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The Inversion of Sampling Solved Algebraically

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Abstract

We show that Shannon's reconstruction formula can be written as $\boldsymbol{a} * (\boldsymbol{b} \cdot \boldsymbol{c}) = \boldsymbol{c} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c}$ with tempered distributions $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{S}'(\mathbb{R}^n)$ where $*$ is convolution, $\cdot$ is multiplication, $\boldsymbol{c}$ is the function being sampled and restored after sampling, $\boldsymbol{b}\cdot$ is sampling and $\boldsymbol{a}*$ its inverse. The requirement $\boldsymbol{a} * \boldsymbol{b} = 1$ which describes a smooth *partition of unity* where $\boldsymbol{b} = \text{III}$ is the *Dirac comb* implies that $\boldsymbol{a}$ is satisfied by *unitary functions* introduced by Lighthill (1958). They form convolution inverses of the Dirac comb. Choosing $\boldsymbol{a} = sinc$ yields Shannon's reconstruction formula where the requirement $\boldsymbol{a} * \boldsymbol{b} = 1$ is met *approximately* and cannot be exact because $sinc$ is not integrable. In contrast, unitary functions satisfy this requirement exactly and stand for the set of functions which solve the problem of inverse sampling algebraically.

Keywords: generalized sampling operators, sampling, interpolation, Dirac comb, Lighthill unitary functions, smooth partitions of unity

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1 Introduction

Let $t \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$, $1 \leq n < \infty$ and $1/W = (1/W_1, \ldots, 1/W_n) \in \mathbb{R}^n$ be such that $W_j > 0$ for $1 \leq j \leq n$. The operation of mapping continuous variables $t \mapsto k/W$, functions $f(t) \mapsto f(k/W)$ or operations $\mathcal{T}\{f(t)\} \mapsto \mathcal{T}\{f(k/W)\}$ on functions to *sequences* of real or complex numbers is commonly known as *discretization* or *sampling* and it raises the question of whether such sequences represent their original variables, functions or operations one-to-one, i.e., whether continuous entities can be reconstructed from samples, either exactly or approximately. In order to reconstruct continuous functions one usually considers (Tamberg [1] and Stens [2], p.130) the generalized sampling series

$$(S_W^{\varphi} f)(t) \;:=\; \sum_{k \in \mathbb{Z}^n} f\left(\frac{k}{W}\right) \varphi(Wt - k) \tag{1}$$

where f must be W-band-limited (or W-band-localized) in some sense and φ must satisfy the condition

$$\sum_{k \in \mathbb{Z}^n} |\varphi(u - k)| < \infty \qquad u \in \mathbb{R}^n \tag{2}$$

which guarantees that the operator $S_W^{\varphi} : f \mapsto S_W^{\varphi} f$ is well-defined. Here, the absolute convergence is understood being uniform on compact intervals of $\mathbb{R}^n$. Obviously, $\varphi(t) = sinc(t)$ yields the classical (Whittaker-Kotel'nikov-)Shannon operator, where $sinc(0) := 1$ and $sinc(t) := sin(\pi t)/(\pi t)$ for $t \neq 0$, for which it is known that (2) is not satisfied because $sinc(t) \notin L^1(\mathbb{R}^n)$, e.g. Butzer and Nessel [3], p.190. For this reason, many suggestions have already been made to replace $sinc$ functions in (1) with the aim of achieving a perfect reconstruction of the sampled function $f(t)$. For example, the idea of replacing $sinc(t)$ by $sinc^2(t)$ can already be found in Theis (1919) [4] and further suggestions can be found in a series of systematic studies on the reversibility of sampling (1) by P. L. Butzer, R. L. Stens and their school since the 1970ies [1].

In this study, we solve the problem of inverse sampling algebraically. This is a completely new approach, it includes (i) the Whittaker-Kotel'nikov-Shannon sampling theorem (using $sinc$-functions), (ii) Campbell's generalized sampling theorem, (using $sinc$-functions and an additional convergence factor) [5] as well as (iii) Theis' approach (using $sinc$-functions and another $sinc$ as convergence factor) as special cases. To see this, we simplify (1) and denote it as

$$(S_W^{\varphi} f)(t) \;=\; \varphi_{\frac{1}{W}}(t) * \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \delta(Wt - k) \;=\; \varphi_{\frac{1}{W}}(t) * \text{III}_{\frac{1}{W}}(t) \cdot f(t) \;=\; \boldsymbol{a}(t) * \boldsymbol{b}(t) \cdot \boldsymbol{c}(t) \tag{3}$$

the sequence of operations $\boldsymbol{a}*$ and $\boldsymbol{b}\cdot$ applied to $\boldsymbol{c}$. First sampling $\boldsymbol{b}\cdot$ is applied to $\boldsymbol{c}$ and then $\boldsymbol{a}*$ is applied to $\boldsymbol{b}\cdot\boldsymbol{c}$ which neutralizes sampling. Here, δ denotes the Dirac delta, $\text{III}_{\frac{1}{W}}$ is the Dirac comb (deltas placed at integral multiples of $\frac{1}{W}$) and $\varphi_{\frac{1}{W}}(t) := \varphi(Wt)$

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is the respective interpolation kernel adapted to the sampling rate $\frac{1}{W}$. Ideally, this kernel is a *rapidly decreasing* function (in $\mathcal{S}$, the Schwartz space) in order to force (1) to converge. In our solution, "unitary functions" $\varphi \in \mathcal{S}$ will be (i) interpolation kernel and (ii) convergence factor (forcing convergence), simultaneously. The property $\varphi \in \mathcal{S}$ replaces thereby condition (2). The *sinc* function is known to decrease in the order of $1/|x|$ which is too slow to guarantee the series convergence [1], i.e., $sinc \notin \mathcal{S}$. One may recall, δ and III are no functions but operators on functions (distributions). For this reason, we choose generalized functions theory (distribution theory) as our setting. We have a look at operator domains and images in distribution theory which is well suited for the purposes of treating operations on functions. The notation $\boldsymbol{b}(t) \equiv \text{III}_{\frac{1}{W}}(t)$ in (3) is meant symbolically. It does not mean that values $\boldsymbol{b}(t) \in \mathbb{C}$ exist for every $t \in \mathbb{R}^n$. In contrast, $\boldsymbol{a}(t)$ and $\boldsymbol{c}(t)$ are ordinary (infinitely differentiable) functions, their values exist for all $t \in \mathbb{R}^n$. We briefly introduce to distribution theory, have a look at the preliminaries in Section 2 and present our main result in Section 3.

2 Preliminaries

A disadvantage in conventional function theory is the fact that the function that is constantly 1, which is neutral with respect to multiplication, and the Dirac δ, which is neutral with respect to convolution, are no integrable functions and hence their Fourier transforms do not exist in the usual sense. The latter is not even a function in the conventional sense—but an operator applied to functions. Distribution theory [6, 7, 8], the theory of generalized functions [9, 10], overcomes these difficulties. Here, functions represent operators but operators are not necessarily represented by functions. In this theory, every (ordinary or generalized) function has a derivative, anti-derivative and a Fourier transform. In particular, 1 and δ map onto each other via the Fourier transform (Figure 1). Distributions (generalized functions) belong to spaces which may not even be normable [11] (Banach spaces) nor metrizible [12]. The merit of Laurent Schwartz (Lützen, [13], p.149) is that the theory of duality between Banach spaces has been extended to a theory of duality on Fréchet spaces (or their inductive limits) and their duals, known as (DF)-spaces [12]. This approach overcomes the difficulty that δ cannot be distinguished (e.g. Rudin [11] p.33) from the *zero function* in ordinary function theory (Lebesgue spaces) where one identifies functions with one another if they coincide (a.e.) almost everywhere (e.g. [3], p.169). In distribution theory, the Fourier transform $\mathcal{F}$ is a structure-preserving isometry between topological vector spaces, not necessarily a mapping between Banach spaces. The value $\langle f, \varphi \rangle \in \mathbb{C}$, called the application of f to φ, does always exist due to the fact that distributions $f \in \mathcal{D}'$ belong to the dual space (continuous linear functionals) of $\mathcal{D}$, the space of compactly supported infinitely differentiable functions $\varphi \in \mathcal{C}^\infty$. The Dirac delta δ is defined to be the operation $\langle \delta, \varphi \rangle := \varphi(0)$ and for ordinary functions one defines $\langle f, \varphi \rangle := \int_{\mathbb{R}^n} f(t)\varphi(t)\,dt^n$ to be the application of f to φ. We may think of $\mathcal{D}'$ as the *transposed space* of $\mathcal{D}$. This older term for the *dual space* (coined by Bourbaki) is due to Schauder (Narici and Beckenstein [14] , p.228). In this terminology, $\delta \in \mathcal{E}'(\mathbb{R}^n)$ is a column and $1 \in \mathcal{E}(\mathbb{R}^n)$ a row vector. Applied to one another they yield $\langle \delta, 1 \rangle = 1$, unity. Here, $\mathcal{E} \equiv \mathcal{C}^\infty$ is Schwartz' notation for the space of infinitely differentiable functions and $\mathcal{E}'$, its dual, is the space of compactly supported (tempered) distributions. Note that $\langle f, 1 \rangle = \int_{\mathbb{R}^n} f(t)\,dt^n$ such that $\langle \delta, 1 \rangle$, the "integral" of δ, is unity. In general, $f \in \mathcal{D}'$ applied to 1 determines its generalized integral if it exists. For that, 1 must belong to the dual space of the operator f. Occasionally, $\langle \cdot, \cdot \rangle$ is understood as generalized *inner product* (e.g. Vladimirov [15], 2002, p.114). In distribution theory, $\mathcal{D}'$ is the largest and $\mathcal{E}'$ the smallest space of distributions (with respect to growth conditions at infinity) and, vice versa, their duals $\mathcal{D}$ and $\mathcal{E}$ are the smallest and largest space of (ordinary, infinitely differentiable) functions (with respect to growth conditions at infinity), respectively. All ordinary functions which are not infinitely differentiable in the ordinary functions sense, i.e., all functions which are not themselves "test functions", are considered being operators on functions (distributions) instead of being functions. Such operators become themselves infinitely differentiable (in the generalized functions sense) because differentiation can always be rolled off to (infinitely differentiable) functions. One defines $\langle f', \varphi \rangle := -\langle f, \varphi' \rangle$ because this holds for ordinary f applied to test functions φ. The space $\mathcal{S}'$ of tempered distributions is $\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$ between $\mathcal{E}'$ and $\mathcal{D}'$, it implies $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$. In this study, we stay (for convenience) in $\mathcal{S}'$, the space of tempered distributions. This can be done for three reasons: (i) the Dirac comb $\text{III} \in \mathcal{S}'$ is a tempered distribution (itself and all its derivatives are bounded by a constant function), (ii) we consider sampling on functions which are at most of polynomial growth at infinity (hence, we exclude exponential growth at infinity) and (iii) the Fourier transform is an automorphism in $\mathcal{S}'$, i.e., $\mathcal{F}(\mathcal{S}') = \mathcal{S}'$, which is convenient. The space $\mathcal{S}'$ together with its (most important) subspaces is depicted in Figure 1 for the reader's convenience.

2.1 Schwartz Space and Tempered Distributions

In an ideal theory, one wishes that (i) any function can be multiplied by any function, (ii) any function can be convolved with any function and (iii) any function has a derivative, anti-derivative and a Fourier transform. Such an ideal setting is the Schwartz space $\mathcal{S}$, defined by Laurent Schwartz [16]. It is an algebra with respect to multiplication and an algebra with respect to convolution (e.g. Larcher [17], p.8), given by the complete metrizable locally convex topological vector space (Fréchet space)

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \sup_{x \in \mathbb{R}^n} |x^\beta D^\alpha \varphi(x)| < \infty\}$$

of infinitely differentiable functions (allowing arbitrary multiplication) which decrease rapidly at ± infinity (allowing arbitrary convolution) together with all their derivatives (allowing arbitrary differentiation and integration). Here, $\alpha, \beta \in \mathbb{N}^n$ are multi-indices, $\mathbb{N}$ the natural numbers including zero, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ the partial differential operator, $D_k := \partial/\partial x_k$ and $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$. The symbol $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the order of the partial differential operator. An order-zero partial differential operator coincides with the identity operator. We say a sequence $(\varphi)_{k\in\mathbb{N}}$ of elements in $\mathcal{S}$ tends to zero as k tends to infinity if

$$\lim_{k\to 0} \sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha \varphi_k(x)| = 0$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ and write $\varphi_k \to 0$ in $\mathcal{S}$, briefly. More generally, $\varphi_k \to \varphi \in \mathcal{S}$ if and only if $(\varphi_k - \varphi) \to 0 \in \mathcal{S}$, see e.g, Gasquet & Witomski [18], 1999, p.173. A generalization of the double-algebra structure of $\mathcal{S}$ can be found in its dual space (continuous linear functionals), the space of *tempered distributions*. Here, $f(\varphi) \equiv \langle f, \varphi\rangle \in \mathbb{C}$ denotes the application of f to φ which is a complex number. For any space $X' \subset \mathcal{D}'$ of distributions, one says that $(f)_k$ converges to $f \in X'$ if $\langle f_k, \varphi\rangle \to \langle f, \varphi\rangle$ converges in $\mathbb{C}$ for all $\varphi \in X$.

2.2 Spaces of Multipliers and Convolutors

In contrast to $\mathcal{S}$ which has a double-algebra structure, $\mathcal{S}'$ has a double-module structure. It is a module with respect to its subspace $\mathcal{O}_\mathcal{M}$, an associative commutative algebra with respect to multiplication, and it is a module with respect to its subspace $\mathcal{O}'_C$, an associative commutative algebra with respect to convolution (e.g. Petersen [19], 1983, p.91). The two "kernel spaces" in $\mathcal{S}'$, see Figure 1, are best described (cf. Dijk [20], p.87, Bargetz & Ortner [21]) by

$$\mathcal{O}_\mathcal{M} := \{\alpha \in \mathcal{S}' \mid \alpha \cdot g \in \mathcal{S}' \ \forall g \in \mathcal{S}'\} \quad \text{and} \quad \mathcal{O}'_C := \{f \in \mathcal{S}' \mid f * g \in \mathcal{S}' \ \forall g \in \mathcal{S}'\}$$

following their module property in $\mathcal{S}'$. Originally, $\mathcal{O}_\mathcal{M}$ and $\mathcal{O}'_C$ were defined by Schwartz [6, 7] as the space of infinitely differentiable functions which do not grow faster than polynomials at infinity, together with all their derivatives, and the space of rapidly decreasing tempered distributions, respectively. $\mathcal{O}$ stands for operators, $\mathcal{M}$ for multiplication, c for convolution, the prime in $\mathcal{O}'_C$ reminds to the fact that it is a space of generalized functions (distributions), i.e., operators which are not necessarily represented by functions and, in contrast to that, no prime in $\mathcal{O}_\mathcal{M}$ means it is a space of (infinitely differentiable) ordinary functions. The Fourier transform is a one-to-one mapping $\mathcal{F}(\mathcal{O}'_C) = \mathcal{O}_\mathcal{M}$ and $\mathcal{F}(\mathcal{O}_\mathcal{M}) = \mathcal{O}'_C$ between multipliers $\mathcal{O}_\mathcal{M}$ and convolutors $\mathcal{O}'_C$, e.g. Petersen [19]. The double-module structure means $\alpha \cdot g \in \mathcal{S}'$ if $\alpha \in \mathcal{O}_\mathcal{M}$ and $f * g \in \mathcal{S}'$ if $f \in \mathcal{O}'_C$ for arbitrary $g \in \mathcal{S}'$. Both products $\alpha \cdot g = g \cdot \alpha$ and $f * g = g * f$ are commutative but not necessarily associative

$$(\alpha \cdot g) \cdot \beta \neq \alpha \cdot (g \cdot \beta) \tag{4}$$
$$(f * g) * h \neq f * (g * h) \tag{5}$$

in a sequence of products where $\alpha, \beta \in \mathcal{O}_\mathcal{M}$ and $f, h \in \mathcal{O}'_C$ if $g \in \mathcal{S}'$ is arbitrary. Counterexamples are commonly known (e.g. Petersen [19], p.54). The products (4) and (5) exist if at most one element is arbitrary ($g \in \mathcal{S}'$). Furthermore, (4) and (5) are associative if $g \in \mathcal{O}_\mathcal{M}$ and $\mathcal{O}'_C$ respectively, because $\mathcal{O}_\mathcal{M}$ and $\mathcal{O}'_C$ are commutative associative algebras, mapped one-to-one onto each other via the *exchange theorem* (e.g. Petersen [19], p.93, Horváth [22], p.424, Edwards [23], p.388). It states that

$$\mathcal{F}(\alpha \cdot g) = \mathcal{F}\alpha * \mathcal{F}g \tag{6}$$
$$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g \tag{7}$$

for $\alpha \in \mathcal{O}_\mathcal{M}$, $f \in \mathcal{O}'_C$ and arbitrary $g \in \mathcal{S}'$. In particular, $1 \in \mathcal{O}_\mathcal{M}$ and the Dirac $\delta \in \mathcal{O}'_C$, the neutral elements with respect to multiplication and convolution, are mapped $\mathcal{F}1 = \delta$ and $\mathcal{F}\delta = 1$ one-to-one onto each other via the Fourier transform $\mathcal{F}$ given by $(\mathcal{F}\varphi)(y) := \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i x \cdot y}\, dx^n$ for $\varphi \in \mathcal{S}$ and $\langle \mathcal{F}f, \varphi\rangle := \langle f, \mathcal{F}\varphi\rangle$ for tempered distributions $f \in \mathcal{S}'$. It reduces to $\langle \hat{f}, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\, dx^n$ if f is an ordinary function. We write $\hat{f} \equiv \mathcal{F}f$ and $\hat{\varphi} \equiv \mathcal{F}\varphi$, briefly.

2.3 Algebraic Structure of the Space of Tempered Distributions

The terms "time" and "frequency" may be misleading, they actually stand for any variable $x = y^{-1} \in \mathbb{R}^n$ and its reciprocal $y = x^{-1} \in \mathbb{R}^n$ which are connected to one another via the Fourier transform in $x \cdot y = x_1 y_1 + \cdots + x_n y_n = 1$. So, in order to use a more neutral terminology we will use the terms "original domain" for $f \in \mathcal{S}'(\mathbb{R}^n)$ on real axes $t \in \mathbb{R}^n$ and "reciprocal domain" for $\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$ on reverse real axes t^{-1}, i.e., their $\hat{f} \equiv \mathcal{F}f$ Fourier transforms. The "reciprocal space" (e.g. Koster [24], eq.(13)) can be traced back at least to Born, e.g. [25], p.69. Note that the Fourier transform (here) is defined such that $\mathcal{F}1 = \delta$ and $\mathcal{F}\delta = 1$, i.e., the neutral elements with respect to multiplication and convolution map one-to-one onto each other (Figure 1). One may recall, this definition of the Fourier transform is the only one among a variety of different Fourier transform definitions to turn $\mathcal{F}$ into both an isometry and an algebra homomorphism (Folland [26], p.5), simultaneously, depicted *red* versus *blue* in Figure 1.

The details used to create Figure 1 go back to (i) Schwartz' diagram of continuous space embeddings [16], p.420 and (ii) commonly known one-to-one mappings [27] between subspaces $\mathcal{F}(\mathcal{S}) = \mathcal{S}$, $\mathcal{F}(\mathcal{D}) = \mathcal{Z}$, $\mathcal{F}(\mathcal{E}') = PW$, $\mathcal{F}(\mathcal{O}'_C) = \mathcal{O}_\mathcal{M}$, $\mathcal{F}(\mathcal{S}') = \mathcal{S}'$ within the space of tempered distributions $\mathcal{S}'$ via the Fourier transform $\mathcal{F}$ where (iii) $\mathcal{F}(\mathcal{O}'_C) = \mathcal{O}_\mathcal{M}$ is an algebra homomorphism, $\mathcal{F}(\mathcal{E}') = PW$ a subalgebra homomorphism, $\mathcal{F}(\mathcal{D}) = \mathcal{Z}$ another subalgebra homomorphism and $\mathcal{F}(\mathcal{S}) = \mathcal{S}$ is a double-algebra automorphism (indicated in *violet* color). Furthermore, $\{\Omega, \hat{\Omega}\}$ is a pole of *smoothness*, $\{\delta, 1\}$ and $\{1, \delta\}$ are

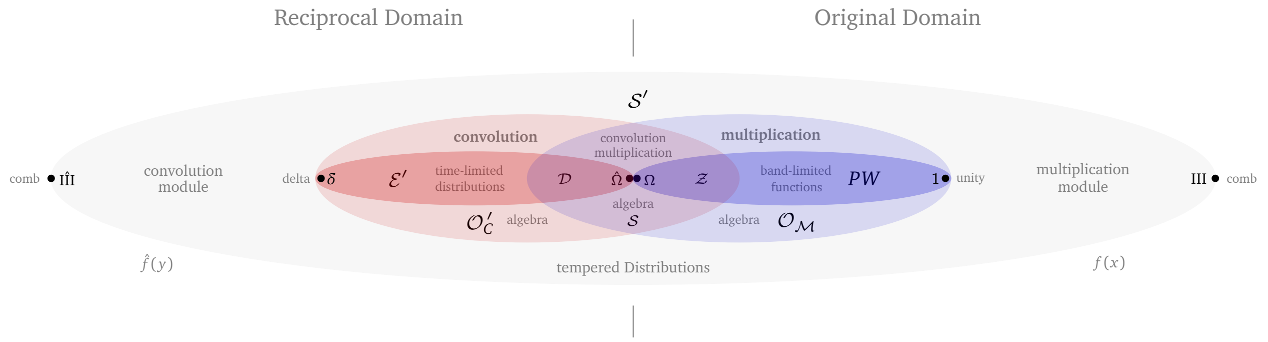

Figure 1: Algebraic structure of tempered distributions.

poles of *semi*-discreteness (time or band-limitness) and $\{\text{III}, \hat{\text{III}}\}$ is a pole of *discreteness* in $\mathcal{S}'$, [28]. However, all four $\{\Omega, \delta, 1, \text{III}\}$ represent *unity*. Note that $\text{III} \equiv \hat{\text{III}}$ is its own Fourier transform and, in contrast to that, there is no (non-trivial) unitary function Ω such that $\Omega \equiv \hat{\Omega}$ [28] due to the fact that being entire (holomorphic everywhere in the complex plane) and being finite (compactly supported) are mutually exclusive function properties $\mathcal{E}' \cap PW = \{0\} = \mathcal{D} \cap \mathcal{Z}$, e.g. Zemanian [27], p.197. The Paley-Wiener space (band-limited functions) is given by the Fourier transform $PW := \mathcal{F}(\mathcal{E}')$ of compactly supported (tempered) distributions, e.g. Berenstein & Gay [29], p.82.

2.4 Translation, Discretization, Periodization, Finitization and Entirization

The translation operator $\tau_a : \mathcal{S}' \to \mathcal{S}', f \mapsto \tau_a f$ is defined as $\tau_a f(t) := f(t-a)$ for ordinary functions, $a \in \mathbb{R}^n$, and for generalized functions one defines $\langle \tau_a f, \varphi \rangle := \langle f, \tau_{-a}\varphi \rangle$ where $f \in X' \subset \mathcal{D}'$ is a distribution and $\varphi \in X \supset \mathcal{D}$ its test functions. For translations of the Dirac delta one briefly writes $\delta_a := \tau_a \delta$ or $\delta(t-a)$ although it is no ordinary function. A property that we need below is the following, e.g. Petersen [19], p.90.

Lemma 2.1. *Let $a \in \mathbb{R}^n$, $f \in \mathcal{O}_C'$ and $g \in \mathcal{S}'$ then $\tau_a(f * g) = (\tau_a f) * g = f * (\tau_a g)$.*

These products exist because $f \in \mathcal{O}_C'$ is a rapidly decreasing (summable) distribution (e.g. Dijk [20], p.87). In particular, because $\delta \in \mathcal{E}' \subset \mathcal{O}_C'$ is compactly supported and $\varphi \in \mathcal{S}'$ may be arbitrary we obtain $\varphi(t-a) = \delta * \varphi(t-a) = \varphi(t) * \delta(t-a)$. More generally, using the linearity of the translation operator

$$\sum_{k \in \mathbb{Z}^n} f(k)\varphi(t-k) \;=\; \varphi(t) * \sum_{k \in \mathbb{Z}^n} f(k)\delta(t-k) \tag{8}$$

which will be needed below. For $f \in \mathcal{S}'$, one may define $f_a := \tau_a f$ such that the following rule holds.

Remark 1 (Translation). Obviously, translation may be defined as the *convolution-type* operator $\tau_a f := \delta_a * f = f_a$.

Remark 2 (Differentiation). Similarly, differentiation may be defined as the *convolution-type* operator $(\frac{d}{dt})^k f := \delta^{(k)} * f = f^{(k)}$.

The idea to interpret $f \in \mathcal{S}'$ as (convolution-type or multiplication-type) operator will come across us many more times. For example, it may be used to define sampling (discretization) or periodic replication (periodization) applied to a function [30]. Useful is the definition of a Dirac comb $\text{III}_\text{W} := \sum_{k \in \mathbb{Z}^n} \delta_{kW}$ which is a tempered distribution (e.g. Zemanian [27], p.106). We write III for $W = [1, \ldots, 1] \in \mathbb{R}^n$ in short. Now, sampling is the following multiplication-type operator [30]. It connects the Fourier transform, Fourier series and the Discrete Fourier Transform (DFT) to one another [31], in the common setting of $\mathcal{S}'$. This definition corresponds (in $\mathcal{S}'$) to what is usually understood (e.g. Butzer et al. [32], eq.(3.7), p.11) as a sampled function.

Definition 2.1 (Discretization). For functions $f \in \mathcal{O}_\mathcal{M}$ one defines an operation $\text{Ш} : \mathcal{O}_\mathcal{M} \to \mathcal{S}', f \mapsto \text{Ш} f$ given by

$$\text{Ш}_{\frac{1}{W}} f \;:=\; \text{III}_{\frac{1}{W}} \cdot f \;=\; \sum_{k \in \mathbb{Z}^n} f\left(\frac{k}{W}\right) \delta\left(t - \frac{k}{W}\right) \;:=\; \sum_{k \in \mathbb{Z}^n} f(k/W)\, \tau_{k/W}\delta$$

called *sampling* or *discretization*. The result $\text{Ш}_{\frac{1}{W}} f \in \mathcal{S}'$ is a discrete function (distribution), sampled at multiples of $\frac{1}{W}$.

Definition 2.2 (Periodization). For distributions $g \in \mathcal{O}_C'$ one defines an operation $\Delta\!\!\Delta\!\!\Delta : \mathcal{O}_C' \to \mathcal{S}', g \mapsto \Delta\!\!\Delta\!\!\Delta g$ given by

$$\Delta\!\!\Delta\!\!\Delta_W g \;:=\; \text{III}_\text{W} * g \;=\; \sum_{k \in \mathbb{Z}^n} g(t - kW) \;:=\; \sum_{k \in \mathbb{Z}^n} \tau_{kW}\, g$$

called *periodic continuation* or *periodization*. The result $\Delta\!\!\Delta\!\!\Delta_W f \in \mathcal{S}'$ is a periodic function or distribution of period W.

These definitions are well-defined because the products $\text{III}_{\frac{1}{W}} \cdot f$ and $\text{III}_\text{W} * g$ exist for $f \in \mathcal{O}_\mathcal{M}$ and $g \in \mathcal{O}_C'$ according to the exchange theorem, $\mathcal{F}(\mathcal{O}_\mathcal{M}) = \mathcal{O}_C'$ and $\mathcal{F}(\mathcal{O}_C') = \mathcal{O}_\mathcal{M}$, in $\mathcal{S}'$ and they are Fourier transforms of one another in the sense that

$$\mathcal{F}(\text{Ш}_{\frac{1}{W}} f) = \mathcal{F}(\text{III}_{\frac{1}{W}} \cdot f) = W\, \text{III}_\text{W} * g = W\, \Delta\!\!\Delta\!\!\Delta_W g \tag{9}$$

$$\mathcal{F}(\Delta\!\!\Delta\!\!\Delta_W g) = \mathcal{F}(\text{III}_\text{W} * g) = \tfrac{1}{W}\, \text{III}_{\frac{1}{W}} \cdot f = \tfrac{1}{W}\, \text{Ш}_{\frac{1}{W}} f \tag{10}$$

where the factors in front, W and $1/W$, are volume balancing. $\mathcal{F}(\text{III}_{\frac{1}{W}}) = W\,\text{III}_{\text{W}}$ and $\mathcal{F}f = g$. In particular, $\mathcal{F}(\text{III}) = \text{III}$ is the Fourier transform of itself if $W \equiv 1$. In other words, (9) and (10) state that *discrete functions* and *periodic functions* are Fourier transforms of one another. Now, let $W > 0$ be some non-zero and (without restriction of generality) positive real number and $\Omega \in \mathcal{D}$, hence, an ordinary (infinitely differentiable) compactly supported function such that

$$\triangle\!\triangle_W \Omega = \sum_{k=-\infty}^{\infty} \Omega(t - kW) = 1 \tag{11}$$

then Ω is called *unitary function*, denoted Ω_W with respect to its parameter W, according to Lighthill (1958), [10], p.61, also Zemanian [27], p.315, Campbell [5], p.635, Walter [33], p.149, King [62], p.509. Unitary functions (with respect to $W > 0$) are those which periodized (with respect to $W > 0$) yield the function that is constantly 1 (Figure 2). The set of functions Ω satisfying (11) form a subspace $\mathcal{U}_W$ of $\mathcal{D}$ (Zemanian [27], p.315). Furthermore, $\hat{\mathcal{U}}_{\frac{1}{W}} := \mathcal{F}(\mathcal{U}_W)$ is a subspace of $\mathcal{Z} := \mathcal{F}(\mathcal{D})$. Now, let us use unitary functions Ω_W and their Fourier transforms $\hat{\Omega}_{\frac{1}{W}}$ to define operations which have the potential to neutralize (reverse) discretizations (sampling) and periodizations (replicating), respectively.

Definition 2.3 (Finitization)**.** For distributions $f \in \mathcal{S}'$ one defines an operation $\sqcap_W : \mathcal{S}' \to \mathcal{E}', f \mapsto \sqcap_W f$ given by

$$\sqcap_W f \;:=\; \Omega_W \cdot f \tag{12}$$

called *truncation* or *finitization*. The result $\sqcap_W f \in \mathcal{S}'$ is a finite (compactly supported) function or distribution.

Definition 2.4 (Entirization)**.** For distributions $g \in \mathcal{S}'$ one defines an operation $\hat{\sqcap}_{\frac{1}{W}} : \mathcal{S}' \to PW, g \mapsto \hat{\sqcap}_{\frac{1}{W}}\, g$ given by

$$\hat{\sqcap}_{\frac{1}{W}}\, g \;:=\; \hat{\Omega}_{\frac{1}{W}} * g \tag{13}$$

called *band-truncation* or *entirization*. The result $\hat{\sqcap}_{\frac{1}{W}}\, g \in \mathcal{S}'$ is an entire (Paley-Wiener) function.

In this way, $\mathcal{E}'_W := \{ f \in \mathcal{E}' \mid \exists\, 0 < W < \infty : \sqcap_W f = f \}$ and $PW_{\frac{1}{W}} := \mathcal{F}(\mathcal{E}'_W) = \{ \alpha \in PW \mid \exists\, 0 < W < \infty : \sqcap_W \hat{\alpha} = \hat{\alpha} \}$ denote the spaces of W-time-limited distributions and W-band-limited (Paley-Wiener) functions (Figure 1), respectively. The factor $1/W$ limits the *resolution* in $\alpha \in PW_{\frac{1}{W}}$ (e.g. [40], Appendix B.2), i.e., no detail can be finer than $1/W$ in $\alpha \in PW_{\frac{1}{W}}$. Finitization (truncation) and entirization (band-truncation) are particular cases of regularization (convolution with a Schwartz function) and localization (multiplication with a Schwartz function), respectively. It has recently been found that *regular functions* and *local functions* are Fourier transforms of one another [34]. They stand for the insight that *infinite differentiability* and *finite summability*, given by $PW \subset \mathcal{O}_{\mathcal{M}}$ and $\mathcal{E}' \subset \mathcal{O}'_{\mathcal{C}}$, are Fourier transforms of one another. The term *finite* reminds to the fact that $\sqcap_W f$ is compactly supported, thus, integrations or summations over it will be finite. This property allows to restore functions exactly rather than approximately. The term *entire* reminds to the fact that $\hat{\sqcap}_{\frac{1}{W}}\, g$ is a Paley-Wiener function which is, in particular, entire (holomorphic everywhere in the complex plane). As above, it is clear that (12) and (13) are Fourier transforms of one another if $\mathcal{F}f = g$. For unitary functions one has $\mathcal{F}\Omega_W = W\,\hat{\Omega}_{\frac{1}{W}}$ and $\mathcal{F}\hat{\Omega}_{\frac{1}{W}} = \frac{1}{W}\,\Omega_W$ in contrast to $\mathcal{F}\,\text{III}_{\text{W}} = \frac{1}{W}\,\text{III}_{\frac{1}{W}}$ and $\mathcal{F}\,\text{III}_{\frac{1}{W}} = W\,\text{III}_{\text{W}}$. In particular, $\mathcal{F}\Omega = \hat{\Omega}$ and $\mathcal{F}\hat{\Omega} = \Omega$ for $W \equiv 1$ is an important special case. Recall $\Omega \not\equiv \hat{\Omega}$ holds for all unitary functions [28] in contrast to $\text{III} \equiv \hat{\text{III}}$. The usefulness of functions satisfying (11) is commonly known in sampling theory, e.g. Higgins & Stens [2], p.137. Note that in distribution theory the usual summability condition (e.g. (6.3.5) in Higgins & Stens [2], p.137) is elegantly replaced by the condition $\Omega \in \mathcal{D}$ which guarantees summability. We will now see that the operational inverse of sampling $\text{III}_{\frac{1}{W}}\cdot$ is the operation $\hat{\Omega}_{\frac{1}{W}} *$ applied to discrete (sampled) functions $f \in \mathcal{S}'$.

2.5 Argumentation in Distribution Spaces

An argumentation via norms $\|\cdot\|$ is not applicable in distribution theory, nor is it required to show that $\langle f, \varphi \rangle$ exist, it is guaranteed by the fact that $f \in X'$ and $\varphi \in X$ belong to dual spaces. Furthermore it is not required to show that sequences of functions or distributions converge within the spaces to which they belong, this is guaranteed by the sequential completeness of Fréchet spaces and the Banach-Steinhaus theorem (e.g. Friedlander [35], p.15, Dijk [20], p.96), respectively. (i) An equality $f = g$ holds in $\mathcal{S}'$ if $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all test functions $\varphi \in \mathcal{S}$, e.g. Gasquet & Witomski [18], p.290. (ii) Once, such equalities have been shown (e.g. $\mathcal{F}1 = \delta$ and $\mathcal{F}\delta = 1$) one may use and combine them without the necessity to show them again on test functions. This is what we do in this study. (iii) As elements in $\mathcal{S}'$ represent operations on functions, it is required that the image of an operator applied first (e.g. sampling) is in the domain of an operator applied next (e.g. interpolation). For that it is convenient to have, on one hand, operator definitions (Section 2.4) and, on the other, (continuous) space embeddings (Figure 1) in mind. (iv) All elements in $\mathcal{S}'$ and, hence, all operations in $\mathcal{S}'$ expressed by its elements have a Fourier transform. This is due to $\mathcal{F}(\mathcal{S}) = \mathcal{S}$ and $\mathcal{F}(\mathcal{S}') = \mathcal{S}'$ which are automorphisms in $\mathcal{S}'$. Any proof in $\mathcal{S}'$ may therefore alternatively be given in reciprocal (Fourier) domain (blue versus red in Figure 1). Their link is the exchange theorem, the double-algebra structure of $\mathcal{S}$, the algebra homomorphism between $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{O}'_{\mathcal{C}}$ as well as the double-module structure of $\mathcal{S}'$ around these two algebraic kernels (Section 2.3).

3 Main Result

3.1 Inversion of Sampling

The following theorem generalizes (i) the Whittaker-Kotel'nikov-Shannon Sampling theorem (where $\varphi \equiv sinc$), (ii) Campbell's sampling theorem (where $\varphi \equiv \theta \cdot sinc$ includes an additional convergence factor θ) and (iii) Theis' sampling theorem (where $\varphi \equiv sinc \cdot sinc = sinc^2$, i.e., $sinc$ plays the role of its own convergence factor, cf. Boyd [36], Table 2). It is furthermore consistent to several other already known results (e.g. [2, 32]) obtained by P. L. Butzer, R. L. Stens and their school on generalized sampling series in ordinary functions theory such as convergence factors or growing conditions imposed on φ, the interpolation kernel.

Theorem 3.1 (Inversion of Sampling)**.** *Let $f\left(\frac{k}{W}\right)$ be the samples of a function $f:\mathbb{R}^n \to \mathbb{C}$ on real axes $t \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$ and $W \in \mathbb{R}^n$ where $W = (W_1, \ldots, W_n)$ are non-zero and (without restriction of generality) positive $W_k > 0$, $1 \leq k \leq n$, then*

$$(S_W^{\varphi} f)(t) \;:=\; \sum_{k \in \mathbb{Z}^n} f(t_k)\,\varphi(t - t_k) \qquad \text{where } t_k = k/W \tag{14}$$

is an operation that restores f from its samples exactly, $S_W^{\varphi} f \equiv f$*, if and only if (14) may be written (applied from left to right) as*

$$\boldsymbol{a} * (\boldsymbol{b} \cdot \boldsymbol{c}) = \boldsymbol{c} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c} \qquad \text{where } \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{S}'(\mathbb{R}^n) \tag{15}$$

and $\boldsymbol{c} \equiv f \in PW_{\frac{1}{W}}(\mathbb{R}^n)$ is a Paley-Wiener function, $\boldsymbol{b} \equiv \text{III}_{\frac{1}{W}} \in \mathcal{S}'$ the Dirac comb and $\boldsymbol{a} \equiv \varphi \in \hat{\mathcal{U}}_{\frac{1}{W}}(\mathbb{R}^n)$ an entire unitary function. In terms of operations, $\boldsymbol{b}\cdot$ is sampling, $\boldsymbol{a}*$ its operational inverse and $\boldsymbol{a} * \boldsymbol{b} = 1$ denotes a smooth partition of unity.

Proof. Due to $\delta \in \mathcal{E}' \subset \mathcal{O}_C'$, Lemma 2.1 and the linearity of the translation operator τ we may formally write

$$(S_W^{\varphi} f)(t) \;:=\; \sum_{k \in \mathbb{Z}^n} f(t_k)\,\varphi(t - t_k) = \varphi(Wt) * \sum_{k \in \mathbb{Z}^n} f(t_k)\,\delta(t - t_k) = \varphi_{\frac{1}{W}}(t) * [\,\text{III}_{\frac{1}{W}}(t) \cdot f(t)\,] = \varphi_{\frac{1}{W}} * \text{III}_{\frac{1}{W}} \cdot f(t)$$

where $\varphi_{\frac{1}{W}}(t) := \varphi(Wt)$ and the sequence of operations applied to f must not be changed. (i) We now assume $S_W^{\varphi} f \equiv f$ in (14) and show (15). The equation $f = \varphi_{\frac{1}{W}} * \text{III}_{\frac{1}{W}} \cdot f$ for all applicable $f \in \mathcal{S}'$ implies $\varphi_{\frac{1}{W}} * \text{III}_{\frac{1}{W}} = 1$. But this means that $\varphi_{\frac{1}{W}} \in \hat{U}_{\frac{1}{W}}(\mathbb{R}^n)$ is an entire unitary function according to its definition. It proves the claim for $\boldsymbol{a} \in \mathcal{S}'$. Furthermore, $\boldsymbol{b} \equiv \text{III}_{\frac{1}{W}} \in \mathcal{S}'$ is commonly known and it remains to show that $\boldsymbol{c} \in \mathcal{S}'$ is a Paley-Wiener function. The comb $\text{III}_{\frac{1}{W}} \cdot$ being applicable to $f \in \mathcal{S}'$ means that $f \in \mathcal{O}_{\mathcal{M}} \subset \mathcal{S}'$ in order to allow sampling to be well-defined according to the exchange theorem. Thus, $\hat{f} \in \mathcal{O}_C'$ and we need to show that $\hat{f} \in \mathcal{E}' \subset \mathcal{O}_C'$ has compact support such that $f \in PW \subset \mathcal{O}_{\mathcal{M}}$ is a Paley-Wiener function. However, this is true due to the truncation in $\hat{f} = (\frac{1}{W}\hat{\varphi}_W) \cdot (W\text{III}_W) * \hat{f}$ caused by the cutout function $\hat{\varphi}_W$ where $\hat{\varphi}_W \cdot \text{III}_W = \delta$. It proves the claim in this direction. (ii) We now assume (15) and show $S_W f \equiv f$ in (14). We have $\boldsymbol{a} * (\boldsymbol{b} \cdot \boldsymbol{c}) = \boldsymbol{c} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c}$ where $\boldsymbol{a} \equiv \varphi \in \hat{\mathcal{U}}_{\frac{1}{W}}(\mathbb{R}^n)$ is an entire unitary function, $\boldsymbol{b} \equiv \text{III}_{\frac{1}{W}} \in \mathcal{S}'$ the Dirac comb and $\boldsymbol{c} \equiv f \in PW_{\frac{1}{W}}(\mathbb{R}^n)$ is a Paley-Wiener function. Now, according to the definition of an entire unitary function $\boldsymbol{a} * \boldsymbol{b} = 1$ forms a partition of unity and this already implies that $S_W^{\varphi} f \equiv f$. □

Corollary 3.2 (Inversion of Periodization)**.** *Let the conditions be as in Theorem 3.1 then (14) and (15) in reciprocal domain are expressed (applied from left to right) as*

$$\boldsymbol{a} \cdot (\boldsymbol{b} * \boldsymbol{c}) = \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{b}) * \boldsymbol{c} \qquad \text{where } \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{S}'(\mathbb{R}^n) \tag{16}$$

and $\boldsymbol{c} \equiv \hat{f} \in \mathcal{E}_W'(\mathbb{R}^n)$ is a compactly supported (tempered) distribution, $\boldsymbol{b} \equiv \text{III}_W \in \mathcal{S}'$ the Dirac comb and $\boldsymbol{a} \equiv \hat{\varphi} \in \mathcal{U}_W(\mathbb{R}^n)$ a finite unitary function. In terms of operations, $\boldsymbol{b}*$ is periodization, $\boldsymbol{a}\cdot$ its operational inverse and $\boldsymbol{a} \cdot \boldsymbol{b} = \delta$ is a smooth cutout of delta.

Proof. By the exchange theorem, (15) exists in $\mathcal{S}'$ if and only if (16) exists in $\mathcal{S}'$. □

Equations (15) and (16) have many applications, see e.g. Liu [57], p.286, eqs. (5) and (6) or Corcoran & Pasch [58], p.464, eqs. (10) and (11). An equation in the sense of tempered distributions holds true if and only if its Fourier transform holds true in the sense of tempered distributions. In particular, the symbolic calculation introduced above can be used to see that

$$\hat{f} \;=\; \sqcap_W(\triangle\!\triangle_W \hat{f}) \;=\; \Omega_W \cdot \text{III}_W * \hat{f} \tag{17}$$

$$f \;=\; \hat{\sqcap}_{\frac{1}{W}}(\sqcup\!\sqcup_{\frac{1}{W}} f) \;=\; \hat{\Omega}_{\frac{1}{W}} * \text{III}_{\frac{1}{W}} \cdot f \tag{18}$$

hold simultaneously in the sense of tempered distributions. We see that $\Omega_W \cdot \text{III}_W = \delta$ if and only if $\hat{\Omega}_{\frac{1}{W}} * \text{III}_{\frac{1}{W}} = 1$ such that f can be reconstructed from its discretization $\sqcup\!\sqcup_{\frac{1}{W}} f$ if and only if $\hat{f}$ can be reconstructed from its periodization $\triangle\!\triangle_W \hat{f}$. Another consequence is that functions which are just continuous (and not infinitely differentiable) cannot be reconstructed exactly from their samples. This is due to the fact that $\hat{f}$ is compactly supported if and only if f is a Paley-Wiener function (which is infinitely differentiable). Another consequence, due to the embedding $PW \subset \mathcal{O}_{\mathcal{M}}$, is that f is "tempered", i.e., it grows (at infinity) at most polynomially. It leads to the insight that functions which grow (at infinity) exponentially cannot be sampled and reconstructed from their samples using equidistant sampling operators. It is obvious that equidistant sampling cannot keep pace with exponentially growing functions.

3.2 Quadruples of Operations

We have seen that $\Phi = \{\text{III}_{\frac{1}{W}}\cdot,\ \hat{\Omega}_{\frac{1}{W}}*,\ \text{III}_W*,\ \Omega_W\cdot\}$ forms a quadruple (family) of equivalent operations (sampling, interpolation, periodization, truncation) generated by sampling. The other three family members are its operational inverse (interpolation), its Fourier transform (periodization) and the operational inverse of its Fourier transform (truncation). In other words, all these operations are equivalent, one element determines all others. The same pattern (cross-type inverses) and a complementary pattern (co-type inverses) may be found for other operations in $\mathcal{S}'$. We distinguish co-type and cross-type families.

3.2.1 Co-Type Families

Co-type families of operations are generated either by multiplication-type operators such that their operational inverse is another multiplication-type operator or they are generated by convolution-type operators such that their operational inverse is another convolution-type operator. Translation, for example, belongs to

$$\Phi_{trans} = \{\ \delta_a*,\ \delta_{-a}*,\ e^{-2\pi iat}\cdot,\ e^{2\pi iat}\cdot\ \} \tag{19}$$

which is a co-type family. Here, δ_a* and $\delta_{-a}*$ are operational inverses of one another because $\delta_a * \delta_{-a} = \delta$. The other two operations are their Fourier transforms, they yield $e^{-2\pi iat}\cdot e^{2\pi iat} = 1$.

3.2.2 Cross-Type Families

A typical cross-type family is the quadruple $\Phi_{samp} = \{\text{III}_{\frac{1}{W}}\cdot,\ \hat{\Omega}_{\frac{1}{W}}*,\ \text{III}_W*,\ \Omega_W\cdot\}$ generated by sampling. Obviously, $\text{III}_{\frac{1}{W}}\cdot$ has no multiplication-type operational inverse. It is clear that locations $t \in \mathbb{R}^n$ deleted by $\text{III}_{\frac{1}{W}}\cdot$ cannot be restored from zero via a multiplication. The way out are convolution-type operators which form a smooth partition of unity. These operations are given by Lighthill's unitary functions (Lighthill [10], p.61, Campbell [5], p.635, Boyd [36]).

3.2.3 Inverses versus Cross-Inverses

The quadruple Φ_{samp} of sampling shows that $\boldsymbol{a}\cdot\boldsymbol{b} = 1$ has no solution $\boldsymbol{a}$ if $\boldsymbol{b}$ is the Dirac comb. In this case, it is convenient to consider its cross-inverse $\boldsymbol{a} * \boldsymbol{b} = \delta$ where $\boldsymbol{a}$ is an inverse of $\boldsymbol{b}$ with respect to convolution. These kind of considerations have, according to our knowledge, never been investigated before and as a result of this many questions arise. For example, can cross-inverses always be found if direct inverses do not exist? Such questions need to be clarified in further studies.

4 Conclusions

The fact that regularity (infinite differentiability) and locality (finite summability) are Fourier transforms of one another is an important insight. It goes back to studies of Paley and Wiener who found that growing conditions of functions are related to regularity conditions of their Fourier transforms (e.g. Walter [33], p.185). If regularity is missing then locality is missing in reciprocal domain and, vice versa, if locality is missing then regularity is missing in reciprocal domain [34]. Many standard problems in ordinary functions theory can be traced back to the fact hat $sinc$ is not local (finitely summable) and, equivalently, $rect$ is not regular (infinitely differentiable). The *Gibbs phenomenon*, for example, disappears if *Cesàro-summation* is applied (e.g. Debnath & Bhatta [37], p.54), that is, $sinc$ functions are replaced by $\theta\cdot sinc$ which include a convergence factor $\theta \equiv sinc$, hence, $rect$ is its own mollifier in $\mathcal{F}(rect * rect) = sinc\cdot sinc$, see Moore [38], Butzer & Nessel [3], p.190, Weisz [39] for details. The fact that $sinc$ is not summable (integrable) is also responsible for the fact that the Fourier transform is no automorphism in $L^1(\mathbb{R}^n)$, the space of Lebesgue integrable functions. One may recall, $rect$ is integrable and $sinc$ is not. Vice versa, $sinc$ is regular and $rect$ is not [28, 40]. All these problems disappear if one uses instead of $rect$ and $sinc$ unitary functions. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$, to which unitary functions belong, may be seen as an idealization of $L^1(\mathbb{R}^n)$, it allows $\mathcal{F}$ to become an automorphism (in $\mathcal{S}$ and, as a consequence of this, in $\mathcal{S}'$). One may say $\mathcal{S}$ lies at the heart of the theory of functions (violet color in Figure 1) and its core is formed by unitary functions. Unitary functions may be used (instead of $rect$ and $sinc$) in either ordinary or generalized functions theory to solve standard problems. It solves, for example, the problem of multiplying distributions ([28], Remark 1). Their use in generalized functions theory, furthermore, includes a calculus for *operations on functions* and their calculation rules play an important role in quantum mechanics (e.g. Susskind & Friedman [41], p.246, Messiah [42], p.474, Becnel & Sengupta [43]), quantum optics (e.g. Schleich [44], p.37) and quantum field theory (e.g. Folland [26], p.3, Glimm & Jaffe [45], p.12, Reed & Simon [46], p.6). The validity of (15) and (16) is moreover of great interest in electrical engineering (e.g. Pfaffelhuber [52, 54], Rao [55]), communication theory (e.g. Blachman [56], eq.(24) is a partition of unity composed of $sinc^2$ functions which can be found as eq.(36) in Theis [4]), optics (e.g. Liu [57], Corcoran & Pasch [58], Wei [59]) and radar (e.g. Woodward [60], p.33, eqs.(28) and (29), Brandwood [61], p.91, eqs.(5.1) and (5.2)). It is therefore important to further explore the rules of validity for a generalized functions calculus.

5 Appendix: Unitary Functions

The use of unitary functions (mollifiers) is required because $rect$ is not a regular (infinitely differentiable) function. It implies that $\text{III} \cdot rect \notin \mathcal{S}'$ is not a tempered distribution. Equivalently, the use of unitary functions (convergence factors) is required because $sinc$ is not a finitely summable (integrable) function. It implies that $\text{III} * sinc \notin \mathcal{S}'$ is not a tempered distribution. For any fix $W > 0$, unitary functions Ω_W may be constructed such that Ω_W and its translates $\tau_{kW}\Omega_W(t) := \Omega_W(t-kW)$ overlap at most twice in the formation of $⧍_W(\Omega_W) = \text{III}_W * \Omega_W = 1$ the function that is constantly 1. Furthermore, $⊔⊔_{\frac{1}{W}}(\hat{\Omega}_{\frac{1}{W}}) = \text{III}_{\frac{1}{W}} \cdot \hat{\Omega}_{\frac{1}{W}} = \delta$, its Fourier transform, is the cutout of one Dirac δ from its periodization known as the Dirac comb (Figure 2).

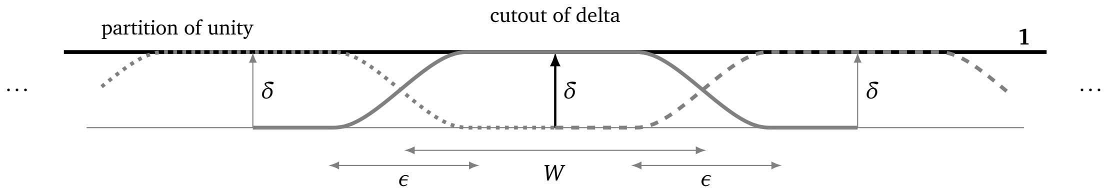

Figure 2: The products $\text{III}_W * \Omega_W = 1$ (**thick black line**) and $\text{III}_W \cdot \Omega_W = \delta$ (**thick black delta**) do both exist in $\mathcal{S}'$.

Another property of unitary functions is that their Fourier transforms are again unitary in the sense that their periodization is 1 and their discretization is δ. We distinguish entire unitary functions and finite unitary functions (Fischer & Stens [28, 40]) due to the circumstance that the function properties of being entire and finite are mutually exclusive.

5.1 Construction via Integration

Let $\epsilon > 0$ be a (small) regularization parameter used in the construction of a bump function $\rho_\epsilon \in \mathcal{D}(\mathbb{R}^n)$, $n = 1$ for simplicity,

$$\rho_\epsilon(t) = e^{-1/(1-(2t/\epsilon)^2)} / \int_{-\infty}^{+\infty} e^{-1/(1-(2t/\epsilon)^2)} dt$$

whose integral is unity. It is non-zero in $]-\epsilon/2, +\epsilon/2[$ and zero else. We use it to construct the derivative

$$\frac{d}{dt}\Omega_W \;=\; \tau_{-W/2}\,\rho_\epsilon - \tau_{+W/2}\,\rho_\epsilon \tag{20}$$

of a unitary function Ω_W of width $W + \epsilon$. Here, $\tau_a\,\rho_\epsilon(t) := \rho_\epsilon(t-a)$ denotes the repositioning of ρ_ϵ at a. Positive and negative bumps moved to $-W/2$ and $+W/2$, respectively, shall not overlap in (20). Hence, the regularization is bounded $0 < \epsilon \le W$ by the targeted interval length W. The case $\epsilon \equiv W$ yields the bump function $\rho_\epsilon \equiv \Omega_W$ that is usually used in the literature (e.g., in [Horvath, p.401]) for the regularization of a distribution. An integration over (20) now yields a unitary function

$$\Omega_W(t) \;=\; \int_{-\infty}^{t} \frac{d}{dt}\Omega_W(\tau)\,d\tau \tag{21}$$

such that $\frac{1}{W}\int_{-\infty}^{+\infty} \Omega_W(t)\,dt = 1$ for any $0 < \epsilon \le W$. Obviously,

$$\Omega_W \xrightarrow{\epsilon \to 0} rect_W \tag{22}$$

where $\epsilon > 0$, the $rect \in \mathcal{E}'(\mathbb{R}^n)$ function being defined as $rect_W(t) := 1$ for $t \in]-W/2, +W/2[$, $rect_W(\pm W/2) := 1/2$ and zero else. We say Ω_W is *double-sided unitary* because it satisfies $\text{III}_W * \Omega_W = 1$ and $\text{III}_W \cdot \Omega_W = \delta$, simultaneously. Figure 2 illustrates that according to construction $\text{III}_W * \Omega_W = 1$ is the periodization of Ω_W that yields 1 (partition of unity) and $\text{III}_W \cdot \Omega_W = \delta$ is the discretization of Ω_W that yields δ (cutout of delta). The latter is another partition of unity $\mathcal{F}\delta = \mathcal{F}(\text{III}_W \cdot \Omega_W) = (\frac{1}{W}\,\text{III}_{\frac{1}{W}}) * (W\,\hat{\Omega}_{\frac{1}{W}}) = \text{III}_W * \hat{\Omega}_W = 1$ in reciprocal domain.

5.2 Construction via Regularization

Unitary functions Ω_W can equivalently be constructed as regularization (mollification) $\rho_\epsilon *$ of the rectangular function

$$\Omega_W := \rho_\epsilon * rect_W \tag{23}$$

where ρ_ϵ is the bump function constructed in the previous section, $0 < \epsilon \le W$. A regularization of $rect$ by ρ_ϵ widens its support by ϵ. The function ρ_ϵ is called mollifier (Friedrichs [47], Schechter [48], Petersen [19]) or regularizer (e.g. Wei [49, 50]) and its Fourier transform $\mathcal{F}\rho_\epsilon = \frac{1}{\epsilon}\,\theta_{\frac{1}{\epsilon}}$ is occasionally called convergence factor (e.g. Sommerfeld [51], p.58, Campbell [5] 1968, p.627, Pfaffelhuber [52], p.654, García et al. [53] 1998, p.50) or just θ-factor (Butzer & Nessel [3], p.190) because it accelerates the respective convergence. Obviously,

$$\hat{\Omega}_{\frac{1}{W}} := \theta_{\frac{1}{\epsilon}} \cdot sinc_{\frac{1}{W}} \tag{24}$$

is the Fourier transform of (23). So, compared to the classical Whittaker-Kotel'nikov-Shannon sampling theorem, these sinc functions (24) are now equipped with a built-in convergence factor. We just used that regular (infinitely differentiable) functions and local (finitely summable) functions are Fourier transforms of one another [34]. For the connection between summability and convergence factors see Moore [38]. Compared to the previous section this is an equivalent construction of Ω_W because

$$\frac{d}{dt}\Omega_W = \frac{d}{dt}(\rho_\epsilon * rect_W) = \rho_\epsilon * \frac{d}{dt}(rect_W) = \rho_\epsilon * (\delta_{-W/2} - \delta_{+W/2}) \;=\; \tau_{-W/2}\,\rho_\epsilon - \tau_{+W/2}\,\rho_\epsilon$$

equals (20). The rule $\frac{d}{dt}(f*g) = f*(\frac{d}{dt}g) = (\frac{d}{dt}f)*g$ is commonly known (e.g. Zemanian [27], p.132). It obeys the same rule as $\tau_a(f*g) = f*(\tau_a g) = (\tau_a f)*g$ (Petersen [19], p.90) and $\triangle\!\triangle\!\triangle_W(f*g) = f*(\triangle\!\triangle\!\triangle_W g) = (\triangle\!\triangle\!\triangle_W f)*g$ (Fischer [30], Lemma 2) for convolution-type operators. Moreover, $\frac{1}{W}(\delta_{-W/2} - \delta_{+W/2})$ is the derivative of δ for $W \to 0$ and $\frac{1}{W}\, rect_W \to \delta$ for $W \to 0$. Obviously, $\frac{1}{W}\,\Omega_W \to \frac{1}{W}\, rect_W$ for $\epsilon \to 0$ where $\frac{1}{W}\int_{\mathbb{R}^n} \Omega_W\, dt^n = 1$ and $\frac{1}{W}\int_{\mathbb{R}^n} rect_W\, dt^n = 1$ independent of $\epsilon > 0$.

Remark 3. An early use of unitary functions U can already be found in Sommerfeld (1947) [51], p.59, where (1) is

$$f(x) = \int_{-\infty}^{\infty} f(\xi)\, U(x-\xi)\, d\xi$$

for $W \to \infty$, symbolically (cf. Mallat [63], p.33, Benedetto [64], p.2, eq.(δ)). The connection between U and Green's function G is [51], p.65, eq.(8). 'We conclude that U has the "character of a δ-function"' [51], p.59. The back and forth switching between unitary functions U and δ given by $⧢_{\frac{1}{W}}(U) = \delta$ and $\hat{\sqcap}_{\frac{1}{W}}(\delta) = U$ using discretization $⧢_{\frac{1}{W}}$ and regularization $\hat{\sqcap}_{\frac{1}{W}}$ expresses a duality (one-to-one correspondence) between *discreteness and smoothness* on Paley-Wiener functions ($0 < W < \infty$).

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