha, \\ \frac{1}{C} & \text{if } \gamma < \alpha. \end{cases}$$

This choice guarantees $\varepsilon^\alpha \leqslant \eta_\varepsilon \leqslant \varepsilon$ is satisfied for all ε sufficiently small. Moreover, choosing $M = C$, the assumption $\mu_0 \geqslant M$ if $\gamma = \alpha$ allows us to drop the second term on the left-hand side in (3.4) in all cases. Therefore, arguing as in the proof above yields

$$\begin{aligned}\|(u_\varepsilon - u)(t)\|^2_{L^2(\Omega_\varepsilon)} \lesssim{}& \|(u_\varepsilon - u)(0)\|^2_{L^2(\Omega_\varepsilon)} + \|(f_\varepsilon - f)\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} \\ &+ \left(\varepsilon^{2\alpha+2\gamma-6} + \varepsilon^{2\alpha-3} + \varepsilon^{2\alpha+\gamma-4} + \varepsilon^{2\gamma}\right).\end{aligned}$$

We observe that $2\alpha + \gamma - 4 \leqslant \max\{2\alpha - 3, 2\alpha + 2\gamma - 6\}$ to finish the proof. □

Proof of proposition 3.1. We focus on the critical case $\gamma = 3 - \alpha$ where u solves (1.7). We discuss the necessary adaptions for the subcritical case $\gamma > 3 - \alpha$ in the last step of the proof. Throughout the proof we write $\lesssim$ for $\leqslant C$ with C as specified in the statement of the proposition.

Step 1: PDE solved by $\check u_\varepsilon := w^\varepsilon u - \mathcal{B}_\varepsilon(u)$: We observe that $\check u_\varepsilon$ satisfies $\check u_\varepsilon = 0$ on $(0,T) \times \partial\Omega_\varepsilon$ and, in $(0,T) \times \Omega_\varepsilon$

$$\begin{aligned}\partial_t \check u_\varepsilon - \varepsilon^\gamma \Delta \check u_\varepsilon + w^\varepsilon (u \cdot \nabla u) + w^\varepsilon \nabla p ={}& w^\varepsilon f + (M_\varepsilon - w^\varepsilon \mathcal{R})u - \varepsilon^\gamma \nabla q^\varepsilon u \\ &- 2\varepsilon^\gamma \nabla w^\varepsilon \nabla u - \varepsilon^\gamma w^\varepsilon \Delta u + \mathcal{B}_\varepsilon(\partial_t u) + \varepsilon^\gamma \Delta \mathcal{B}_\varepsilon(u),\end{aligned} \tag{3.5}$$

with M_ε as in (2.10). Moreover, $\operatorname{div} \check u_\varepsilon = 0$.

Step 2: Relative energy inequality: We consider the relative energy $\frac{1}{2}\|v_\varepsilon\|^2_{L^2}$. We estimate using the energy inequality (1.2) for u_ε as well as $\check u_\varepsilon \in L^\infty(H^1)$, $\partial_t \check u_\varepsilon \in L^1(H^{-1})$

$$\begin{aligned}\frac{1}{2}\|v_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} ={}& \frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} - (\check u_\varepsilon(t), u_\varepsilon(t))_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\check u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} \\ \leqslant{}& \frac{1}{2}\|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} - \varepsilon^\gamma \int_0^t \|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \,\mathrm{d}s + \int_0^t \int_{\Omega_\varepsilon} f_\varepsilon \cdot u_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t \int_{\Omega_\varepsilon} (\partial_t \check u_\varepsilon \cdot u_\varepsilon + \partial_t u_\varepsilon \cdot \check u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_\varepsilon} \partial_t \check u_\varepsilon \cdot \check u_\varepsilon \,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{3.6}$$

Using the equation solved by u_ε, we have

$$-\int_0^t \int_{\Omega_\varepsilon} \partial_t u_\varepsilon \cdot \check{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\Omega_\varepsilon} \left((u_\varepsilon \cdot \nabla u_\varepsilon) \cdot \check{u}_\varepsilon + \varepsilon^\gamma \nabla u_\varepsilon \cdot \nabla \check{u}_\varepsilon - f_\varepsilon \cdot \check{u}_\varepsilon \right) \mathrm{d}x\,\mathrm{d}s \tag{3.7}$$

and likewise, using the equation of $\check{u}_\varepsilon$

$$\begin{aligned} \int_0^t \int_{\Omega_\varepsilon} \partial_t \check{u}_\varepsilon \cdot v_\varepsilon = & -\int_0^t \int_{\Omega_\varepsilon} \Big(\varepsilon^\gamma \nabla \check{u}_\varepsilon \cdot \nabla v_\varepsilon + (w^\varepsilon (u \cdot \nabla u)) \cdot (\check{u}_\varepsilon - u_\varepsilon) \\ & - (w^\varepsilon f + F_\varepsilon) \cdot v_\varepsilon \Big) \,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \tilde{F}_\varepsilon = & - w^\varepsilon \nabla p + (M_\varepsilon - w^\varepsilon \mathcal{R}) u - \varepsilon^\gamma \nabla q^\varepsilon u - 2\varepsilon^\gamma \nabla w^\varepsilon \nabla u \\ & - \varepsilon^\gamma w^\varepsilon \Delta u + \mathcal{B}_\varepsilon (\partial_t u) + \varepsilon^\gamma \Delta \mathcal{B}_\varepsilon (u). \end{aligned}$$

inserting (3.7) and (3.8) in (3.6) and denoting

$$F_\varepsilon = \tilde{F}_\varepsilon + (w^\varepsilon f - f_\varepsilon)$$

yields

$$\begin{aligned} & \frac{1}{2} \|v_\varepsilon\|_{L^2}^2 (t) + \varepsilon^\gamma \int_0^t \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 \,\mathrm{d}s \\ & \quad \leqslant \int_0^t \int_{\Omega_\varepsilon} \left((u_\varepsilon \cdot \nabla u_\varepsilon) \cdot \check{u}_\varepsilon - (w^\varepsilon (u \cdot \nabla u)) \cdot v_\varepsilon + F_\varepsilon \cdot v_\varepsilon \right) \mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

Thus, we deduce

$$\frac{1}{2} \|v_\varepsilon (t)\|_{L^2(\Omega_\varepsilon)}^2 + \varepsilon^\gamma \|\nabla v_\varepsilon\|_{L^2((0,t)\times\Omega_\varepsilon)}^2 \leqslant \frac{1}{2} \|v_\varepsilon (0)\|_{L^2(\Omega_\varepsilon)}^2 + |I_1| + |I_2| \tag{3.10}$$

where

$$\begin{aligned} I_1 &= \int_0^t \int_{\Omega_\varepsilon} \left((u_\varepsilon \cdot \nabla u_\varepsilon) \cdot \check{u}_\varepsilon - (w^\varepsilon (u \cdot \nabla u)) \cdot v_\varepsilon \right) \mathrm{d}x\,\mathrm{d}s, \\ I_2 &= \int_0^t \int_{\Omega_\varepsilon} F_\varepsilon \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Step 3: Bound of I_1: We first manipulate the first term in I_1. Using $u_\varepsilon = \check{u}_\varepsilon = 0$ on $\partial\Omega_\varepsilon$ as well as $\operatorname{div} u = \operatorname{div} u_\varepsilon = 0$ yields by integration by parts

$$\begin{aligned} \int_0^t \int_{\Omega_\varepsilon} (u_\varepsilon \cdot \nabla u_\varepsilon) \cdot \check{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}s &= -\int_0^t \int_{\Omega_\varepsilon} (u_\varepsilon \cdot \nabla \check{u}_\varepsilon) \cdot (u_\varepsilon - \check{u}_\varepsilon) \,\mathrm{d}x\,\mathrm{d}s \\ &= -\int_0^t \int_{\Omega_\varepsilon} (v_\varepsilon \cdot \nabla \check{u}_\varepsilon) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\ & \quad + \int_0^t \int_{\Omega_\varepsilon} (\check{u}_\varepsilon \cdot \nabla \check{u}_\varepsilon) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.11}$$

This allows us to rewrite

$$
\begin{aligned}
I_1 = &- \int_0^t \int_{\Omega_\varepsilon} (v_\varepsilon \cdot \nabla \check{u}_\varepsilon) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_\varepsilon} (\mathrm{Id} - w^\varepsilon)(u \cdot \nabla u) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\Omega_\varepsilon} ((\check{u}_\varepsilon - u) \cdot \nabla u) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_{\Omega_\varepsilon} (\check{u}_\varepsilon \cdot \nabla (\check{u}_\varepsilon - u)) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s =: I_1^1 + I_1^2 + I_1^3 + I_1^4.
\end{aligned}
$$

We recall $\check{u}_\varepsilon = w^\varepsilon u - \mathcal{B}_\varepsilon(u)$ to estimate by the regularity assumptions of u, (2.6) and (2.20) combined with (2.3) and another integration by parts

$$
\begin{aligned}
|I_1^1| &\lesssim \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \|w_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \|\nabla u\|_{L^\infty(0,t;L^\infty(\mathbb{R}^3))} \\
&\quad + \|\nabla w_\varepsilon |v_\varepsilon|^2\|_{L^1(0,t;L^1(\Omega_\varepsilon))} \|u\|_{L^\infty(0,t;L^\infty(\mathbb{R}^3))} \\
&\quad + \|\nabla v_\varepsilon\|_{L^2(0,t;L^2(\Omega_\varepsilon))} \|v_\varepsilon\|_{L^2(0,t;L^6(\Omega_\varepsilon))} \|\mathcal{B}_\varepsilon(u)\|_{L^\infty(0,T;L^3(\Omega_\varepsilon)} \qquad (3.12) \\
&\lesssim \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \eta_\varepsilon (1 + \varepsilon^{\alpha-1} |\log \varepsilon|^{\frac{1}{3}}) \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \\
&\lesssim \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \eta_\varepsilon \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))},
\end{aligned}
$$

where we used $\alpha > 1$ in the last estimate.

By the regularity assumptions of u and (2.2), we have

$$
|I_1^2| \lesssim \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \eta_\varepsilon \varepsilon^{2\alpha-3}.
$$

Similarly, relying additionally on (2.20),

$$
|I_1^3| \lesssim \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \eta_\varepsilon \varepsilon^{2\alpha-3}.
$$

Finally, we estimate by another integration by parts

$$
|I_1^4| \leqslant \frac{1}{4} \varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \varepsilon^{-\gamma} \|\check{u}_\varepsilon |\check{u}_\varepsilon - u|\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))}.
$$

We estimate using that u and w^ε are uniformly bounded in L^∞ as well as (2.20) and (2.2) and Sobolev embedding

$$
\begin{aligned}
\|\check{u}_\varepsilon |\check{u}_\varepsilon - u|\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} &\lesssim \int_0^t \left(\|(w^\varepsilon - \mathrm{Id}) u\|^2_{L^2(\Omega_\varepsilon))} + \|\mathcal{B}_\varepsilon(u)\|^2_{L^2(\Omega_\varepsilon)} + \|\mathcal{B}_\varepsilon(u)\|^2_{L^4(\Omega_\varepsilon))} \right) \mathrm{d}s \\
&\lesssim \int_0^t \left(\|(w^\varepsilon - \mathrm{Id}) u\|^2_{L^2(\Omega_\varepsilon))} + \|\mathcal{B}_\varepsilon(u)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla \mathcal{B}_\varepsilon(u)\|^2_{L^2(\Omega_\varepsilon))} \right) \mathrm{d}s \\
&\lesssim \eta_\varepsilon \varepsilon^{2\alpha-3}.
\end{aligned}
$$

In summary, we find,

$$
|I_1| \leqslant C \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \left(\frac{1}{4} \varepsilon^\gamma + C\eta_\varepsilon \right) \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C\eta_\varepsilon \varepsilon^{2\alpha-\gamma-3}. \qquad (3.13)
$$

Step 4: Bound of I_2: We split

$$
I_2 = I_2^1 + I_2^2 + I_2^3 + I_2^4
$$

where

$$\begin{aligned}
I_2^1 &= \int_0^t \int_{\Omega_\varepsilon} \left((\mathrm{Id} - w^\varepsilon)(\nabla p - f) + f - f_\varepsilon\right) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s, \\
I_2^2 &= \int_0^t \int_{\Omega_\varepsilon} \left((M_\varepsilon - w^\varepsilon \mathcal{R}) u\right) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s, \\
I_2^3 &= -\varepsilon^\gamma \int_0^t \int_{\Omega_\varepsilon} \left(2\nabla w^\varepsilon \nabla u + w^\varepsilon \Delta u + \nabla q^\varepsilon u\right) \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}s, \\
I_2^4 &= \int_0^t \int_{\Omega_\varepsilon} \left(\mathcal{B}_\varepsilon(\partial_t u) \cdot v_\varepsilon + \varepsilon^\gamma \nabla \mathcal{B}_\varepsilon(u) \nabla v_\varepsilon\right) \mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

We estimate

$$\begin{aligned}
|I_2^1| &\lesssim \|(w^\varepsilon - \mathrm{Id}) \nabla p\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|(w^\varepsilon - \mathrm{Id}) f\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\
&\quad + \|f_\varepsilon - f\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} \\
&\lesssim \eta_\varepsilon \varepsilon^{2\alpha-3} + \|f_\varepsilon - f\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \|v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}.
\end{aligned}$$

We rewrite

$$I_2^2 = \int_0^t \int_{\Omega_\varepsilon} (w^\varepsilon - \mathrm{Id}) \mathcal{R} u \cdot v_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_0^t \langle (M_\varepsilon - \mathcal{R}) u, v_\varepsilon \rangle \,\mathrm{d}s.$$

The first term on the right-hand side is estimated as above. Combining this with (2.11) to estimate the second term on the right-hand side yields for some $\delta > 0$ to be chosen later

$$\begin{aligned}
|I_2^2| &\leqslant C\eta_\varepsilon \varepsilon^{2\alpha-3} + C\eta_\varepsilon^{-2}\varepsilon^{2\alpha} + \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \\
&\quad + C_\delta \eta_\varepsilon^{-1} \varepsilon^3 C \varepsilon^{-\gamma} + \delta \varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \\
&\leqslant C\eta_\varepsilon \varepsilon^{2\alpha-3} + C_\delta \eta_\varepsilon^{-1} \varepsilon^{3-\gamma} + \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \delta\varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))}.
\end{aligned}$$

where we used that $\eta_\varepsilon \geqslant \varepsilon^\alpha$ and $\alpha \geqslant 3 - \gamma$ to absorb the term $\eta_\varepsilon^{-2}\varepsilon^{2\alpha}$. Next, we estimate using (2.5) and (2.6)

$$\begin{aligned}
|I_2^3| &\leqslant C\varepsilon^\gamma \int_0^t \left(\|(|\nabla w^\varepsilon|^{\frac12} + |q_k^\varepsilon|^{\frac12}) \nabla u\|_{L^2} \|(|\nabla w^\varepsilon|^{\frac12} + |q_k^\varepsilon|^{\frac12}) v_\varepsilon\|_{L^2} + \|w^\varepsilon\|_\infty \|v_\varepsilon\|_{L^2} \right) \mathrm{d}s \\
&\lesssim C_\delta \varepsilon^\gamma \eta_\varepsilon^2 \varepsilon^{\alpha-3} + \delta \varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C\varepsilon^{2\gamma} + \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon)} \\
&\lesssim C_\delta \eta_\varepsilon^2 + \delta\varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + \varepsilon^{2\gamma} + \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon)},
\end{aligned}$$

where we used $\alpha + \gamma \geqslant 3$ in the last inequality.

Finally, we estimate, relying on (2.20) and (2.2)

$$|I_2^4| \leqslant C\eta_\varepsilon^2 \eta_\varepsilon \varepsilon^{2\alpha-3} + \|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C_\delta \varepsilon^\gamma \eta_\varepsilon \varepsilon^{2\alpha-3} + \delta\varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))}.$$

Thus, choosing δ sufficiently small, we obtain in summary, after absorbing some higher order terms,

$$\begin{aligned}
|I_2| &\leqslant \frac14 \varepsilon^\gamma \|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C\|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C\|f_\varepsilon - f\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} \\
&\quad + C\left(\eta_\varepsilon \varepsilon^{2\alpha-3} + \eta_\varepsilon^{-1}\varepsilon^{3-\gamma} + \varepsilon^{2\gamma} + \eta_\varepsilon^2\right).
\end{aligned} \tag{3.14}$$

Step 5: Conclusion: Inserting the bounds for I_1 from (3.13) and I_2 from (3.14) into (3.10) yields (3.3).

Step 6: Adaptations in the subcritical case: Let now $\gamma > 3-\alpha$ and let u solves the the Euler equations (1.8). There are only very little changes in the proof in this case. In Step 1, the only differences are that in the PDE solved by $\breve{u}$, (3.5) all inctances of ε^γ should be replaced by $\mu_0\varepsilon^\gamma$ (in the critical case, we assumed $\mu_0 = 1$) and that $(M_\varepsilon - w^\varepsilon\mathcal{R})u$ has to be replaced by $\mu_0\varepsilon^{\gamma+\alpha-3}M_\varepsilon u$. Consequently, estimate (3.10) still holds up to replacing all instances of ε^γ by $\mu_0\varepsilon^\gamma$ and where in the source F_ε (appearing in I_3) the term $(M_\varepsilon - w^\varepsilon\mathcal{R})u$ is likewise replaced by $\mu_0\varepsilon^{\gamma+\alpha-3}M_\varepsilon u$. In particular, the estimates for I_1 in steps 3 still apply, and all the estimates of step 4 for I_2 are unaffected except for the estimate of I_2^2 which now takes the form

$$
\begin{aligned}
I_2^2 &= \mu_0\varepsilon^{\gamma+\alpha-3}\int_0^t\int_{\Omega_\varepsilon}(M_\varepsilon u)\cdot v_\varepsilon\,\mathrm{d}x\,\mathrm{d}s\\
&= \mu_0\varepsilon^{\gamma+\alpha-3}\int_0^t\int_{\Omega_\varepsilon}((M_\varepsilon-\mathcal{R})u)\cdot v_\varepsilon\,\mathrm{d}x\,\mathrm{d}s + \mu_0\varepsilon^{\gamma+\alpha-3}\int_0^t\int_{\Omega_\varepsilon}(\mathcal{R}u)\cdot v_\varepsilon\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Thus, we estimate with lemma 2.2

$$
\begin{aligned}
|I_2^2| &\leqslant \mu_0^2\varepsilon^{2\gamma+2\alpha-6}\left(\eta_\varepsilon^{-2}\varepsilon^{2\alpha}+1\right) + \|v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\\
&\quad + C_\delta\mu_0\eta_\varepsilon^{-1}\varepsilon^3\varepsilon^{\gamma+2\alpha-6} + \delta\mu_0\varepsilon^\gamma\|\nabla v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)}\\
&\leqslant \|v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + \delta\varepsilon^\gamma\|\nabla v_\varepsilon\|^2_{L^2((0,t)\times\Omega_\varepsilon)} + C_{\mu_0,\delta}\left(\varepsilon^{2\gamma+2\alpha-6}+\eta_\varepsilon^{-1}\varepsilon^{2\alpha+\gamma-3}\right)
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
|I_2| \leqslant{}& \frac{1}{4}\mu_0\varepsilon^\gamma\|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C\|v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))} + C\|(f_\varepsilon-f)\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))}\\
&+ C_{\mu_0}\left(\eta_\varepsilon\varepsilon^{2\alpha-3}+\eta_\varepsilon^{-1}\varepsilon^{2\alpha+\gamma-3}+\varepsilon^{2\alpha+2\gamma-6}+\varepsilon^{2\gamma}+\eta_\varepsilon^2\right).
\end{aligned}
$$

Combining this estimate as before with the estimates for I_1, (3.13), yields (3.4). □

3.2. *Proof of theorems 1.3 and 1.5*

In this subsection, we consider u_ε a Leray solution to (1.9) and u the solution to (1.10).

Proof of theorem 1.3. We follow closely the proof of proposition 3.1 to obtain an estimate for $v_\varepsilon = \breve{u}_\varepsilon - u_\varepsilon$, where $\breve{u}_\varepsilon := w^\varepsilon u - \mathcal{B}_\varepsilon(u)$ with w^ε as in section 2 and with $\mathcal{B}_\varepsilon$ as in lemma 2.3.

Recall that w^ε depends on a parameter η_ε. We take $\eta_\varepsilon = \varepsilon^\beta$ for some $1 \leqslant \beta \leqslant \alpha$ to be chosen later.

Step 1: PDE solved by $\breve{u}_\varepsilon$: We have $\breve{u}_\varepsilon = 0$ on $(0,T)\times\partial\Omega_\varepsilon$, and, in $(0,T)\times\Omega_\varepsilon$

$$
\begin{aligned}
\varepsilon^{6-2\alpha-2\gamma}\partial_t\breve{u}_\varepsilon - \varepsilon^{3-\alpha}\Delta\breve{u}_\varepsilon ={}& f - \nabla p + (M_\varepsilon-\mathcal{R})u + \varepsilon^{6-2\alpha-2\gamma}\partial_t\breve{u}_\varepsilon\\
&- \varepsilon^{3-\alpha}\nabla q^\varepsilon u - 2\varepsilon^{3-\alpha}\nabla w^\varepsilon\nabla u - \varepsilon^{3-\alpha}w^\varepsilon\Delta u + \varepsilon^{3-\alpha}\Delta\mathcal{B}_\varepsilon(u),
\end{aligned}
$$

with M_ε as in (2.10). Moreover, $\operatorname{div}\breve{u}_\varepsilon = 0$.

Step 2: Relative energy inequality: Thanks to the energy inequality (1.11) as well as the PDEs solved by u_ε and $\breve{u}_\varepsilon$, we have, correspondingly to (3.9),

$$\begin{aligned}
&\frac{\varepsilon^{6-2\alpha-2\gamma}}{2}\|v_\varepsilon(T)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{3-\alpha}\int_0^T \|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\,\mathrm{d}t \\
&\quad\leqslant \frac{\varepsilon^{6-2\alpha-2\gamma}}{2}\|v^\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \int_0^T\int_{\Omega_\varepsilon}(F_\varepsilon + f_\varepsilon - f)\cdot v_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \varepsilon^{6-2\alpha-2\gamma}\int_0^T\int_{\Omega_\varepsilon}\left((u_\varepsilon\cdot\nabla u_\varepsilon)\cdot\check{u}_\varepsilon + \partial_t\check{u}_\varepsilon\cdot v_\varepsilon\right)\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where

$$F_\varepsilon = (M_\varepsilon - \mathcal{R})u - \varepsilon^{3-\alpha}\nabla q^\varepsilon u - 2\varepsilon^{3-\alpha}\nabla w^\varepsilon\nabla u - \varepsilon^{3-\alpha}w^\varepsilon\Delta u + \varepsilon^{3-\alpha}\Delta\mathcal{B}_\varepsilon(u).$$

Thus, using the Poincaré inequality (2.21) and Young's inequality,

$$\begin{aligned}
\frac{1}{2}\varepsilon^{3-\alpha}\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} &\leqslant \varepsilon^{6-2\alpha-2\gamma}\|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} \\
&\quad + \|f_\varepsilon - f\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + |I_1| + |I_2|, \qquad (3.15)\\
I_1 &= \int_0^T \langle F_\varepsilon, v_\varepsilon\rangle\,\mathrm{d}t,\\
I_2 &= \varepsilon^{6-2\alpha-2\gamma}\int_0^T\int_{\Omega_\varepsilon}\left((u_\varepsilon\cdot\nabla u_\varepsilon)\cdot\check{u}_\varepsilon + \partial_t\check{u}_\varepsilon\cdot v_\varepsilon\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Step 3: Estimate of I_1: We estimate with (2.2) as well as lemma 2.2 and the Poincaré inequality (2.21)

$$\begin{aligned}
\left|\int_0^T\langle(M_\varepsilon - \mathcal{R})u, v_\varepsilon\rangle\,\mathrm{d}t\right| &\lesssim \varepsilon^{\alpha-\beta}\|v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \varepsilon^{\frac{3-\beta}{2}}\|v_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))}\\
&\lesssim \left(\varepsilon^{\frac{3-\alpha}{2}}\varepsilon^{\alpha-\beta} + \varepsilon^{\frac{3-\beta}{2}}\right)\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\\
&\lesssim \varepsilon^{\frac{3-\beta}{2}}\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))},
\end{aligned}$$

where we used $\alpha \geqslant \beta$ in the last inequality. Moreover, since $\operatorname{div} v_\varepsilon = 0$ and using (2.5) and (2.6),

$$\begin{aligned}
\left|\varepsilon^{3-\alpha}\int_0^T\int_{\Omega_\varepsilon}(v_\varepsilon\cdot\nabla q^\varepsilon)\cdot u\,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\varepsilon^{3-\alpha}\int_0^T\int_{\Omega_\varepsilon} q^\varepsilon\cdot(v_\varepsilon\cdot\nabla u)\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\lesssim \varepsilon^{3-\alpha}\varepsilon^{\frac{\beta}{2}}\varepsilon^{\frac{\alpha+\beta-3}{2}}\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\\
&= \varepsilon^{\frac{3-\alpha}{2}}\varepsilon^{\beta}\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left|\varepsilon^{3-\alpha}\int_0^T\int_{\Omega_\varepsilon} v_\varepsilon\cdot\left(\nabla w^\varepsilon\nabla u + \varepsilon^{3-\alpha}w^\varepsilon\Delta u\right)\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad\lesssim \varepsilon^{\frac{3-\alpha}{2}}\varepsilon^{\beta}\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \varepsilon^{3-\alpha}\|v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\\
&\quad\lesssim \left(\varepsilon^{\frac{3-\alpha}{2}}\varepsilon^{\beta} + \varepsilon^{\frac{9-3\alpha}{2}}\right)\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.
\end{aligned}$$

Finally, by (2.20) and (2.2)

$$\left|\varepsilon^{3-\alpha}\int_0^T\int_{\Omega_\varepsilon}\nabla v_\varepsilon : \nabla\mathcal{B}_\varepsilon(u)\,\mathrm{d}x\,\mathrm{d}t\right|$$
$$\lesssim \varepsilon^{3-\alpha}\varepsilon^{\alpha-\frac{3-\beta}{2}}\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = \varepsilon^{\frac{3+\beta}{2}}\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.$$

Since $\alpha \geqslant \beta \geqslant 1$ and $\alpha < 3$, we observe that $\varepsilon^{\frac{3+\beta}{2}} \lesssim \varepsilon^{\frac{3-\alpha}{2}}\varepsilon^{\beta} \lesssim \varepsilon^{\frac{3-\beta}{2}}$ to conclude

$$|I_1| \leqslant C\left(\varepsilon^{\frac{3-\beta}{2}} + \varepsilon^{\frac{9-3\alpha}{2}}\right)\|\nabla v_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}$$
$$\leqslant \frac{1}{8}\varepsilon^{3-\alpha}\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\left(\varepsilon^{\alpha-\beta} + \varepsilon^{9-3\alpha}\right). \tag{3.16}$$

Step 4: Estimate of I_2: Using the identity (3.11) that still holds since $u_\varepsilon = \check{u}_\varepsilon = 0$ on $\partial\Omega_\varepsilon$ and $\mathrm{div} u = \mathrm{div} u_\varepsilon = 0$, we can decompose

$$I_2 = \varepsilon^{6-2\alpha-2\gamma}\int_0^T\int_{\Omega_\varepsilon}(v_\varepsilon\cdot\nabla\check{u}_\varepsilon)\cdot v_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$+\varepsilon^{6-2\alpha-2\gamma}\int_0^T\int_{\Omega_\varepsilon}\left((\check{u}_\varepsilon\cdot\nabla\check{u}_\varepsilon)\cdot v_\varepsilon + \partial_t\check{u}_\varepsilon\cdot v_\varepsilon\right)\mathrm{d}x\,\mathrm{d}t$$
$$=: I_2^1 + I_2^2$$

Combining the estimate (3.12) with the Poincaré inequality (2.21), we have

$$|I_2^1| \lesssim \varepsilon^{6-2\alpha-2\gamma}\left(\varepsilon^{3-\alpha} + \varepsilon^{\beta}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}.$$

Moreover, we estimate using again (2.21) as well as (2.6), (2.4) and (2.20) combined with (2.2)

$$|I_2^2| \leqslant C\varepsilon^{6-2\alpha-2\gamma}\int_0^T\Big(\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \||\nabla w^\varepsilon|^{\frac12}v_\varepsilon\|_{L^2(\Omega_\varepsilon)}\||\nabla w^\varepsilon|^{\frac12}\check{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$
$$+\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\mathcal{B}_\varepsilon(u)\|_{L^2(\Omega_\varepsilon)}\Big)\,\mathrm{d}t$$
$$\leqslant C\varepsilon^{6-2\alpha-2\gamma}\int_0^T\Big(\varepsilon^{\frac{3-\alpha}{2}}\|\nabla v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{\beta}\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\check{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$
$$+\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}\varepsilon^{\beta}\varepsilon^{\beta+\alpha-\frac32}\Big)\,\mathrm{d}t$$
$$\leqslant \frac{1}{8}\varepsilon^{3-\alpha}\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} + C\left(\varepsilon^{9-3\alpha-4\gamma}\varepsilon^{2\beta}\varepsilon^{\alpha-3} + \varepsilon^{12-4\alpha-4\gamma}\right).$$

Combining these estimates yields

$$|I_2| \leqslant \varepsilon^{3-\alpha}\left(C\varepsilon^{6-2\alpha-2\gamma} + C\varepsilon^{3-\alpha-2\gamma+\beta} + \frac{1}{8}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}$$
$$+C\left(\varepsilon^{6-2\alpha-4\gamma+2\beta} + \varepsilon^{12-4\alpha-4\gamma}\right). \tag{3.17}$$

Step 5: Conclusion: Inserting (3.16) and (3.17) into (3.15) yields

$$\begin{aligned}\varepsilon^{3-\alpha}\left(\frac{1}{4}-C\varepsilon^{6-2\alpha-2\gamma}-C\varepsilon^{3-\alpha-2\gamma+\beta}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}\\ \lesssim \varepsilon^{6-2\alpha-2\gamma}\|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)}+\|f_\varepsilon-f\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}\\ +\varepsilon^{6-2\alpha-4\gamma+2\beta}+\varepsilon^{\alpha-\beta}+\varepsilon^{9-3\alpha}+\varepsilon^{12-4\alpha-4\gamma}.\end{aligned}$$

We choose

$$\beta=\min\left\{1,\alpha-\frac{6-4\gamma}{3}\right\}.$$

Then, for all ε sufficiently small, using the assumptions $\gamma<3/2$ and $\alpha+\gamma<3$, the left-hand side is positive and, combination with the Poincaré inequality (2.21) yields

$$\begin{aligned}\|v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}\lesssim \varepsilon^{6-2\alpha-2\gamma}\|v_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)}+\|f_\varepsilon-f\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))}\\ +\varepsilon^{\frac{6-4\gamma}{3}}+\varepsilon^{\alpha-1}+\varepsilon^{9-3\alpha}+\varepsilon^{12-4\alpha-4\gamma}.\end{aligned}$$

Applying (3.2) and observing that this only produces a higher order error since $2\alpha+\beta-3\geqslant\alpha-\beta$ thanks to $\alpha\geqslant\beta\geqslant1$, we find

$$\begin{aligned}\|u_\varepsilon-u\|^2_{L^2((0,T)\times\Omega_\varepsilon)}\lesssim \varepsilon^{6-2\alpha-2\gamma}\|u^0_\varepsilon-u_0\|^2_{L^2(\Omega_\varepsilon)}+\|f_\varepsilon-f\|^2_{L^2((0,T)\times L^2(\Omega_\varepsilon)}\\ +\varepsilon^{\frac{6-4\gamma}{3}}+\varepsilon^{\alpha-1}+\varepsilon^{9-3\alpha}+\varepsilon^{12-4\alpha-4\gamma}.\end{aligned}$$

This concludes the proof. □

Proof of theorem 1.5. For simplicity of the notation, we write u_ε instead of $\tilde{u}_\varepsilon$ for the extension of u_ε by 0 to $\mathbb{R}^3$. Note that the energy inequality (1.11) does not immediately provide uniform *a priori* estimates for u_ε. The first step of the proof therefore consists in combining the energy inequality with the Poincaré inequality from proposition 2.4 to deduce a uniform *a priori* bound for u_ε in $L^2(0,T;L^2(\mathbb{R}^3)$. Then, $u_\varepsilon\rightharpoonup u$ for some $u\in L^2(0,T;L^2(\mathbb{R}^3))$ along subsequences and it suffices to show that u solves (1.10).

Step 1: Uniform a priori estimate We claim that,

$$\|u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3)}+\varepsilon^{\frac{3-\alpha}{2}}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3)}\lesssim\varepsilon^{3-\alpha}\varepsilon^{-\gamma}\|u^\varepsilon_0\|_{L^2(\mathbb{R}^3)}+\|f_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3)}\lesssim 1. \tag{3.18}$$

By the energy inequality (1.11) and the Poincaré inequality (2.21) we have

$$\begin{aligned}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon^{2\gamma}\varepsilon^{\alpha-3}\|\nabla u_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_\varepsilon))}\\ \lesssim\|u^\varepsilon_0\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon^{2\gamma}\varepsilon^{\frac{3\alpha-9}{2}}\|f_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3)}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3)}.\end{aligned}$$

Applying Young's inequality, this establishes the estimate for ∇u_ε, and the estimate for u_ε follows by another application of the Poincaré inequality (2.21).

Step 2: Testing with $w^\varepsilon\varphi-\mathcal{B}_\varepsilon(\varphi)$: Let $\varphi\in C^\infty_c((0,T)\times\mathbb{R}^3)$ with $\operatorname{div}\varphi=0$. Then, we test the equation (1.9) of u_ε with

$$\varphi_\varepsilon:=w^\varepsilon\varphi-\mathcal{B}_\varepsilon(\varphi),$$

where w^ε is as in section 2 and depends on a parameter η_ε which we take as $\eta_\varepsilon = \varepsilon^\beta$ for some $1 \leqslant \beta < \alpha$ to be chosen late. This yields

$$\varepsilon^{3-\alpha} \int_0^T \int_{\mathbb{R}^3} \nabla u_\varepsilon : \nabla \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} f_\varepsilon \cdot \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + \varepsilon^{6-2\alpha-2\gamma} \int_0^T \int_{\mathbb{R}^3} (u_\varepsilon \cdot \partial_t \varphi_\varepsilon + u_\varepsilon \cdot (u_\varepsilon \cdot \nabla \varphi_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t.$$

It remains to show

$$\begin{aligned} I_1 &:= \int_0^T \int_{\mathbb{R}^3} f_\varepsilon \cdot \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_{\mathbb{R}^3} f \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t, \\ I_2 &:= \varepsilon^{6-2\alpha-2\gamma} \int_0^T \int_{\mathbb{R}^3} (u_\varepsilon \cdot \partial_t \varphi_\varepsilon + u_\varepsilon \cdot (u_\varepsilon \cdot \nabla \varphi_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t \to 0, \\ I_3 &:= \varepsilon^{3-\alpha} \int_0^T \int_{\mathbb{R}^3} \nabla u_\varepsilon : \nabla \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \to \int \mathcal{R} u \cdot \varphi. \end{aligned}$$

Step 2: Convergence of I_1: Recalling the assumption that $f_\varepsilon \rightharpoonup f$ in $L^2(0,T;L^2(\mathbb{R}^3)))$ and that $w_\varepsilon \to \mathrm{Id}$ strongly in $L^2(\operatorname{supp}\varphi)$ by lemma 2.1 (ii), we have

$$\int_0^T \int_{\mathbb{R}^3} f_\varepsilon \cdot (w^\varepsilon \varphi) \, \mathrm{d}x \, \mathrm{d}s \to \int_0^T \int_{\mathbb{R}^3} f \cdot \varphi \, \mathrm{d}x \, \mathrm{d}s.$$

Moreover, by (2.2) and (2.20)

$$\left| \int_0^T \int_{\mathbb{R}^3} f_\varepsilon \mathcal{B}_\varepsilon(\varphi) \, \mathrm{d}x \, \mathrm{d}t \right| \lesssim \varepsilon^\beta \varepsilon^{\alpha - \frac{3-\beta}{2}} = \varepsilon^\alpha \varepsilon^{3\frac{\beta-1}{2}} \to 0$$

as $\beta \geqslant 1$.

Step 3: Convergence of I_2:

We have by the regularity of u, using (2.6), the *a priori* estimate (3.18) and the estimates (2.20), (2.2) and (2.3)

$$\begin{aligned} |I_2| &\lesssim \varepsilon^{6-2\alpha-2\gamma} \int_0^t (\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} (\|\partial_t \varphi\|_{L^2(\Omega_\varepsilon)} + \|\mathcal{B}_\varepsilon(\partial_t \varphi)\|_{L^2(\Omega_\varepsilon)}) \\ &\quad + \|u_\varepsilon\|_{L^6(\Omega_\varepsilon)} \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\mathcal{B}_\varepsilon(\varphi)\|_{L^3(\Omega_\varepsilon)} \\ &\quad + \|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \|\nabla \varphi\|_{L^\infty(\Omega_\varepsilon)} + \||u_\varepsilon|^2 \nabla w^\varepsilon\|_{L^1(\Omega_\varepsilon)} \|\varphi\|_{L^\infty(\Omega_\varepsilon)}) \, \mathrm{d}x \, \mathrm{d}s \\ &\lesssim \varepsilon^{6-2\alpha-2\gamma} (1 + \varepsilon^\beta \varepsilon^{\alpha + \frac{\beta-3}{2}} + \varepsilon^{\alpha-3} \varepsilon^\beta (1 + \varepsilon^{\alpha-1} |\log \varepsilon|^{\frac{1}{3}})) \\ &\lesssim \varepsilon^{6-2\alpha-2\gamma} + \varepsilon^{3+\beta-\alpha-2\gamma}. \end{aligned}$$

Thanks to the assumption $\alpha > 1$, $\gamma < 3 - \alpha$ and $\gamma < 3/2$, we may choose $\beta \geqslant 1$ such that $\beta \in (\alpha + 2\gamma - 3, \alpha$, which implies $I_2 \to 0$ as $\varepsilon \to 0$.

Step 4: Convergence of I_3: With M_ε as in lemma 2.2, we rewrite

$$\begin{aligned}\varepsilon^{3-\alpha}\int_0^T\int_{\mathbb{R}^3}\nabla u_\varepsilon:\nabla\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\langle\varphi M_\varepsilon,u_\varepsilon\rangle\,\mathrm{d}t+\varepsilon^{3-\alpha}\int(u_\varepsilon\cdot\nabla q^\varepsilon)\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\varepsilon^{3-\alpha}\int_0^T\int_{\mathbb{R}^3}\nabla u_\varepsilon:\nabla\mathcal{B}_\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\varepsilon^{3-\alpha}\int_0^T\int_{\mathbb{R}^3}u_\varepsilon\cdot(2\nabla w^\varepsilon\nabla\varphi_\varepsilon+w^\varepsilon\Delta\varphi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\ &=: I_3^1+I_3^2+I_3^3+I_3^4.\end{aligned}$$

By lemma 2.2 and (3.18), we have

$$\begin{aligned}\left|I_3^1-\int_0^T\int_{\mathbb{R}^3}\mathcal{R}u\cdot\varphi\right| &\lesssim\left(\varepsilon^{\alpha-\beta}\|u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3))}+\varepsilon^{\frac{3-\beta}{2}}\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\mathbb{R}^3))}\right)\|\varphi\|_{L^2(0,T;H^2(\mathbb{R}^3))}\\ &\lesssim\varepsilon^{\alpha-\beta}+\varepsilon^{\frac{\alpha-3}{2}}\varepsilon^{\frac{3-\beta}{2}}\to 0\end{aligned}$$

since $\beta<\alpha$. Moreover, we estimate using (2.4)

$$|I_3^2|=\left|\varepsilon^{3-\alpha}\int_0^T\int_{\mathbb{R}^3}(q^\varepsilon\cdot\nabla\varphi)\cdot u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\right|\lesssim\varepsilon^{\frac{3-\alpha}{2}}\to 0.$$

Furthermore, by lemma (3.18) and (2.20) and (2.2)

$$|I_3^3|\lesssim\varepsilon^{\frac{3-\alpha}{2}}\varepsilon^{\alpha-\frac{3-\beta}{2}}=\varepsilon^{\frac{\alpha+\beta}{2}}\to 0.$$

Finally, by (2.4) and (2.1)

$$|I_3^4|\lesssim\varepsilon^{\frac{3-\alpha}{2}}\to 0.$$

Therefore, the desired convergence of I_3 is established and this finishes the proof. □

Data availability statement

No new data were created or analysed in this study.

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ORCID iD

Richard M Höfer https://orcid.org/0000-0002-8907-3219

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