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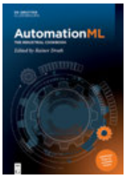

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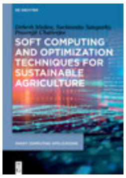

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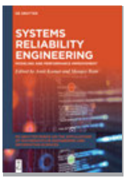

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Editors
Prof. Dr. Kathrin Dörfler
Technical University of Munich
School of Engineering and Design
Department of Architecture
Arcisstr. 21
80333 München
Germany
doerfler@tum.de

Prof. Dr.-Ing. Jan Knippers
University of Stuttgart
Institute of Building Structures
and Structural Design
Keplerstr. 11
70174 Stuttgart
Germany
info@itke.uni-stuttgart.de

Prof. Achim Menges
University of Stuttgart
Institute of Computational Design
and Construction
Keplerstr. 11
70174 Stuttgart
Germany
mail@icd.uni-stuttgart.de

Prof. Dr. Stefana Parascho
École Polytechnique Fédérale de Lausanne
ENAC IA CRCL
GC G1 567 (Bâtiment GC)
Station 18
1015 Lausanne
Switzerland
stefana.parascho@epfl.ch

Prof. Dr. Helmut Pottmann
Vienna University of Technology
Applied Geometry
Wiedner Hauptstraße 8–10/104
1040 Vienna
Austria
pottmann@geometrie.tuwien.ac.at

Prof. Dr. Thomas Wortmann
University of Stuttgart
Institute of Computational Design
and Construction
Keplerstr. 11
70174 Stuttgart
Germany
thomas.wortmann@icd.uni-stuttgart.de

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Contents

Jelena Milošević and Ognjen Graovac

An Approach to Designing Architectural Structures Using 3D Graphic Statics

Abstract: Graphic statics is a well-established method for designing diverse typologies of architectural structures. Although this method has been in use since the 19th century, the advent of digital technologies contributed to its current, more widespread usage. This paper presents the application of 3D graphic statics for finding structural forms. Parametric design tools based on 3D graphic statics for explorations in a generative design process was employed. The explorations were implemented by the 3D Graphic Statics plugin developed for Grasshopper visual programming editor for the commercial computer-aided design (CAD) system Rhinoceros, commonly used by architects. The design outcomes confirm the effectiveness of the approach based on geometrical operation for designing efficient structural forms composed of elements in perfect geometrical equilibrium with external forces. Furthermore, graphical methods are intuitive and facilitate rapid statical calculations in the conceptual stage of the architectural design process. Moreover, the description of the design approach also allows for reflection on its methods, outcomes, and the interactions of designers with design tools throughout the design process. Finally, this study informs architects about the increasingly sophisticated design tools made available by digital technology to support architectural design and how these tools compare to older design mediums.

Keywords: architectural design, computational design tools, spatial structures, form-finding, graphic statics, design research

1 Introduction

Graphic Statics (GS) includes a set of methods for efficiently designing diverse structural typologies in architecture including arches, vaults, and bridges. The method is based on geometrical operation, and the principle that forces in the structural elements are in equilibrium with the external forces. The method is intuitive and facilitates rapid statical calculations (in comparison to, for example, Finite Element Analysis), which makes it relevant for the conceptual design stage.

There are only indications that the Romans and later the Byzantines were familiar with specific graphic techniques. Even though the late Gothic is full of geometrically intricate structures like vaults, arches, rosettes, and buttresses, very few recorded sources provide thorough information on the used graphic techniques. Moreover, until the 20th century, no techniques could reliably describe the equilibrium state of Gothic vaults. Several ancient concepts came back into prominence during the Renaissance and Baroque periods, and structural design approaches based on mathematics and

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graphics underwent extensive development. The equilibrium state of arches and domes based on a closed polygon of forces is graphically described by Simon Stevin's rule on the parallelogram of forces (1586). More elaborate elliptical forms were also built during the Renaissance and Baroque periods, along with significant structures including the St. Peter's Church (1506) by Bramante and Michelangelo, and Santa Maria del Fiore (1436) by Brunelleschi (1436).

The contributions of various authors allowed GS to become an established scientific discipline. Hooke (1676) described the connection between the geometry of the tensioned chain and compressed arch with the anagram: *As hangs the flexible line, so but inverted will stand the rigid arch*. Gregory independently came to the same conclusion and developed effective equations in 1689. Poleni (1743) calculated the equilibrium state of the dome of St. Peter's during the restoration using an inverted chain model. Rankine introduced the idea of the reciprocity of the diagram of forces and form in 1864. Maxwell established geometric procedures for constructing reciprocal diagrams in 1864, while Culmann (1866) established mathematical proofs. Also, Cremona (1879, 1880) shaped GS as a method for solving specific structural problems in 1890. In addition to these studies, the thrust line theory was developed in the 19th century (Mosely 1843; Heyman 1989; Benvenuto 1991). Milankovitch (1907) contributed to the field by formulating a comprehensive thrust line theory and a mathematical knowledge of the equilibrium state of structures.

Gaudi instrumentalized his knowledge of graphical methods for determining structures to develop distinctive architectural design methodology at the beginning of the 20th century (Burrey et al. 2005). He applied graphic and experimental methods in designing and constructing his unconventional equilibrium forms (Huerta 2003). Gaudi's work and design approach came back in focus in recent years due to the emergence of complex architectural forms and interest in generative form-finding. The term "form-finding" refers to the process of finding the best shape that best represents (or approximates) the state of static equilibrium design (Lewis 2003) as an alternative to random free-form. In the later decades of the 20th century, physical models as instruments for form-finding were replaced by digital tools, facilitating further development and implementation of more complex concepts.

Although GS represents an intuitive technique for the exploration of structural forms in two dimensions (Zalewski and Allen 1998), three-dimensional graphic static (3DGS) methods were also developed (Foppl 1892). The application of 3DGS methods was constrained because of rather complicated procedures that require knowledge of several specific rules of projective geometry. Developments in computational technologies motivated the recent interest of researchers in this topic. The modeling of vaults as a discrete network of forces in equilibrium with gravity loads were proposed based on Heyman's (1982) principle (O'Dweyer 1999), expanded further through the Thrust Network Analysis (TNA) (Block 2009), various related methods (Angelillo et al. 2010; Fraternali 2010), and Thrust Surface Method (Fraddosio 2020). In addition, in recent years, many authors have explored the topic of 3DGS. The results of the literature analysis (Fig. 1) show increased interest in GS in recent years, covering a wide specter of topics from new methods and design tools to digital fabrications.

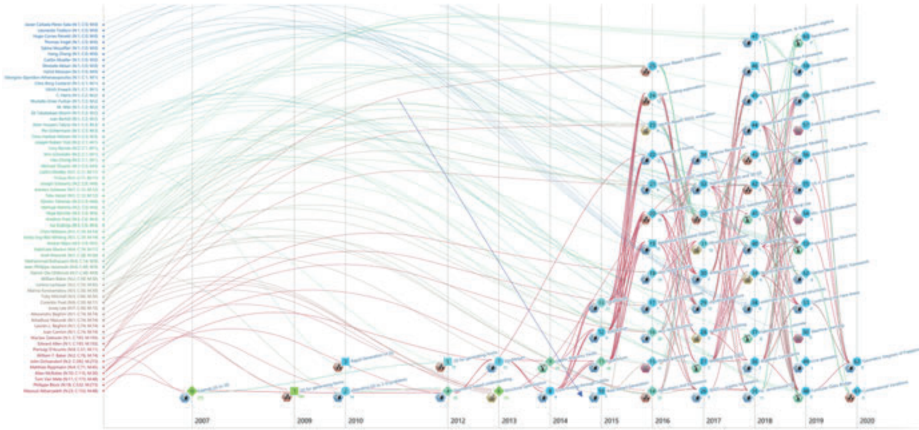


Fig. 1: Literature analysis including information on number of papers (N), citation (C, M), number of authors, their impact, and connections; thematic frame; type of publication.

Various authors developed computational tools. Tools such as software modules for computational-aided design (CAD), which is standardly used in architecture, have increased interest in using GS in the early design stages. An example is *RhinoVault* by Rippmann (2012; 2014) developed as a plug-in for *Rahinoceros* commercial CAD system for 3D modeling. Using the same benefits of Graphic Statics, *RhinoVAULT* uses reciprocal diagrams to provide an intuitive, rapid form-finding tool for structural design in three dimensions. The application of the tool in structural design is best exemplified by Armadillo Vault, constructed for the Architecture Biennale in Venice in 2016 (Rippmann et al. 2016). Also, a form-finding tool developed for *Rhinoceros* using the principles of 3DGS is *PolyFrame* by Akbarzadeh and Nejur (2018). This computational tool allows the construction and manipulation of reciprocal polyhedrons for compression-only structural form-finding. Finally, the *3D Graphic Statics* by Graovac (2019), a parametric tool for the *Rhinoceros/Grasshopper* that facilitates a parametric approach to structural design based on 3DGS, is tested in this paper, in an educational context, for its use in early design stage.

A previous brief overview of the GS history illustrates the constant development of the method supported with new theoretical knowledge and technological advances, as well as recurring interest in its architectural application. This suggests that it is relevant for architects to learn about GS methods and tools.

2 Design Application

The premise of this study is that using the 3DGS computational design tool, which is integrated into a standard CAD modeler, throughout the architectural design process enables designers to explore a variety of effective structural forms and broaden their design vocabulary intuitively and interactively. To test the previous premise, design by research method was used in an educational context. The first part of the research was a study conducted within the course Research Methodologies in the first year of Ph.D. studies at the University of Belgrade – Faculty of Architecture (UB–FA). The study focused extensive review of the development of GS methods and procedures and aimed to test that knowledge practically by developing a computational design tool based on 3DGS. The second part of the research included user study of the *3D Graphic Statics* plugin for *Rhinoceros/Grasshopper* by Graovac (Fig. 2) through design experiments conducted with the group of master architectural students at the UB–AF within the elective course Form-finding of Spatial Structures.

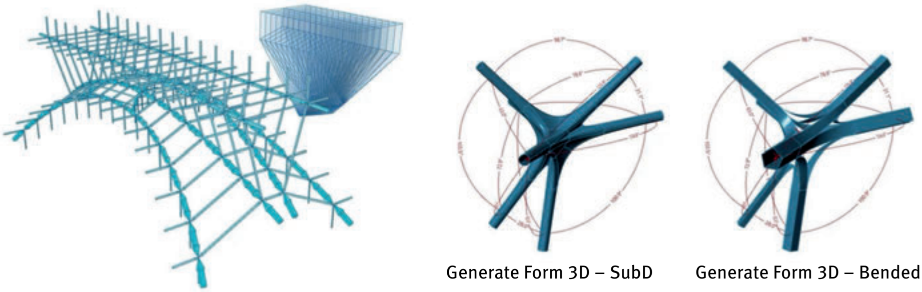


Fig. 2: An example of form-finding of a bridge structure using 3D Graphic Statics plug-in.

2.1 Computational Implementation of 3DGS Concepts

The developed plugin implements common GS concepts (Clarke 2018). In GS, the forces acting on a structure are represented by lines or vectors, and their magnitudes and directions are accurately depicted. The graphical approach allows visualization and analysis of complex interactions of forces in a structure without complex mathematical calculations. The fundamental principle of graphic statics is based on the equilibrium of forces. By constructing reciprocal force and form diagrams, the magnitude and direction of unknown forces can be determined, and the overall equilibrium of the structure can be found. Traditional application of GS implies using 2D reciprocal diagrams; however, the method was extended in 3D using the principle of the equilibrium of polyhedral forms and is known as 3D/polyhedral/spatial GS. While traditional GS primarily focuses on 2D structures, 3DGS allows for analyzing more complex spatial structures (including trusses, frames, and arches). Table 1 summarizes the differences between the two graphical representations and equilibrium representations in 2DGS and 3DGS.

Tab. 1: Differences between key graphical representations in 2DGS and 3DGS.

	2DGS	**3DGS**
Graphical representations	**Force diagram/polygon** Represents a magnitudes and directions of the forces acting on a structure. It is constructed by drawing lines or vectors to scale, where the length of each line represents the magnitude of the corresponding force, and the direction of the line represents the direction of the force.	**Force diagram/polyhedral** They represent the magnitudes and directions of forces acting on a 3D structure. The edges or faces of the polyhedron correspond to the forces acting on the structure, and their lengths or areas represent the magnitudes of those forces.
	Form diagram/polygon (also called funicular polygon, or an influence line diagram) Represents the distribution of internal forces within a structure. It shows the path of the resultant force for a given set of external lodes applied to the structure.	**Funicular diagram/polyhedral** Represents the flow or forces through a spatial structure and shows the paths of resultant forces for different loading conditions.
Equilibrium	**Closure of polygon** The lines forming force polygons in 2DGS must close to indicate force equilibrium. The closure of the force polygon ensures that the sum of forces and moments acting on the structure is zero.	**Closure of polyhedral** Force polyhedral in 3DGS must satisfy equilibrium conditions. The closure of the force polyhedron ensures that the sum of forces and moments acting on the structure is zero. The polyhedral must be convex.

In GS, force and from diagrams are two key graphical representations of the forces acting on a structure and aid in determining the internal forces and stability. The external and internal forces acting on the structure are considered to construct a force diagram. External forces include applied loads, reactions at supports, and other externally applied forces. Internal forces result from the resistance of the structure to external forces and are usually represented by lines within the structure. The form diagram is constructed considering a particular force or load applied to the structure. The lines or vectors representing the internal forces are drawn to scale and connected to form a polygon, known as a form polygon or form diagram. The form diagram provides insights into the load paths and the magnitude of internal forces at different points along the structure. By analyzing different load cases and constructing corresponding form diagrams, engineers can understand how the structure responds to different loading conditions and optimize its design for strength and stability.

The force and the form diagram are fundamental tools in GS, allowing analysis and understanding of the equilibrium of forces in structures (Fig. 3) without the need for complex mathematical calculations. Furthermore, these graphical representations facilitate the visualization and optimization of structures, leading to efficient and safe designs. Since force and form diagrams are topological duals, design explorations could be done by manipulating either force or form diagrams.

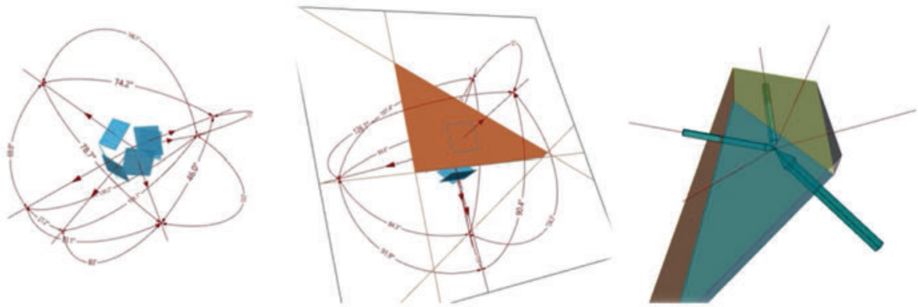


Fig. 3: Representation of statical equilibrium in 3DGS.

Unlike 2D methods, 3DGS is challenging to perform manually, so it was implemented in the *3D Graphic Statics* computational tool for generating funicular structures. The development of the tool had several goals, including automation of the process of force diagrams' construction and acceleration of the design exploration process by facilitating rapid production of alternative formal solutions. The tool is implemented as a set of components within the *Grasshopper* environment for visual programming. It uses the advantages of real-time geometry preview, which facilitates form exploration based on simulation results. The increased popularity of the parametric modeling design approach caused the development of diverse software; however, *Rhinoceros/Grasshopper* interface and strong community support set this system apart, and the development of the tool as a module within this system facilitates exploration connecting of *3D Graphic statics* with other *Grasshopper* tools. Most of the implemented procedures are based on the research conducted by Block Research Group (2009–2023) at ETH Zurich and Polyhedral Structures Laboratory (n.d.) at Weitzman School of Design, University of Pennsylvania. These approaches are repackaged and generalized in the new plugin for design.

Implemented form-finding algorithm of 3D Graphic Statics incorporates two phases: preparation (phase 1) and simulation (phase 2). Phase 1 includes subsequent steps:

Step 1.1 Construct a node (point) inside each polyhedron.
Step 1.2 Determine the interconnections of polyhedral and connect the nodes with lines.

Phase 2 is iterative problem-solving and includes the next steps:

Step 2.1 Rotate the lines around the center so they are parallel to the side normal.
Step 2.2 Connect the separated nodes with the arithmetic mean.
Step 2.3 Repeat until all corner deviations are below tolerance.

Implemented algorithm enables interactive structural form-finding by manipulating both form and force diagrams in parametric settings (Fig. 4). Regarding previous, the plug-in provides a set of components including components for subdividing polyhedral cells, form-finding processes, computing forces, generating cross sections.

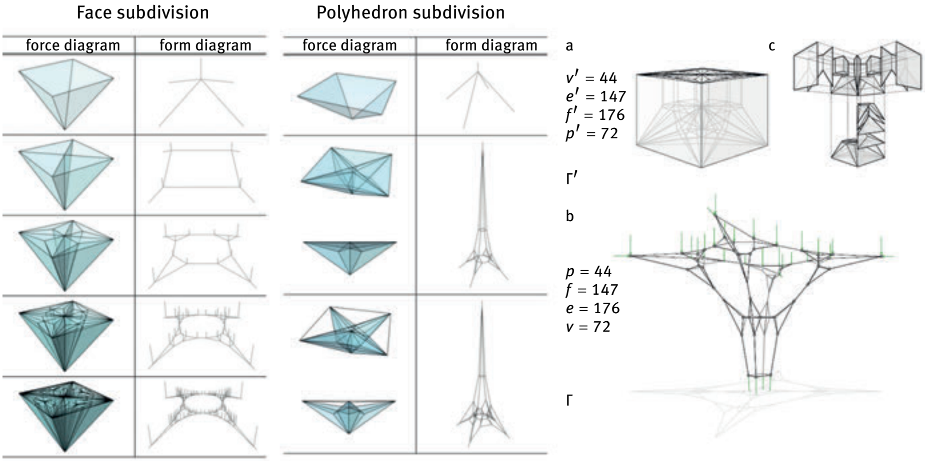


Fig. 4: Form-finding using implemented 3D Graphic Statics algorithm.

In the general structure of the plugin (Fig. 5), components are organized in several pallets (corresponding form-finding algorithm). The first pallet contains pre-processing components which enable initial geometry description and manipulation. The second pallet is the processor and contains components that implement constructing methods and solvers. The third pallet comprises components for post-processing that enable extraction and representation of geometric data. Besides, the plugin contains two additional pallets of utilities and presets containing components that are generally useful for diverse modeling operations. With this digital tool, architects, engineers, students, and researchers can create a parametrically driven system to generate optimal structures of aesthetics.

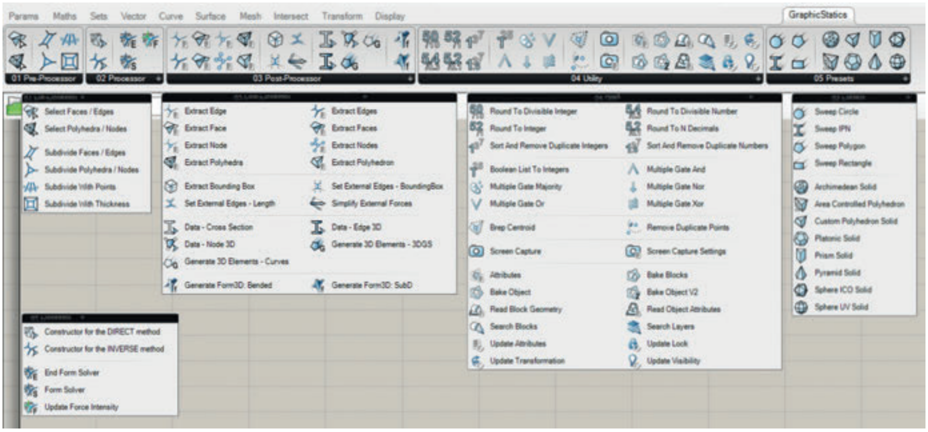


Fig. 5: Structure of 3D Graphic Statics plugin for Grasshopper.

A Grasshopper definition for form-finding of equilibrium structures that uses 3D Graphic Static components (Fig. 6), includes following steps:

Step 1 Set parameter system that generates polyhedral.
Step 2 Convert polyhedral to 3DGS data.
Step 3 Use one of the shape simulation solvers.
Step 4 Based on one known force, match the intensities of the others.

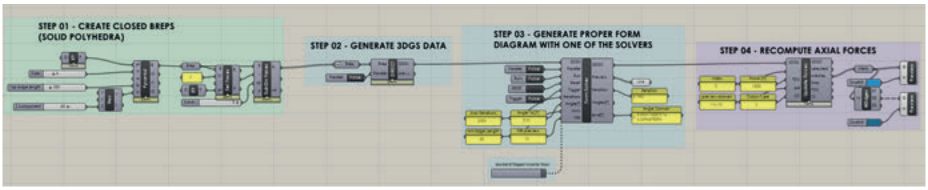


Fig. 6: Grasshopper definition for form-finding equilibrium structures using 3D Graphic statics components.

2.2 Educational/Design Applications

The architectural design application of 3DGS and the developed tool were tested in an educational context. The design project was implemented as a task within the elective course for master architectural (MA) students. The course introduces students to form-finding methods and tools and their application in designing spatial structures (particularly form-active structures) in architecture. Since the course is primarily theoretical, the practical design activity was organized as time constrained workshop during second part of the autumn semester of the academic year 2022/2023. Students working in groups were assigned to design lightweight spatial structures. The workshop's goal was also to connect PhD research and MA education and engage students in a collaborative manner of work. The learning outcomes of this practical course activity were to acquire theoretical and methodological knowledge on 3DGS in architecture, improve digital modeling skills, and exceed themselves in suggesting novel design proposals.

Workshop was implemented as a one-day block of classes that included three parts:

- A lecture on the theoretical basis of graphic statics,
- Instruction related to the application of the 3D Graphic Statics plugin, and
- Designing of spatial structure.

This organization facilitated students to gain procedural experience related to the specific design process and acquisition of competencies through working on a specific design task with the help of workshop tutors. Furthermore, problem-solving design activity motivated the development of knowledge and skills in a creative environment. Also, students took advantage of digital tools to produce interactive, collaborative products. In the continuation of this paper, three design proposals produced by students during the workshop are presented. The results of the workshop, and three design proposals, are presented in Fig. 7.

Previous user study generally validates the practical application of the 3DGS in the early stage of the architectural design process. However, based on the artifacts produced during the workshop and conversation with participants following potentials and limitations were identified concerning the application of the 3D Graphic Statics plugin for the design:

- Using a tool based on 3DGS provides valuable insights into the behavior and stability of spatial structures, allowing designers to optimize their designs.
- The tool enables the production, testing, and elaboration of diverse design concepts.
- The tool enables rapid and easy form explorations and the creation of alternative designs.
- The plugin represents an intuitive design tool that enables structural evaluations adequate for the conceptual design stage.
- Implementing 3DGS as a plugin of a standard CAD environment familiar to architects makes this tool more receptive to designers.
- The tool can support the creative process and lead to the emergence of structurally rational and aesthetically pleasing forms.
- The workshop enabled students to familiarize themselves with 3DGS, improve modeling skills, and creatively apply these competencies to the design activity.

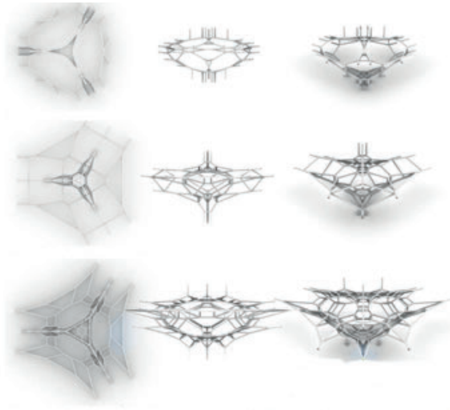
Variation

Class students: Đorđe Milisavljević, Luka Marković, Lazar Žikić and Danilo Radovanović

Scope: Design of a canopy with the rational number of nod connections

Process: (1) Selection of global geometries and experimentations with several types of closed bodies (e. g., cubes and pyramids); (2) Selection of capped pyramid as a geometry for further elaboration; (3) Design elaboration through further form segmentation and generation of new structural formations; (4) Optimization of number of rods

3DGS application: Using plugin for production of design alternatives and fast structural evaluation



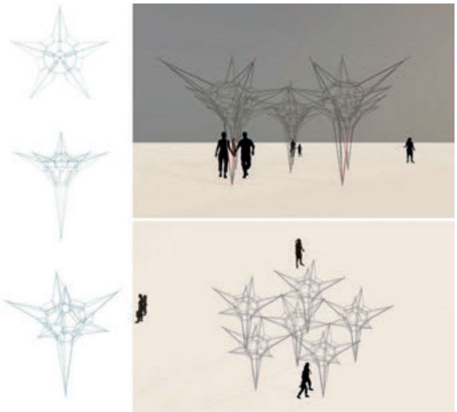
Modularity

Class students: Petar Rakić, Đorđe Stanojlović, Nemanja Papić and Sara Brkić

Scope: Design of a modular canopy and exploration of construction efficiency

Process: (1) Form-finding of a modular element; (2) Design explorations of a architectural composition made of modular elements; (3) optimizing number of rods and modular units

3DGS application: Using plugin for form-finding of a modular element and fast structural evaluation



Assembling – disassembling

Class students: Jovana Lazarević, Tijana Pejić, Tamara Bojović, Filip Vasić and Jovana Stakić

Scope: Design traveling pavilion out of a demountable kit of architectural elements

Process: (1) Form-exploration by varying parameters concerning polyhedral (number of sides, number of divisions); (2) Selection of a geometry for further elaboration based on fabrication efficiency demands; (3) Design elaboration and production of the catalogue of elements (rods and nod connections)

3DGS application: Using plugin for design explorations and fast structural evaluation

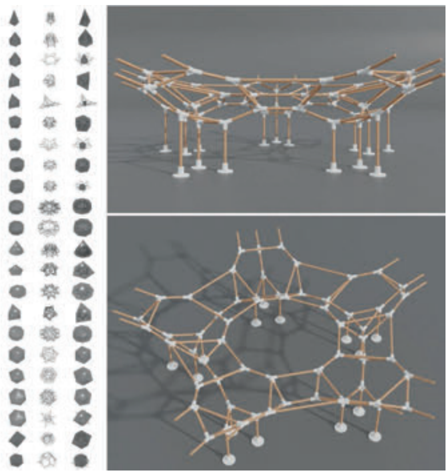


Fig. 7: Topics and projects produced during workshop.

- Different levels of digital modeling skills and time constraints affected the work process and result. A more personalized approach to teaching students computational modeling skills should be considered in the future.
- Organization of workshops at masters' studies based on the results of research conducted within Ph.D. studies proved to be very motivating for students.
- Learning by working on a design problem interested architectural students in structural engineering concepts and familiarizing them with the tool that could be applied in work on different architectural design projects in the future.

3 Conclusion

This paper contributes to the studies of using computational tools in creative design processes. By focusing 3DGS method, its computational implementation and practical application study illustrate an approach to designing architectural structures. The design results support the efficacy of the geometrical operation-based method for creating structural forms made up of parts that are perfectly in geometrical equilibrium with external forces. This approach is a sustainable way of designing complex spatial structures, particularly from-active structural typologies.

Standard CAD software supports architectural design creation through geometric operation without information on the tectonic quality of obtained forms. On the other hand, structural analysis is usually performed based on these geometries in specialized engineering software after the design is established. This analytical step requires a relatively long time, and structural analysis feedback may ask for design modifications. Since the structural behavior of complex forms is difficult to predict, the possibility of early feedback on the structure is essential.

In contrast, generative form-finding, that become lately widespread in architectural design due to the development of various digital tools, enables a structurally informed design process. Interacting with structures characterized by a force-geometry relationship, the designer can directly observe structural behavior while exploring possible forms. This encourages an exploratory approach to design and supports unconventional solutions that integrate and respond to design intent. The structural and dimensional evaluation of the form is not an afterthought but an essential part of the design process.

The field of GS continues to evolve, and some of the future perspectives of the development of this field that could be considered may include integration with computational methods relying on hybrid approaches that leverage the benefits of both graphical and computational methods. Previous could facilitate more efficient and accurate analysis of complex structures. Also, developing (specialized) digital software tools for GS will likely continue. Besides tools that can be integrated within CAD systems, tools that can be integrated with the Building Information Modelling (BIM) systems could be developed to provide real-time feedback during the design process and enable seamless integration of GS with other engineering disciplines. Furthermore, advancements in computer graphics and visualization techniques could enable research of more immersive and interactive force and form diagrams using virtual reality (VR) and augmented reality (AR) technologies.

Furthermore, GS principles could be applied to design, analyze, and optimize advanced structural systems that involve more complex geometries and behaviors, including tensegrity structures, deployable structures, and adaptive structures. GS can play a role in analyzing and optimizing structures for material efficiency and structural performances and developing sustainable solutions, contributing to the overall goal of sustainable, environmentally conscious design. Moreover, GS can be applied in the design and optimization of 3D-printed structures for the production of innovative and efficient designs that leverage the unique capabilities of digital fabrication processes. Finally, although GS has a research history and strong theoretical foundations, future perspectives include further research and educational efforts to explore and expand the application of GS in various fields, including architecture.

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