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Admissible replacements for simplicial monoidal model categories

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Using Dugger's construction of universal model categories, we produce replacements for simplicial and combinatorial symmetric monoidal model categories with better operadic properties. Namely, these replacements admit a model structure on algebras over any given colored operad.

As an application, we show that in the stable case, such symmetric monoidal model categories are classified by commutative ring spectra when the monoidal unit is a compact generator. In other words, they are strong monoidally Quillen equivalent to modules over a uniquely determined commutative ring spectrum.

1 Introduction

A symmetric monoidal model category is a setting to study structured objects such as monoids, commutative monoids or modules using homotopy theoretic tools. However, one does not always have a model structure on these structured objects. For example, commutative monoids in chain complexes (commutative DGAs) are not known to carry a model structure induced from the underlying model category of chain complexes. The general problem of transferring a model structure to the categories of monoids and modules is studied by Schwede and Shipley in [19] and it is shown that these lifts exist under mild hypotheses. In [11, Section 4.5.4], Lurie studies this transfer problem for commutative monoids and this lifting problem requires a stronger hypothesis, which may be verified, though, for symmetric spectra with the positive stable model structure. More generally, one considers the algebras over a colored symmetric operad in a given symmetric monoidal model category. The transfer problem in this generality was studied by Pavlov and Scholbach [14].

The main result of this paper is that any combinatorial simplicial symmetric monoidal model category may be replaced, up to strong symmetric monoidal Quillen equivalence, by a model category allowing for algebras over any symmetric operad to have a model structure transferred from the underlying category. As an application, we show that, under mild conditions, any stable symmetric monoidal model category is strong symmetric monoidally Quillen equivalent to a category of modules over *commutative* ring spectrum. This result generalizes the theorem by Schwede and Shipley [21, Theorem 3.1.1]. In the framework of stable ∞–categories, a similar result was obtained by Lurie [11, Proposition 7.1.2.7]. Our approach provides a refinement of Lurie's theorem to the realm of stable combinatorial model categories.

Pavlov and Scholbach [16] call a symmetric monoidal model category *admissible* if for every colored symmetric operad $\mathcal{O}$, the category of $\mathcal{O}$–algebras carry a model structure transferred from the underlying category. According to Pavlov and Scholbach, a monoidal model category is *nice* if it is h–monoidal, pretty small, flat and tractable. Detailed definitions may be found in Section 3.4.

Pavlov and Scholbach's main result in [16] is that for a nice symmetric monoidal model category $\mathcal{M}$, there exists an admissible replacement of $\mathcal{M}$, up to a strong symmetric monoidal Quillen equivalence. This replacement is the category of symmetric spectra over $\mathcal{M}$.

In this paper, we show that any simplicial combinatorial symmetric monoidal model category is strong symmetric monoidally Quillen equivalent to a nice symmetric monoidal model category. Combining with the result of Pavlov and Scholbach, we obtain that such model categories have replacements with admissible symmetric monoidal model categories.

These replacements have many applications. For example, our admissible replacement is a Goerss–Hopkins context. This is again due to the results of Pavlov and Scholbach [16, Theorem 1.6]. This means that Goerss–Hopkins obstruction theory can be applied in this replacement to obtain commutative monoids from commutative monoids in the homotopy category. Furthermore, one has a strictification for E_∞–algebras. In other words, E_∞–algebras are Quillen equivalent to commutative monoids in this setting.

Notation 1.0.1   All our monoidal categories are symmetric monoidal. Therefore we say monoidal when we mean symmetric monoidal. Similarly for model categories, functors and Quillen pairs.

Notation 1.0.2 There are two notions of monoidal Quillen equivalences defined by Schwede and Shipley [20]: weak monoidal and strong monoidal. All the monoidal Quillen equivalences we talk about are strong monoidal Quillen equivalences.

Theorem 1.0.3 *Every combinatorial and simplicial symmetric monoidal model category is symmetric monoidally Quillen equivalent to an admissible symmetric monoidal model category. An analogous result holds for monoidal model categories that are not necessarily symmetric.*

Remark 1.0.4 In [22], Shipley constructs a model structure for commutative ring spectra where cofibrant commutative ring spectra forget to spectra that are cofibrant in a nonpositive model structure. The admissible replacement provided by the proof of the theorem above also satisfies a similar property. In this admissible replacement, cofibrant commutative monoids forget to cofibrant objects in a nonpositive model structure. Indeed, this is true for algebras over various operads. This is a consequence of Pavlov and Scholbach [16, Proposition 2.3.10 and Theorem 4.4].

The admissible replacement we construct satisfies further properties due to Pavlov and Scholbach [16, Theorems 4.6 and 4.9]. These are stated in the following theorems.

Theorem 1.0.5 *Every combinatorial and simplicial symmetric monoidal model category is symmetric monoidally Quillen equivalent to an admissible symmetric monoidal model category $\mathcal{C}$ where a weak equivalence of operads $O \to P$ in $\mathcal{C}$ induces a Quillen equivalence between the model categories of O–algebras and P–algebras.*

Remark 1.0.6 We prove Theorems 1.0.3 and 1.0.5 in a more general setting; see Theorem 3.5.5. Namely, these theorems are true for combinatorial $\mathcal{V}$ symmetric monoidal model categories for every symmetric monoidal model category $\mathcal{V}$ satisfying the axioms listed in Notation 3.0.1. For instance, the model categories of pointed simplicial sets and chain complexes over a field satisfy these axioms.

In the theorem below, $\mathcal{O}$ denotes an operad in simplicial sets and $N^{\otimes}\mathcal{O}$ denotes the operadic nerve of $\mathcal{O}$.

Theorem 1.0.7 *Every combinatorial and simplicial symmetric monoidal model category is symmetric monoidally Quillen equivalent to an admissible symmetric monoidal category $\mathcal{C}$ where the underlying ∞–category of the model category of $\mathcal{O}$–algebras in $\mathcal{C}$ is equivalent to the ∞–category of $N^{\otimes}\mathcal{O}$–algebras on the underlying ∞–category of $\mathcal{C}$.*

Remark 1.0.8 The resulting symmetric monoidal model categories in Theorems 1.0.3, 1.0.5 and 1.0.7 are the same. In other words, for a given combinatorial and simplicial symmetric monoidal model category, we construct a replacement that satisfies all the properties given in Theorems 1.0.3, 1.0.5 and 1.0.7.

Remark 1.0.9 Applying our constructions to combinatorial and simplicial braided monoidal model categories, one obtains the following. Every combinatorial and simplicial braided monoidal model category is monoidally Quillen equivalent to another braided monoidal model category that satisfies the properties given in Theorems 1.0.3, 1.0.5 and 1.0.7 with respect to nonsymmetric operads.

To prove these results, we first replace the given monoidal model category by another one with a cofibrant monoidal unit if necessary. This is obtained using the results of Muro [12]; see Section 3.1. The replacement also satisfies the hypothesis of the theorems above.

For a monoidal model category $\mathcal{M}$ as in the Theorem 1.0.3 with cofibrant unit, we obtain a zig-zag of Quillen equivalences,

$$(1) \qquad \mathcal{M} \underset{E}{\overset{D}{\rightleftarrows}} L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}] \underset{Ev_0}{\overset{F_0}{\rightleftarrows}} \mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}], E).$$

Here, $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}]$ denotes the universal model category construction of Dugger. This is a localization of the simplicial presheaves on a set of λ–presentable cofibrant objects of $\mathcal{M}$. We equip this category with a monoidal structure using the Day convolution. The arrows at the top denote the left adjoints. The right adjoint functor E is the restricted Yoneda embedding and D is the left adjoint to E defined in (3).

We show that $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}]$ satisfies further properties with respect to its monoidal structure. This allows us to use the results of Pavlov and Scholbach [16] to show that the monoidal model category $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}], E)$ is admissible. To be precise, let E denote the commutative monoid in symmetric sequences in $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}]$ given by the monoidal unit of $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}]$ at each degree. Here, $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}], E)$ denotes the monoidal model category of E–modules. Indeed, this is the same as the category of I–spaces in $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}]$ of Sagave and Schlichtkrull [18]. The adjoint pair on the right-hand side is the standard one given by Hovey in [9, Definition 7.3]; for instance, Ev_0 is the evaluation at degree 0. The pair $F_0 \dashv Ev_0$ is a Quillen pair when $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}], E)$ is given the stable model structure. Furthermore,

$\text{Sp}(L_S[\mathcal{M}^{\text{op}}_{\lambda,\text{cof}}, \mathcal{S}], E)$ is admissible with the positive stable model structure which is Quillen equivalent via the identity functor to the stable model structure. We omit this detail in (1).

Gabriel's theorem provides a classification of cocomplete abelian categories with a single small projective generator as categories of modules. Schwede and Shipley prove a similar result for stable homotopy theory. They show that every stable, simplicial, cofibrantly generated and proper model category with a compact generator is Quillen equivalent to modules over a ring spectrum [21, Theorem 3.1.1]. We use Theorem 1.0.3 to prove a monoidal version of this result.

Theorem 1.0.10 *Let $\mathcal{M}$ be a stable, combinatorial and simplicial symmetric monoidal model category whose monoidal unit is a compact generator. In this situation, $\mathcal{M}$ is strong symmetric monoidally Quillen equivalent to R–modules where R is a commutative ring spectrum.*

Furthermore, R is uniquely determined in the following sense. If the monoidal unit of $\mathcal{M}$ is cofibrant and if $\mathcal{M}$ is (strong or weak) symmetric monoidally Quillen equivalent to R'–modules for another commutative ring spectrum R' where each monoidal model category in the zig-zag has a cofibrant unit, then R and R' are weakly equivalent as commutative ring spectra.

The proof of this result makes essential use of the proof of Theorem 1.0.3. First, in the zig-zag of Quillen equivalences in (1), instead of E, we use the symmetric sequence corresponding to the simplicial tensor $S^1 \otimes -$. This provides a spectral replacement of the given model category which allows us to obtain the derived endomorphism spectrum of the unit. To make sure this derived endomorphism spectrum is a commutative ring spectrum, one needs to take a fibrant replacement of the unit as a commutative ring spectrum. For this, we make use of admissibility.

As we mentioned earlier, the ∞–categorical analogue of this result is due to Lurie [11, Proposition 7.1.2.7]. To our knowledge, there is no straightforward way of obtaining our result from Lurie's. Nikolaus and Sagave [13] show that presentably symmetric monoidal ∞–categories come from symmetric monoidal model categories. However, if the given presentably symmetric monoidal ∞–category is obtained from a symmetric monoidal model category, it is not known if the construction of Nikolaus and Sagave gives back the monoidal model category one starts with. On the other hand, the uniqueness part of Theorem 1.0.10 follows by Lurie's theorem.

Outline In Section 2, we start with a discussion on monoidal model categories and monoidal Quillen equivalences. After that, we define Day convolution and combinatorial monoidal model categories. Section 3 is devoted to the proof of Theorems 1.0.3, 1.0.5 and 1.0.7. In Section 4, we prove Theorem 1.0.10.

2 Preliminaries

2.1 Monoidal model categories

We recall the theory of monoidal model categories and enriched monoidal model categories. As mentioned earlier, we say monoidal when we mean symmetric monoidal for categories, model categories and functors. Furthermore, by a monoidal category, we mean a closed symmetric monoidal category. See Definitions 4.1.1, 4.1.4 and 4.1.12 in Hovey [8] for the definition of a closed symmetric monoidal category. In the following definition, our unit axiom is stronger than that of [8] since we don't assume X to be cofibrant. This is what is called the very strong unit axiom by Muro in [12].

Definition 2.1.1 A *monoidal model category* $\mathcal{M}$ is a model category whose underlying category is a monoidal category $(\mathcal{M}, \otimes, \mathbb{I})$ with a product $\otimes$ and a unit $\mathbb{I}$ such that the monoidal structure satisfies the following compatibility conditions with respect to the model structure on $\mathcal{M}$.

(1) **Pushout-product axiom** For two cofibrations $f\colon U \to V$ and $g\colon X \to Y$, the map

$$f \,\square\, g\colon U \otimes Y \amalg_{U \otimes X} V \otimes X \to V \otimes Y$$

is a cofibration. Furthermore, this is a weak equivalence if either f or g is a weak equivalence.

(2) **Unit axiom** There is a cofibrant replacement $c\mathbb{I} \xrightarrow{\sim} \mathbb{I}$ of the unit such that for every X in $\mathcal{M}$, the map

$$(c\mathbb{I}) \otimes X \to \mathbb{I} \otimes X \cong X$$

is a weak equivalence.

Remark 2.1.2 The second axiom above is satisfied if $\mathbb{I}$ is cofibrant.

Remark 2.1.3 Schwede and Shipley define weak and strong monoidal Quillen equivalences in [20]. They show that monoidal Quillen equivalences induce Quillen equivalences at the level of monoids and modules; see [20, Theorem 3.12]. All our Quillen equivalences are indeed strong monoidal Quillen equivalences; therefore, when we say monoidal Quillen equivalence, we mean strong monoidal Quillen equivalence. Except in Section 3.1, we only consider monoidal Quillen equivalences between monoidal model categories with cofibrant units.

A *monoidal functor* $F \colon \mathcal{C} \to \mathcal{D}$ between two monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{I}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{I}_{\mathcal{D}})$ is a functor that is equipped with natural isomorphisms

$$F(C_1) \otimes_{\mathcal{D}} F(C_2) \cong F(C_1 \otimes_{\mathcal{C}} C_2) \quad \text{and} \quad I_{\mathcal{D}} \cong F(I_{\mathcal{C}}),$$

which are coherently symmetric, associative and unital; see Borceux [3, 6.4.1].

Definition 2.1.4 (Schwede–Shipley [20, Definition 3.6]) A *monoidal Quillen equivalence* between monoidal model categories with cofibrant units is a Quillen equivalence where the left adjoint is a monoidal functor.

For a symmetric monoidal model category $\mathcal{V}$, a $\mathcal{V}$ model category is a model category with an action of $\mathcal{V}$ that is compatible with the model structures; see Hovey [8, Definition 4.2.18]. A $\mathcal{V}$ monoidal model category is a $\mathcal{V}$ model category where the $\mathcal{V}$ action is compatible with the monoidal structure. This can be formulated as in the following definition.

Definition 2.1.5 (Hovey [8, Definition 4.2.20]) Let $\mathcal{V}$ and $\mathcal{C}$ be monoidal model categories. We say that $\mathcal{C}$ is a V monoidal model category if there is a left Quillen monoidal functor

$$F \colon \mathcal{V} \to \mathcal{C}.$$

In this situation, the action of an object V in $\mathcal{V}$ on C in $\mathcal{C}$ is given by

$$V \otimes C := F(V) \otimes_{\mathcal{C}} C.$$

2.2 Day convolution for monoidal categories

Let $\mathcal{V}$ be a monoidal category. Given a small monoidal $\mathcal{V}$ enriched category $(\mathcal{C}, \otimes, \mathbb{I})$, we consider the category of V–enriched functors $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. We use the monoidal structure on $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ due to Day [5],

$$F \otimes_{\mathrm{Day}} G = \int^{C_1, C_2 \in \mathcal{C}} \mathrm{hom}_{\mathcal{C}}(-, C_1 \otimes C_2) \otimes F(C_1) \otimes G(C_2) \quad \text{for all } F, G \in \mathcal{V}^{\mathcal{C}^{\mathrm{op}}}.$$

This is called the Day convolution. With this monoidal product, $([\mathcal{C}^{op}, \mathcal{V}], \otimes_{Day}, Y(\mathbb{I}))$ becomes a closed symmetric monoidal category where

$$Y : \mathcal{C} \to [\mathcal{C}^{op}, \mathcal{V}]$$

denotes the Yoneda embedding given by

$$Y(C)(-) = \hom_{\mathcal{C}}(-, C)$$

for every C in $\mathcal{C}$. We use Day convolution as our monoidal product on the presheaf category because Y becomes a strong monoidal functor in this situation.

There is also the pointwise monoidal structure on the presheaf category $[\mathcal{C}^{op}, \mathcal{V}]$ but this does not suit our purposes. This is because this monoidal product makes no reference to the monoidal structure on $\mathcal{C}$ and therefore does not render Y into a strong monoidal functor in general.

2.3 Combinatorial monoidal model categories

Here, we provide an overview of combinatorial model categories and we define what we mean by combinatorial monoidal model categories.

Let $\mathcal{C}$ be a category and let C be an object in $\mathcal{C}$. For a regular cardinal λ, we say C is *λ–presentable* if mapping out of C commutes with λ–filtered colimits.

A category $\mathcal{C}$ is said to be *locally λ–presentable* if it is cocomplete and if there is a set of λ–presentable objects in $\mathcal{C}$ such that every object of $\mathcal{C}$ is a λ–filtered colimit of objects in this set. We say $\mathcal{C}$ is *locally presentable* if it is *locally λ–presentable* for some regular cardinal λ.

We say a model category $\mathcal{M}$ is *λ–combinatorial* if it admits sets of generating (acyclic) cofibrations with λ–presentable (co)domains and if it is locally λ–presentable; see Barwick [1, Definition 1.21]. We say $\mathcal{M}$ is *combinatorial* if it is λ–combinatorial for some regular cardinal λ.

In a λ–locally presentable category $\mathcal{C}$, isomorphism classes of λ–presentable objects form a set [6, Section 2] we call the corresponding small full subcategory $\mathcal{C}_\lambda$. For a λ–combinatorial model category $\mathcal{M}$, we denote the cofibrant objects of $\mathcal{M}_\lambda$ by $\mathcal{M}_{\lambda,cof}$.

For monoidal model categories, we use a stronger notion of combinatoriality. For this, we use the definition of locally λ–presentable base due to Borceux, Quinteiro and Rosický [4, Definition 1.1]. A monoidal category C is said to be a *locally λ–presentable*

base if it is a locally λ–presentable category, if $\mathcal{C}_\lambda$ contains the monoidal unit and if $\mathcal{C}_\lambda$ is closed under monoidal products.

Definition 2.3.1 A monoidal model category $\mathcal{M}$ is λ–combinatorial if it is cofibrantly generated and its underlying monoidal category is a locally λ–presentable base. In other words, if $\mathcal{M}$ is λ–combinatorial as a model category and if the monoidal structure on $\mathcal{M}$ gives a monoidal structure on $\mathcal{M}_\lambda$, then we say the monoidal model category $\mathcal{M}$ is λ–combinatorial. This is equivalent to $\mathcal{M}$ being λ–combinatorial as a category enriched over itself.

Furthermore, $\mathcal{M}$ is said to be combinatorial if it is λ–combinatorial for some regular cardinal λ.

Lemma 2.3.2 *Let $\mathcal{N}$ be a model category and J be a regular cardinal. Assume that $\mathcal{N}$ admits a set of generating cofibrations with J–presentable (co)domains. Given a J–filtered colimit $\mathrm{colim}_{J'}\, N_j$ for some $J \leq J'$, the canonical map*

$$\operatorname*{hocolim}_{J'} N_j \xrightarrow{\ \sim\ } \operatorname*{colim}_{J'} N_j$$
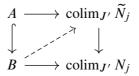
is a weak equivalence.

Proof There is an adjunction

$$\mathcal{N} \xleftarrow[\mathrm{cnst}]{\mathrm{colim}} [J', \mathcal{N}]$$

where the right adjoint cnst provides the constant diagram and the left adjoint denoted by colim takes the corresponding colimit in $\mathcal{N}$. Furthermore, the functor category is given the projective model structure. Let N_* denote the corresponding diagram in the diagram category and let a trivial fibration $\widetilde{N}_* \xrightarrow{\sim} N_*$ provide a cofibrant replacement of N_*. We need to show that the induced map

$$\mathrm{colim}(\widetilde{N}_*) = \operatorname*{hocolim}_{J'} N_j \to \operatorname*{colim}_{J'} N_j$$

is a weak equivalence where the equality above follows by the definition of homotopy colimits. Indeed, we show that the map above is a weak equivalence by showing that it satisfies the left lifting property with respect to the generating cofibrations of $\mathcal{N}$. Let $A \hookrightarrow B$ denote a generating cofibration of $\mathcal{N}$. We need to solve the lifting problem

$$
\begin{array}{ccc}
A & \longrightarrow & \mathrm{colim}_{J'}\, \widetilde{N}_j \\
\downarrow & \nearrow & \downarrow \\
B & \longrightarrow & \mathrm{colim}_{J'}\, N_j
\end{array}
$$

Since A and B are J–presentable, we obtain that the horizontal maps out of them in the diagram above factors through $\widetilde{N}_j \to N_j$ for some j. Since $\widetilde{N}_* \xrightarrow{\sim} N_*$ is a levelwise trivial fibration, the map $\widetilde{N}_j \to N_j$ is a trivial fibration and therefore one obtains the desired lifting. □

The following corollary should be compared to [6, Proposition 7.3].

Corollary 2.3.3 *Let $\mathcal{N}$ be a model category and J be a regular cardinal. Assume that $\mathcal{N}$ admits a set of generating cofibrations with J–presentable (co)domains. Furthermore, let J' be a regular cardinal with $J \leq J'$. Given a levelwise equivalence $N_* \xrightarrow{\sim} N_*'$ of diagrams over J', the induced map*

$$\operatorname*{colim}_{J'} N_j \to \operatorname*{colim}_{J'} N_j'$$

is a weak equivalence.

Proof There is a commuting diagram

$$
\begin{array}{ccc}
\operatorname{hocolim}_{J'} N_j & \xrightarrow{\;\cong\;} & \operatorname{hocolim}_{J'} N_j' \\
\downarrow & & \downarrow \\
\operatorname{colim}_{J'} N_j & \xrightarrow{\hspace{2cm}} & \operatorname{colim}_{J'} N_j'
\end{array}
$$

where the vertical maps are weak equivalences due to Lemma 2.3.2. The result follows by the two out of three property of weak equivalences. □

3 Admissible replacement

In this section, we prove Theorem 1.0.3. In other words, we show that every combinatorial and simplicial monoidal model category is monoidally Quillen equivalent to an admissible model category. More generally, we prove this result for combinatorial $\mathcal{V}$ monoidal model categories where $\mathcal{V}$ denotes a symmetric monoidal model category satisfying the axioms stated in the following.

Notation 3.0.1 For the rest of this work, let $\mathcal{V}$ denote a combinatorial symmetric monoidal model category satisfying the following properties.

(1) Every object of $\mathcal{V}$ is cofibrant.

(2) There is a set of generating cofibrations of $\mathcal{V}$ where (co)domains of the generating cofibrations are $\aleph_0$–presentable.

(3) The model category $\mathcal{V}$ is left proper.

The axioms above are satisfied by the model categories of simplicial sets, pointed simplicial sets and chain complexes over a field k.

For our constructions, it is important that we start with a monoidal model category whose monoidal unit is cofibrant. In Section 3.1, we use a theorem of Muro [12] to show that a given monoidal model category satisfying the hypothesis of Theorem 1.0.3 can be replaced with a monoidal model category whose unit is cofibrant. Furthermore, this replacement is also a combinatorial $\mathcal{V}$ symmetric monoidal model category. Therefore, for the rest of this section, we assume that we start with a monoidal model category with a cofibrant unit.

Let $\mathcal{M}$ be a monoidal model category as in Theorem 1.0.3 whose monoidal unit is cofibrant. To prove Theorem 1.0.3, we need to construct the zig-zag of monoidal Quillen equivalences in (1) and prove that $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}])$ is admissible. The first Quillen equivalence is constructed in Section 3.2. Section 3.4 is devoted to the proof of the fact that $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is nice in the sense of Pavlov and Scholbach [16, Definition 2.3.1]. Proposition 3.5.3 provides the Quillen equivalence on the right-hand side and the admissibility of $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}])$ is given in Theorem 3.5.1.

3.1 Cofibrant monoidal unit

Using Muro's results, we show that every combinatorial and $\mathcal{V}$ monoidal model category carries a monoidally Quillen equivalent model structure where the monoidal unit is cofibrant [12].

Let $\mathcal{M}$ be a combinatorial $\mathcal{V}$ monoidal model category. Theorem 1 in Muro [12] provides a new model structure $\widetilde{\mathcal{M}}$ on the same underlying category whose monoidal unit is cofibrant. The weak equivalences of $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ are the same but $\widetilde{\mathcal{M}}$ possibly has more cofibrations than $\mathcal{M}$, ie cofibrations of $\mathcal{M}$ are also cofibrations in $\widetilde{\mathcal{M}}$. In particular, the identity functor is a left Quillen functor

$$\mathcal{M} \to \widetilde{\mathcal{M}}$$

which is the left adjoint of a monoidal Quillen equivalence. Since the unit of $\mathcal{M}$ is not cofibrant, we refer the reader to Schwede and Shipley [20, Definition 3.6] for the definition of monoidal Quillen equivalences instead of the one given in Section 2.1.

Furthermore, $\widetilde{\mathcal{M}}$ is combinatorial and monoidal; see Muro [12, Theorem 1]. We only need to show that $\widetilde{\mathcal{M}}$ is also a $\mathcal{V}$ monoidal model category. This amounts to having a monoidal left Quillen functor $\mathcal{V} \to \widetilde{\mathcal{M}}$. Since $\mathcal{M}$ is $\mathcal{V}$ monoidal, there is a monoidal left Quillen functor

$$F : \mathcal{V} \to \mathcal{M}$$

and composing this with the left Quillen functor induced by the identity functor, we obtain the desired functor $\mathcal{V} \to \widetilde{\mathcal{M}}$. This shows that $\widetilde{\mathcal{M}}$ is a $\mathcal{V}$ monoidal model category. We obtain the following version of Muro's theorem.

Theorem 3.1.1 (Muro [12]) *Every combinatorial $\mathcal{V}$ symmetric monoidal model category is symmetric monoidally Quillen equivalent to a combinatorial $\mathcal{V}$ symmetric monoidal model category whose unit is cofibrant.*

3.2 Replacement with the presheaf category

Here, we construct the first monoidal Quillen equivalence in (1). This is analogous to Dugger's construction of universal model categories. We obtain a replacement with a localization of the $\mathcal{V}$-enriched presheaves on the cofibrant λ–presentable objects of the given monoidal model category (for a sufficiently large cardinal λ). This presheaf category equipped with the Day convolution satisfies further properties on its monoidal structure and this guarantees that the symmetric spectra on the presheaf category (with the positive stable model structure) is admissible.

Let $(\mathcal{M}, \wedge, \mathbb{I}_{\mathcal{M}})$ be a combinatorial $\mathcal{V}$ monoidal model category with a cofibrant monoidal unit. Let λ denote a regular cardinal for which the following are satisfied:

(1) The symmetric monoidal model category $\mathcal{M}$ is λ–combinatorial.

(2) The cofibrant replacement functor in $\mathcal{M}$ preserves λ–filtered colimits.

(3) A cofibrant replacement of a λ–presentable object is λ–presentable.

The last two items above follow by Dugger [6, Proposition 2.3].

Suppose $\mathcal{M}$ is λ–combinatorial for some cardinal λ and let $\mathcal{M}_{\lambda,\mathrm{cof}}$ denote the subcategory of λ–presentable cofibrant objects.

We consider the category of $\mathcal{V}$–enriched functors and $\mathcal{V}$–natural transformations from $\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}$ to $\mathcal{V}$. Let $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ denote this category. There is a fully faithful functor

$$Y : \mathcal{M}_{\lambda,\mathrm{cof}} \to [\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$$

given by the Yoneda embedding. In other words, $Y(M) = \hom(-, M)$ for every $M \in \mathcal{M}_{\lambda,\mathrm{cof}}$. For the inclusion functor $I : \mathcal{M}_{\lambda,\mathrm{cof}} \to \mathcal{M}$, there is the left Kan extension

(2)

$$
\begin{array}{c}
\mathcal{M}_{\lambda,\mathrm{cof}} \\
\downarrow{\scriptstyle I} \qquad \searrow{\scriptstyle Y} \\
\mathcal{M} \xleftarrow{\ \ \ D\ \ \ } [\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]
\end{array}
$$

which makes the above diagram commute up to a natural isomorphism; see Kelly [10, Proposition 4.23]. Furthermore by Kelly [10, Equation 4.25] and the Yoneda lemma, D is given by the coend

(3)
$$
D(F) = \int^{M \in \mathcal{M}_{\lambda,\mathrm{cof}}} F(M) \otimes I(M).
$$

By Kelly [10, Equations 3.5 and 3.70], D is the left adjoint of the restricted Yoneda functor E given by $E(M)(M_i) = \hom(M_i, M)$ for each $M \in \mathcal{M}$ and $M_i \in \mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}$. We obtain the adjoint pair

$$
\mathcal{M} \underset{E}{\overset{D}{\rightleftarrows}} [\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}].
$$

There is a model structure on $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ where weak equivalences and fibrations are given by levelwise weak equivalences and levelwise fibrations. This is called the projective model structure. The generating cofibrations and trivial cofibrations of the projective model structure are given by

(4)
$$
\begin{aligned}
I' &= \{Y(A) \otimes i \mid A \in \mathcal{M}_{\lambda,\mathrm{cof}}, i \in I\}, \\
J' &= \{Y(A) \otimes j \mid A \in \mathcal{M}_{\lambda,\mathrm{cof}}, j \in J\},
\end{aligned}
$$

respectively, where I and J are generating sets of cofibrations and trivial cofibrations of $\mathcal{V}$, respectively. By (3), it is clear that D commutes with tensoring with morphisms in $\mathcal{V}$. Therefore,

$$
D(Y(A) \otimes i) = D(Y(A)) \otimes i \cong A \otimes i
$$

for every object A in $\mathcal{M}_{\lambda,\mathrm{cof}}$ and morphism i in $\mathcal{V}$. Since $\mathcal{M}$ is a $\mathcal{V}$–model category and A is cofibrant, $A \otimes i$ is a (trivial) cofibration for every (trivial) cofibration i. This shows that D preserves generating cofibrations and trivial cofibrations. We obtain that $D \dashv E$ is indeed a Quillen pair.

Now we prove that the left Quillen functor D is homotopically surjective; see Dugger [6, Definition 3.1]. We start with the following lemma.

Lemma 3.2.1 *The generating cofibrations of $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ given above have $\aleph_0$–presentable (co)domains.*

Proof We need to show that the morphisms in I' have $\aleph_0$–presentable (co)domains. Due to our standing assumptions, there is a set of generating cofibrations I of $\mathcal{V}$ consisting of maps with $\aleph_0$–presentable (co)domains. In what follows, for a given category $\mathcal{C}$, we denote the set of morphisms in $\mathcal{C}$ by $\mathcal{C}(-,-)$.

Given a filtered colimit $\mathrm{colim}_{j \in J} M_j$ in $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ for some $\aleph_0$–filtered category J, an $\aleph_0$–presentable K in $\mathcal{V}$ and an object A in $\mathcal{M}_{\lambda,\mathrm{cof}}$, we have

$$
\begin{aligned}
[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}](R_A \otimes K, \operatorname*{colim}_{j \in J} M_j) &\cong \mathcal{V}(K, \hom(R_A, \operatorname*{colim}_{j \in J} M_j)) \\
&\cong \mathcal{V}(K, (\operatorname*{colim}_{j \in J} M_j)(A)) \\
&\cong \mathcal{V}(K, \operatorname*{colim}_{j \in J} \hom(R_A, M_j)) \\
&\cong \operatorname*{colim}_{j \in J} \mathcal{V}(K, \hom(R_A, M_j)) \\
&\cong \operatorname*{colim}_{j \in J} [\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}](R_A \otimes K, M_j).
\end{aligned}
$$

The second and the third equalities follow by the Yoneda lemma, and the first and the last equalities follow because $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is a $\mathcal{V}$–model category. The fourth equality follows by the assumption that K is $\aleph_0$–presentable. In the equalities above, $\hom(-,-)$ denotes the $\mathcal{V}$–enriched maps in $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$. Since the (co)domains of the maps in I' are of the form $R_A \otimes K$, the chain of isomorphisms above provides the desired result. $\square$

Proposition 3.2.2 *The left Quillen functor D is homotopically surjective.*

Proof We need to show that for every fibrant $M \in \mathcal{M}$, the natural map

$$D[E(M)]^{\mathrm{cof}} \to M$$

is a weak equivalence where $[E(M)]^{\mathrm{cof}}$ denotes a cofibrant replacement of $E(M)$. There is a J–filtered colimit
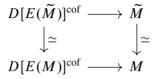
$$M = \operatorname*{colim}_J M_j$$

for some $\lambda \leq J$ such that each M_j is in $\mathcal{M}_\lambda$. Using the functorial cofibrant replacement in $\mathcal{M}$ that preserves λ–filtered colimits, one obtains another J–diagram $\tilde{M}_*$ with compatible trivial fibrations $\tilde{M}_j \overset{\sim}{\twoheadrightarrow} M_j$ such that each $\tilde{M}_j$ is in $\mathcal{M}_{\lambda,\mathrm{cof}}$. This factorization

guarantees that the induced map

$$\operatorname*{colim}_{J} \tilde{M}_j \xrightarrow{\sim}\twoheadrightarrow \operatorname*{colim}_{J} M_j$$

is a trivial fibration. In particular, $\operatorname*{colim}_{J} \tilde{M}_j$ is fibrant. Let $\tilde{M}$ denote $\operatorname*{colim}_{J} \tilde{M}_j$.

Due to the diagram

$$
\begin{array}{ccc}
D[E(\tilde{M})]^{\mathrm{cof}} & \longrightarrow & \tilde{M} \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
D[E(M)]^{\mathrm{cof}} & \longrightarrow & M
\end{array}
$$

it is sufficient to show that the top horizontal map above is a weak equivalence. The left-hand vertical map above is a weak equivalence as E (resp. D) preserves weak equivalences between fibrant (resp. cofibrant) objects.

We obtain a cofibrant replacement of $E(\tilde{M})$ as

$$\operatorname*{hocolim}_{J} \hom(-, \tilde{M}_j) \xrightarrow{\sim} \operatorname*{colim}_{J} \hom(-, \tilde{M}_j) \cong E(\tilde{M}),$$

where the map above is a weak equivalence due to Lemmas 3.2.1 and 2.3.2. The isomorphism above follows because

$$E(\tilde{M})(N) = \hom\big(N, \operatorname*{colim}_{J} \tilde{M}_j\big)$$

and because mapping out of the objects of $\mathcal{M}_{\lambda,\mathrm{cof}}$ preserves λ–filtered colimits.

The top horizontal map in the square above is given by composing in the following commuting diagram:

$$
\begin{array}{ccc}
D[E(\tilde{M})]^{\mathrm{cof}} = D(\operatorname*{hocolim}_{J} \hom(-, \tilde{M}_i)) \xrightarrow{\cong} \operatorname*{hocolim}_{J} D(\hom(-, \tilde{M}_j)) \\
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\simeq} \\
D(\operatorname*{colim}_{J} \hom(-, \tilde{M}_j)) \xrightarrow{\cong} \operatorname*{colim}_{J} D(\hom(-, \tilde{M}_j)) \xrightarrow{\cong} \operatorname*{colim}_{J} \tilde{M}_j = \tilde{M}
\end{array}
$$

Each $\hom(-, \tilde{M}_j)$ above is cofibrant; see (4). Therefore, $D(\hom(-, \tilde{M}_j))$ provides the correct homotopy type and one obtains that the top horizontal arrow is an equivalence as D preserves homotopy colimits between cofibrant objects. The vertical arrow on the right-hand side is a weak equivalence due to Lemma 2.3.2. Finally, the isomorphisms $D(\hom(-, \tilde{M}_j)) \cong \tilde{M}_j$ follows by diagram (2). This provides the desired result. $\qquad\square$

Furthermore, $D \dashv E$ becomes a Quillen equivalence after a localization of $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$. Since $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ is left proper and combinatorial, this follows by Dugger [6, Proposition 3.2]. Indeed, this localization can be described as follows. Let λ' be a regular cardinal and let $S_{\lambda'}$ be the set of maps given by the first factor in the factorization of the natural maps

(5) $$N \to E(fD(N))$$

as a cofibration followed by a trivial fibration for every cofibrant and N in $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]_{\lambda'}$ where f denotes a fibrant replacement functor. As before, $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]_{\lambda'}$ denotes the subcategory of λ'–presentable objects in $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$. The proof of Dugger [6, Proposition 3.2] shows that there is a large enough λ' for which the localization with respect to $S_{\lambda'}$ renders $D \dashv E$ into a Quillen equivalence. Let S denote $S_{\lambda'}$ for a chosen large enough λ'. We obtain the following.

Proposition 3.2.3 *For the set of maps S defined above, the Quillen adjoint pair $D \dashv E$ induces a Quillen equivalence*

$$\mathcal{M} \xrightleftharpoons[E]{D} L_S[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}].$$

3.3 The monoidal structure on the presheaf category

We equip $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ with a symmetric monoidal product using the Day convolution; see Section 2.2. This makes $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ a monoidal model category; see Batanin and Berger [2, Theorem 4.1].

The levelwise monoidal structure on $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ is not suitable for our purposes because it makes no reference to the monoidal structure on $\mathcal{M}$ and therefore does not render $D \dashv E$ into a monoidal Quillen pair in general.

In order to use the Day convolution, we need to show that $\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}$ is a monoidal category.

Proposition 3.3.1 *The monoidal structure on $\mathcal{M}$ induces a monoidal structure on $\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}$.*

Proof It is sufficient to note that $\mathcal{M}_{\lambda,\mathrm{cof}}$ is closed under the monoidal product and that the unit is in $\mathcal{M}_{\lambda,\mathrm{cof}}$. This is true for $\mathcal{M}_\lambda$ because it is part of our definition of combinatorial monoidal model categories; see Definition 2.3.1. The unit of the monoidal structure is also assumed to be cofibrant and due to the pushout product axiom, the monoidal product of cofibrant objects is cofibrant. □

Finally, we show that $D \dashv E$ is a monoidal Quillen equivalence.

Proposition 3.3.2 *The Quillen equivalence $D \dashv E$ between $\mathcal{M}$ and $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is a strong monoidal Quillen equivalence.*

Proof It is sufficient to show that D is a strong symmetric monoidal functor; see Definition 2.1.4. The unit map of the monoidal structure of D is provided by (2).

For the monoidal structure map of D, we consider two functors F_1 and F_2 that solve the extension problem in the diagram

(6)
$$
\begin{array}{ccc}
\mathcal{M}_{\lambda,\mathrm{cof}} \times \mathcal{M}_{\lambda,\mathrm{cof}} & & \\
\Big\downarrow {\scriptstyle \wedge \circ (I \times I)} & \overset{Y \times Y}{\searrow} & \\
\mathcal{M} & \overset{F_i}{\dashleftarrow} & [\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}] \times [\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]
\end{array}
$$

where $\wedge$ denotes the smash product functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ of $\mathcal{M}$. Furthermore, the functors F_i are defined by the formulas

$$
F_1(A, B) = D(A \otimes_{\mathrm{Day}} B) \quad \text{and} \quad F_2(A, B) = D(A) \wedge D(B).
$$

Using the fact that Y is strong monoidal and the natural isomorphism that makes diagram (2) commute, one observes that for each i, there is a canonical natural isomorphism that makes diagram (6) commute for F_i. Furthermore, the functors F_i are separately enriched cocontinuous in each variable. By the universal property of the category of presheaves, there is a unique such dashed arrow in diagram (6); see Kelly [10, Theorem 4.51]. In particular, we obtain that there is a natural isomorphism $F_2 \cong F_1$ that is compatible with the natural isomorphisms that make the diagram above commute. This isomorphism serves as the monoidal structure map

$$
D(A) \wedge D(B) \cong D(A \otimes_{\mathrm{Day}} B)
$$

of D. This map is commutative, associative and unital as desired. $\qquad\square$

3.4 The localized presheaf category is nice

Here, we prove that $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is nice in the sense of Pavlov and Scholbach [16, Definition 2.3.1]. This ensures that symmetric spectra in $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is an admissible monoidal model category. A monoidal model category is *nice* if it is left proper, pretty small (Pavlov and Scholbach [15, Definition 2.1]), *h*–monoidal (Batanin and Berger [2,

Definition 1.11]), flat [15, Definition 3.2.4] and tractable. We first show that $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is nice.
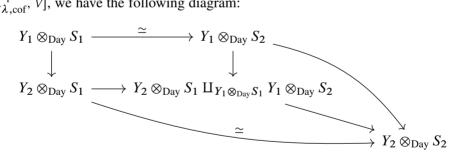
A cofibrantly generated model category $\mathcal{C}$ is pretty small if a set of generating cofibrations for $\mathcal{C}$ have $\aleph_0$–presentable domains and codomains. The presheaf category $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is pretty small due to Lemma 3.2.1.

Recall that every object is cofibrant in $\mathcal{V}$; see Notation 3.0.1. This implies that $\mathcal{V}$ is strongly h–monoidal — see Batanin and Berger [2, Lemma 1.12] — ie $\mathcal{V}$ is h–monoidal (Batanin and Berger [2, Definition 1.11]) and the monoidal product preserves weak equivalences between all objects. Therefore, by Theorem 4.1 of Batanin and Berger [2], $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is also strongly h–monoidal and left proper.

The tractability of $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ follows by (4). The following proposition completes the proof of our claim that $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is nice.

Proposition 3.4.1 *The monoidal model category $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is flat in the sense of Pavlov and Scholbach [15, Definition 3.2.4]. In other words, the pushout product of a cofibration and a weak equivalence is a weak equivalence in $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$.*

Proof Given a cofibration $y\colon Y_1 \hookrightarrow Y_2$ and a weak equivalence $s\colon S_1 \xrightarrow{\sim} S_2$ in $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$, we have the following diagram:

$$
\begin{array}{ccc}
Y_1 \otimes_{\mathrm{Day}} S_1 & \xrightarrow{\ \simeq\ } & Y_1 \otimes_{\mathrm{Day}} S_2 \\
\downarrow & & \downarrow \\
Y_2 \otimes_{\mathrm{Day}} S_1 & \longrightarrow \ Y_2 \otimes_{\mathrm{Day}} S_1 \amalg_{Y_1 \otimes_{\mathrm{Day}} S_1} Y_1 \otimes_{\mathrm{Day}} S_2 \\
 & \xrightarrow[\quad\simeq\quad]{} \qquad\qquad Y_2 \otimes_{\mathrm{Day}} S_2
\end{array}
$$

The morphisms marked as weak equivalences are weak equivalences because $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is strongly h–monoidal and therefore the monoidal product on it preserves weak equivalences between all objects. This also implies that $y \wedge S_1$ is an h–cofibration. Pushouts along h–cofibrations preserve weak equivalences; therefore the bottom horizontal map is a weak equivalence. By two out of three property of weak equivalences, we deduce that the pushout product of y and s is also a weak equivalence. $\square$

Let $\mathcal{C}$ be a left proper, pretty small, tractable and flat monoidal model category and let C be a set of morphisms between cofibrant objects in $\mathcal{C}$. In this case, we say that

the left Bousfield localization L_C is a *monoidal left Bousfield localization* if $f \otimes A$ is a C–local equivalence for every cofibrant A in $\mathcal{C}$ and morphism $f \in C$. This in particular guarantees that $L_C \mathcal{C}$ is a monoidal model category; see Gorchinskiy and Guletskiĭ [7, Lemma 31]. The property of being nice is preserved by monoidal left Bousfield localizations. Pretty smallness only depends on the cofibrations of the given category and left Bousfield localization preserve the properties of being left proper and tractable; see Barwick [1, Proposition 4.12]. Furthermore, flatness and h–monoidality are preserved due to Pavlov and Scholbach [15, Proposition 6.4].

Proposition 3.4.2 *The left Bousfield localization L_S defined in (5) is a monoidal left Bousfield localization. Furthermore, $L_S[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ is a $\mathcal{V}$–model category.*

Proof We start with the first statement. Let $f \in S$ and A be a cofibrant object in $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$. Since $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ is a left proper, pretty small, tractable and flat monoidal model category, it is sufficient to show that $f \otimes_{\mathrm{Day}} A$ is a weak equivalence after localization. By definition, S only contains cofibrations between cofibrant objects. Therefore $f \otimes_{\mathrm{Day}} A$ is a map between cofibrant objects.

Since the Quillen pair $D \dashv E$ is a Quillen equivalence after localization, a map between cofibrant objects in $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ is an S–local equivalence if and only if its image under D is a weak equivalence in $\mathcal{M}$. Therefore it is sufficient to show that $D(f \otimes_{\mathrm{Day}} A)$ is a weak equivalence.

By Proposition 3.3.2, D is a monoidal functor. We have

$$D(f \otimes_{\mathrm{Day}} A) \cong D(f) \wedge D(A).$$

Since f is an S–local equivalence between cofibrant objects, $D(f)$ is a weak equivalence between cofibrant objects. In a monoidal model category, the monoidal product with a cofibrant object is a left Quillen functor; therefore it preserves weak equivalences between cofibrant objects. Furthermore $D(A)$ is cofibrant; therefore $D(f) \wedge D(A)$ is a weak equivalence.

Now we prove the second statement. Since the cofibrations of $L_S[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ are the same as the cofibrations of $[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$, we only need to prove the case of SM7 for a generating trivial cofibration $f: A_1 \to A_2$ in $L_S[\mathcal{M}_{\lambda,\mathrm{cof}}^{\mathrm{op}}, \mathcal{V}]$ and a generating cofibration $g: B_1 \to B_2$ in $\mathcal{V}$. Indeed, we only need to show that the map $f \square g$, the pushout product of f and g, is an S–local equivalence.

Since $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ and $\mathcal{V}$ are tractable, we assume that A_1, A_2, B_1 and B_2 are cofibrant objects. Due to SM7 in $[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$, this guarantees that $f \,\square\, g$ is also a map between cofibrant objects. Therefore, it is sufficient to show that $D(f \,\square\, g)$ is a weak equivalence. Since D is a left adjoint functor that commutes with $\mathcal{V}$–tensor — see (3) — it preserves pushout products. In particular,

$$D(f \,\square\, g) \cong D(f) \,\square\, g.$$

Since D is a left Quillen functor, $D(f)$ is a trivial cofibration in $\mathcal{M}$ and therefore $D(f) \,\square\, g$ is a weak equivalence. $\square$

3.5 Admissible replacement

We have shown that the given simplicial and combinatorial monoidal model category $\mathcal{M}$ (with cofibrant unit) is monoidally Quillen equivalent to the nice monoidal model category $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$. Therefore, in order to prove Theorem 1.0.3, we need to show that $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is monoidally Quillen equivalent to an admissible model category. For this, we use the main result of Pavlov and Scholbach [16].

Pavlov and Scholbach work in a more general setting than Hovey. They consider modules over a commutative monoid in symmetric sequences. Let $\mathbb{I}$ denote the monoidal unit of $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$. Let E be the symmetric sequence given by $\mathbb{I}$ at each degree equipped with the trivial action. Let $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ denote the category of E–modules in the category of symmetric sequences in $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$. Indeed, the category $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ is equivalent to the category of I–spaces in $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ (Sagave and Schlichtkrull [18]); see Pavlov and Scholbach [16, Proposition 3.2.2].

Theorem 3.5.1 (Pavlov and Scholbach [16, Theorem 1.1]) *For a nice monoidal model category $\mathcal{C}$, $\mathrm{Sp}(\mathcal{C}, R)$ with the positive stable model structure is an admissible model category, where R denotes a commutative monoid in symmetric sequences in $\mathcal{C}$.*

In particular, $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ equipped with the positive stable model structure is admissible. The following proposition is due to Theorem 9.1 of Hovey [9]; see also Pavlov and Scholbach [16, Example 3.3.2].

Remark 3.5.2 The proposition below does not follow Hovey's notation. In Hovey's notation, one would replace E with the monoidal unit of $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$.

Proposition 3.5.3   *The left Quillen functor*

$$F_0 \colon L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}] \to \mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$$

— see Hovey [9, *Definition 7.3*] *— is the left adjoint of a symmetric monoidal Quillen equivalence. Here,* $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ *is given the stable model structure.*

Remark 3.5.4   Hovey works in the setting of left proper cellular model categories in [9]. The cellularity assumption is needed in order to make sure that certain left Bousfield localizations exists. Namely, in order to localize the projective model structure on symmetric spectra to obtain the stable model structure. In our case, $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is indeed cellular but we do not need this. Since $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is left proper and combinatorial, the relevant left Bousfield localizations are guaranteed to exist in $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}])$ which is also left proper and combinatorial.

Finally, we prove Theorems 1.0.3, 1.0.5 and 1.0.7. The following theorem is a generalization of Theorems 1.0.3 and 1.0.5 since the model category of simplicial sets satisfy the axioms for $\mathcal{V}$ given in Notation 3.0.1.

Theorem 3.5.5   *Let* $\mathcal{V}$ *be as in* Notation 3.0.1. *Every combinatorial* $\mathcal{V}$ *symmetric monoidal model category is symmetric monoidally Quillen equivalent to an admissible symmetric monoidal model category where a weak equivalence of operads induce a Quillen equivalence between the model categories of the corresponding algebras.*

Proof of Theorems 3.5.5 and 1.0.7   We start with the proof of Theorem 3.5.5. Let $\mathcal{N}$ be a symmetric monoidal model category satisfying the hypothesis of the theorem. Theorem 3.1.1 states that $\mathcal{N}$ is strong monoidally Quillen equivalent to another symmetric monoidal model category $\mathcal{M}$ that also satisfies the hypothesis of the theorem and has a cofibrant monoidal unit.

Propositions 3.2.3 and 3.3.2 state that $\mathcal{M}$ is strong monoidally Quillen equivalent to the presheaf category $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$. Furthermore, we proved in Section 3.4 that $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is nice.

Finally, $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}]$ is strong monoidally Quillen equivalent to $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ with the stable model structure due to Proposition 3.5.3. With the stable model structure, $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ may not be admissible. However, this model structure is monoidally Quillen equivalent to the positive stable model structure through the identity functor; see Pavlov and Scholbach [16, Proposition 3.3.1]. As desired, the positive

stable model structure on $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ is admissible due to Theorem 3.5.1. This provides the admissible replacement of $\mathcal{N}$.

Furthermore, weak equivalence of operads induce a Quillen equivalence of the corresponding model categories of algebras in $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{V}], E)$ due to Pavlov and Scholbach [16, Theorem 4.6]. This finishes the proof of Theorem 3.5.5.

For Theorem 1.0.7, we work in the setting where $\mathcal{V}$ is the model category of simplicial sets. In this situation, $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}], E)$ satisfies the desired property due to Pavlov and Scholbach [16, Theorem 4.9].   □

4   Categories of modules

This section is devoted to the proof of Theorem 1.0.10. In other words, we show that every stable, combinatorial and simplicial monoidal model category whose monoidal unit is a compact generator is monoidally Quillen equivalent to modules over a commutative ring spectrum. Furthermore, we show that this commutative ring spectrum is unique.

4.1   Replacement with a spectral model category

Here, we show that every stable, combinatorial and simplicial symmetric monoidal model category $\mathcal{N}$ is monoidally Quillen equivalent to a spectral one.

As before, one replaces $\mathcal{N}$ with another stable, combinatorial and simplicial symmetric monoidal model category $\mathcal{M}$ whose monoidal unit is cofibrant; see Theorem 3.1.1. Since $\mathcal{N}$ is pointed and since $\mathcal{M}$ is equivalent to $\mathcal{N}$ as a category, $\mathcal{M}$ is also pointed.

Since $\mathcal{M}$ is a pointed simplicial model category, it is also an $\mathcal{S}_*$–model category in a natural way where $\mathcal{S}_*$ denotes the category of pointed simplicial sets; see Hovey [8, Definition 4.2.19]. As before, $\mathcal{M}$ is strong monoidally Quillen equivalent to $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}_*]$ where λ is as in Section 3.2 and S is as in Proposition 3.2.3. This follows by Propositions 3.2.3 and 3.3.2.

Notation 4.1.1   For a given nice $\mathcal{S}_*$ symmetric monoidal model category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{I}_{\mathcal{C}})$, let K denote the symmetric sequence in $\mathcal{C}$ given by $K_n = (S^1 \otimes \mathbb{I}_{\mathcal{C}})^{\otimes_{\mathcal{C}} n}$ at degree n. We denote the model category of K–modules equipped with the stable model structure by $\mathrm{Sp}(\mathcal{C})$. Let A be a monoid in $\mathrm{Sp}(\mathcal{C})$. We denote the category of A–modules by A–mod and unless otherwise stated, we assume that this category is equipped with the

stable model structure. The category of A–modules equipped with the *positive* stable model structure is denoted by A–mod$^+$. Similarly, we write $\mathrm{Sp}(\mathcal{C})^+$ when we are using the positive stable model structure on $\mathrm{Sp}(\mathcal{C})$.

Notation 4.1.2 For the rest of this work, let $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{I}_\mathcal{D})$ denote a stable, combinatorial and nice $\mathcal{S}_*$ symmetric monoidal model category. Furthermore, assume that $\mathbb{I}_\mathcal{D}$ is cofibrant and assume that $\mathbb{I}_\mathcal{D}$ is a compact generator of the homotopy category of $\mathcal{D}$. We denote the monoidal product and the monoidal unit of $\mathrm{Sp}(\mathcal{D})$ as in $(\mathrm{Sp}(\mathcal{D}), \wedge, \mathbb{I})$.

The following is a consequence of Hovey [9, Theorem 9.1].

Proposition 4.1.3 *Let $\mathcal{D}$ be as in Notation 4.1.2. In this situation, $\mathrm{Sp}(\mathcal{D})$ is strong symmetric monoidally Quillen equivalent to $\mathcal{D}$. Furthermore, the monoidal unit of $\mathrm{Sp}(\mathcal{D})$ is a compact generator of the homotopy category of $\mathrm{Sp}(\mathcal{D})$.*

Ultimately, we are interested in the case $\mathcal{D} = L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}_*]$. We have the following.

Proposition 4.1.4 *Let $\mathcal{N}$ be a symmetric monoidal model category satisfying the hypothesis of Theorem 1.0.10. Also, let $\mathcal{M}$, λ and S be as above. The presheaf category $L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}_*]$ satisfies the assumptions for $\mathcal{D}$ given in Notation 4.1.2. Furthermore, $\mathcal{N}$ is strong symmetric monoidally Quillen equivalent to $\mathrm{Sp}(L_S[\mathcal{M}^{\mathrm{op}}_{\lambda,\mathrm{cof}}, \mathcal{S}_*])$.*

In order to prove Theorem 1.0.10, we need to show that $\mathrm{Sp}(\mathcal{D})$ is monoidally Quillen equivalent to the model category of R–modules for some commutative ring spectrum R. For this, we need to consider a spectral enrichment of $\mathrm{Sp}(\mathcal{D})$.

Construction 4.1.5 Since $\mathcal{D}$ is an $\mathcal{S}_*$ monoidal model category, there is a left Quillen functor $T \colon \mathcal{S}_* \to \mathcal{D}$. Following the discussion after Definition 7.2 of Hovey [9] and Hovey [9, Theorem 8.11], one obtains that $\mathrm{Sp}(\mathcal{D})$ is a spectral symmetric monoidal model category. In other words, there is a canonical monoidal left Quillen functor

$$\widetilde{T} \colon \mathrm{Sp}(\mathcal{S}_*) \to \mathrm{Sp}(\mathcal{D}).$$

We denote the right adjoint of $\widetilde{T}$ by $\widetilde{U}$.

Proposition 4.1.6 *The adjoint pair $\widetilde{T} \dashv \widetilde{U}$ in Construction 4.1.5 is also a Quillen pair between $\mathrm{Sp}(\mathcal{S}_*)^+$ and $\mathrm{Sp}(\mathcal{D})^+$.*

Proof We start by describing the functor $\widetilde{T}$ in detail. Given a category $\mathcal{C}$, let $\Sigma\mathcal{C}$ denote the category of symmetric sequences in $\mathcal{C}$. There is a symmetric sequence S in

$\mathcal{S}_*$ given by $(S^1)^{\wedge k}$ in degree k. Furthermore, recall that K is the symmetric sequence in $\mathcal{D}$ given by $K_k = (S^1)^{\wedge k} \otimes \mathbb{I}_{\mathcal{D}}$. With this notation, the categories of S–modules and K–modules are $\mathrm{Sp}(\mathcal{S}_*)$ and $\mathrm{Sp}(\mathcal{D})$, respectively.

Recall that there is a monoidal left Quillen functor

$$T \colon \mathcal{S}_* \to \mathcal{D}.$$

Let U denote the right adjoint of T. Applying T levelwise makes $\Sigma\mathcal{D}$ into a $\Sigma\mathcal{S}_*$ monoidal model category; see Hovey [9, Section 7]. In other words, the functor $\Sigma\mathcal{S}_* \to \Sigma\mathcal{D}$ given by T at each level is monoidal. Furthermore, the right adjoint to this functor is also given by U at each level as explained in Hovey [9, Section 7]. The levelwise application of T to S gives K and therefore there is an adjoint pair between S–modules and K–modules. This is indeed the adjoint pair $\widetilde{T} \dashv \widetilde{U}$. This shows that $\widetilde{U}$ is given by levelwise application of U on the underlying symmetric sequences.

Since localizations do not change trivial fibrations, the trivial fibrations in the positive stable model structure of $\mathrm{Sp}(\mathcal{S}_*)$ (resp. $\mathrm{Sp}(\mathcal{D})$) are given by the maps that are trivial fibrations at each level of $\Sigma\mathcal{S}_*$ (resp. $\Sigma\mathcal{D}$) except possibly in degree zero; see Pavlov and Scholbach [16, Notation 2.3.5]. Since $\widetilde{U}$ applies the right Quillen functor U levelwise, it preserves the trivial fibrations of the positive stable model structure. This also shows that $\widetilde{T}$ preserves the cofibrations of the positive stable model structure.

Let f be an acyclic cofibration in the positive stable model structure on $\mathrm{Sp}(\mathcal{S}_*)$. We already showed that $\widetilde{T}(f)$ is a cofibration in the positive stable model structure. Therefore it is sufficient to show that $\widetilde{T}(f)$ is a weak equivalence in the positive stable model structure. Since there are more trivial fibrations in the positive stable model structure than the stable model structure, a cofibration in the positive stable model structure is also a cofibration in the stable model structure. Furthermore, the weak equivalences of the positive stable model structure and the stable model structure agree; see Pavlov and Scholbach [16, Proposition 3.3.1]. In particular, f is also an acyclic cofibration in the stable model structure. Therefore $\widetilde{T}(f)$ is an acyclic cofibration in the stable model structure. This shows that $\widetilde{T}(f)$ is a weak equivalence in the positive stable model structure as desired. $\qquad\square$

4.2   Proof of Theorem 1.0.10

To prove Theorem 1.0.10, we need to show that $\mathrm{Sp}(\mathcal{D})$ is monoidally Quillen equivalent to the model category R–modules for some commutative ring spectrum R. Here, $\mathcal{D}$

is as in Notation 4.1.2. The definition of the symmetric monoidal model category $(\mathrm{Sp}(\mathcal{D}), \wedge, \mathbb{I})$ is given in Notation 4.1.1. Our conventions on stable and positive stable model structures are also given in Notation 4.1.1.

The commutative ring spectrum R should be considered as the derived endomorphism ring spectrum of the monoidal unit of $\mathrm{Sp}(\mathcal{D})$. However, in order to ensure that the endomorphism ring spectrum is a commutative ring spectrum, one needs a fibrant model of $\mathbb{I}$ as a commutative monoid. For this, we take a fibrant replacement

$$(7) \qquad\qquad \varphi : \mathbb{I} \xrightarrow{\sim} f\mathbb{I}$$

of $\mathbb{I}$ in the model category of commutative monoids in $\mathrm{Sp}(\mathcal{D})^+$. Since the weak equivalences of the positive stable model structure agree with those of the stable model structure — see Pavlov and Scholbach [16, Proposition 3.3.1] — φ is also a weak equivalence in $\mathrm{Sp}(\mathcal{D})$.

Since $\mathrm{Sp}(\mathcal{D})$ is flat — Pavlov and Scholbach [16, Proposition 3.4.2] — Quillen invariance holds for $\mathrm{Sp}(\mathcal{D})$. This means that the Quillen adjunction between the model categories of $\mathbb{I}$–modules and $f\mathbb{I}$–modules induced by φ is a Quillen equivalence; see Schwede and Shipley [19, Theorem 4.3]. Similarly, this adjunction is also a Quillen equivalence between $\mathbb{I}$–mod$^+$ and $f\mathbb{I}$–mod$^+$. Since $\mathbb{I}$ denotes the monoidal unit of $\mathrm{Sp}(\mathcal{D})$, $\mathbb{I}$–mod is simply another name for $\mathrm{Sp}(\mathcal{D})$. Because $f\mathbb{I}$ is a commutative monoid, $f\mathbb{I}$–mod is also a closed symmetric monoidal model category. The left adjoint of the Quillen equivalence between $\mathbb{I}$–mod and $f\mathbb{I}$–mod is a monoidal Quillen functor; see the discussion after Definition 4.1.14 in Hovey [8]. This functor is given by

$$f\mathbb{I} \wedge - : \mathrm{Sp}(\mathcal{D}) \to f\mathbb{I}\text{–mod}.$$

The monoidality of this functor follows from the natural isomorphism

$$f\mathbb{I} \wedge (X \wedge Y) \cong (f\mathbb{I} \wedge X) \wedge_{f\mathbb{I}} (f\mathbb{I} \wedge Y)$$

that holds for every $X, Y \in \mathrm{Sp}(\mathcal{D})$. Here, $\wedge_{f\mathbb{I}}$ denotes the monoidal product in $f\mathbb{I}$–modules.

This shows that $\mathrm{Sp}(\mathcal{D})$ is monoidally Quillen equivalent to the model category of $f\mathbb{I}$–modules. Therefore, it is sufficient to show that the model category of $f\mathbb{I}$–modules is monoidally Quillen equivalent to modules over a commutative ring spectrum. We obtain the following.

Proposition 4.2.1 *Let $f\mathbb{I}$ be the fibrant replacement of $\mathbb{I}$ given in* (7). *The functor $f\mathbb{I} \wedge -$ is the left adjoint of a monoidal Quillen equivalence between* $\mathrm{Sp}(\mathcal{D})$ *(resp.* $\mathrm{Sp}(\mathcal{D})^+$*) and* $f\mathbb{I}$*–mod (resp.* $f\mathbb{I}$*–mod$^+$).*

Construction 4.2.2 In Construction 4.1.5, we showed that there is a monoidal left Quillen functor $\widetilde{T}: \mathrm{Sp}(\mathcal{S}_*) \to \mathrm{Sp}(\mathcal{D})$. Composing this with the monoidal left Quillen functor $f\mathbb{I} \wedge -$, we obtain a monoidal left Quillen functor

$$F := (f\mathbb{I} \wedge -) \circ \widetilde{T}: \mathrm{Sp}(\mathcal{S}_*) \to f\mathbb{I}\text{–mod}.$$

Let R denote the right adjoint of F.

Since F is monoidal, R is lax monoidal and $R(f\mathbb{I})$ is a commutative ring spectrum. Again because it is monoidal, F induces a functor

$$F': R(f\mathbb{I})\text{–mod} \to FR(f\mathbb{I})\text{–mod}.$$

Here, F' is given by F and the right adjoint R' of F' is again given by R. Since $F \dashv R$ is a strong monoidal Quillen pair, the counit map of this adjunction provides a map $FR(f\mathbb{I}) \to f\mathbb{I}$ of commutative monoids. Through this map, we obtain a left Quillen functor

$$f\mathbb{I} \wedge_{FG(f\mathbb{I})} -: FR(f\mathbb{I})\text{–mod} \to f\mathbb{I}\text{–mod}.$$

Finally, let $L = (f\mathbb{I} \wedge_{FG(f\mathbb{I})} -) \circ F'$. We obtain a Quillen pair $L \dashv G$,

$$L: R(f\mathbb{I})\text{–mod} \rightleftarrows f\mathbb{I}\text{–mod}: G.$$

Going through the definition of these functors above, one observes that G is given by R.

To prove Theorem 1.0.10, we need to show that $L \dashv G$ is a monoidal Quillen equivalence.

Remark 4.2.3 In the construction above, although we do not make it explicit, we make use of the spectral enrichment of $f\mathbb{I}$–mod as in the proof of Schwede and Shipley [21, Theorem 3.9.3]. The mapping spectrum $\hom(-, -)$ in the category of $f\mathbb{I}$–modules is given by $R(\mathrm{Hom}(-, -))$ where $\mathrm{Hom}(-, -)$ denotes the internal hom in $f\mathbb{I}$–mod. The functor G above is given by

$$G(-) = R(-) = R(\mathrm{Hom}(f\mathbb{I}, -)) = \hom(f\mathbb{I}, -).$$

In particular, one observes that the adjoint pair $L \dashv G$ above agrees with the one constructed in the proof of Schwede and Shipley [21, Theorem 3.9.3].

Proposition 4.2.4 *The Quillen pair $L \dashv G$ in Construction 4.2.2 is also a Quillen pair between $R(f\mathbb{I})$–mod$^+$ and $f\mathbb{I}$–mod$^+$.*

Proof It is sufficient to show that G preserves positive (trivial) fibrations. In the model category of modules over a given monoid in a nice monoidal model category, (trivial) fibrations are those of the underlying model category. Since the G is given by R, it is sufficient to show that R preserves positive (trivial) fibrations. For this, it is sufficient to show that F is a left Quillen functor between $\mathrm{Sp}(\mathcal{S}_*)^+$ and $f\mathbb{I}$–mod$^+$. This follows by Propositions 4.1.6 and 4.2.1. $\square$

Proposition 4.2.5 *The left Quillen functor L is monoidal. In other words, $L \dashv G$ is a strong monoidal Quillen pair.*

Proof Since F is a composite of monoidal functors, it is a monoidal functor. Since F preserves coequalizers and monoidal products, one observes that it also carries monoidal products of $R(f\mathbb{I})$–modules to monoidal products of $FR(f\mathbb{I})$–modules. In other words, F' is also monoidal. Finally L is monoidal as it is given by a composite of monoidal functors. $\square$

The proof of Schwede and Shipley [21, Theorem 3.9.3] goes through to show that $L \dashv G$ is a Quillen equivalence between the respective stable model structures. The only difference in our situation and that of Schwede and Shipley [21, Theorem 3.9.3] is that our compact generator $f\mathbb{I}$ may not be bifibrant in $f\mathbb{I}$–mod. It is fibrant in $f\mathbb{I}$–mod$^+$ but it may not be fibrant in $f\mathbb{I}$–mod. Similarly, $f\mathbb{I}$ is cofibrant in $f\mathbb{I}$–mod but it may not be cofibrant in $f\mathbb{I}$–mod$^+$.

Theorem 4.2.6 (Schwede–Shipley [21, Theorem 3.9.3]) *As in (7), let $\mathbb{I} \xrightarrow{\sim} f\mathbb{I}$ be a fibrant replacement of the monoidal unit $\mathbb{I}$ of $\mathrm{Sp}(\mathcal{D})$ in the model category of commutative monoids in $\mathrm{Sp}(\mathcal{D})^+$. The Quillen pair*

$$L \dashv G$$

given in Construction 4.2.2 is a Quillen equivalence between $R(f\mathbb{I})$–mod and $f\mathbb{I}$–mod.

Proof We first show that the derived unit map

$$R(f\mathbb{I}) \to Gf'Lc'(R(f\mathbb{I}))$$

of the adjunction $L \dashv G$ (evaluated at $R(f\mathbb{I})$) is a weak equivalence. Here, f' denotes a fibrant replacement functor in $f\mathbb{I}$–mod and c' denotes a cofibrant replacement functor in $R(f\mathbb{I})$–mod. As $R(f\mathbb{I})$ is cofibrant in $R(f\mathbb{I})$–mod, the cofibrant replacement functor may be omitted. Also, $L(R(f\mathbb{I})) = f\mathbb{I}$ as L is monoidal. The monoidal unit $f\mathbb{I}$ is only assumed to be fibrant in the positive stable model structure and therefore it may not be fibrant in the nonpositive stable model structure. However, G is also a right Quillen functor between the respective positive stable model structures and the weak equivalences of the positive and the nonpositive stable model structures agree Pavlov and Scholbach [16, Proposition 3.3.1]. Due to this, the fibrant replacement f' above may also be omitted. Since G is given by R and since L is monoidal, one obtains that

$$GL(R(f\mathbb{I})) = G(f\mathbb{I}) = R(f\mathbb{I})$$

as desired. Similarly, one shows that the derived counit map evaluated at $f\mathbb{I}$

$$Lc'Gf'(f\mathbb{I}) \to f\mathbb{I}$$

is also a weak equivalence where f' and c' denote the relevant fibrant and cofibrant replacement functors respectively.

The rest of the proof follows as in Schwede and Shipley [21, Theorem 3.9.3]. For the sake of completeness, we provide a sketch of the proof of Schwede and Shipley [21, Theorem 3.9.3] in our case, ie in the case of a single compact generator. For this, it is sufficient to show that the induced adjoint pair at the level of homotopy categories is indeed an equivalence of categories. The unit and the counit maps of the derived adjunction are isomorphisms when evaluated at the respective monoidal units as we show above.

It follows from Proposition 4.1.3 and our constructions that $f\mathbb{I}$ is a compact generator of the homotopy category of $f\mathbb{I}$–mod.

The derived functor of L is a left adjoint and therefore it preserves coproducts. Using the compactness of $f\mathbb{I}$, one can show as in the proof of Schwede and Shipley [21, Theorem 3.9.3] that the derived functor of G also preserves coproducts. Furthermore, both derived functors preserve shifts and triangles. Therefore, the unit and the counit maps of the derived adjunction are isomorphisms on the categories generated by the monoidal units under coproducts, triangles and shifts. Since both monoidal units are generators, this shows that the counit and the unit maps of the derived adjunction are isomorphisms on all objects. □

The following provides the uniqueness part of Theorem 1.0.10. This is a consequence of Lurie's results.

Proposition 4.2.7 *Let $\mathcal{M}$ be a monoidal model category as in Theorem 1.0.10 with a cofibrant monoidal unit. Furthermore, assume that $\mathcal{M}$ is (strong or weak) symmetric monoidally Quillen equivalent to R_i–modules via a zig-zag of symmetric monoidal Quillen equivalences where each symmetric monoidal model category involved in the zig-zag has a cofibrant monoidal unit. Here, R_i denotes a commutative ring spectrum for $i = 1$ and $i = 2$. In this situation, R_1 and R_2 are weakly equivalent as commutative ring spectra.*

Proof The hypothesis implies that the symmetric monoidal ∞–categories corresponding to the model categories of R_1–modules and R_2–modules are equivalent; see Péroux [17, Theorem 2.13]. It follows from Proposition 7.1.2.7 of Lurie [11] that R_1 and R_2 are weakly equivalent as commutative ring spectra. $\square$

We are ready to prove Theorem 1.0.10.

Proof of Theorem 1.0.10 Let $\mathcal{N}$ be a stable, combinatorial and simplicial monoidal model category whose monoidal unit is a compact generator, ie let $\mathcal{N}$ satisfy the hypothesis of Theorem 1.0.10. Due to Proposition 4.1.4, $\mathcal{N}$ is monoidally Quillen equivalent to $\mathrm{Sp}(\mathcal{D})$ for some monoidal model category $\mathcal{D}$ as in Notation 4.1.2. Let $\mathbb{I} \xrightarrow{\sim} f\mathbb{I}$ be a fibrant replacement of the monoidal unit $\mathbb{I}$ of $\mathrm{Sp}(\mathcal{D})$ in the model category of commutative monoids in $\mathrm{Sp}(\mathcal{D})^+$. Due to Proposition 4.2.1, $\mathrm{Sp}(\mathcal{D})$ is monoidally Quillen equivalent to $f\mathbb{I}$–mod. Construction 4.2.2 provides a Quillen adjunction $L \dashv G$ between the model categories of $f\mathbb{I}$–modules and $R(f\mathbb{I})$–modules where $R(f\mathbb{I})$ is a commutative ring spectrum. This Quillen adjunction is monoidal due to Proposition 4.2.5 and it is indeed a Quillen equivalence due to Theorem 4.2.6. This proves that the given $\mathcal{N}$ is monoidally Quillen equivalent to the model category of modules over $R(f\mathbb{I})$ (with the stable model structure) as desired.

The uniqueness part of Theorem 1.0.10 follows by Proposition 4.2.7. $\square$

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