A family of integrable maps associated with the Volterra lattice

A.N.W. Hone [*] and J.A.G. Roberts
School of Mathematics & Statistics
University of New South Wales
Sydney, NSW 2052, Australia.
A.N.W.Hone@kent.ac.uk   j.a.g.roberts@unsw.edu.au

P. Vanhaecke
Laboratoire de Mathématiques et Applications
UMR 7348 CNRS-Université de Poitiers
86360 Chasseneuil-du-Poitou, France.
Pol.Vanhaecke@univ-poitiers.fr

September 5, 2023

Abstract

Recently Gubbiotti, Joshi, Tran and Viallet classified birational maps in four dimensions admitting two invariants (first integrals) with a particular degree structure, by considering recurrences of fourth order with a certain symmetry. The last three of the maps so obtained were shown to be Liouville integrable, in the sense of admitting a non-degenerate Poisson bracket with two first integrals in involution. Here we show how the first of these three Liouville integrable maps corresponds to genus 2 solutions of the infinite Volterra lattice, being the $g = 2$ case of a family of maps associated with the Stieltjes continued fraction expansion of a certain function on a hyperelliptic curve of genus $g \geqslant 1$. The continued fraction method provides explicit Hankel determinant formulae for tau functions of the solutions, together with an algebro-geometric description via a Lax representation for each member of the family, associating it with an algebraic completely integrable system. In particular, in the elliptic case ($g = 1$), as a byproduct we obtain Hankel determinant expressions for the solutions of the Somos-5 recurrence, but different to those previously derived by Chang, Hu and Xin. By applying contraction to the Stieltjes fraction, we recover integrable maps associated with Jacobi continued fractions on hyperelliptic curves, that one of us considered previously, as well as the Miura-type transformation between the Volterra and Toda lattices.

1   Introduction

In classical mechanics, the study of integrable Hamiltonian systems, given by Hamiltonian vector fields with a sufficient number of independent first integrals in involution with respect to a Poisson bracket, has a long history that goes back to the origins of calculus. It was further enriched in the latter half of the last century by the discovery of the method of inverse scattering for solving certain Hamiltonian partial differential equations, which gave new perspectives and new techniques for deriving finite-dimensional integrable systems obtained as reductions of the latter. The case of discrete integrable systems, in the form of difference equations or maps preserving a symplectic (or Poisson) structure and satisfying the conditions for a discrete analogue of Liouville's theorem, soon began to attract attention [7, 34, 54], but it is fair to say that, despite the fact

---

[*]Work begun while on leave from School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury CT2 7NF, UK.

that many examples are now known, the theory of discrete integrability is much less well developed. For integrable birational maps in the plane, the archetypal example is provided by the QRT family of maps [44], whose level sets are biquadratic curves (generically, of genus one), which are associated with elliptic fibrations [15]. If one imposes a requirement of subexponential degree growth (zero algebraic entropy, in the terminology of [6]), then in two dimensions the only possibilities are maps that preserve a pencil of genus one curves (like QRT), maps that preserve a pencil of rational curves, or completely periodic maps [13]. This fits in with an observation of Veselov [54], that for an infinite order birational map of the plane with an algebraic invariant, the level curves can have genus at most one (as a consequence of the Hurwitz theorem on the automorphism group of a Riemann surface).

Poisson maps in three dimensions with two first integrals, of which one is a Casimir, can be reduced to the two-dimensional case by restricting to symplectic leaves, and the common level sets are curves, so in an algebro-geometric setting this will typically lead to elliptic fibrations. Thus, in order to see new geometrical features, with invariant tori of dimension greater than one, it is necessary to look to integrable maps in four dimensions. Building on the work [30] and [20], which was based on considering autonomous versions of the fourth-order members of hierarchies of discrete Painlevé I/II equations from [11], in [21] Gubbiotti et al. presented a classification of four-dimensional birational maps of recurrence type, that is

$$\varphi: \qquad (w_0, w_1, w_2, w_3) \mapsto \Big(w_1, w_2, w_3, \mathrm{F}(w_0, w_1, w_2, w_3)\Big), \qquad (1.1)$$

for a suitable rational function F of affine coordinates $(w_0, w_1, w_2, w_3) \in \mathbb{C}^4$, with φ being invariant under the involution $\iota: (w_0, w_1, w_2, w_3) \mapsto (w_3, w_2, w_1, w_0)$ and having two independent polynomial invariants, K_1, K_2 say, with specific degree patterns $(\deg_{w_0} K_j, \deg_{w_1} K_j, \deg_{w_2} K_j, \deg_{w_3} K_j) = (1, 3, 3, 1)$ and $(2, 4, 4, 2)$ for $j = 1, 2$, respectively. The result of this classification was six maps with parameters, labelled (P.i-vi), together with six associated maps, labelled (Q.i-vi), dual to them in the sense of [45], meaning that they arise as discrete integrating factors for linear combinations of the first integrals. (Note that, aside from the original connection with [11], the letter P in this nomenclature has nothing to do with the usual labelling of continuous Painlevé equations.)

1.1 The map (P.iv): an integrable map in 4D

From our point of view, the most interesting examples among those presented in [21] are the maps labelled (P.iv), (P.v) and (P.vi). According to Table 1 in [21], these are the only ones arising from a discrete variational principle (Lagrangian), which leads to a non-degenerate Poisson bracket in four dimensions, such that the two first integrals K_1, K_2 are in involution, and this means that in the real case the Liouville tori are two-dimensional (cf. Fig.1). In this paper, our main concern will be the case of (P.iv), which is the birational map given in affine form by the recurrence

$$\begin{aligned} &w_{n+4}w_{n+3}w_{n+2} + w_{n+2}w_{n+1}w_n + 2w_{n+2}^2(w_{n+3} + w_{n+1}) \\ &+w_{n+2}(w_{n+3}^2 + w_{n+3}w_{n+1} + w_{n+1}^2) + w_{n+2}^3 + \nu w_{n+2}(w_{n+3} + w_{n+2} + w_{n+1}) + bw_{n+2} + a = 0. \end{aligned} \qquad (1.2)$$

The above map depends on three essential parameters a, b, ν (compared with [21], by rescaling we have set the parameter $d = 1$), and it can be written in the form (1.1), with

$$\mathrm{F} = -\frac{w_0w_1w_2 + w_1w_2w_3 + w_1^2w_2 + w_2w_3^2 + 2w_1w_2^2 + 2w_2^2w_3 + w_2^3 + \nu(w_1w_2 + w_2w_3 + w_2^2) + bw_2 + a}{w_2w_3},$$

which is the rational function of w_0, w_1, w_2, w_3 obtained by solving for w_4 in the recurrence (1.2) with $n = 0$. More recently, Gubbiotti, showed how the equation (1.2) also arises from a classification of additive fourth-order difference equations, based on the requirement of a discrete Lagrangian structure alone [22].

The first integral denoted $I_{\mathrm{low}}^{\mathrm{P.iv}}$ in [21] is given in affine coordinates by

$$K_1 = w_1w_2\Big(w_2w_3 + w_0w_1 - w_0w_3 + (w_1 + w_2)^2 + \nu(w_1 + w_2) + b\Big) + a(w_1 + w_2). \qquad (1.3)$$

The latter has the degree pattern $(1, 3, 3, 1)$. In particular, it is linear in w_3, which implies that, on each three-dimensional level set $K_1 = k_1 = \mathrm{const}$, the map (1.2) reduces to a birational map in three dimensions, given by the recurrence

$$
\begin{aligned}
w_{n+3}w_{n+2}w_{n+1}(w_{n+2} - w_n) + w_{n+2}w_{n+1}^2 w_n + w_{n+2}w_{n+1}(w_{n+1} + w_{n+2})^2 & \\
+\nu\, w_{n+2}w_{n+1}(w_{n+1} + w_{n+2}) + b\, w_{n+2}w_{n+1} + a\,(w_{n+1} + w_{n+2}) & = k_1.
\end{aligned}
$$

A second independent invariant for (1.2), with degree pattern $(2, 4, 4, 2)$, is given by

$$
K_2 = w_1 w_2 \left(
\begin{array}{c}
w_0^2 w_1 + w_3^2 w_2 + w_0 w_3 (w_1 + w_2) + w_0 (w_2^2 + 2w_1^2) + w_3 (w_1^2 + 2w_2^2) \\
+ 3(w_0 + w_3)w_1 w_2 + (w_1 + w_2)^3 \\
+\nu \left(w_0 w_3 + (w_0 + w_3)(w_1 + w_2) + (w_1 + w_2)^2 \right) + b\,(w_0 + w_1 + w_2 + w_3) \\
+ a\left(w_0 w_1 + w_3 w_2 + (w_1 + w_2)^2 \right).
\end{array}
\right) \tag{1.4}
$$

This differs slightly from the second invariant presented in [21], which is $I_{\mathrm{high}}^{\mathrm{P.iv}} = K_2 - \nu K_1$.

The non-degenerate Poisson bracket between the coordinates, which was obtained in [21] by making use of a discrete Lagrangian for (1.2), is given by

$$
\{\, w_n, w_{n+1}\,\} = 0, \;\; \{\, w_n, w_{n+2}\,\} = \frac{1}{w_{n+1}}, \;\; \{\, w_n, w_{n+3}\,\} = -\frac{w_n + 2w_{n+1} + 2w_{n+2} + w_{n+3} + \nu}{w_{n+1}w_{n+2}}, \tag{1.5}
$$

for all n. So the 4D map of the form (1.1) defined by (1.2) is a Poisson map, in the sense that $\{\, \varphi^* G, \varphi^* H\,\} = \varphi^* \{\, G, H\,\}$ for all functions G, H on $\mathbb{C}^4$. Equivalently, the fact that φ is Poisson with respect to a non-degenerate bracket means that it preserves a symplectic form ω, such that $\varphi^* \omega = \omega$. The two independent invariants given in [21] are in involution with respect to the Poisson bracket, which is equivalent to the involutivity of functions (1.3) and (1.4), that is

$$
\{\, K_1, K_2\,\} = 0.
$$

Hence the four-dimensional map defined by (1.2) is integrable in the Liouville sense.

Computing the Hamiltonian vector field for the first flow, generated by K_1, we find that on the phase space $\mathbb{C}^4$ with coordinates (w_0, w_1, w_2, w_3) this takes the form

$$
\frac{\mathrm{d}w_n}{\mathrm{d}t} = w_n(w_{n+1} - w_{n-1}) \qquad \text{for} \quad n = 1, 2, \tag{1.6}
$$

while the components of the vector field for $n = 0, 3$ appear to be more complicated rational functions of these 4 coordinates and the parameters a, b, ν. However, since (1.2) is a Poisson map it commutes with this flow, so it follows that the relation (1.6) extends to all $n \in \mathbb{Z}$. To see this, note that the vecor field $\frac{\mathrm{d}}{\mathrm{d}t} = \{\cdot, K_1\}$ commutes with the action of φ, and $\varphi^*(w_n) = w_{n+1}$; hence, if (1.6) holds for some particular n, then

$$
\frac{\mathrm{d}w_{n+1}}{\mathrm{d}t} = \varphi^* \left(\frac{\mathrm{d}w_n}{\mathrm{d}t} \right) = \varphi^* \big(w_n(w_{n+1} - w_{n-1}) \big) = w_{n+1}(w_{n+2} - w_n),
$$

which is just the same equation with $n \to n + 1$. Thus the combined solutions of the iterated map and the flow, which are compatible with one another, generate a sequence of functions $\big(w_n(t) \big)_{n \in \mathbb{Z}}$ satisfying (1.6), which is the Volterra lattice equation, first considered by Kac and van Moerbeke [31]. We will see that, in a certain sense to be made precise, these are genus 2 solutions of this lattice equation.

A wide variety of difference equations admitting Lax pairs and explicit formulae for first integrals have been presented by Svinin [49, 50], including a family that arises as reductions of the hierarchy of symmetries of the Volterra lattice. By eliminating the parameter b from (1.2) we get an equation of fifth order, that is

$$
w_{n+4}\left(\sum_{j=2}^{5} w_{n+j} + \nu\right) + \frac{a}{w_{n+3}} = w_{n+1}\left(\sum_{j=0}^{3} w_{n+j} + \nu\right) + \frac{a}{w_{n+2}}, \tag{1.7}
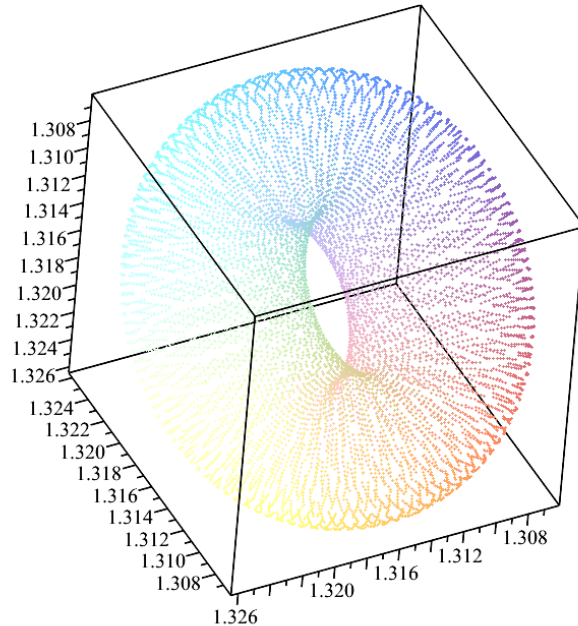$$

Figure 1: Plot of the 3D projection of 10000 points on the orbit of (1.2) with initial values $(\frac{21}{16}, \frac{21}{16}, \frac{452}{343}, \frac{3124}{2373})$ and parameters $a = -9$, $b = 29$, $\nu = -10$.

and upon setting $a = 0$ this reduces to equation (1) in [51] when $s = 4$ (cf. also the case $N = 4$ in [27], where an equivalent equation is obtained via a periodic reduction of the lattice KdV equation); thus the map (1.2) can be viewed as a 1-parameter generalization of one of Svinin's symmetry reductions of the Volterra lattice hierarchy, which in turn is a generalization of one of the maps considered in [12].

1.2 Outline of the paper

The purpose of this article is to give a complete description of the complex geometry of the solutions of the map defined by (1.2). In reaching this goal, we found that all of the structures we obtained could naturally be extended to analogous constructions associated with a family of curves of arbitrary genus g (elliptic for $g = 1$, hyperelliptic for $g \geqslant 2$).

Section 2 presents a set of empirical observations, numerical examples and standalone results about the (P.iv) map. Originally, these were the specific clues that led us to uncover the geometrical structure of the solutions of (1.2). To begin with, we use a p-adic method to identify the singularity pattern of the solutions, leading us to introduce a tau function τ_n, which lifts (1.2) to a recurrence of order 7 with the Laurent property; this is a Laurentification of the original map, in the sense of [23]. By considering a pattern of initial values that approaches a singularity, and substituting this set of initial data into the expressions (1.3) and (1.4) on the level set $K_j = k_j$ for $j = 1, 2$, in the limit that the singularity is reached we find a hyperelliptic curve of genus 2, isomorphic to the Weierstrass quintic

$$y^2 = (1 + \nu x + bx^2)^2 + 4a(1 + \nu x + bx^2)x^3 + 4k_1x^4 + 4(k_2 + \nu k_1)x^5. \tag{1.8}$$

We also show that the tau function τ_n satisfies a Somos-9 recurrence with coefficients that depend on a, b, ν and the values of K_1, K_2 along each orbit of (1.2). It turns out that both the singularity pattern, and the corresponding tau function substitution $w_n = \tau_n\tau_{n+3}/(\tau_{n+1}\tau_{n+2})$ found for (1.2), are the same as for the

QRT map associated with the Somos-5 recurrence [24], which is associated with a family of elliptic curves; so this was a strong initial hint that analogues of the (P.iv) map should exist for any genus g. For enthusiasts of detective stories, the results in this section provide motivation and insight into how we made the first steps on the trail that led to the rest of the paper. However, a reader who is not particularly fond of experimental mathematics can safely omit Section 2 on first reading, since the subsequent sections are not logically dependent on it, and are written in a more linear, deductive style.

In Section 3, we start from a hyperelliptic curve Γ_f of arbitrary genus $g \geqslant 1$, given by a Weierstrass equation $y^2 = f(x)$ where $f \in \mathbb{C}[x]$ is of odd degree $2g + 1$, analogous to (1.8), together with a particular choice of rational function F_0 on the curve, and show how a Stieltjes continued fraction (S-fraction) expansion of this function, of the form

$$F_0 = 1 - \frac{w_1 x}{F_1} = 1 - \cfrac{w_1 x}{1 - \cfrac{w_2 x}{F_2}} = 1 - \cfrac{w_1 x}{1 - \cfrac{w_2 x}{1 - \cfrac{w_3 x}{1 - \cdots}}} \,, \tag{1.9}$$

leads to a birational map on the coefficients w_j of the fraction, in dimension $3g + 1$, which we refer to as the *Volterra map* $\mathcal{V}_g$. As we shall see, iterating $\mathcal{V}_g$ for generic initial data produces the infinite sequence of coefficients w_n for $n \geqslant 1$ that appear in the fraction (1.9), while applying the inverse map $\mathcal{V}_g^{-1}$ extends this sequence to $n \leqslant 0$. Furthermore, the recursion for the S-fraction can be rewritten in the form of a discrete Lax equation. In this setting, the hyperelliptic curve Γ_f is the spectral curve, and the polynomial f has $2g + 1$ non-trivial coefficients which provide conserved quantities (first integrals) for the map. In particular, for $g = 1$ it reduces to a QRT map whose tau functions satisfy the Somos-5 recurrence, while when $g = 2$ we find that, by fixing the values of three of the first integrals to reduce it to four dimensions, the map is precisely (1.2). We also show from the S-fraction that, for any g, the solutions of the map can be written explicitly in terms of tau functions that (up to gauge transformations) are expressed as Hankel determinants.

Next, in Section 4, we introduce a family of compatible Poisson brackets for the map $\mathcal{V}_g$: it is a Poisson map with respect to any of these brackets, and the conserved quantities provide a sufficient number of invariants in involution, so we have a Liouville integrable map for any positive integer g. There is a considerable literature on algebraic completely integrable (a.c.i.) Hamiltonian systems [1, 53], and in this vein we prove a stronger result on the integrability of the Volterra maps, namely that they can be considered as discrete analogues of a.c.i. systems: for each g, the generic level set of the invariants is an affine part of an Abelian variety, namely the Jacobian $\mathrm{Jac}(\bar{\Gamma}_f)$ of the (completion of the) corresponding spectral curve, and the restriction of the map to any of these complex tori is given by translation over a fixed vector which we describe explicitly. In particular, for the map (P.iv) given by (1.2), each generic level set defined by fixing $K_j = k_j$ for $j = 1, 2$ is an affine part of an Abelian surface, that is the Jacobian of the curve (1.8).

The whole basis of our construction is the S-fraction expansion (1.9), which may appear to be a deus ex machina in Section 3, but in fact has many antecedents in the literature on integrable systems, and especially in the development of van der Poorten's results on Jacobi fraction (J-fraction) expansions in elliptic [41] and hyperelliptic function fields [40, 42, 43], as presented in recent work by one of us [28]. The latter revealed the integrable structure of maps generated by J-fractions of the form

$$Y_0 = \alpha_0(X) + \frac{1}{Y_1} = \alpha_0(X) + \cfrac{1}{\alpha_1(X) + \cfrac{1}{Y_2}} = \alpha_0(X) + \cfrac{1}{\alpha_1(X) + \cfrac{1}{\alpha_2(X) + \cfrac{1}{\ddots}}} \,, \tag{1.10}$$

where $Y_0 = \big(Y + P_0(X)\big)/Q_0(X)$ is a rational function on a hyperelliptic curve $\mathcal{C}$ defined by a polynomial of even degree $2g + 2$, that is $\mathcal{C} : Y^2 = P_0^2 + Q_0 Q_{-1}$, for polynomials P_j, Q_j of degrees $g+1, g$ in X, respectively, with the coefficients $\alpha_j = \alpha_j(X)$ in (1.10) being linear in X. It was shown in [28] that, for appropriate such Y_0, the shift from one line of the J-fraction to the next defines a Liouville integrable map on a phase space

of dimension $3g + 1$, which (on a generic level set of the first integrals) corresponds to a fixed translation on the Jacobian of (the completion of) $\mathcal{C}$.

There are classical results going back to Abel on the continued fraction expansion of the square root of an even degree polynomial (i.e. the function Y on an even hyperelliptic curve $\mathcal{C}$), although the fact that the sequence of degrees of the coefficients $\alpha_j(X)$ in such an expansion is eventually periodic was proved only very recently [56] (they need not all be linear in X, as per the above assumption about the function Y_0 in (1.10)). This is intimately related to elliptic [2] and hyperelliptic analogues of orthogonal polynomials [3], as well as more general types of Padé approximation problems connected with integrable systems [5, 14]. In fact, Stieltjes continued fractions (of finite type) were already used in the solution of the finite Volterra lattice by Moser [37], and such fractions were applied to obtain Hankel determinant solutions for non-isospectral extensions more recently [9].

Section 5 of the paper starts by considering the continuous Hamiltonian system that shares the same phase space with the Volterra map $\mathcal{V}_g$. After proving that this continuous system is a.c.i., we show that iteration of $\mathcal{V}_g$ (and its inverse) leads to an infinite sequence of meromorphic functions $\left(w_n(t)\right)_{n \in \mathbb{Z}}$ of t, the time associated with one of the commuting Hamiltonian flows, providing a meromorphic solution of the Volterra lattice equation (1.6) (hence reproducing the above observation about (P.iv) in the particular case $g = 2$). Next we show that this also produces a meromorphic solution of the Toda lattice, taken in the form

$$\frac{\mathrm{d}d_n}{\mathrm{d}t} = d_n(v_{n-1} - v_n) , \qquad \frac{\mathrm{d}v_n}{\mathrm{d}t} = d_n - d_{n+1} , \qquad n \in \mathbb{Z}, \tag{1.11}$$

by applying the well-known Miura transformation between the Volterra and Toda lattices. We further show that the latter transformation arises naturally via the contraction procedure for continued fractions, due to Stieltjes [48], which combines successive pairs of lines in an S-fraction into a single line in a J-fraction, and thereby maps a generic solution of the Volterra map $\mathcal{V}_g$ to an associated solution of the map generated by van der Poorten's construction in genus g. The paper ends in Section 6, with a few conclusions and observations concerning transformations relating solutions of (P.v) and (P.vi) to solutions of the map (P.iv), which we plan to discuss in detail elsewhere. There is also an appendix (Section 7), in which a birational Poisson isomorphism is established between the genus g even Mumford system (see [53]) and the Hamiltonian system associated with the Volterra map $\mathcal{V}_g$.

2 Laurentification and tau functions for the map (P.iv)

In this section we exhibit certain phenomena displayed by the iterates of the map (P.iv), which are related to its discrete integrability. Firstly, we describe the singularity pattern of the iterates, which is found from an empirical p-adic approach, and leads to the introduction of a sequence of tau functions τ_n for these iterates. On the one hand, these tau functions satisfy a homogeneous recursion relation of order 7 with the Laurent property; so this is a Laurentification of (P.iv), as we state here and prove in Section 3.4. On the other hand, these tau functions are also shown to satisfy a Somos-9 relation, with invariants of (P.iv) as coefficients. Secondly, by considering the limit where a solution of (P.iv) approaches a singularity, we are led to a family of genus two curves which will turn out to be at the core of the Stieltjes continued fractions (Section 3) and the algebraic integrability of (P.iv) (Section 4).

The Laurent property is a very special feature of certain birational transformations, appearing in cluster algebras and their generalizations [17, 33], which a priori is unrelated to integrability [25]. However, it turns out that the solutions of discrete integrable systems are often encoded by tau functions satisfying relations that have the Laurent property, such as bilinear equations of discrete Hirota type [35]. Despite the fact that integrable maps occurring "in the wild" typically do not exhibit the Laurent phenomenon, it nevertheless seems to be a common feature of such maps that they admit Laurentification, that is, a lift to a higher-dimensional relation that does have the Laurent property [23]. For some time, singularity analysis has been used as a tool to detect integrability of maps (see [36] and references), and when the pattern of places where the solutions have a zero or pole is sufficiently simple, this can further suggest an appropriate way to introduce tau functions and perform Laurentification [27].

To start with, we apply the p-adic approach described in [27] (see also [32]) to the map (P.iv) defined by (1.2), and derive a singularity pattern from it. This empirical approach is based on examining the prime factorization of the terms of specific orbits $(w_n)_{n\in\mathbb{N}}$ defined in $\mathbb{Q}$, chosen arbitrarily, and considering the behaviour of the p-adic norms $|w_n|_p$ for particular primes p. As a concrete example, upon picking the specific parameter values $\nu = 3$, $a = 5$, $b = 7$ and setting all four initial values to be 1, we then find a sequence of rational numbers given by

$$-30, \tfrac{743}{30}, \tfrac{10541}{22290}, \tfrac{3819540}{7831963}, -\tfrac{4315187227}{1342059038}, \tfrac{6624290612327}{739436079902}, -\tfrac{23965197528782842}{3649794341246183}, -\tfrac{304709076970269230792}{118290200741883010693}, \ldots, \tag{2.1}$$

where the latter terms factorize as $-2 \cdot 3 \cdot 5$, $\tfrac{743}{2\cdot3\cdot5}$, $\tfrac{83\cdot127}{2\cdot3\cdot5\cdot743}$, $\tfrac{2^2\cdot3\cdot5\cdot63659}{83\cdot127\cdot743}$, $-\tfrac{13\cdot743\cdot446753}{2\cdot83\cdot127\cdot63659}$, $\tfrac{19\cdot83\cdot127\cdot1579\cdot20947}{2\cdot13\cdot63659\cdot446753}$, $-\tfrac{2\cdot59\cdot51593\cdot61837\cdot63659}{13\cdot19\cdot1579\cdot20947\cdot446753}$, $-\tfrac{2^3\cdot13\cdot967\cdot446753\cdot6782004923}{19\cdot59\cdot1579\cdot20947\cdot51593\cdot61837}, \ldots$ For several different primes, e.g. $p = 3$, 5, 83, 127, $743, \ldots$, this reveals a common pattern whereby, for some n,

$$|w_n|_p = p^{-1}, \quad |w_{n+1}|_p = p, \quad |w_{n+2}|_p = p, \quad |w_{n+3}|_p = p^{-1}, \tag{2.2}$$

with the prime p being absent from the factorization on the previous and on the next terms: $|w_{n-1}|_p = |w_{n+4}|_p = 1$. The p-adic norms (2.2) identify places where the orbit of the map over the finite field $\mathbb{F}_p$ has a zero or pole, as well as the order of these [32]. Since the recurrence (1.2) defines a birational map, any orbit defined for $n \geqslant 0$ can be extended to $n < 0$ (at least, provided that it does not reach a singularity, where $w_n = 0$ for some n; but see Corollary 2.6 below). Hence, the pattern (2.2) suggests that for $n \in \mathbb{Z}$ one should make the tau function substitution

$$w_n = \frac{\tau_n \tau_{n+3}}{\tau_{n+1}\tau_{n+2}}, \tag{2.3}$$

so that the places where a prime factor p appears in the numerators or denominators of the sequence $(w_n)_{n\in\mathbb{Z}}$ can be encoded by the appearance of the factor p in the terms of the tau function sequence $(\tau_n)_{n\in\mathbb{Z}}$. These tau functions can be defined recursively, in two quite different ways:

Proposition 2.1. *Suppose that $(w_n)_{n\in\mathbb{Z}}$ is a solution of (1.2). Then the corresponding sequence $(\tau_n)_{n\in\mathbb{Z}}$ satisfies*

(1) A homogeneous recurrence of order 7 and degree 8:

$$\begin{aligned}
&\tau_{n+7}\tau_{n+4}^2\tau_{n+3}^3\tau_{n+2}^2 + \tau_{n+6}^2\tau_{n+3}^4\tau_{n+2}^2 + 2\tau_{n+6}\tau_{n+5}^2\tau_{n+3}^2\tau_{n+2}^3 + \tau_{n+6}\tau_{n+5}\tau_{n+4}^2\tau_{n+3}^2\tau_{n+2}\tau_{n+1} \\
&\quad + \tau_{n+5}^4\tau_{n+2}^4 + 2\tau_{n+5}^3\tau_{n+4}^2\tau_{n+2}^2\tau_{n+1} + \tau_{n+5}^2\tau_{n+4}^4\tau_{n+1}^2 + \tau_{n+5}^2\tau_{n+4}^3\tau_{n+3}^2\tau_n \\
&\quad + \nu\big(\tau_{n+6}\tau_{n+5}\tau_{n+4}\tau_{n+3}^3\tau_{n+2} + \tau_{n+5}^3\tau_{n+4}\tau_{n+3}\tau_{n+2}^2 + \tau_{n+5}^2\tau_{n+4}^3\tau_{n+3}\tau_{n+2}\tau_{n+1}\big) \\
&\quad + b\,\tau_{n+5}^2\tau_{n+4}^2\tau_{n+3}^2\tau_{n+2}^2 + a\,\tau_{n+5}\tau_{n+4}^3\tau_{n+3}^3\tau_{n+2} = 0.
\end{aligned} \tag{2.4}$$

(2) A (generalized) Somos-9 recurrence:

$$\alpha_1\,\tau_{n+9}\tau_n + \alpha_2\,\tau_{n+8}\tau_{n+1} + \alpha_3\,\tau_{n+7}\tau_{n+2} + \alpha_4\,\tau_{n+6}\tau_{n+3} + \alpha_5\,\tau_{n+5}\tau_{n+4} = 0. \tag{2.5}$$

The coefficients α_i are polynomial functions of the parameters of the map and of the values k_1, k_2 of the invariants K_1, K_2, hence are constant along each orbit $(w_n)_{n\in\mathbb{Z}}$. They are given by

$$\alpha_1 = k_1, \qquad \alpha_2 = ak_2 - k_1^2, \qquad \alpha_3 = a\big(ak_2 - 2k_1^2\big),$$
$$\alpha_4 = a\big(k_2^2 + \nu k_1 k_2 + bk_1^2 + a^2 k_1\big), \qquad \alpha_5 = -k_1\big(k_2^2 + \nu k_1 k_2 + bk_1^2 + a^2 k_1\big).$$

Proof. The proof of *(1)* is by a direct substitution of (2.3) in (1.2), while *(2)* is derived directly by implementing the method from [26], which involves computing determinants of matrices with entries that are of degree two in tau functions. More precisely, for *(2)* one should write five copies of the Somos-9 relation as a matrix equation $\mathrm{M}_n\,\boldsymbol{\alpha} = \mathbf{0}$, and then verify that there is a non-zero vector of coefficients $\boldsymbol{\alpha}$, lying in the kernel of M_n, that is independent of n; the latter calculations require extensive use of computer algebra. $\square$

The recursion defined by (1.2) requires 4 initial values, while (2.4) requires 7, and the discrepancy between the two is described by the three-parameter group $(\mathbb{C}^*)^3$ of gauge transformations, with action given by

$$\tau_n \mapsto A_\pm \, B^n \, \tau_n \,, \qquad A_+, A_-, B \in \mathbb{C}^* \,, \tag{2.6}$$

corresponding to the freedom to rescale even/odd terms by a different factor $A_\pm$, and apply a rescaling B^n that is exponential in n to all terms. Any 4 non-zero initial values $w_0, \dots, w_3$ of (1.2) allow a corresponding set of non-zero initial data to be determined for (2.4), up to this gauge freedom; for example, we may take as corresponding initial data $\tau_n = 1$ for $n = 0, 1, 2$ and $\tau_3 = w_0$, $\tau_4 = w_0 w_1$, $\tau_5 = w_0^2 w_1 w_2$ and $\tau_6 = w_0^2 w_1^2 w_2 w_3$, which are polynomials (in fact, monomials) in w_0, w_1, w_2, w_3. Notice that (2.4) can also be solved rationally for τ_n in terms of $\tau_{n+1}, \dots, \tau_{n+7}$ so that the sequence (τ_n) is actually defined for all $n \in \mathbb{Z}$.

Example 2.2. In the case $\nu = 3$, $a = 5$, $b = 7$, taking the initial values $\tau_j = 1$ for $0 \leqslant j \leqslant 6$ in (2.4) yields the sequence of tau functions beginning

$$1, 1, 1, 1, 1, 1, 1, -30, -743, 10541, 127318, 5807789, 628430947, -188231024119, 52465590084328, \dots, \tag{2.7}$$

which consists of integers, and the terms after the initial 1s have prime factorizations given by $-2 \cdot 3 \cdot 5, -743, 83 \cdot 127, 2 \cdot 63659, 13 \cdot 446753, 19 \cdot 1579 \cdot 20947, -59 \cdot 61837 \cdot 51593, 2^3 \cdot 967 \cdot 6782004923$, etc. These correspond to the prime factors appearing in the numerators and denominators in the particular sequence of rational values of w_n illustrated in (2.1) above. Due to a reversing symmetry of the recurrence (2.4), this sequence extends backwards to $n < 0$ in such a way that the property $\tau_{6-n} = \tau_n$ holds for all $n \in \mathbb{Z}$, since the 7 initial data have this symmetry. Furthermore, these tau functions also satisfy the Somos-9 recurrence

$$28\, \tau_{n+9}\tau_n - 239\, \tau_{n+8}\tau_{n+1} - 5115\, \tau_{n+7}\tau_{n+2} + 136125\, \tau_{n+6}\tau_{n+3} - 762300\, \tau_{n+5}\tau_{n+4} = 0, \tag{2.8}$$

which corresponds to (2.5) with the coefficients α_j being fixed (up to overall rescaling) by the specified choices of ν, a, b, together with the fact that the first integrals take the values $K_1 = 28$, $K_2 = 109$. For any solution of (2.4), the subsequences consisting of even/odd index terms, that is $\hat{\tau}_n = \tau_{2n}$ or τ_{2n+1}, respectively, also satisfy a Somos-8 relation, of the form

$$\hat{\alpha}_1 \, \hat{\tau}_{n+8}\hat{\tau}_n + \hat{\alpha}_2 \, \hat{\tau}_{n+7}\hat{\tau}_{n+1} + \hat{\alpha}_3 \, \hat{\tau}_{n+6}\hat{\tau}_{n+2} + \hat{\alpha}_4 \, \hat{\tau}_{n+5}\hat{\tau}_{n+3} + \hat{\alpha}_5 \, \hat{\tau}_{n+4}^2 = 0 \,. \tag{2.9}$$

For the particular integer sequence above, up to overall scale the coefficients are given by

$$\hat{\alpha}_1 = 195848, \qquad \hat{\alpha}_2 = -61660241775, \qquad \hat{\alpha}_3 = 13236763233189375,$$
$$\hat{\alpha}_4 = -8064076031989579800, \qquad \hat{\alpha}_5 = -3603810041796109733 \,.$$

The relation (2.9) can be regarded as an ordinary difference reduction of a constraint for a tau function defined on a multidimensional lattice, which arises from a Hermite-Padé approximation problem (cf. equation (2.10) in [14]). An explanation for why this Somos-8 relation must hold will be provided in Section 5, via the connection with the Toda lattice and the results in [28].

The following proposition shows that the recurrence (2.4) is a Laurentification of (P.iv). In particular, this explains why the tau functions in the preceding example are all integers.

Proposition 2.3. *The recurrence (2.4) has the Laurent property. More precisely, for all $n \in \mathbb{Z}$,*

$$\tau_n \in \mathbb{Z}[a, b, \nu, \tau_0, \tau_1, \tau_2^{\pm 1}, \tau_3^{\pm 1}, \tau_4^{\pm 1}, \tau_5, \tau_6] \,.$$

In principle, for $n \geqslant 0$ the proof of the proposition is a direct application of Theorem 2 in [23], and the same method of proof applies for $n < 0$, because the map defined by (2.4) is birational. However, the formal verification to be done by this method quickly gets out of hand, since the rational functions obtained as the formulae for the first few iterations of the map defined by (2.4) soon become too complicated for the checks to be carried out by a simple computer algebra program. A general proof of Proposition 2.3 that does not require any computer algebra will be given in Section 3.4.

Remark 2.4. As mentioned in the introduction, through the relation (1.7), the special case $a = 0$ of the (P.iv) map is closely linked to a difference equation appearing in the work of Svinin, and to a reduction of the lattice KdV equation. Thus, when the parameter $a = 0$, the result of Proposition 2.3 is subtly related to the fact that the order 7, degree 6 recurrence

$$
\begin{aligned}
&\tau_{n+7}\tau_{n+4}^2\tau_{n+3}\tau_{n+2}\tau_{n+1} + \tau_{n+6}\tau_{n+5}\tau_{n+4}\tau_{n+3}^2\tau_n + \tau_{n+6}^2\tau_{n+3}^2\tau_{n+2}\tau_{n+1} + \tau_{n+6}\tau_{n+5}\tau_{n+4}^2\tau_{n+1}^2 \\
&+\tau_{n+6}\tau_{n+5}^2\tau_{n+2}^2\tau_{n+1} - \alpha\tau_{n+5}\tau_{n+4}^2\tau_{n+3}^2\tau_{n+2} + \beta\tau_{n+6}\tau_{n+5}\tau_{n+4}\tau_{n+3}\tau_{n+2}\tau_{n+1} = 0
\end{aligned}
\tag{2.10}
$$

has the Laurent property, in the sense that it generates Laurent polynomials in $\tau_0, \ldots, \tau_6$ with coefficients in $\mathbb{Z}[\alpha, \beta]$ (the case $N = 4$ of Proposition 2.3 in [27]). The point is that (2.3) coincides with the tau function substitution found for maps related with lattice KdV reductions in [27], and the relation (2.10) holds for solutions of (2.4) obtained by setting $a = 0$, $\nu = \beta$, and taking an orbit for which the value of the first integral K_1 is fixed to be $k_1 = -\alpha$.

Example 2.5. Upon taking $a = 0$, $b = -17$, $\nu = -11$ and making a specific choice of 7 integer initial values for (2.4), with the 3 central values fixed to be 1, we generate an integer sequence that begins as follows:

$$3, 2, 1, 1, 1, 4, 5, 699, -25626, 453024, -112570254, 23354432973, 61327997061471, -35520663450983076, \ldots.$$

Then we see that this sequence also satisfies the relation (2.10) with $\beta = -11$ and $\alpha = -k_1 = 327$. Also, since three of the coefficients in (2.5) vanish when $a = 0$, we see that the Somos-9 relation for this sequence takes the shorter (three-term) Gale-Robinson form

$$\tau_{n+9}\tau_n + 327\,\tau_{n+8}\tau_{n+1} + 3850083\,\tau_{n+5}\tau_{n+4} = 0,$$

which is a reduction of the discrete Hirota equation, and is in agreement with the $N = 4$ case of Theorem 1.1 in [27].

The Laurent property for (2.4), together with the formula (2.3), immediately implies that the generic orbit (w_n) of (P.iv) is well-defined, as stated in the following corollary.

Corollary 2.6. *For generic non-zero initial values $(w_0, w_1, w_2, w_3) \in (\mathbb{C}^*)^4$, the orbit $(w_n)_{n \in \mathbb{Z}}$ exists, with $w_n \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.*

Proof. As we shall see in Section 3.4, the initial tau functions can be chosen so that all τ_n are polynomials in the initial data for (1.2). For a fixed index n it is then clear from (2.3) that w_n is an indeterminate element of $\mathbb{P}^1$ only when at least two out of three successive tau functions in (2.3) vanish, i.e. belong to a certain proper Zariski closed subset of the space of (non-zero) initial data for (1.2). Considering this condition for all n yields a subset of this space of initial data, which is the intersection of a countable family of Zariski open subsets. Such an intersection is a residual, hence dense, subset so that for generic initial data, the orbit is well-defined. □

Note that (P.iv) was originally defined as a birational affine map in $\mathbb{C}^4$, but the above corollary allows the existence of certain orbits defined in $(\mathbb{P}^1)^4$. Notice also that this corollary does not state that the subset of initial data for which the orbit exists is open. This stronger statement will follow from algebraic integrability, without use of the tau functions (see Section 4).

As an initial foray into the geometry of the map, defined by (1.2), we now consider the singularity pattern in more detail, by taking three non-zero initial values $w_0, w_1, w_2 \in \mathbb{C}^*$ followed by a fourth value proportional to a small parameter $\epsilon \in \mathbb{C}$, and consider the behaviour of the solution in the limit $\epsilon \to 0$. To reformulate this in terms of tau functions, we set

$$\tau_0 = Z, \ \tau_1 = \tau_2 = \tau_3 = 1, \tau_4 = X, \ \tau_5 = Y, \ \tau_6 = \epsilon, \quad XYZ \neq 0,$$

where three adjacent values have been set equal to 1 by a choice of gauge. This gives, using (2.3), four non-zero initial values

$$w_0 = Z, \quad w_1 = X, \quad w_2 = \frac{Y}{X}, \quad w_3 = \frac{\epsilon}{XY},
\tag{2.11}$$

for the map (1.2), such that the fourth value $w_3 \to 0$ as $\epsilon \to 0$. Upon substituting these values in (1.2) we find as subsequent values

$$w_4 = C_4 \, \epsilon^{-1} + O(1) \, , \quad w_5 = C_5 \, \epsilon^{-1} + O(1) \, , \quad w_6 = C_6 \, \epsilon + O(\epsilon^2) \, , \quad w_7 = C_7 + O(\epsilon) \, ,$$

for certain coefficients C_j which are rational functions of X, Y, Z. Notice that the leading order behaviours of w_3, w_4, w_5, w_6 are $\epsilon, \epsilon^{-1}, \epsilon^{-1}, \epsilon$, respectively, with terms of $O(1)$ on either side, which corresponds to the singularity pattern (2.2) obtained above by the p-adic method. Now if we substitute the initial values (2.11) into $K_1 = k_1, \ K_2 = k_2$, where K_1 and K_2 are the invariants (1.3) and (1.4), and take the limit $\epsilon \to 0$, then (after clearing denominators) we get two polynomial relations between X, Y, Z, which define an affine algebraic curve. Upon taking resultants with respect to Z, this yields a single relation between X and Y, namely

$$\begin{aligned}
&\left(aY^2 - (\nu k_1 + k_2)Y - ak_1 \right) X^4 + \left((a\nu - k_1)Y^2 + (a^2 - bk_1)Y + k_1^2 \right) X^3 \\
&+ \left(2aY^3 + (ab - \nu k_1)Y^2 - 2ak_1 Y \right) X^2 + Y^2 \left((a\nu - k_1)Y + a^2 \right) X + aY^4 = 0 \, .
\end{aligned} \tag{2.12}$$

This plane curve is birationally equivalent to the Weierstrass quintic (1.8), which can be seen from the transformation

$$x = \frac{X(aY - k_1 X)}{Y(aX^2 - k_1 X + aY)} \, , \tag{2.13}$$

together with a corresponding formula for $y = y(X, Y)$, which is rather unwieldy and so is omitted. For generic values of k_1, k_2, the curve (2.12) is a smooth hyperelliptic curve of genus 2. As already said, in its Weierstrass form (1.8), this family of curves will play an important role in all that follows.

3 S-fractions on hyperelliptic curves and Volterra maps

In this section, for a fixed integer $g > 0$, we introduce an affine space of triplets of polynomials, reminiscent of the phase space of the Mumford system [38, 53] and use Stieltjes continued fractions (S-fractions) to construct a series of birational automorphisms of the affine space, indexed by g, which we describe in several ways. When $g = 1$ we recover several known integrable maps, and for $g = 2$ we recover the map (P.iv), which was the primary motivation for this study, while the maps for $g > 2$ appear to be new. We also give solutions in terms of Hankel determinants of the iterates of these maps, i.e., of the corresponding recursion relations.

For a fixed $g > 0$ we consider the affine space

$$M_g := \left\{ (\mathcal{P}(x), \mathcal{Q}(x), \mathcal{R}(x)) \in \mathbb{C}[x]^3 \ \middle| \ \begin{array}{ll} \deg \mathcal{P}(x) \leqslant g \, , & \mathcal{P}(0) = 1 \, , \\ \deg \mathcal{Q}(x) \leqslant g \, , & \mathcal{Q}(0) = 2 \\ \deg \mathcal{R}(x) \leqslant g + 1 \, , & \mathcal{R}(0) = 0 \end{array} \right\} \, . \tag{3.1}$$

It is clear that M_g is an affine space of dimension $3g + 1$: writing

$$\mathcal{P}(x) = 1 + \sum_{i=1}^{g} p_i x^i \, , \qquad \mathcal{Q}(x) = 2 + \sum_{i=1}^{g} q_i x^i \, , \qquad \mathcal{R}(x) = \sum_{i=1}^{g+1} r_i x^i \, , \tag{3.2}$$

the coefficients $p_1, \ldots, p_g, q_1, \ldots, q_g, r_1, \ldots, r_{g+1}$ provide a natural system of linear coordinates on M_g. We will often write an element $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ of M_g as a traceless 2×2 matrix

$$\mathbf{L}(x) := \begin{pmatrix} \mathcal{P}(x) & \mathcal{R}(x) \\ \mathcal{Q}(x) & -\mathcal{P}(x) \end{pmatrix} \, , \tag{3.3}$$

which will later serve as a Lax operator, and think of M_g as an affine space of matrices (Lax operators). It is then natural to consider the map μ defined by

$$
\mu \quad : \quad
\begin{array}{ccc}
M_g & \to & \mathbb{C}[x] \\
\mathbf{L}(x) = \begin{pmatrix} \mathcal{P}(x) & \mathcal{R}(x) \\ \mathcal{Q}(x) & -\mathcal{P}(x) \end{pmatrix} & \mapsto & -\det \mathbf{L}(x) = \mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x) \ .
\end{array}
\tag{3.4}
$$

In view of the degree constraints on the entries of $\mathbf{L}(x)$, the polynomial $-\det\mathbf{L}(x)$ has degree at most $2g+1$ and its constant term is 1; moreover, every such polynomial is contained in the image of μ. In the $g = 2$ case these curves are precisely the ones encountered in Section 2 in the singularity analysis of (P.iv), see (1.8), (2.12) and (2.13), which in part motivates the choice of constraints on the polynomials $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$. (When M_g is endowed with a Poisson structure, as in Section 4, then μ can be viewed a momentum map.)

Throughout this section, $g > 0$ is fixed. In each of the following subsections, the results and phenomena being discussed will be specialized and illustrated for the cases of $g = 1$ and $g = 2$, when M_g has dimension 4 and 7, respectively.

3.1 Stieltjes continued fractions

We start from a hyperelliptic curve Γ_f, defined by an odd Weierstrass equation

$$
\Gamma_f : y^2 = f(x) \ , \quad \text{with} \quad f(x) := 1 + \sum_{j=1}^{2g+1} c_j x^j \in \mathbb{C}[x] \ .
\tag{3.5}
$$

When f has degree $2g+1$ and has no multiple roots, Γ_f is non-singular and its genus is g, which explains the notation used. Let $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ be any point in $\mu^{-1}(f)$, the fiber of μ over f, so that $f(x) = \mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x)$, and the spectral curve Γ_f is the characteristic equation $\det(\mathbf{L}(x) - y\mathbf{1}) = 0$. We consider on Γ_f the rational function, given by

$$
F := \frac{y + \mathcal{P}(x)}{\mathcal{Q}(x)} = \frac{\mathcal{R}(x)}{y - \mathcal{P}(x)} \ .
\tag{3.6}
$$

In preparation for constructing the Stieltjes continued fraction of F, we show in the following lemma how the triplet $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ leads to another triplet $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})$ in the same fiber of μ, under the assumption that $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ is *regular*, meaning that

$$
\frac{2\mathcal{P}(x) - \mathcal{Q}(x) + \mathcal{R}(x)}{x}\bigg|_{x=0} = 2p_1 - q_1 + r_1 \neq 0 \ .
\tag{3.7}
$$

Lemma 3.1. *Given a regular triplet $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ in $\mu^{-1}(f)$, there exists a unique $w \in \mathbb{C}^*$ and a unique triplet $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})$ in $\mu^{-1}(f)$ such that*

$$
\frac{y + \mathcal{P}(x)}{\mathcal{Q}(x)} = 1 - \frac{w\,x}{\frac{y + \tilde{\mathcal{P}}(x)}{\tilde{\mathcal{Q}}(x)}} \ .
\tag{3.8}
$$

The two triplets are related by

$$
\tilde{\mathcal{P}}(x) = \mathcal{Q}(x) - \mathcal{P}(x) \ , \quad \tilde{\mathcal{Q}}(x) = \frac{2\mathcal{P}(x) - \mathcal{Q}(x) + \mathcal{R}(x)}{-wx} \ , \quad \tilde{\mathcal{R}}(x) = -wx\mathcal{Q}(x) \ ,
\tag{3.9}
$$

and

$$
w = -\frac{2\,p_1 - q_1 + r_1}{2} = -\frac{\tilde{r}_1}{2} \ .
\tag{3.10}
$$

Proof. We will constructively show that we can achieve (3.8) with $w \in \mathbb{C}^*$ and $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}}) \in \mu^{-1}(f)$ uniquely determined. Clearing the denominators in (3.8) and using $y^2 = f(x)$ we get

$$
y(\mathcal{P}(x) + \tilde{\mathcal{P}}(x) - \mathcal{Q}(x)) + f(x) + \mathcal{P}(x)\tilde{\mathcal{P}}(x) - \tilde{\mathcal{P}}(x)\mathcal{Q}(x) + wx\mathcal{Q}(x)\tilde{\mathcal{Q}}(x) = 0 \ ,
\tag{3.11}
$$

an equality which holds in the field of fractions of $\mathbb{C}[x, y]/(y^2 - f(x))$, so that the coefficients of y and of y^0 in (3.11) must be zero. The vanishing of the former coefficient gives the first equation in (3.9) and guarantees $\tilde{\mathcal{P}}(0) = 1$ and $\deg \tilde{\mathcal{P}} \leqslant g$. The vanishing of the y^0 coefficient gives

$$-w \, \tilde{\mathcal{Q}}(x) = \frac{f(x) + (\mathcal{P}(x) - \mathcal{Q}(x))\tilde{\mathcal{P}}(x)}{x\mathcal{Q}(x)} = \frac{f(x) - (\mathcal{P}(x) - \mathcal{Q}(x))^2}{x\mathcal{Q}(x)} = \frac{2\mathcal{P}(x) - \mathcal{Q}(x) + \mathcal{R}(x)}{x} \,, \qquad (3.12)$$

where we have used in the last step that $f(x) = \mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x)$. Notice that in view of the values of the constant terms in (3.2), the last numerator in (3.12) vanishes for $x = 0$ and is of degree at most $g + 1$. Also, as the triplet $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ is assumed to be regular, the polynomial $(2\mathcal{P} - \mathcal{Q} + \mathcal{R})/x$ does not vanish at $x = 0$, hence we can (uniquely) choose $w \in \mathbb{C}^*$ such that $\tilde{\mathcal{Q}}(0) = 2$. This gives the first equality in (3.10) and the second equation in (3.9). The first equality in (3.12) also shows that $f(x) - \tilde{\mathcal{P}}^2(x)$ is divisible by $\tilde{\mathcal{Q}}(x)$, with quotient $-wx\mathcal{Q}(x)$; thus, if we take the third equality in (3.9) to define $\tilde{\mathcal{R}}$, then $\deg \tilde{\mathcal{R}} = g + 1$, $\tilde{\mathcal{R}}(0) = 0$ and $f(x) = \tilde{\mathcal{P}}^2(x) + \tilde{\mathcal{Q}}(x)\tilde{\mathcal{R}}(x)$, completing the proof that $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})$ belongs to $\mu^{-1}(f)$. Notice that the third equality in (3.9) implies the alternative formula for w in (3.10), since $\mathcal{Q}(0) = 2$. □

Applying the lemma to all regular points of the fiber $\mu^{-1}(f)$ yields a rational map of the fiber to itself, given by $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) \mapsto (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})$ with $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})$ given by (3.9). Since the latter can also be solved rationally for $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ in terms of $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}})$ by using the second expression for w in (3.10), this rational map is actually a birational automorphism of the fiber. Iterating this map starting from a triplet $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0) = (\mathcal{P}, \mathcal{Q}, \mathcal{R}) \in \mu^{-1}(f)$, we get an infinite sequence $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$ of triplets as well as an infinite sequence $(w_n)_{n \in \mathbb{Z}}$ in $\mathbb{C}^*$, such that $(\mathcal{P}_{n+1}, \mathcal{Q}_{n+1}, \mathcal{R}_{n+1})$ and w_{n+1} are related to $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)$ as dictated by the lemma:

$$\mathcal{P}_{n+1}(x) = \mathcal{Q}_n(x) - \mathcal{P}_n(x) \,, \quad \mathcal{Q}_{n+1}(x) = \frac{2\mathcal{P}_n(x) - \mathcal{Q}_n(x) + \mathcal{R}_n(x)}{-w_{n+1}x} \,, \quad \mathcal{R}_{n+1}(x) = -w_{n+1}x\mathcal{Q}_n(x) \,. \quad (3.13)$$

Writing, as in (3.2), $\mathcal{P}_n(x) = 1 + \sum_{i=1}^{g} p_{n,i}x^i$, and similarly for $\mathcal{Q}_n(x)$ and $\mathcal{R}_n(x)$, the value of w_{n+1} is given, according to (3.10), by

$$w_{n+1} = -\frac{2p_{n,1} - q_{n,1} + r_{n,1}}{2} = -\frac{r_{n+1,1}}{2} \,, \qquad (3.14)$$

for all $n \in \mathbb{Z}$.

It is clear that $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)$ is obtained from $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$ by repeating the map n times (or, when $n < 0$, repeating the inverse of the map $-n$ times). As pointed out in the lemma, the starting triplet $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$ must be regular in order for $(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1)$ and $w_1 \in \mathbb{C}^*$ to exist. But nothing guarantees that $(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1)$ will also be regular, and in general it need not be so; assuming $(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1)$ to be a regular triplet puts an open linear condition on the coefficients of $(\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1)$, namely that $2p_{1,1} - q_{1,1} + r_{1,1} \neq 0$, which amounts to an open polynomial condition on $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$, and for the existence of every extra term of the sequence such an extra condition is to be added to the triplet $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$. However, since this amounts to a countable number of open conditions on the latter, this means that when $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$ is *generic*, in the sense that it belongs to a residual subset of the fiber $\mu^{-1}(f)$, the sequence of triplets of polynomials $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n) \in \mu^{-1}(f)$ and the sequence of constants $w_n \in \mathbb{C}^*$, both indexed by $n \in \mathbb{Z}$, exist. For generic $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$, iterating (3.8) gives

$$F_0 := \frac{y + \mathcal{P}_0(x)}{\mathcal{Q}_0(x)} = 1 - \frac{w_1 x}{\frac{y + \mathcal{P}_1(x)}{\mathcal{Q}_1(x)}} = \cdots = 1 - \cfrac{w_1 x}{1 - \cfrac{w_2 x}{1 - \cfrac{w_3 x}{1 - \cdots}}} \,, \qquad (3.15)$$

yielding the *Stieltjes continued fraction*, also called *S-fraction*, of F_0. Similarly, each triple $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)$, $n \in \mathbb{Z}$, is associated with a rational function F_n, with a corresponding S-fraction obtained by shifting each of the indices in (3.15), which for $n > 0$ appears on the nth line below the top.

Example 3.2. Suppose that $g = 1$. Then the entries of the triplets $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ and sequences $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$ in $M_g = M_1$ take the form

$$
\begin{aligned}
\mathcal{P}(x) &= 1 + p_1 x , & \mathcal{P}_n(x) &= 1 + p_{n,1} x , \\
\mathcal{Q}(x) &= 2 + q_1 x , & \text{and} \qquad \mathcal{Q}_n(x) &= 2 + q_{n,1} x , \\
\mathcal{R}(x) &= r_1 x + r_2 x^2 , & \mathcal{R}_n(x) &= r_{n,1} x + r_{n,2} x^2 .
\end{aligned}
\tag{3.16}
$$

The birational automorphism (3.9), constructed in Lemma 3.1, and its iterates are given by

$$
\begin{aligned}
\tilde{p}_1 &= q_1 - p_1 , & p_{n+1,1} &= q_{n,1} - p_{n,1} , \\
\tilde{q}_1 &= -r_2/w , & q_{n+1,1} &= -r_{n,2}/w_{n+1} , \\
\tilde{r}_1 &= -2w , & \text{and} \qquad r_{n+1,1} &= -2w_{n+1} , \\
\tilde{r}_2 &= -wq_1 , & r_{n+1,2} &= -w_{n+1}q_{n,1} ,
\end{aligned}
\tag{3.17}
$$

where $w = -\frac{2p_1 - q_1 + r_1}{2}$ and $w_{n+1} = -\frac{2p_{n,1} - q_{n,1} + r_{n,1}}{2}$ for $n \in \mathbb{Z}$.

Example 3.3. Suppose now that $g = 2$. The entries of the triplets $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ and sequences $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$ in M_2 now take the form

$$
\begin{aligned}
\mathcal{P}(x) &= 1 + p_1 x + p_2 x^2 , & \mathcal{P}_n(x) &= 1 + p_{n,1} x + p_{n,2} x^2 , \\
\mathcal{Q}(x) &= 2 + q_1 x + q_2 x^2 , & \text{and} \qquad \mathcal{Q}_n(x) &= 2 + q_{n,1} x + q_{n,2} x^2 , \\
\mathcal{R}(x) &= r_1 x + r_2 x^2 + r_3 x^3 , & \mathcal{R}_n(x) &= r_{n,1} x + r_{n,2} x^2 + r_{n,3} x^3 .
\end{aligned}
\tag{3.18}
$$

From the construction in Lemma 3.1, the birational automorphism (3.9) and its iterates are given by (3.17), with the expression for $\tilde{q}_1$ modified to

$$
\tilde{q}_1 = -(2p_2 - q_2 + r_2)/w , \quad \text{and} \quad q_{n+1,1} = -(2p_{n,2} - q_{n,2} + r_{n,2})/w_{n+1},
$$

further supplemented with the following formulae:

$$
\begin{aligned}
\tilde{p}_2 &= q_2 - p_2 , & p_{n+1,2} &= q_{n,2} - p_{n,2} , \\
\tilde{q}_2 &= -r_3/w , & \text{and} \qquad q_{n+1,2} &= -r_{n,3}/w_{n+1} , \\
\tilde{r}_3 &= -wq_2 , & r_{n+1,3} &= -w_{n+1}q_{n,2} ,
\end{aligned}
\tag{3.19}
$$

where, as in the genus 1 case, $w = -\frac{2p_1 - q_1 + r_1}{2}$ and $w_{n+1} = -\frac{2p_{n,1} - q_{n,1} + r_{n,1}}{2}$ for $n \in \mathbb{Z}$.

3.2   Lax equation and invariants

In Section 3.1 we defined a birational automorphism of the fiber $\mu^{-1}(f) \subset M_g$, where $f = f(x)$ is any polynomial of degree at most $2g + 1$, satisfying $f(0) = 1$. This map, given by (3.9), is not just defined on $\mu^{-1}(f)$, but is also as it stands a well-defined birational automorphism of M_g. In view of its relation to the Volterra lattice (see Section 5), we call it the *Volterra map*, denoted $\mathcal{V}_g$; explicitly,

$$
\mathcal{V}_g : \qquad (\mathcal{P}, \mathcal{Q}, \mathcal{R}) \mapsto (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}}),
$$

where the entries of the latter are given by (3.9); also, we can write $\mathcal{V}_g(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n) = (\mathcal{P}_{n+1}, \mathcal{Q}_{n+1}, \mathcal{R}_{n+1})$ for all $n \in \mathbb{Z}$. For a fixed initial triplet $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$, the entire sequence of triplets $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)_{n \in \mathbb{Z}}$ in M_g is called the *orbit* of $\mathcal{V}_g$ through $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$. We also refer to a sequence of triplets that satisfies the recursion relations (3.13) for all n as a *solution*. The equations (3.9) for the Volterra map, as well as the recursion relations (3.13) for its iterates, are easily rewritten as discrete Lax equations; this fact has many important consequences which will be worked out in what follows.

13

Proposition 3.4. *The Volterra map $\mathcal{V}_g$ can be written in the compact form*

$$\mathcal{V}_g : \qquad \mathbf{L}(x)\mathbf{M}(x) = \mathbf{M}(x)\widetilde{\mathbf{L}}(x) , \qquad (3.20)$$

where $\mathbf{L}(x)$ is given by (3.3) and $\mathbf{M}(x) := \begin{pmatrix} 1 & -wx \\ 1 & 0 \end{pmatrix}$, with w given by (3.10). As a consequence, the $2g + 1$ polynomial functions $H_1, \ldots, H_{2g+1}$ on M_g, defined by

$$\mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x) = 1 + \sum_{i=1}^{2g+1} H_i x^i \qquad (3.21)$$

are invariants of the Volterra map, i.e. $\tilde{H}_i = H_i$ for $i = 1, \ldots, 2g + 1$.

Proof. It is easily checked by direct computation that (3.9) and (3.20) are the same set of equations. Since the latter says that $\widetilde{\mathbf{L}}(x)$ is obtained from $\mathbf{L}(x)$ by conjugation with $\mathbf{M}(x)$, the spectrum of $\mathbf{L}(x)$ is preserved, hence also all coefficients of the determinant of $\mathbf{L}(x)$, i.e., the coefficients H_i of $\mu(\mathbf{L}(x)) = \mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x)$. $\square$

Upon iterating the Volterra map, as discussed in Section 3.1, starting from a generic triplet $\mathbf{L}_0(x)$ of M_g we get a sequence of triplets

$$\mathbf{L}_n(x) := \begin{pmatrix} \mathcal{P}_n(x) & \mathcal{R}_n(x) \\ \mathcal{Q}_n(x) & -\mathcal{P}_n(x) \end{pmatrix}$$

of M_g. According to (3.20), a discrete Lax equation for this sequence is given by

$$\mathbf{L}_n(x)\mathbf{M}_n(x) = \mathbf{M}_n(x)\mathbf{L}_{n+1}(x) , \qquad (3.22)$$

where

$$\mathbf{M}_n(x) := \begin{pmatrix} 1 & -w_{n+1}x \\ 1 & 0 \end{pmatrix} , \qquad \text{with} \qquad w_{n+1} = -\frac{2p_{n,1} - q_{n,1} + r_{n,1}}{2} = -\frac{r_{n+1,1}}{2} . \qquad (3.23)$$

Example 3.5. When $g = 1$, respectively when $g = 2$, the invariants H_i can be computed from (3.21):

$$
\begin{aligned}
H_1 &= 2(p_1 + r_1) , \\
H_2 &= p_1^2 + q_1 r_1 + 2r_2 , \\
H_3 &= q_1 r_2 ,
\end{aligned}
\qquad
\begin{aligned}
H_1 &= 2(p_1 + r_1) , \\
H_2 &= 2p_2 + p_1^2 + q_1 r_1 + 2r_2 , \\
H_3 &= 2p_1 p_2 + 2r_3 + q_1 r_2 + q_2 r_1 , \\
H_4 &= p_2^2 + q_1 r_3 + q_2 r_2 , \\
H_5 &= q_2 r_3 .
\end{aligned}
\qquad (3.24)
$$

The formulae on the left, which correspond to $g = 1$, can be obtained from the first three formulae on the right by setting $p_2 = q_2 = r_3 = 0$ in them.

3.3   The Volterra map and its reductions

The invariants H_i can be used to reduce the Volterra map to the submanifolds obtained by fixing the values of some of these invariants. Here we will use this to express the Volterra map in terms of the variables w_i which we introduced when constructing the S-fraction (3.15).

We start from the linear coordinates $p_1, \ldots, p_g, q_1, \ldots, q_g, r_1, \ldots, r_{g+1}$ of M_g, which we identify with $p_{0,1}, \ldots, p_{0,g}, q_{0,1}, \ldots, q_{0,g}, r_{0,1}, \ldots, r_{0,g+1}$. The latter functions are used to define recursively $p_{n,1}, \ldots, p_{n,g}$, $q_{n,1}, \ldots, q_{n,g}, r_{n,1}, \ldots, r_{n,g+1}$, as well as w_n, for all $n \in \mathbb{Z}$. Recall that this is done using (3.13) and (3.14).

In a first step, we will use a birational map to replace our linear coordinates for M_g by $p_{0,1}, \ldots, p_{0,g}$ and some of its iterates $p_{n,1}, \ldots p_{n,g}$. To do this, we fix the value of the invariant $H_1 = 2(p_1 + r_1)$ to an arbitrary

14

constant c_1. It means that we consider the hyperplane $H_1 = c_1$ of M_g, which denote by $M_g^{c_1}$. On it, we can take $p_1, \ldots, p_g, q_1, \ldots, q_g, r_2, \ldots, r_{g+1}$ as linear coordinates (we left out r_1). The invariance of H_1 implies that $2(p_{n,1} + r_{n,1}) = c_1$ which, combined with $r_{n,1} = -2w_n$ (see (3.14)), leads to

$$w_n = \frac{1}{2}\left(p_{n,1} - \frac{c_1}{2}\right) \tag{3.25}$$

for all $n \in \mathbb{Z}$. Using this and the first and last equations in (3.13), we can express the above variables in terms of $p_{0,1}, \ldots, p_{0,g}$ and their iterates:

$$q_{n,k} = p_{n+1,k} + p_{n,k}, \quad \text{and} \quad r_{n,k+1} = -w_n q_{n-1,k} = \frac{1}{2}\left(\frac{c_1}{2} - p_{n,1}\right)(p_{n,k} + p_{n-1,k}), \tag{3.26}$$

where $k = 1, \ldots, g$. Taking $n = 0$ we get

$$q_{0,k} = p_{1,k} + p_{0,k}, \quad \text{and} \quad r_{0,k+1} = \frac{1}{2}\left(\frac{c_1}{2} - p_{0,1}\right)(p_{0,k} + p_{-1,k}), \tag{3.27}$$

and we have expressed all coordinates of $M_g^{c_1}$ in terms of the $3g$ coefficients of the polynomials $\mathcal{P}_{-1}, \mathcal{P}_0$ and $\mathcal{P}_1$. It is clear that the (3.27) can be solved rationally for $p_{1,k}$ and $p_{-1,k}$ so that (3.27) defines a birational morphism from $M_g^{c_1}$ to the space of triplets $(\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_1)$. For later use, we also express the iterates of the Volterra map as a recursion relation on the polynomials $\mathcal{P}_n$. To do this, we apply (3.13) several times to get

$$\mathcal{P}_{n+2} = \mathcal{Q}_{n+1}(x) - \mathcal{P}_{n+1}(x) = \frac{2\mathcal{P}_n(x) - \mathcal{Q}_n(x) + \mathcal{R}_n(x)}{-w_{n+1}x} - \mathcal{P}_{n+1}(x)$$

$$= -\mathcal{P}_{n+1}(x) + \frac{1}{w_{n+1}x}(\mathcal{P}_{n+1}(x) - \mathcal{P}_n(x)) + \frac{w_n}{w_{n+1}}(\mathcal{P}_n(x) + \mathcal{P}_{n-1}(x)). \tag{3.28}$$

We will now go one step further and show how the above coordinates $p_{0,k}, p_{1,k}$ and $p_{-1,k}$ can be expressed birationally in terms of $p_{0,1}$ and some of its iterates $p_{n,1}$. To do this we do a further reduction, namely we also fix the value of each one of the invariants $H_2, \ldots, H_g$ to an arbitrary constant $c_2, \ldots, c_g$ and consider the (possibly singular) subvariety $\cap_{i=1}^g (H_i = c_i)$ of M_g, which we denote by M_g^c, so c stands now for $(c_1, \ldots, c_g)$. Using (3.26), we get the following formula for the invariants H_i in terms of $\mathcal{P}_n, \mathcal{P}_{n+1}$ and $\mathcal{P}_{n-1}$, valid for any $n \in \mathbb{Z}$:

$$1 + \sum_{k=1}^{2g+1} H_i x^i = \mathcal{P}_n^2(x) + \mathcal{Q}_n(x)\mathcal{R}_n(x) = \mathcal{P}_n^2(x) - w_n x(\mathcal{P}_n(x) + \mathcal{P}_{n+1}(x))(\mathcal{P}_n(x) + \mathcal{P}_{n-1}(x)),$$

with w_n given by (3.25). Upon comparing the coefficient of x^k on both sides, for $k = 1$ we recover (3.25), while for $k = 2, \ldots, g$ we recursively obtain $p_{n,k}$ in terms of $p_{0,1}$ and its iterates, via the following formulae:

$$c_k = 2p_{n,k} + \sum_{i=1}^{k-1} p_{n,i} p_{n,k-i} - 2w_n(2p_{n,k-1} + p_{n-1,k-1} + p_{n+1,k-1})$$

$$- w_n \sum_{i=1}^{k-2}(p_{n,i} + p_{n+1,i})(p_{n,k-1-i} + p_{n-1,k-1-i}). \tag{3.29}$$

Indeed, aside from the linear term in $p_{n,k}$, the above equation contains only the variables $p_{n,i}$ and $p_{n\pm1,i}$ with $1 \leqslant i < k$. For $k = 2$ one gets

$$c_2 = 2p_{n,2} + p_{n,1}^2 - 2w_n(2p_{n,1} + p_{n-1,1} + p_{n+1,1}),$$

from which it is clear that $p_{n,2}$ depends (polynomially) only on $p_{n-1,1}, p_{n,1}$ and $p_{n+1,1}$ (see (3.25) for the formula for w_n). An easy recursion on k using (3.29) shows that $p_{n,k}$ depends on $p_{n-k+1,1}, \ldots, p_{n+k-1,1}$

only. Taking $n = -1$, $n = 0$ and $n = -1$ we get that the coefficients of $\mathcal{P}_{-1}, \mathcal{P}_0$ and $\mathcal{P}_1$ depend only on $p_{-g,1}, \ldots, p_{g,1}$. Conversely, it is obvious from (3.28) that the coefficients of $\mathcal{P}_2$, and hence of all $\mathcal{P}_n$ with $n \geqslant 2$, are rational functions of the coefficients of $\mathcal{P}_{-1}, \mathcal{P}_0$ and $\mathcal{P}_1$. This applies in particular to $p_{n,1}$ with $n \geqslant 2$. Using the inverse recursion, which yields a formula similar to (3.28) expressing $\mathcal{P}_{n-2}$ in terms of $\mathcal{P}_{n+1}, \mathcal{P}_n$ and $\mathcal{P}_{n-1}$ one obtains similarly that all $p_{n,1}$ with $n \in \mathbb{Z}$ are rational functions of the coefficients of $\mathcal{P}_{-1}, \mathcal{P}_0$ and $\mathcal{P}_1$. The upshot is that we have a birational map between M_g^c and $\mathbb{C}^{2g+1}$, equipped with the coordinates $p_{-g,1}, \ldots, p_{g,1}$. In view of (3.25), which we now write as

$$p_{n,1} = 2w_n + \frac{c_1}{2} , \tag{3.30}$$

it amounts to a birational map between M_g^c and $\mathbb{C}_w^{2g+1}$ where the latter denotes $\mathbb{C}^{2g+1}$, equipped with the coordinates $w_{-g}, \ldots, w_g$.

Hence we can use the birational map between M_g^c and $\mathbb{C}_w^{2g+1}$ to write the Volterra map on M_g, restricted to M_g^c, as a birational automorphism of $\mathbb{C}^{2g+1}$. Since we already gave in (3.28) the Volterra map and its iterates in terms of the variables $\mathcal{P}_i$, we set $n = 0$ therein, which gives the Volterra map itself, and take the leading terms of both sides of (3.28):

$$w_1(p_{2,g} + p_{1,g}) = w_0(p_{0,g} + p_{-1,g}) . \tag{3.31}$$

In view of the dependence of $p_{-1,g}, \ldots, p_{2,g}$ on the variables w_i, (3.31) gives an equation for w_{g+1}, which appears linearly in it, and the birational automorphism $(w_{-g}, w_{1-g}, \ldots, w_g) \mapsto (w_{1-g}, \ldots, w_g, w_{g+1})$ is the Volterra map on $\mathbb{C}_w^{2g+1}$. Explicit expressions for it will be given in the examples below.

However, we can do a further reduction, restricting the Volterra map to the subvariety $H_k = c_k$ for some k with $g < k \leqslant 2g + 1$. The relation

$$H_k(w_{-g}, \ldots, w_0, \ldots, w_g) = c_k$$

defines w_g as a rational function of $w_{-g}, \ldots, w_{g-1}$ because by inspection w_g appears linearly in it (the same applies to w_{-g}). As we will see in the examples below, we can therefore take any of the invariants H_k, with $g < k \leqslant 2g + 1$ which will give a birational automorphism which is an incarnation of the Volterra map $\mathcal{V}_g$ on $M_g^c \cap (H_k = c_k)$; precisely it is conjugate, via the above birational map, to the Volterra map, restricted to $M_g^c \cap (H_k = c_k)$, where values of $c = (c_1, \ldots, c_g)$ and c_k are arbitrary.

Example 3.6. We first consider the case $g = 1$. In this case, we only need to consider $k = 1$ in (3.27), which combined with (3.30) yields the following birational map between $M_g^c = M_g^{c_1}$ and $\mathbb{C}_w^3$:

$$
\begin{aligned}
p_1 &= p_{0,1} = 2w_0 + c_1/2 , \\
q_1 &= q_{0,1} = p_{1,1} + p_{0,1} = 2(w_0 + w_1) + c_1 , \\
r_1 &= r_{0,1} = c_1/2 - p_{0,1} , \\
r_2 &= r_{0,2} = -w_0(p_{0,1} + p_{-1,1}) = -2w_0(w_0 + w_{-1} + c_1/2) .
\end{aligned} \tag{3.32}
$$

In terms of the polynomials $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ this can also be written as

$$
\begin{aligned}
\mathcal{P}(x) &= 1 + (2w_0 + c_1/2)x , \\
\mathcal{Q}(x) &= 2 + 2(w_1 + w_0 + c_1/2)x , \\
\mathcal{R}(x) &= -2w_0 x(1 + (w_0 + w_{-1} + c_1/2)x) .
\end{aligned} \tag{3.33}
$$

For $g = 1$ the formula (3.31) takes the form $w_1(p_{2,1} + p_{1,1}) = w_0(p_{0,1} + p_{-1,1})$, and can be expressed immediately in terms of the quantities $w_{-1}, \ldots, w_2$ since $p_{n,1} = 2w_n + c_1/2$, for all n:

$$w_1(2w_2 + 2w_1 + c_1) = w_0(2w_0 + 2w_{-1} + c_1) . \tag{3.34}$$

16

It defines the Volterra map $(w_{-1}, w_0, w_1) \mapsto (w_0, w_1, w_2)$ on $\mathbb{C}_w^3$, being the same as equation (2) in [51], where Svinin used continued fraction expansions to construct particular solutions; equation (3.34) also appears in [24]. Substituting (3.33) in $\mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x) = 1 + H_1 x + H_2 x^2 + H_3 x^3$ we get the following formulas for the invariants H_2 and H_3 of the Volterra map in terms of the variables w_{-1}, w_0 and w_1:

$$H_2 = -4w_0(w_1 + w_0 + w_{-1} + c_1/2) + c_1^2/4 \ , \tag{3.35}$$

$$H_3 = -4w_0(w_1 + w_0 + c_1/2)(w_0 + w_{-1} + c_1/2) \ . \tag{3.36}$$

We now fix c_3 and consider the Volterra map on the subvariety $H_3 = c_3$ of M_g^c. According to (3.36) we get

$$4w_0(w_1 + w_0 + c_1/2)(w_0 + w_{-1} + c_1/2) + c_3 = 0 \ , \tag{3.37}$$

which defines a $2D$ map $(w_{-1}, w_0) \mapsto (w_0, w_1)$, where w_1 is computed from (3.37). It has H_2 (from which w_1 is eliminated using (3.37)) as invariant, given by

$$H_2 = \frac{c_3}{w_0 + w_{-1} + c_1/2} - 4w_0 w_{-1} + \frac{c_1^2}{4} \ . \tag{3.38}$$

We next fix c_2 and consider the Volterra map on the subvariety $H_2 = c_2$ of M_1^c. From (3.35) we now get

$$4w_0(w_1 + w_0 + w_{-1} + c_1/2) - c_1^2/4 + c_2 = 0 \ . \tag{3.39}$$

It defines a $2D$ map $(w_{-1}, w_0) \mapsto (w_0, w_1)$ with H_3 as invariant, which (after using (3.39)) to eliminate w_1) takes the form

$$H_3 = \left(w_0 + w_{-1} + \frac{c_1}{2}\right)\left(4w_0 w_{-1} + c_2 - \frac{c_1^2}{4}\right) \ . \tag{3.40}$$

To finish the $g = 1$ example we will present a slightly more involved reduction, leading to a map which is closely related to Somos-5. To do this, we first compare two different ways of writing of the genus 1 curve $y^2 = f(x)$, which lead to an alternative generating set of invariants of the recursion. If we write

$$y^2 = 1 + c_1 x + c_2 x^2 + c_3 x^3 = (1 - c_1' x)(1 - c_1' x + 4c_2' x^2) - 4c_3' x^3 \ , \tag{3.41}$$

then the constants c_i and c_i' are related by

$$
\begin{array}{llll}
c_1 &=& -2c_1' \ , & \qquad c_1' &=& -c_1/2 \ , \\
c_2 &=& 4c_2' + c_1'^2 \ , & \qquad c_2' &=& \frac{1}{4}\left(c_2 - \frac{c_1^2}{4}\right) \ , \\
c_3 &=& -4(c_1' c_2' + c_3') \ , & \qquad c_3' &=& \frac{c_1}{8}\left(c_2 - \frac{c_1^2}{4}\right) - \frac{c_3}{4} \ .
\end{array}
\tag{3.42}
$$

Next, if we write the recursion relations (3.37) and (3.39) in terms of the constants c_i' using (3.42), we get respectively

$$w_0\left(w_1 + w_0 - c_1'\right)\left(w_0 + w_{-1} - c_1'\right) = c_1' c_2' + c_3' \ , \tag{3.43}$$

$$w_0\left(w_1 + w_0 + w_{-1} - c_1'\right) + c_2' = 0 \ . \tag{3.44}$$

Notice that c_1' now appears linearly in (3.44), so we can easily eliminate c_1' between (3.43) and (3.44), which yields the following simple relation

$$w_1 w_{-1} = c_2' + \frac{c_3'}{w_0} \tag{3.45}$$

on the generic level surface $(H_2' = c_2') \cap ((H_3' = c_3')$, which is also birational with $\mathbb{C}^2$. It defines a $2D$ map $(w_{-1}, w_0) \mapsto (w_0, w_1)$ which is a (multiplicative) QRT map with c_2' and c_3' as parameters. To get an invariant for this map, we eliminate w_1 between (3.43) and (3.44), to get

$$c_1' w_0 w_{-1} = (w_0 + w_{-1})w_0 w_{-1} + c_2'(w_0 + w_{-1}) + c_3'. \tag{3.46}$$

It leads upon division by $w_0 w_{-1}$ to the following explicit formula for the invariant:

$$H_1' = w_0 + w_{-1} + c_2' \left(\frac{1}{w_0} + \frac{1}{w_{-1}} \right) + \frac{c_3'}{w_0 w_{-1}} . \tag{3.47}$$

Also, the tau function substitution

$$w_n = \frac{\tau_n \tau_{n+3}}{\tau_{n+1} \tau_{n+2}}$$

in (3.45), which we now write in the form of the recursion relation $w_{n+1} w_{n-1} = c_2' + c_3'/w_n$, yields the general form of the Somos-5 recursion relation, namely

$$\tau_{n+5} \tau_n = c_2' \tau_{n+4} \tau_{n+1} + c_3' \tau_{n+3} \tau_{n+2} , \qquad n \in \mathbb{Z} . \tag{3.48}$$

In our previous work [24] we showed how to solve the initial value problem for (3.48) explicitly in terms of the Weierstrass sigma function, but in Section 3.4 below we will show how it can also be solved in terms of Hankel determinants, using the S-fraction (3.15).

Note that, apart from (3.45), the maps (3.37) and (3.39) are also examples of symmetric QRT maps [44], and the orbits of all three maps can be identified by restricting to particular level sets of their invariants, which is a common feature of families of these maps [29]. To see how the orbits of these different 2D maps coincide, it is necessary to identify the parameters and values of the invariants (3.38), (3.40) and (3.47) in an appropriate way, from which it can be seen that H_2, H_3 and H_1' define an identical biquadratic curve in the (w_{-1}, w_0) plane, namely (3.46), whose coefficients can be rewritten in terms of c_1, c_2, c_3 using (3.42). Moreover, this curve is birationally equivalent to the Weierstrass cubic (3.41).

Example 3.7. Using (3.31) when $g = 2$, on M_2^c (which is birational to $\mathbb{C}^5$) the formula for the Volterra map takes the form

$$w_1 (p_{2,2} + p_{1,2}) = w_0 (p_{0,2} + p_{-1,2}) , \tag{3.49}$$

so we need to express $p_{n,2}$ for $n = -1, \ldots, 2$ in terms of the variables w_i. To do this, we use (3.29), keeping in mind that $p_{n,1} = 2w_n + c_1/2$ for all n:

$$2p_{n,2} = -p_{n,1}^2 + 2w_n(p_{n+1,1} + 2p_{n,1} + p_{n-1,1}) + c_2$$

$$= -\left(2w_n + \frac{c_1}{2} \right)^2 + 4w_n(w_{n+1} + 2w_n + w_{n-1} + c_1) + c_2$$

$$= 4w_n \left(w_{n+1} + w_n + w_{n-1} + \frac{c_1}{2} \right) + c_2 - \frac{c_1^2}{4} . \tag{3.50}$$

Substituted in (3.49) and slightly reordering the terms, we get the following symmetric relation:

$$w_1 \left(2w_2(w_3 + w_2) + 2w_1(w_1 + w_0) + 4w_2 w_1 + (w_2 + w_1)c_1 + c_2 - \frac{c_1^2}{4} \right)$$

$$= w_0 \left(2w_0(w_1 + w_0) + 2w_{-1}(w_{-1} + w_{-2}) + 4w_0 w_{-1} + (w_0 + w_{-1})c_1 + c_2 - \frac{c_1^2}{4} \right) . \tag{3.51}$$

This defines a 5D map $(w_{-2}, w_{-1}, w_0, w_1, w_2) \mapsto (w_{-1}, w_0, w_1, w_2, w_3)$, where w_3 is computed from (3.51). Using the first equation in (3.9), the above formulae for $p_{n,1}$ and $p_{n,2}$ lead at once to the following expressions for the coefficients of $\mathcal{Q}_n$:

$$q_{n,1} = 2w_{n+1} + 2w_n + c_1 ,$$

$$q_{n,2} = 2(w_n w_{n-1} + w_{n+1} w_{n+2}) + 2(w_{n+1} + w_n)^2 + (w_{n+1} + w_n)c_1 + c_2 - \frac{c_1^2}{4} . \tag{3.52}$$

The formulae for $r_{n,k}$ then follow from $r_{n,1} = -2w_n$ and $r_{n,k} = -w_n q_{n-1,k-1}$ for $k > 1$, by applying the third equation in (3.9). With these formulae we can express the invariants $H_3, \ldots, H_5$ in terms of the variables

w_i. We write this out for H_3, in order to find a recursion relation of order 4. According to (3.24), and using the above expressions for $r_{n,k}$,

$$H_3 = 2p_1p_2 + 2r_3 + q_1r_2 + q_2r_1 = p_{0,1}p_{0,2} + 2r_{0,3} + q_{0,1}r_{0,2} + q_{0,2}r_{0,1}$$
$$= 2p_{0,1}p_{0,2} - w_0(q_{0,1}q_{-1,1} + 2q_{-1,2} + 2q_{0,2}) ,$$

which can be written completely in terms of the variables w_i using (3.50), (3.52) and the fundamental formula $p_{n,1} = 2w_n + c_1/2$. After some simplification, we get on the hypersurface $H_3 = c_3$, which is birational to $\mathbb{C}^4$,

$$w_2w_1w_0 + w_0w_{-1}w_{-2} + 2w_0^2(w_1 + w_{-1}) + w_0^3 + w_0(w_1^2 + w_1w_{-1} + w_{-1}^2)$$
$$+ \frac{c_1}{2}w_0(w_1 + w_0 + w_{-1}) + \frac{1}{2}\left(c_2 - \frac{c_1^2}{4}\right)w_0 + \frac{c_3}{4} - \frac{c_1c_2}{8} + \frac{c_1^3}{32} = 0 .$$

This is exactly the equation (1.2) defining the $4D$ map (P.iv), after setting $n = -2$ and

$$\nu = \frac{c_1}{2} , \quad a = \frac{c_3}{4} - \frac{c_1c_2}{8} + \frac{c_1^3}{32} , \quad b = \frac{1}{2}\left(c_2 - \frac{c_1^2}{4}\right) . \tag{3.53}$$

The invariants H_4 and H_5 yield the invariants for (P.iv), given in (1.3) and (1.4).

3.4 Hankel determinant solutions

The function F_0 in (3.15) that defines the S-fraction admits a series expansion in x around $(0,1) \in \Gamma_f$ that we shall use to give explicit solutions to the recurrence relation defined by the Volterra map. For a generic point $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ in $\mu^{-1}(f)$, with f as in (3.5), we introduce new variables $s_1, s_2, \ldots$ by writing

$$1 - \cfrac{w_1x}{1 - \cfrac{w_2x}{1 - \cfrac{w_3x}{1 - \cdots}}} = 1 - \sum_{j=1}^{\infty} s_j x^j = 1 - S(x) , \tag{3.54}$$

where the latter equality defines the power series $S(x)$, which can be regarded as a generating function for the moments s_j. The moments can be defined from the integral

$$s_j = \frac{1}{2\pi i} \oint \frac{(1 - F_0)}{x^{j+1}} \, \mathrm{d}x,$$

for a sufficiently small contour around the point $(0,1)$ on Γ_f, and this leads to a linear functional (defined on polynomials in x^{-1}), and the connection with the classical theory of orthogonal polynomials [46], but we shall not pursue this further here. In view of (3.15), $F_0 = 1 - S(x)$ in the sense that the latter is a Taylor series expansion of the rational function F_0 at $(0,1)$. It was shown by Stieltjes [48] that the variables w_i can be expressed as Hankel determinants of the variables s_i. Precisely, he showed that

$$w_n = \frac{\Delta_{n-3}\Delta_n}{\Delta_{n-2}\Delta_{n-1}} \qquad \text{for } n \geqslant 1 , \tag{3.55}$$

where $\Delta_{2k-1} = \det(s_{i+j-1})_{i,j=1,\ldots,k}$ and $\Delta_{2k} = \det(s_{i+j})_{i,j=1,\ldots,k}$, for $k \geqslant 1$, that is,

$$\Delta_{2k-1} = \begin{vmatrix} s_1 & s_2 & \cdots & s_k \\ s_2 & & \cdot^{\cdot^{\cdot}} & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ s_k & \cdots & \cdots & s_{2k-1} \end{vmatrix} \quad \text{and} \quad \Delta_{2k} = \begin{vmatrix} s_2 & s_3 & \cdots & s_{k+1} \\ s_3 & & \cdot^{\cdot^{\cdot}} & \vdots \\ \vdots & \cdot^{\cdot^{\cdot}} & & \vdots \\ s_{k+1} & \cdots & \cdots & s_{2k} \end{vmatrix} . \tag{3.56}$$

Also, by definition, $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$. For example, $w_1 = s_1$, $w_2 = s_2/s_1$, $w_3 = (s_1 s_3 - s_2^2)/(s_1 s_2)$, and so on. It is clear from (3.54) that, conversely, s_i can be expressed as a polynomial in $w_1, \ldots, w_i$, for example $s_1 = w_1$, $s_2 = w_1 w_2$, $s_3 = w_1 w_2 (w_2 + w_3)$, and so on. (While these expressions can be found by expanding geometric series, a systematic method using continuants is presented in Section 3.5.)

Theorem 3.8. *The terms w_n, $n \geqslant 1$, of the recurrence sequence defined by the Volterra map on $\mu^{-1}(f)$ can be written as*

$$w_n = \frac{\Delta_{n-3} \Delta_n}{\Delta_{n-2} \Delta_{n-1}} \ ,$$

where the entries of the Hankel matrices satisfy the recursion relation

$$s_j = \sum_{k=1}^{g} \left(p_k - q_k \right) s_{j-k} + \sum_{i=1}^{j-1} s_i s_{j-i} + \frac{1}{2} \sum_{k=1}^{g} q_k \sum_{i=1}^{j-k-1} s_i s_{j-k-i}, \quad j \geqslant g+2 \ . \tag{3.57}$$

The initial values $s_1, s_2, \ldots, s_{g+1}$ and the coefficients for the recursion (3.57) are provided by a generic triple $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) \in \mu^{-1}(f)$.

Proof. In order to prove the recursion formula (3.57), we will derive a quadratic formula for $S(x) = 1 - F_0$, introduced in (3.54). We use (3.15) to write

$$y = -\mathcal{P}(x) + F_0 \mathcal{Q}(x) = -\mathcal{P}(x) + (1 - S(x)) \mathcal{Q}(x) \ , \tag{3.58}$$

which we substitute in

$$y^2 = f(x) = \mathcal{P}(x)^2 + \mathcal{Q}(x) \mathcal{R}(x) \tag{3.59}$$

to get the following quadratic equation for $S(x)$:

$$\left(\mathcal{Q}(x) - \mathcal{P}(x) \right) S(x) - \frac{1}{2} \mathcal{Q}(x) S^2(x) + \mathcal{P}(x) - \frac{1}{2} \mathcal{Q}(x) + \frac{1}{2} \mathcal{R}(x) = 0 \ . \tag{3.60}$$

Substituting the power series for $S(x)$ into the quadratic, as well as the polynomials $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, the coefficients of x^j for $1 \leqslant j \leqslant g+1$ allow the $g+1$ initial values $s_1, \ldots, s_{g+1}$ to be determined from these polynomials. Then, upon taking the coefficient of x^j for $j \geqslant g+2$, the recursion relation (3.57) is obtained directly. Observe that the number of initial values plus independent coefficients appearing linearly in the recursion is $g + 1 + 2g = 3g + 1 = \dim M_g$. $\qquad \square$

As we will see in the examples, it is often more practical not to fix the curve $y^2 = f(x)$, i.e. not to fix the values of all invariants, but only fix some of them and take $w_0, \ldots, w_g$ as extra initial conditions.

Example 3.9. We specialize the above results to $g = 1$, fixing arbitrary constants c_1 and c_2 and taking the Volterra map on the surface $H_1 = c_1$, $H_2 = c_2$, as in one of the reductions considered in Example 3.6. For the recursion (3.57) we can take p_1, q_1 as initial conditions, since given p_1 and q_1, specifying the values of H_1 and H_2 is equivalent to specifying the values of r_1 and r_2 (see the explicit formulas for H_1 and H_2 in the left column of (3.24)). It follows from (3.32) that

$$p_1 - q_1 = -2w_1 - \frac{c_1}{2} \ , \quad \text{and} \quad q_1 = 2 \left(w_1 + w_0 + \frac{c_1}{2} \right) = -2w_2 + \frac{1}{2w_1} \left(\frac{c_1^2}{4} - c_2 \right) \ ,$$

where we obtained the last equality by using the recursion relation (3.39), shifted by 1, to replace w_0 by w_2. Substituted in (3.57), we get the following recursive formula for s_j ($j \geqslant 3$) in terms of w_1, w_2, which we can take as initial conditions, instead of p_1 and q_1:

$$s_j = \left(-2w_1 - \frac{c_1}{2} \right) s_{j-1} + \sum_{i=1}^{j-1} s_i s_{j-i} + \left(\frac{1}{4w_1} \left(\frac{c_1^2}{4} - c_2 \right) - w_2 \right) \sum_{i=1}^{j-2} s_i s_{j-1-i} \ . \tag{3.61}$$

Then specifying the two initial values w_1, w_2 fixes the initial conditions s_1, s_2 for the above, as $s_1 = w_1$ and $s_2 = w_1 w_2$. Notice that s_j is a polynomial of degree j in w_1, w_2, with $w_1 | s_j$ for all j.

As a concrete example, consider the curve $y^2 = 1 - 10x + 29x^2 - 24x^3$, with initial conditions $s_1 = 1$, $s_2 = 2$ (or, equivalently, $w_1 = 1$, $w_2 = 2$). Since $c_1 = -10$ and $c_2 = 29$, the recursion (3.61) becomes

$$s_j = 3s_{j-1} + \sum_{i=1}^{j-1} s_i s_{j-i} - 3 \sum_{i=1}^{j-2} s_i s_{j-i-1}, \quad j \geqslant 3,$$

which generates the sequence

$$(s_j)_{j \geqslant 1} : \quad 1, 2, 7, 27, 109, 456, 1969, 8746, 39825, 185266, \ldots,$$

producing $\Delta_1 = 1$, $\Delta_2 = 2$,

$$\Delta_3 = \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 3, \ \Delta_4 = \begin{vmatrix} 2 & 7 \\ 7 & 27 \end{vmatrix} = 5, \ \Delta_5 = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 7 & 27 \\ 7 & 27 & 109 \end{vmatrix} = 11, \ \Delta_6 = \begin{vmatrix} 2 & 7 & 27 \\ 7 & 27 & 109 \\ 27 & 109 & 456 \end{vmatrix} = 37, \ldots,$$

which extends symmetrically to all $n \in \mathbb{Z}$ to produce the original Somos-5 sequence [39],

$$\ldots, 3, 2, 1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, \ldots,$$

generated by the bilinear recurrence

$$\Delta_{n+5} \Delta_n = \Delta_{n+4} \Delta_{n+1} + \Delta_{n+3} \Delta_{n+2}. \tag{3.62}$$

It is a particular case of (3.48) with $c_2' = c_3' = 1$, where the latter are computed from $c_1 = -10$, $c_2 = 29$, $c_3 = -24$, using (3.42).

Remark 3.10. Note that Hankel determinant formulae for Somos-5 were previously obtained in the work of Chang, Hu and Xin, using a bilinear Bäcklund transformation for Somos-4. We will return to the connection with Somos-4 in Section 5, but for now we just point out that the Hankel determinant expressions found in [8] are more complicated than the above, because two different moment sequences are required for the terms with even/odd indices. In particular, for the original Somos-5 sequence, there are two sequences of moments, namely $(\bar{s}_j)_{j \geqslant 0} : 1, -1, 4, -8, 25, -65, 197, -571, 1753, -5351, 16746, -52626, \ldots$, and $(\hat{s}_j)_{j \geqslant 0} : 2, -1, 3, -1, 12, 2, 61, 39, 352, 374, 2210, 3162, \ldots$, which are defined by

$$\bar{s}_0 = 1, \ \bar{s}_1 = -1, \ \bar{s}_{j+1} = -\bar{s}_j + 2\bar{s}_{j-1} + \sum_{i=0}^{j-1} \bar{s}_i \bar{s}_{j-i-1}, \quad \text{and} \quad \hat{s}_0 = 2, \ \hat{s}_1 = -1, \ \hat{s}_{j+1} = \hat{s}_j + \sum_{i=0}^{j-1} \hat{s}_i \hat{s}_{j-i-1},$$

respectively, where the recursions hold for $j \geq 1$, and (with the indexing convention of Theorem 1.2 in [8]) the terms of the Somos-5 sequence [39] are then given by $S_0 = 1, S_1 = 1, S_2 = \bar{s}_0 = 1, S_3 = \hat{s}_0 = 2$, and

$$S_4 = \begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} = 3, \ S_5 = \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5, \ S_6 = \begin{vmatrix} 1 & -1 & 4 \\ -1 & 4 & -8 \\ 4 & -8 & 25 \end{vmatrix} = 11, \ S_7 = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & 12 \end{vmatrix} = 37, \ldots,$$

which are not related to the determinants in Example 3.9 in a straightforward manner.

Example 3.11. We now specialize the above results to $g = 2$, thereby continuing Example 3.7. From (3.53) it is clear that fixing the values c_1, c_2, c_3 is equivalent to specifying the parameters a, b and ν, which we fix, since these are the coefficients in the recurrence relation (P.iv), and we can take w_0, w_1, w_2, w_3 (or w_1, w_2, w_3, w_4) as initial data for the latter. We now have

$$s_1 = w_1, \quad s_2 = w_1 w_2, \quad s_3 = w_1 w_2 (w_2 + w_3),$$

21

providing the 3 initial values for the recursion (3.57), which takes the form

$$s_j = \hat{\alpha}\, s_{j-1} + \hat{\beta}\, s_{j-2} + \sum_{i=1}^{j-1} s_i s_{j-i} + \hat{\gamma} \sum_{i=1}^{j-2} s_i s_{j-i-1} + \hat{\delta} \sum_{i=1}^{j-3} s_i s_{j-i-2}\ , \quad j \geqslant 4\ . \tag{3.63}$$

While s_1, s_2, s_3 are determined by w_1, w_2, w_3 only, w_0 and a, b, ν are required to find the coefficients $\hat{\alpha}, \ldots, \hat{\delta}$, which are computed using $p_1 = 2w_0 + \nu$ (recall that $\nu = c_1/2$) and (3.50), (3.52) for $n = 1$, to give

$$\begin{aligned}
\hat{\alpha} &= p_1 - q_1 = -2w_1 - \nu\ , \\
\hat{\beta} &= p_2 - q_2 = -2w_1(w_2 + w_1 + w_0 + \nu) - b\ , \\
\hat{\gamma} &= \frac{1}{2}q_1 = w_1 + w_0 + \nu\ , \\
\hat{\delta} &= \frac{1}{2}q_2 = w_{-1}w_0 + w_1 w_2 + (w_0 + w_1)^2 + \nu(w_0 + w_1) + b \\
&= -\left(w_2 w_3 + w_1 w_2 + w_2^2 + w_0 w_2 + \nu w_2 + \frac{a}{w_1}\right)\ ,
\end{aligned} \tag{3.64}$$

where, in the last equality, we have used the recurrence relation (1.2) to replace w_{-1} by w_3. If desired, one can apply (1.2) once again to replace w_0 by w_4 in the above expressions, but we have not done this.

As a particular numerical example, we take the rational orbit (2.1) of (P.iv) considered in Section 2. Upon fixing $w_0 = w_1 = w_2 = w_3 = 1$ and $\nu = 3$, $a = 5$, $b = 7$, we see that for $j \geq 4$ the recursion (3.63) becomes

$$s_j = -5\, s_{j-1} - 19\, s_{j-2} + \sum_{i=1}^{j-1} s_i s_{j-i} + 5 \sum_{i=1}^{j-2} s_i s_{j-i-1} - 12 \sum_{i=1}^{j-3} s_i s_{j-i-2}\ ,$$

and the three initial values $s_1 = s_2 = 1$, $s_3 = 2$ lead to the following moment sequence and Hankel determinants:

$$(s_j)_{j \geqslant 1}:\quad 1, 1, 2, -26, 45, 11, -116, 553, 1151, -26727, 108897, -169157, -310959, 3004412, -4722005, \ldots,$$

$$\Delta_1 = 1,\ \Delta_2 = 1,\ \Delta_3 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1,\ \Delta_4 = \begin{vmatrix} 1 & 2 \\ 2 & -26 \end{vmatrix} = -30,\ \Delta_5 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -26 \\ 2 & -26 & 45 \end{vmatrix} = -743,$$

$$\Delta_6 = \begin{vmatrix} 1 & 2 & -26 \\ 2 & -26 & 45 \\ -26 & 45 & 11 \end{vmatrix} = 10541, \qquad \Delta_7 = \begin{vmatrix} 1 & 1 & 2 & -26 \\ 1 & 2 & -26 & 45 \\ 2 & -26 & 45 & 11 \\ -26 & 45 & 11 & -116 \end{vmatrix} = 127318,\ \ldots.$$

This reproduces the sequence of tau functions in Example 2.2, if we identify $\Delta_{n-3} = \tau_n$ in (2.7).

The preceding explicit form of the recursion for the entries of the Hankel determinants when $g = 2$ yields a simple proof of the Laurent property for (2.4).

Proof of Proposition 2.3 (reprise): For $n \geqslant 1$ we have

$$w_n = \frac{\tau_n \tau_{n+3}}{\tau_{n+1}\tau_{n+2}} = \frac{\Delta_{n-3}\Delta_n}{\Delta_{n-2}\Delta_{n-1}}\ ,$$

where τ_n satisfies (2.4). Hence the tau functions are given by Hankel determinants, up to a shift of index and a gauge transformation of the form (2.6), with a different scaling for even/odd n. Comparing with the values $\Delta_{-2} = \Delta_{-1} = \Delta_0 = 1$, we see that the relation between the two sequences is

$$\tau_{2k+1} = \tau_1 \left(\frac{\tau_3}{\tau_1}\right)^k \Delta_{2k-2}, \qquad \tau_{2k+2} = \frac{\tau_1 \tau_2}{\tau_3}\left(\frac{\tau_3}{\tau_1}\right)^{k+1} \Delta_{2k-1} \tag{3.65}$$

for $k \geqslant 0$. Recall that $\mathcal{R}$ denotes the ring formed of Laurent polynomials in τ_2, τ_3, τ_4 and polynomials in $\tau_0, \tau_1, \tau_5, \tau_6$ with coefficients in $\mathbb{Z}[a, b, \nu]$. Upon rewriting the formulae (3.64) for the coefficients in (3.63) in terms of the 7 initial tau functions, we see that $\hat{\alpha}, \hat{\beta} \in \mathcal{R}$, but (due to the presence of terms involving w_0 and $1/w_1$), $\hat{\gamma}$ and $\hat{\delta}$ both have τ_1 appearing in the denominator, so instead $\hat{\gamma}, \hat{\delta} \in \tau_1^{-1}\mathcal{R}$. However, the three initial values are

$$s_1 = \frac{\tau_1 \tau_4}{\tau_2 \tau_3}, \quad s_2 = \frac{\tau_1 \tau_5}{\tau_3^2}, \quad s_3 = \tau_1 \left(\frac{\tau_2 \tau_5^2}{\tau_3^3 \tau_4} + \frac{\tau_6}{\tau_3 \tau_4} \right),$$

so $s_j \in \tau_1 \mathcal{R}$ for $j = 1, 2, 3$. Then by induction, since $\hat{\gamma}$ and $\hat{\delta}$ appear in front of terms of degree 2 in s_i in the recursion (3.63), it follows that $s_j \in \tau_1 \mathcal{R}$ for all $j \geqslant 1$. Then since Δ_{2k-1} and Δ_{2k} are $k \times k$ determinants, a factor of τ_1 can be taken out of each row (or column), so they are each given by an overall factor of τ_1^k times an element of $\mathcal{R}$. Thus the powers of τ_1 exactly cancel in (3.65), and hence $\tau_n \in \mathcal{R}$ for all $n \geqslant 1$. □

Remark 3.12. Hankel determinant formula for negative indices: As was previously noted, the Laurent property for negative indices n follows automatically from the birationality and reversing symmetry of (2.4), but it can also be shown directly from an appropriate extension of (3.65) to $k < 0$. In fact, for any g there is a version of the Hankel determinant formula (3.55) which is valid for $n \leqslant 0$. Indeed, the Volterra map arises from the S-fraction expansion (3.15) of the function F_0, based on the power series $S(x)$, with $F_0 = 1 - S$, as in (3.54), but more precisely this is the expansion around the point $(0, 1)$ on the curve Γ_f given by (3.5), with x being a local parameter. The inverse Volterra map arises from another S-fraction, associated with a power series $S^*(x)$, corresponding to the expansion of the same function F_0 around the point $(0, -1) \in \Gamma_f$, that is

$$F_0 = \cfrac{w_0 x}{1 - \cfrac{w_{-1} x}{1 - \cfrac{w_{-2} x}{1 - \cdots}}} = \sum_{j=1}^{\infty} s_j^* x^j =: S^*(x). \tag{3.66}$$

Then the extension of (3.55) to non-positive values of the index is

$$w_n = \frac{\Delta_{-n-2}^* \Delta_{-n+1}^*}{\Delta_{-n}^* \Delta_{-n-1}^*} \qquad \text{for } n \leqslant 0, \tag{3.67}$$

where $\Delta_{2k-1}^* = \det(s_{i+j-1}^*)_{i,j=1,\ldots,k}$ and $\Delta_{2k}^* = \det(s_{i+j}^*)_{i,j=1,\ldots,k}$ for $k \geqslant 1$, with $\Delta_{-2}^* = \Delta_{-1}^* = \Delta_0^* = 1$. Mutatis mutandis, this is proved in the same way as Theorem 3.8, and the moments s_j^* satisfy another recursion of the form (3.57). The two sets of Hankel determinants combine to produce a single sequence of tau functions $(\tau_n)_{n \in \mathbb{Z}}$, consistently defined by taking $\tau_n = \Delta_{n-3}$ for $n \geqslant 1$, and $\tau_n = \Delta_{-n+1}^*$ for $n \leqslant 3$.

3.5 The birational map $M_g \to \mathbb{C}_w^{3g+1}$

Elaborating further on the S-fraction of $F = \frac{y + \mathcal{P}(x)}{\mathcal{Q}(x)}$, we construct a birational map between M_g and $\mathbb{C}_w^{3g+1}$, where the latter stands for $\mathbb{C}^{3g+1}$ equipped with $w_1, \ldots, w_{3g+1}$ as affine coordinates. We may call it the *unreduced* birational map, in view of the birational map $M_g^c \to \mathbb{C}_w^{2g+1}$ which we obtained by reduction (fixing $c = (c_1, \ldots, c_g)$). This unreduced map turns out to be less convenient for deriving the Volterra map in the w_i coordinates, but it is nevertheless useful for obtaining abstract results in these coordinates.

We start from the equation (3.60), which we can view as a linear relation for $\mathcal{P}(x), \mathcal{Q}(x)$ and $\mathcal{R}(x)$. It amounts to an infinite linear system of equations for the coefficients $p_1, \ldots, p_g, q_1, \ldots, q_g, r_1, \ldots, r_{g+1}$ of these polynomials, given in terms of the coefficients s_j of the power series $S(x)$ defined in (3.54). We show that its solution can be expressed rationally in terms of $s_1, \ldots, s_{3g+1}$, and hence in terms of $w_1, \ldots, w_{3g+1}$, yielding the birational map. To do this, we investigate the first $3g + 1$ equations only, namely the ones corresponding to the coefficient of x^j in (3.60). For convenience, we set

$$\rho_k := \sum_{i+j=k} s_i s_j,$$

23

for $k \geqslant 2$, so that $S^2 = \sum_{i+j=k} s_i s_j x^k = \sum_{k=2}^{\infty} \rho_k x^k$, and then consider the terms of (3.60) at each order in x. At leading order, the constant term cancels, and the coefficients of $x, x^2, \ldots, x^{g+1}$ can be used to obtain $r_1, \ldots, r_{g+1}$ in terms of $p_1, \ldots, p_g, q_1, \ldots, q_g$, and s_j for $1 \leqslant j \leqslant g+1$, so it is sufficient to show that we can solve the coefficients of x^j with $g + 2 \leqslant j \leqslant 3g + 1$ for $p_1, \ldots, p_g, q_1, \ldots, q_g$ in terms of $s_1, \ldots, s_{3g+1}$. For each such j, the equation at order x^j in (3.60) is given by (3.57), which we can rewrite as

$$\sum_{i=1}^{g} s_{j-i}(p_i - q_i) + \frac{1}{2} \sum_{i=1}^{g} \rho_{j-i} q_i = s_j - \rho_j , \qquad \text{for} \quad g + 2 \leqslant j \leqslant 3g + 1 .$$

Viewed as a linear system in the $2g$ variables $p_j - q_j$ and $q_j/2$, with $i = 1, \ldots, g$, the associated matrix takes the 2×2 block form, each block being a Toeplitz matrix,

$$\begin{pmatrix} T_g(s,g) & T_g(\rho,g) \\ T_g(s,2g) & T_g(\rho,2g) \end{pmatrix} \quad \text{where} \quad T_g(\sigma, k) := \begin{pmatrix} \sigma_{k+1} & \sigma_k & \cdots & \sigma_{k-g+1} \\ \sigma_{k+2} & \sigma_{k+1} & \ddots & \\ \vdots & \ddots & \ddots & \sigma_k \\ \sigma_{k+g} & & \sigma_{k+2} & \sigma_{k+1} \end{pmatrix} \tag{3.68}$$

Our claim that $p_1, \ldots, p_g, q_1, \ldots, q_g$ can be solved rationally in terms of $s_1, \ldots, s_{3g+1}$ then follows from the fact that its determinant is non-zero, which is shown in the following lemma:

Lemma 3.13. *The block 2×2 matrix in* (3.68) *is non-degenerate.*

Proof. Since the entries of the matrix are polynomials in $s_1, \ldots, s_{3g+1}$, so is their determinant. To show that the determinant is non-zero for generic values of the s_i, it is therefore sufficient to show it for one particular set of values of the s_i. We pick

$$s_i := \begin{cases} i & 1 \leqslant i \leqslant g , \\ 1/4 & 1 \leqslant i = g + 1 , \\ 0 & g + 1 \leqslant i \leqslant 3g + 1 . \end{cases}$$

Then $T_g(s, g)$ is upper triangular, with all diagonal entries $1/4$, while $T_g(s, 2g)$ is the zero matrix. So we only need to show that

$$V_g := T_g(\rho, 2g) = \begin{pmatrix} 1 & 2 & 3 & \cdots & \cdots & g \\ 1/4 & 1 & 2 & 3 & \cdots & g-1 \\ 0 & 1/4 & 1 & 2 & \cdots & g-2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 1/4 & 1 & 2 \\ 0 & \cdots & \cdots & 0 & 1/4 & 1 \end{pmatrix}$$

is non-singular. We show by complete induction that $\det V_g = 2^{2-2g} g$ for all $g \geqslant 1$. The formula being true for $g = 1$, we assume it to be true for all values smaller than g and compute by successive development with respect to the first column, to find that $\det V_g$ equals

$$\det V_{g-1} - \frac{1}{4} \begin{vmatrix} 2 & 3 & 4 & \cdots & g \\ 1/4 & 1 & 2 & \cdots & g-2 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 1/4 & 1 \end{vmatrix} = \det V_{g-1} - \frac{1}{4} \times 2 \det V_{g-2} + \left(\frac{1}{4}\right)^2 \det \begin{vmatrix} 3 & 4 & 5 & \cdots & g \\ 1/4 & 1 & 2 & \cdots & g-3 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2 \\ 0 & \cdots & 0 & 1/4 & 1 \end{vmatrix} ,$$

and hence (defining $\det V_0 := 1$ for convenience)

$$\det V_g = \sum_{i=1}^{g} \left(-\frac{1}{4}\right)^{i-1} i \det V_{g-i} = \left(\frac{1}{4}\right)^{g-2} \sum_{i=1}^{g-1}(-1)^{i-1}i(g-i) + (-1)^{g-1}2^{2-2g}g \ ,$$

where we have used the induction hypothesis in the last step. It remains to be shown that

$$\sum_{i=1}^{g-1}(-1)^{i-1}i(g-i) = \begin{cases} g/2 & \text{when } g \text{ is even,} \\ 0 & \text{when } g \text{ is odd.} \end{cases}$$

The even case follows from an easy induction, while the odd case is completely trivial by symmetry. $\qquad\square$

Example 3.14. The simplest example is the case of $g = 1$. Then $3g + 1 = 4$. The birational map between $s_1, \ldots, s_4$ and $w_1, \ldots, w_4$ is given by

$$s_1 = w_1 \ , \quad s_2 = w_1 w_2 \ , \quad s_3 = w_1 w_2 (w_2 + w_3) \ , \quad s_4 = w_1 w_2 \left((w_2 + w_3)^2 + w_3 w_4\right) \ . \tag{3.69}$$

The linear system for p_1, q_1, r_1, r_2 then corresponds to the coefficients of x^j in (3.60) with $j = 1, \ldots, 4$. They are explicitly given by

$$\begin{aligned}
q_1 - 2p_1 - 2s_1 - r_1 &= 0 \ , \\
2p_1 s_1 - 2q_1 s_1 - 2s_2 + 2\rho_2 - r_2 &= 0 \ , \\
2\rho_3 - 2s_3 + (\rho_2 - 2s_2)q_1 + 2s_2 p_1 &= 0 \ , \\
2\rho_4 - 2s_4 + (\rho_3 - 2s_3)q_1 + 2s_3 p_1 &= 0 \ .
\end{aligned} \tag{3.70}$$

As in the general case, we first solve the last $2g = 2$ equations for p_1, q_1, which gives, upon writing each ρ_k in terms of the s_j,

$$p_1 = 2\frac{-3s_1^2 s_2^2 - 2s_2^3 + 2(s_1 s_2 + s_1^3 - s_3)s_3 + (2s_2 - s_1^2)s_4}{s_1(2s_2^2 - s_1 s_3)} = -w_1 - w_2 + w_3 + \frac{w_3(w_1 - w_4)(w_1 - 2w_2)}{w_1(w_2 - w_3)} \ ,$$

$$q_1 = 2\frac{s_2 s_4 - s_3^2 - s_2^3}{s_1(2s_2^2 - s_1 s_3)} = -2w_2 \left(1 + \frac{w_3(w_1 - w_4)}{w_1(w_2 - w_3)}\right) \ ,$$

where we have used (3.69) to write the formulae in terms of $w_1, \ldots, w_4$. The first two equations can now be solved for r_1 and r_2, leading to the following expressions:

$$r_1 = -2w_3 \left(1 + \frac{(w_1 - w_2)(w_1 - w_4)}{w_1(w_2 - w_3)}\right) \ , \qquad r_2 = 2w_1 w_3 \left(1 + \frac{w_1 - w_4}{w_2 - w_3}\right) \ .$$

Note also that in this case the unreduced version of the Volterra map is the birational map of $\mathbb{C}_w^4$ defined by

$$\begin{aligned}
(w_1, w_2, w_3, w_4) &\mapsto (w_2, w_3, w_4, w_5) \ , \\
\frac{w_5 w_4 + w_4^2 - w_3^2 - w_3 w_2}{w_4 - w_3} &+ \frac{w_1 w_2 + w_2^2 - w_3^2 - w_3 w_4}{w_3 - w_2} = 0 \ ,
\end{aligned} \tag{3.71}$$

which (after replacing each $w_i \to w_{i-2}$) is the relation obtained by eliminating c_1 from (3.34).

For any g, we can describe an explicit algorithmic procedure for obtaining the exact expressions for the birational map between $s_1, \ldots, s_{3g+1}$ and $w_1, \ldots, w_{3g+1}$. One way round, this is given by the Hankel determinant formula (3.55), but to describe the inverse map more explicitly we must recall how this relates to the approximation problem originally considered by Stieltjes [48], as well as other classical results on convergents of continued fractions in terms of continuants (for a concise review of the latter, see [16]).

The convergents of the S-fraction (3.15) are the sequence of rational functions of x obtained by truncating the continued fraction at some finite line n, which approximate the series for $F_0 = 1 - S$ exactly up to and including the coefficient of x^n, that is

$$\mathcal{F}_n(x) := 1 - \cfrac{w_1 x}{1 - \cfrac{w_2 x}{1 - \cfrac{\cdots}{1 - w_n x}}} = 1 - S(x) + \mathcal{O}(x^{n+1}), \tag{3.72}$$

Thus, for the first few convergents we have

$$\mathcal{F}_0 = 1, \quad \mathcal{F}_1 = 1 - w_1 x, \quad \mathcal{F}_2 = \frac{1 - (w_1 + w_2)x}{1 - w_2 x}, \quad \mathcal{F}_3 = \frac{1 - (w_1 + w_2 + w_3)x + w_1 w_3 x^2}{1 - (w_2 + w_3)x},$$

and so on. By the usual correspondence between convergents and 2×2 matrices, we see that the monodromy product over the conjugation matrices (3.23) appearing in the discrete Lax equation is given by

$$\boldsymbol{\Phi}_{n+1} := \mathbf{M}_1(x)\mathbf{M}_2(x)\cdots\mathbf{M}_{n+1}(x) = \begin{pmatrix} \mathcal{K}_{n+1}(w_1,\ldots,w_n;x) & -w_n x\,\mathcal{K}_n(w_1,\ldots,w_{n-1};x) \\ \mathcal{K}_n(w_2,\ldots,w_n;x) & -w_n x\,\mathcal{K}_{n-1}(w_2,\ldots,w_{n-1};x) \end{pmatrix}, \tag{3.73}$$

with the nth convergent being the ratio of the entries in the first column, that is

$$\mathcal{F}_n(x) = \frac{\mathcal{K}_{n+1}(w_1,\ldots,w_n;x)}{\mathcal{K}_n(w_2,\ldots,w_n;x)},$$

where the polynomial $\mathcal{K}_n$ is a continuant of size n, that is

$$\mathcal{K}_n(w_1,\ldots,w_{n-1};x) := \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ -w_1 x & 1 & -1 & \ddots & \vdots \\ 0 & -w_2 x & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -w_{n-1} x & 1 \end{vmatrix}.$$

From (3.73), the continuants are generated recursively from the linear relation $\boldsymbol{\Phi}_{n+1} = \boldsymbol{\Phi}_n \mathbf{M}_{n+1}$, starting from $\boldsymbol{\Phi}_0 = \mathbf{1}$ (the identity matrix). Hence, by replacing the series for $F_0 = 1 - S$ in (3.72) by its truncation at the nth term, then multiplying by $\mathcal{K}_n(w_2,\ldots,w_n;x)$ on both sides, we find the relation

$$\left(1 - \sum_{j=1}^{n} s_j x^j\right) \mathcal{K}_n(w_2,\ldots,w_n;x) = \mathcal{K}_{n+1}(w_1,\ldots,w_n;x) + \mathcal{O}(x^{n+1}),$$

and comparing the coefficients of x^j for $j = 1,\ldots,n$ allows $s_1, s_2,\ldots,s_n$ to be calculated recursively as polynomial expressions in $w_1,\ldots,w_n$. For instance, when $n = 4$ the expressions (3.69) are obtained from the numerator and denominator of

$$\mathcal{F}_4 = \frac{1 - (w_1 + w_2 + w_3 + w_4)x + (w_1 w_3 + w_1 w_4 + w_2 w_4)x^2}{1 - (w_2 + w_3 + w_4)x + w_2 w_4 x^2}.$$

Observe that these expressions are universal, in the sense that they depend only on the structure of the S-fraction, so that each s_n is always given by the same polynomial in w_j for $1 \leqslant j \leqslant n$, independent of g.

For $g \geqslant 2$, the corresponding formulae for the coefficients of $\mathcal{P}(x), \mathcal{Q}(x), \mathcal{R}(x)$ in terms of $w_1,\ldots,w_{3g+1}$ become very complicated, and we do not have a compact way to write them. However, we will not need the explicit form of these formulae in what follows, even when dealing with the example of genus 2.

4   Discrete integrability

We show in this section that the map (P.iv) is a discrete a.c.i. system, in a sense which will be defined below. To do this, we will first show that the affine space M_g which was constructed in the previous section has the same integrability properties as the odd and even Mumford systems (see [38, 53]): the invariants H_i of the Volterra map are in involution with respect to a large family of compatible Poisson structures and their generic level sets are affine parts of Jacobi varieties. We then show that the Volterra map is a Poisson map with respect to these Poisson structures and that it is a translation on the latter complex tori. In the Appendix (Section 7) we further show the precise relation between the Mumford-like system, introduced here, and the even Mumford system.

4.1   Compatible Poisson structures for the Mumford-like system

Recall that g is a fixed positive integer, and M_g denotes the affine space defined by

$$M_g := \left\{ (\mathcal{P}(x), \mathcal{Q}(x), \mathcal{R}(x)) \in \mathbb{C}[x]^3 \;\middle|\; \begin{array}{ll} \deg \mathcal{P}(x) \leqslant g\,, & \mathcal{P}(0) = 1 \\ \deg \mathcal{Q}(x) \leqslant g\,, & \mathcal{Q}(0) = 2 \\ \deg \mathcal{R}(x) \leqslant g+1\,, & \mathcal{R}(0) = 0 \end{array} \right\}\,, \tag{4.1}$$

with the $3g + 1$ non-constant coefficients of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ being linear coordinates on M_g.

We first introduce a $g + 1$-dimensional family of compatible Poisson structures on M_g. The family is parametrized by the $g+1$-dimensional vector space of polynomials $\phi \in \mathbb{C}[x]$ of degree at most $g+1$, vanishing at 0, so $\phi(0) = 0$. For such a polynomial ϕ, the corresponding Poisson structure is most naturally written by viewing $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ as generating functions, and is defined by

$$\{\mathcal{P}(x), \mathcal{P}(y)\}^\phi = \{\mathcal{Q}(x), \mathcal{Q}(y)\}^\phi = 0\,, \quad \{\mathcal{R}(x), \mathcal{R}(y)\}^\phi = \phi(y)\mathcal{R}(x) - \phi(x)\mathcal{R}(y)\,,$$

$$\{\mathcal{P}(x), \mathcal{Q}(y)\}^\phi = \frac{\phi(x)y\mathcal{Q}(y) - \phi(y)x\mathcal{Q}(x)}{x - y}\,, \quad \{\mathcal{P}(x), \mathcal{R}(y)\}^\phi = -y\frac{\phi(x)\mathcal{R}(y) - \phi(y)\mathcal{R}(x)}{x - y}\,, \tag{4.2}$$

$$\{\mathcal{Q}(x), \mathcal{R}(y)\}^\phi = 2y\frac{\phi(x)\mathcal{P}(y) - \phi(y)\mathcal{P}(x)}{x - y} - \phi(y)\mathcal{Q}(x)\,.$$

Of course, one needs to verify that the above definition is coherent, in the sense that the right-hand side of each of these formulae is indeed a polynomial in x and y, and also that the right-hand side does not contain any monomials $x^i y^j$ which are absent in the left-hand side. For example, $\{\mathcal{P}(x), \mathcal{Q}(y)\}^\phi$ is a polynomial in x, y with only non-zero coefficients of $x^i y^j$ when $1 \leqslant i, j \leqslant g$, while $\phi(x)y\mathcal{Q}(y) - \phi(y)x\mathcal{Q}(x)$ is clearly divisible by $x - y$ and the quotient has only non-zero coefficients of $x^i y^j$ in the same range. The same argument applies to the other formulae. Moreover, we need to verify that $\{\cdot, \cdot\}^\phi$ is a Poisson bracket, i.e., that it satisfies the Jacobi identity. This follows easily from the formulae (4.2). For example,

$$\left\{\{\mathcal{Q}(x), \mathcal{Q}(y)\}^\phi, \mathcal{R}(z)\right\}^\phi + \left\{\{\mathcal{Q}(y), \mathcal{R}(z)\}^\phi, \mathcal{Q}(x)\right\}^\phi + \left\{\{\mathcal{R}(z), \mathcal{Q}(x)\}^\phi, \mathcal{Q}(y)\right\}^\phi$$

$$= \left\{2z\frac{\phi(y)\mathcal{P}(z) - \phi(z)\mathcal{P}(y)}{y - z} - \phi(z)\mathcal{Q}(y), \mathcal{Q}(x)\right\}^\phi - \langle x \leftrightarrow y\rangle$$

$$= \frac{2z\phi(y)}{y - z}\frac{\phi(z)x\mathcal{Q}(x) - \phi(x)z\mathcal{Q}(z)}{z - x} - \frac{2z\phi(z)}{y - z}\frac{\phi(y)x\mathcal{Q}(x) - \phi(x)y\mathcal{Q}(y)}{y - x} - \langle x \leftrightarrow y\rangle$$

$$= \left(\frac{1}{(y - z)(z - x)} + \frac{1}{(x - y)(y - z)} + \frac{1}{(z - x)(x - y)}\right)2xz\phi(y)\phi(z)\mathcal{Q}(x) - \langle x \leftrightarrow y\rangle = 0\,.$$

Notice that since $\{\cdot, \cdot\}^\phi$ is a Poisson bracket for any $\phi \in \mathbb{C}[x]$ with $\deg \phi \leqslant g + 1$ and $\phi(0) = 0$, the Poisson brackets $\{\cdot, \cdot\}^\phi$ are compatible, simply because $\{\cdot, \cdot\}^\phi + \{\cdot, \cdot\}^{\phi'} = \{\cdot, \cdot\}^{\phi+\phi'}$ for any such polynomials ϕ, ϕ'.

We show in the next proposition that the Volterra map $\mathcal{V}_g$ is a Poisson map with respect to each one of these Poisson brackets. Recall from (3.9) and (3.10) that $\mathcal{V}_g$ is given by

$$\tilde{\mathcal{P}}(x) = \mathcal{Q}(x) - \mathcal{P}(x) \,, \qquad \tilde{\mathcal{Q}}(x) = \frac{2\mathcal{P}(x) - \mathcal{Q}(x) + \mathcal{R}(x)}{-wx} \,, \qquad \tilde{\mathcal{R}}(x) = -wx\mathcal{Q}(x) \,, \tag{4.3}$$

where $w = -\frac{2\,p_1 - q_1 + r_1}{2}$.

Proposition 4.1. *For any $\phi = \sum_{i=1}^{g+1} \phi_i x^i$, the Volterra map $\mathcal{V}_g : M_g \to M_g$ is a (birational) Poisson map with respect to $\{\cdot,\cdot\}^\phi$.*

Proof. We need to check that $\mathcal{V}_g$ preserves the Poisson bracket $\{\cdot,\cdot\}^\phi$. In formulae, this means that $\left\{\tilde{\mathcal{S}}(x), \tilde{\mathcal{T}}(y)\right\}^\phi = \widetilde{\{\mathcal{S}(x), \mathcal{T}(y)\}}^\phi$, where $\mathcal{S}$ and $\mathcal{T}$ stand for any of the polynomials $\mathcal{P}, \mathcal{Q}, \mathcal{R}$. To do this, it helps to first use (4.2) to derive the following formulae for the Poisson brackets of w with the polynomials $\mathcal{P}, \mathcal{Q}, \mathcal{R}$:

$$\{w, \mathcal{P}(y)\}^\phi = \{w, \mathcal{Q}(y)\}^\phi = \frac{\phi_1}{2}\mathcal{Q}(y) - \frac{\phi(y)}{y} \,, \quad \{w, \mathcal{R}(y)\}^\phi = \frac{\phi(y)}{y} - w\phi(y) - \frac{\phi_1}{2}(2\mathcal{P}(y) + \mathcal{R}(y)) \,. \tag{4.4}$$

To derive these formulae from (4.2), it suffices to use that $-2w$ is the linear term of $2\mathcal{P}(x) - \mathcal{Q}(x) + \mathcal{R}(x)$. For example,

$$\{-2w, \mathcal{R}(y)\}^\phi = \lim_{x \to 0} \frac{1}{x} \{2\mathcal{P}(x) - \mathcal{Q}(x) + \mathcal{R}(x), \mathcal{R}(y)\}^\phi$$

$$= \lim_{x \to 0} \frac{1}{x} \left[-2y\frac{\phi(x)\mathcal{R}(y) - \phi(y)\mathcal{R}(x)}{x - y} - 2y\frac{\phi(x)\mathcal{P}(y) - \phi(y)\mathcal{P}(x)}{x - y} + \phi(y)(\mathcal{Q}(x) + \mathcal{R}(x)) - \phi(x)\mathcal{R}(y) \right]$$

$$= \phi_1\mathcal{R}(y) - \phi(y)r_1 + 2\phi_1\mathcal{P}(y) + \phi(y)\lim_{x \to 0}\frac{2y\mathcal{P}(x) + (x - y)\mathcal{Q}(x)}{x(x - y)}$$

$$= \phi_1\mathcal{R}(y) - \phi(y)r_1 + 2\phi_1\mathcal{P}(y) - 2\frac{\phi(y)}{y} - (2p_1 - q_1)\phi(y)$$

$$= \phi_1(2\mathcal{P}(y) + \mathcal{R}(y)) - 2\frac{\phi(y)}{y} + 2w\phi(y) \,,$$

which proves the last formula in (4.4). It is now easy to prove that the Volterra map is a Poisson map. For example, it is clear from (4.4) that $\{\mathcal{P}(x) - \mathcal{Q}(x), w\}^\phi = 0$, and so

$$\left\{\tilde{\mathcal{P}}(x), \tilde{\mathcal{R}}(y)\right\}^\phi = \{\mathcal{Q}(x) - \mathcal{P}(x), -wy\mathcal{Q}(y)\}^\phi = wy\{\mathcal{P}(x), \mathcal{Q}(y)\}^\phi = wy\frac{\phi(x)y\mathcal{Q}(y) - \phi(y)x\mathcal{Q}(x)}{x - y}$$

$$= -y\frac{\phi(x)\tilde{\mathcal{R}}(y) - \phi(y)\tilde{\mathcal{R}}(x)}{x - y} = \widetilde{\{\mathcal{P}(x), \mathcal{R}(y)\}}^\phi \,.$$

The other verifications are done in the same way. This shows that the Volterra map is a Poisson map. $\square$

We show in the next proposition that the invariants of the Volterra map, as introduced in the previous section, are in involution with respect to any of the Poisson brackets $\{\cdot,\cdot\}^\phi$, where we recall that $\deg\phi \leqslant g+1$ and $\phi(0) = 0$.

Proposition 4.2. *The $2g + 1$ invariants $H_1, \ldots, H_{2g+1}$ of the Volterra map, defined by*

$$\mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x) = 1 + \sum_{i=1}^{2g+1} H_i x^i \,. \tag{4.5}$$

are in involution with respect to $\{\cdot,\cdot\}^\phi$.

28

Proof. We first compute from (4.2)

$$\{\mathcal{P}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi = \mathcal{Q}(y)\{\mathcal{P}(x), \mathcal{R}(y)\}^\phi + \mathcal{R}(y)\{\mathcal{P}(x), \mathcal{Q}(y)\}^\phi$$

$$= -y\mathcal{Q}(y)\frac{\cancel{\phi(x)\mathcal{R}(y)} - \phi(y)\mathcal{R}(x)}{x - y} + \mathcal{R}(y)\frac{y\cancel{\phi(x)\mathcal{Q}(y)} - x\phi(y)\mathcal{Q}(x)}{x - y}$$

$$= \phi(y)\frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{x - y} , \tag{4.6}$$

and similarly

$$\{\mathcal{Q}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi = 2\phi(y)\frac{x\mathcal{P}(y)\mathcal{Q}(x) - y\mathcal{P}(x)\mathcal{Q}(y)}{x - y} - \phi(y)\mathcal{Q}(x)\mathcal{Q}(y) , \tag{4.7}$$

$$\{\mathcal{R}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi = 2x\phi(y)\frac{\mathcal{P}(x)\mathcal{R}(y) - \mathcal{P}(y)\mathcal{R}(x)}{x - y} + \phi(y)\mathcal{Q}(y)\mathcal{R}(x) . \tag{4.8}$$

These formulae imply

$$\{\mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x), \mathcal{P}^2(y) + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi$$

$$= 2\mathcal{P}(x)\{\mathcal{P}(x), \mathcal{P}^2(y) + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi + \mathcal{Q}(x)\{\mathcal{R}(x), \mathcal{P}^2(y) + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi + \mathcal{R}(x)\{\mathcal{Q}(x), \mathcal{P}^2(y) + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi$$

$$= 2\mathcal{P}(x)\phi(y)\frac{y\cancel{\mathcal{Q}(y)\mathcal{R}(x)} - x\cancel{\mathcal{Q}(x)\mathcal{R}(y)}}{x - y} + \mathcal{Q}(x)2x\phi(y)\frac{\cancel{\mathcal{P}(x)\mathcal{R}(y)} - \cancel{\mathcal{P}(y)\mathcal{R}(x)}}{x - y} + \mathcal{Q}(x)\phi(y)\mathcal{Q}(y)\mathcal{R}(x)$$

$$+ \mathcal{R}(x)2\phi(y)\frac{x\cancel{\mathcal{P}(y)\mathcal{Q}(x)} - y\cancel{\mathcal{P}(x)\mathcal{Q}(y)}}{x - y} - \mathcal{R}(x)\phi(y)\mathcal{Q}(x)\mathcal{Q}(y) = 0 ,$$

so that the invariants H_i are in involution, $\{H_i, H_j\}^\phi = 0$ for $1 \leqslant i, j \leqslant 2g + 1$. $\qquad\square$

As before, we often fix the Hamiltonians $H_1, \ldots, H_g$ to generic values so that we actually work on a (non-singular) subvariety of M_g^c, which is birational with $\mathbb{C}^{2g+1}$. We show in the following proposition that M_g^c is a bi-Hamiltonian manifold, i.e., that it is equipped with a pencil of compatible Poisson structures.

Proposition 4.3. *When $\phi = \phi_g x^g + \phi_{g+1} x^{g+1}$ the Hamiltonians $H_1, \ldots, H_g$ are Casimirs of the Poisson structure $\{\cdot, \cdot\}^\phi$. Moreover,*

$$C_\phi := \sum_{i=0}^g (-1)^i \phi_g^{g-i} \phi_{g+1}^i H_{2g+1-i} \tag{4.9}$$

is also a Casimir function of $\{\cdot, \cdot\}^\phi$.

Proof. Consider for $i = 1, \ldots, g$ the Hamiltonian vector field X_{H_i}, which is given as the coefficient of y^i in $\{\cdot, \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\}^\phi$, which we computed in (4.6) – (4.8). These Hamiltonians are Casimir functions for $\{\cdot, \cdot\}^\phi$ if and only if their Hamiltonian vector fields are zero, which is in turn equivalent to the fact that the right hand sides in (4.6) – (4.8) are divisible by y^{g+1}. Notice that the latter right hand sides, without the factor $\phi(y)$, are divisible by y since $\mathcal{P}(0) = 1$, $\mathcal{Q}(0) = 2$ and $\mathcal{R}(0) = 0$, without being divisible by y^2. Therefore, we need $\phi(y)$ to be divisible by y^g; since $\deg \phi \leqslant g + 1$, it follows that the Poisson structures which make M_g^c into a bi-Hamiltonian manifold are the restrictions to M_g^c of the Poisson pencil $\{\cdot, \cdot\}^\phi$ with $\phi(x) = \phi_g x^g + \phi_{g+1} x^{g+1}$. It is easily shown by direct computation that C_ϕ, given by (4.9) is a Casimir of

the Poisson pencil. For example,

$$\left\{ \mathcal{P}(x), \sum_{i=0}^{g} (-1)^i \phi_g^{g-i} \phi_{g+1}^i H_{2g+1-i} \right\}^{\phi}$$

$$= \operatorname*{Res}_{y=0} \frac{1}{y^{2g+2}} \left\{ \mathcal{P}(x), H_{2g+1-i} y^{2g+1} \right\}^{\phi} \sum_{i=0}^{g} (-1)^i \phi_g^{g-i} \phi_{g+1}^i$$

$$= \operatorname*{Res}_{y=0} \frac{1}{y^{2g+2}} \left\{ \mathcal{P}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y) \right\}^{\phi} \sum_{i=0}^{g} (-1)^i \phi_g^{g-i} \phi_{g+1}^i y^i$$

$$= \operatorname*{Res}_{y=0} \frac{1}{y^{2g+2}} \left[\frac{\phi(x)y\mathcal{Q}(y) - \phi(y)x\mathcal{Q}(x)}{x-y} \mathcal{R}(y) - y\mathcal{Q}(y)\frac{\phi(x)\mathcal{R}(y) - \phi(y)\mathcal{R}(x)}{x-y} \right] \sum_{i=0}^{g} (-1)^i \phi_g^{g-i} \phi_{g+1}^i y^i$$

$$= \operatorname*{Res}_{y=0} \frac{1}{y^{2g+2}} \frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{x-y} \sum_{i=0}^{g} (-1)^i \phi_g^{g-i} \phi_{g+1}^i y^i \phi(y)$$

$$= \operatorname*{Res}_{y=0} \frac{1}{y^{2g+2}} \frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{x-y} \left(\phi_g^{g+1} y^g + (-1)^g \phi_{g+1}^{g+1} y^{2g+1} \right)$$

$$= \phi_g^{g+1} \operatorname*{Res}_{y=0} \frac{1}{y^{g+2}} \frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{x-y} = 0 \;,$$

where we have used in the last two equalities respectively that $\mathcal{R}(y)$ is divisible by y and that the polynomial $\frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{x-y}$ has degree at most g in y. $\qquad\square$

Example 4.4. Continuing Example 3.6, we specialize the above results to $g = 1$ and make them more explicit. Recall that M_1 is the 4-dimensional vector space of polynomials $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ with

$$\mathcal{P}(x) = 1 + p_1 x \;, \quad \mathcal{Q}(x) = 2 + q_1 x \;, \quad \mathcal{R}(x) = r_1 x + r_2 x^2 \;, \tag{4.10}$$

and the invariants H_1, H_2, H_3 are given in (3.24). The Poisson structures $\{\cdot,\cdot\}^{\phi}$ on M_1 are parametrized by $\phi(x) = \phi_1 x + \phi_2 x^2$ and they all have H_1 as a Casimir function. The Poisson matrices of the basic Poisson structures $\{\cdot,\cdot\}^x$ and $\{\cdot,\cdot\}^{x^2}$ (with the coordinates taken in the following order: p_1, q_1, r_1, r_2) are easily determined from (4.2) and are given by

$$\{\cdot,\cdot\}^x = \begin{pmatrix} 0 & -q_1 & 0 & r_2 \\ q_1 & 0 & -q_1 & 0 \\ 0 & q_1 & 0 & -r_2 \\ -r_2 & 0 & r_2 & 0 \end{pmatrix} \;, \quad \{\cdot,\cdot\}^{x^2} = \begin{pmatrix} 0 & 2 & 0 & -r_1 \\ -2 & 0 & 2 & 2q_1 - p_1 \\ 0 & -2 & 0 & r_1 \\ r_1 & p_1 - 2q_1 & -r_1 & 0 \end{pmatrix} \;. \tag{4.11}$$

For a generic $c_1 \in \mathbb{C}$, the subvariety M_1^c is defined by $H_1 = c_1$. The pencil of Poisson structures $\{\cdot,\cdot\}^{\phi}$ can be restricted to M_1^c and the Casimir function C_{ϕ} on $(M_1^c, \{\cdot,\cdot\}^{\phi})$, given by (4.9), takes the simple form $C_{\phi} = \phi_2 H_2 - \phi_1 H_3$. In particular, H_2 and H_3 are Casimir functions of $\{\cdot,\cdot\}^{x^2}$ and $\{\cdot,\cdot\}^x$, respectively, and $\{\cdot,\cdot\}^x$ can be restricted to $H_3 = c_3$ while $\{\cdot,\cdot\}^{x^2}$ can be restricted to $H_2 = c_2$ (for generic c_2, c_3). Since, by Proposition 4.1, the Volterra map is a Poisson map with respect to any such ϕ (of degree at most $g+1$ with $\phi(0) = 0$), it follows that

- (3.34) defines a Poisson map with respect to the full Poisson pencil $\{\cdot,\cdot\}^{\phi}$;

- (3.37) defines a Poisson map with respect to $\{\cdot,\cdot\}^x$;

- (3.39) defines a Poisson map with respect to $\{\cdot,\cdot\}^{x^2}$.

It follows from the Poisson matrices (4.11) and from (3.33) that the Poisson structures $\{\cdot,\cdot\}^x$ and $\{\cdot,\cdot\}^{x^2}$ are respectively given by $\{w_{n-1}, w_n\}^x = -(w_{n-1} + w_n + c_1/2)/2$, and $\{w_{n-1}, w_n\}^{x^2} = 1/2$.

In Example 3.6 we also considered the recursion relation on the surface $H_2' = c_2'$, $H_3' = c_3'$. None of the Poisson structures $\{\cdot,\cdot\}^\phi$ considered above can be restricted to these surfaces, but a Nambu-Poisson structure with H_2' and H_3' as Casimirs can be so restricted. It leads to the quadratic Poisson bracket

$$\{w_{n-1}, w_n\} = w_{n-1} w_n \tag{4.12}$$

with respect to which (3.45) is a Poisson map. The above bracket can also be derived via reduction of a presymplectic structure for the tau functions, by regarding (3.48) as a mutation in a cluster algebra [18].

To see how the quadratic bracket arises here, recall from [52] that, for some fixed choice of m-form Ω, a Nambu-Poisson bracket of order m is defined by

$$\{f_1, f_2, \ldots, f_m\}\, \Omega = \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \cdots \wedge \mathrm{d}f_m.$$

In the case at hand (taking $m = 4$), observe that the unreduced version (3.71) of the Volterra map φ on $\mathbb{C}_w^4$ with coordinates (w_{-1}, w_0, w_1, w_2) preserves the rational volume form

$$\Omega = \frac{w_0 w_1}{w_0 - w_1}\, \mathrm{d}w_{-1} \wedge \mathrm{d}w_0 \wedge \mathrm{d}w_1 \wedge \mathrm{d}w_2, \qquad \varphi^*(\Omega) = \Omega.$$

Then the corresponding Nambu-Poisson bracket defines a Poisson bracket on $\mathbb{C}_w^4$ according to

$$\{f_1, f_2\} := \{f_1, f_2, H_2', H_3'\},$$

which by construction has Casimirs H_2' and H_3', and restricts to (4.12) on the surface $H_2' = c_2'$, $H_3' = c_3'$. The same Nambu-Poisson bracket also produces any member of the pencil $\{\cdot,\cdot\}^\phi$ for $g = 1$: in particular, the Poisson structures $\{\cdot,\cdot\}^x$ and $\{\cdot,\cdot\}^{x^2}$ arise in this way, by taking (up to scaling) $\{f_1, f_2, H_1, H_3\}$ and $\{f_1, f_2, H_1, H_2\}$, respectively. However, this construction does not extend to $g > 1$ in a straightforward manner.

Example 4.5. Continuing Example 3.7, taking $\phi = x^3$ when $g = 2$ we find that the Poisson brackets on M_2 (with coordinates $p_1, p_2, q_1, q_2, r_1, r_2, r_3$) take the form

$$\{r_1, r_3\}^{x^3} = r_1, \qquad\qquad \{r_2, r_3\}^{x^3} = r_2, \qquad \{p_1, q_2\}^{x^3} = 2 = \{p_2, q_1\}^{x^3},$$

$$\{p_2, q_2\}^{x^3} = q_1, \qquad\qquad \{p_2, r_3\}^{x^3} = -r_2, \qquad \{p_1, r_3\}^{x^3} = -r_1 = \{p_2, r_2\}^{x^3},$$

$$\{q_1, r_2\}^{x^3} = 2 = \{q_2, r_1\}^{x^3}, \qquad \{q_1, r_3\}^{x^3} = 2p_1 - q_1, \quad \{q_2, r_2\}^{x^3} = 2p_1, \quad \{q_2, r_3\}^{x^3} = 2p_2 - q_2,$$

where only the non-zero brackets are specified here. By restricting this Poisson structure to $M_g^c \cap (H_3 = c_3)$, we find that (up to an overall factor of $1/2$) it coincides with Poisson bracket (1.5) for the (P.iv) map that was derived from a discrete Lagrangian in [21], where the parameters ν, a, b are fixed according to (3.53) in terms of the values c_1, c_2, c_3 of the 3 Casimirs of $\{\cdot,\cdot\}^{x^3}$.

4.2   The generic fibers of the momentum map

We will now give an algebro-geometric description of the generic fibers of the momentum map $\mu : M_g \to \mathbb{C}[x]$, which we recall is given by $\mu(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = \mathcal{P}^2 + \mathcal{Q}\mathcal{R}$. Precisely, we will describe the fiber over any polynomial $f \in \mathbb{C}[x]$ of degree $2g + 1$, without multiple roots, and satisfying $f(0) = 1$. Notice that such polynomials are exactly those f in the image of μ for which $y^2 = f(x)$ defines a non-singular affine curve Γ_f of genus g; we use $\bar{\Gamma}_f$ to denote the completion of the latter, which can be thought of as a compact Riemann surface.

Proposition 4.6. *Let f be polynomial of degree $2g + 1$, without multiple roots, such that $f(0) = 1$. Denote by Γ_f the non-singular affine curve of genus g, defined by $y^2 = f(x)$, with $\bar{\Gamma}_f$ being its completion. Then $\mu^{-1}(f)$ is isomorphic to an affine part of the Jacobian variety of $\bar{\Gamma}_f$ minus three translates of the theta divisor,*

$$\mu^{-1}(f) \cong \mathrm{Jac}(\bar{\Gamma}_f) \setminus (\Theta \cup \Theta_+ \cup \Theta_-) \,. \tag{4.13}$$

The (-1)-involution on $\mathrm{Jac}(\bar{\Gamma}_f)$ leaves Θ invariant and permutes Θ_+ and Θ_-.

Proof. Let $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) \in \mu^{-1}(f)$, so that

$$\mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x) = f(x) \,, \quad \deg \mathcal{P}, \deg \mathcal{Q}, \deg \mathcal{R} - 1 \leqslant g \,, \quad (\mathcal{P}(0), \mathcal{Q}(0), \mathcal{R}(0)) = (1, 2, 0) \,. \tag{4.14}$$

Since f has degree $2g + 1$, $\deg \mathcal{Q} = g$ and $\deg \mathcal{R} = g + 1$; it also implies that $\bar{\Gamma}_f$ is obtained from Γ_f by adding a single point, which we denote by ∞. The hyperelliptic involution on $\bar{\Gamma}_f$ is denoted by $\imath$; it fixes ∞ and sends $(x, y) \in \Gamma_f$ to $(x, -y)$. To $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ we associate a divisor $\sum_{i=1}^{g}(x_i, y_i) - g\infty$ on $\bar{\Gamma}_f$ as follows: $x_1, x_2, \ldots, x_g$ are the roots of $\mathcal{Q}(x)$ and $y_i := \mathcal{P}(x_i)$ for $i = 1, \ldots, g$. It is indeed a divisor on $\bar{\Gamma}_f$ since for $j = 1, \ldots, g$,

$$y_j^2 - f(x_j) = \mathcal{P}^2(x_j) - (\mathcal{P}^2(x_j) + \mathcal{Q}(x_j)\mathcal{R}(x_j)) = 0 \,.$$

Of course, $(x_i, y_i) \neq \infty$ for all i. Notice that when $\mathcal{Q}(x)$ has multiple roots, say $x_1 = \cdots = x_k$, then $y_1 = y_2 = \cdots = y_k$. We show by contradiction that if x_i is a root of $\mathcal{Q}(x)$ and $y_i = 0$ (so that x_i is also a root of $\mathcal{P}(x)$), then x_i is a simple root of $\mathcal{Q}(x)$. Indeed, if x_i is a multiple root of $\mathcal{Q}(x)$ and is also a root of $\mathcal{P}(x)$, then x_i is a multiple root of $f(x) = \mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x)$, so that f is not square-free, contrary to the assumptions. The upshot is that the obtained divisors are of the form $\sum_{i=1}^{g} P_i - g\infty$, where $P_i \in \Gamma_f$ for $i = 1, \ldots, g$ and $P_i \neq \imath(P_j)$ when $i \neq j$. It is well-known that two such divisors are linearly equivalent if and only if they are the same; also, that none of these divisors are equivalent to a divisor of the form $\sum_{i=1}^{g-1} Q_i - (g-1)\infty$, with $Q_i \in \bar{\Gamma}_f$ for $i = 1, \ldots, g-1$. Since $\mathrm{Jac}(\bar{\Gamma}_f)$ is the group of degree zero divisors on $\bar{\Gamma}_f$, modulo linear equivalence, this shows that the map $\mu^{-1}(f) \to \mathrm{Jac}(\bar{\Gamma}_f)$, which associates to $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ the divisor class $[\sum_{i=1}^{g} P_i - g\infty]$, is injective. This map is of course not surjective. In order to determine the image, let

$$\Theta := \left\{ \left[\sum_{i=1}^{g-1} P_i - (g-1)\infty \right] \mid \forall i \; P_i \in \bar{\Gamma}_f \right\} \,,$$

$$\Theta_+ := \left\{ \left[(0, 1) + \sum_{i=1}^{g-1} P_i - g\infty \right] \mid \forall i \; P_i \in \bar{\Gamma}_f \right\} = \Theta + [(0, 1) - \infty] \,,$$

$$\Theta_- := \left\{ \left[(0, -1) + \sum_{i=1}^{g-1} P_i - g\infty \right] \mid \forall i \; P_i \in \bar{\Gamma}_f \right\} = \Theta + [(0, -1) - \infty] \,.$$

The first one is the theta divisor and the other two are translates of it. Notice that $\imath(\Theta) = \Theta$ and $\imath(\Theta_+) = \Theta_-$, since $\imath(\infty) = \infty$. As we already said, the image contains no point of the form $\left[\sum_{i=1}^{g-1} P_i - (g-1)\infty \right]$, i.e., is disjoint from Θ. Since $P_i = (x_i, y_i)$ where x_i is a root of $\mathcal{Q}(x)$ and since $\mathcal{Q}(0) = 2$, it is clear that $x_i \neq 0$, so that every $P_i = (x_i, y_i)$ is different from $(0, 1)$ and from $(0, -1)$; hence the image is also disjoint from Θ_+ and Θ_-. Take now any point in $\mathrm{Jac}(\bar{\Gamma}_f) \setminus (\Theta \cup \Theta_+ \cup \Theta_-)$. It can as above be written uniquely as $[\sum_{i=1}^{g} P_i - g\infty]$ with $P_i \notin \{\infty, (1, 0), (-1, 0)\}$ and $P_i \neq \imath(P_j)$ for all $i \neq j$. When all $P_i = (x_i, y_i)$ are different, there is a unique polynomial $\mathcal{Q}(x)$ whose roots are the x_i and with $\mathcal{Q}(0) = 2$, and there is a unique polynomial $\mathcal{P}(x)$ of degree g, with $\mathcal{P}(x_i) = y_i$ for $i = 1, \ldots, g$ and $\mathcal{P}(0) = 1$: setting $(x_0, y_0) = (0, 1)$, they are given by

$$\mathcal{Q}(x) = 2 \prod_{i=1}^{g} \left(1 - \frac{x}{x_i} \right) \,, \qquad \mathcal{P}(x) = \sum_{i=0}^{g} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \,. \tag{4.15}$$

This also works in the limiting case when some of the P_i are the same upon adding in the definition of $\mathcal{P}(x)$ a tangency condition (see [38, page 3.18]), which assures that $f(x) - \mathcal{P}^2(x)$ is divisible by $\mathcal{Q}(x)$. The quotient is a polynomial $\mathcal{R}(x)$ of degree g satisfying $f(x) = \mathcal{P}(x)^2 + \mathcal{Q}(x)\mathcal{R}(x)$; by uniqueness, $\mu(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = [\sum_{i=1}^{g} P_i - g\infty]$, as required. $\qquad\square$

Example 4.7. We specialize Proposition 4.6 to the case of $g = 1$. Notice that in this case we should say *elliptic* rather than *hyperelliptic*, and in this case the (hyper-) elliptic involution is not unique. Another peculiarity about $g = 1$ is the well-known fact that a complete non-singular genus one curve (i.e., any compact elliptic Riemann surface) is isomorphic to its Jacobian, a fact that we will be able to illustrate here. Let $f(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3$ be an arbitrary polynomial of degree 3 with no multiple roots. We investigate $\mu^{-1}(f)$, which is the affine curve defined by the following equations, which are found by expressing that $\mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x) = f(x)$:

$$2(p_1 + r_1) = c_1 \ ,$$
$$p_1^2 + q_1 r_1 + 2r_2 = c_2 \ ,$$
$$q_1 r_2 = c_3 \ .$$

The curve $\mu^{-1}(f)$ can be equivalently written as a plane algebraic curve by first eliminating r_1 from the first two equations and then r_2 from the two remaining equations; writing p_1 and q_1 simply as p and q, the final equation takes the simple form

$$\mu^{-1}(f) \ : \ \frac{pq}{2}(p - q) + c_1 \frac{q^2}{4} - c_2 \frac{q}{2} + c_3 = 0 \ . \tag{4.16}$$

It is easy to see that, thanks to the conditions on f, this curve is non-singular, just like Γ_f. In fact, if we denote the left-hand side of (4.16) by F then a singular point (q_0, p_0) of $\mu^{-1}(f)$ must satisfy

$$\frac{\partial F}{\partial p}(q_0, p_0) = \frac{q_0}{2}(q_0 - 2p_0) = 0 \ , \quad \frac{\partial F}{\partial q}(q_0, p_0) = \frac{1}{2}(p_0^2 - 2p_0 q_0 + c_1 q_0 - c_2) = 0 \ . \tag{4.17}$$

Since $q_0 \neq 0$ (as $F(q_0, p_0) = 0$ and $c_3 \neq 0$), we get $q_0 = 2p_0$; substituted in $F(q_0, p_0) = 0$ and in the second equation of (4.17) we get

$$p_0^3 - c_1 p_0^2 + c_2 p_0 - c_3 = 0 \ , \quad 3p_0^2 - 2c_1 p_0 + c_2 = 0 \ ,$$

which can be written as $f(-p_0) = f'(-p_0) = 0$. Since f has no multiple roots these equations have no common solution, which shows that $\mu^{-1}(f)$ is non-singular.

To see that $\mu^{-1}(f)$ and Γ_f are birationally isomorphic, it suffices to consider the following rational map:

$$q = -\frac{2}{x} \ , \ p = \frac{y-1}{x} \ , \quad \text{with inverse} \quad x = -\frac{2}{q} \ , \ y = \frac{q - 2p}{q} \ . \tag{4.18}$$

Notice that, despite the appearance of q in the denominator, the inverse map in 4.18 is actually regular, because $q \neq 0$ on $\mu^{-1}(f)$. The rational map and its inverse extend (uniquely) to an isomorphism of the completions $\bar{\Gamma}_f$ and $\overline{\mu^{-1}(f)}$, which can be thought of respectively as an elliptic curve and its Jacobian. It allows us to determine the number and nature of the points at infinity of $\overline{\mu^{-1}(f)}$, i.e., the points needed to complete $\mu^{-1}(f)$ into $\overline{\mu^{-1}(f)}$; they are the points corresponding to affine points (x, y) for which the map is not defined, to wit $(x, y) = (0, \pm 1)$, and to the point at infinity ∞ of $\bar{\Gamma}_f$, making a total of three points, as asserted by Proposition 4.6. More specifically, by using the map we can determine a local parametrisation around these points from local parametrisations around the points $(0, \pm 1)$ and ∞. For the latter, we can take $(x, y) = (t, 1 \pm \frac{c_1 t}{2} + \mathcal{O}(t^2))$ and $(x, y) = (t^{-2}, \sqrt{c_3}t^{-3}(1 + \frac{c_2}{2c_3}t^2 + \mathcal{O}(t^4))$ to obtain, again using the map, the following local parametrisations at the three points at infinity of $\overline{\mu^{-1}(f)}$:

$$\infty_0 : \left(-2t^2, \frac{\sqrt{c_3}}{t}\left(1 + \frac{c_2}{2c_3}t^2 + \mathcal{O}(t^3)\right)\right) \ , \quad \infty_1 : \left(-\frac{2}{t}, \frac{c_1}{2} + \mathcal{O}(t)\right) \ , \quad \infty_2 : \left(-\frac{2}{t}, -\frac{2}{t} - \frac{c_1}{2} + \mathcal{O}(t)\right) \ .$$

Again, using the map we can derive that the (hyper-) elliptic involution on Γ_f, which is given by $(x, y) \mapsto (x, -y)$, is given on $\mu^{-1}(f)$ by $(q, p) \mapsto (q, q - p)$. It permutes the points ∞_1 and ∞_2 while leaving ∞_0 fixed (together with the points $(2p, p)$ where $-p$ is a root of f). The points ∞_0, ∞_1 and ∞_2 correspond to Θ, Θ_+ and Θ_-, respectively.

Example 4.8. We also specialize Proposition 4.6 to the case of $g = 2$ and provide some extra information. The polynomial f is now of degree 5, taking the form $f(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$, with no multiple roots, which is equivalent to the curve (1.8) associated with the map (P.iv). In this case, the theta divisor and its translates are genus 2 curves, isomorphic to $\bar{\Gamma}_f$. This general fact can also be seen here directly from the description that $\Theta = \left[\bar{\Gamma}_f - \infty\right]$ and similarly for $\Theta_+ = \left[\bar{\Gamma}_f + (0, 1) - 2\infty\right]$ and $\Theta_- = \left[\bar{\Gamma}_f + (0, -1) - 2\infty\right]$. We show that these curves in $\overline{\mu^{-1}(f)}$ meet according to the intersection pattern in Figure 2.
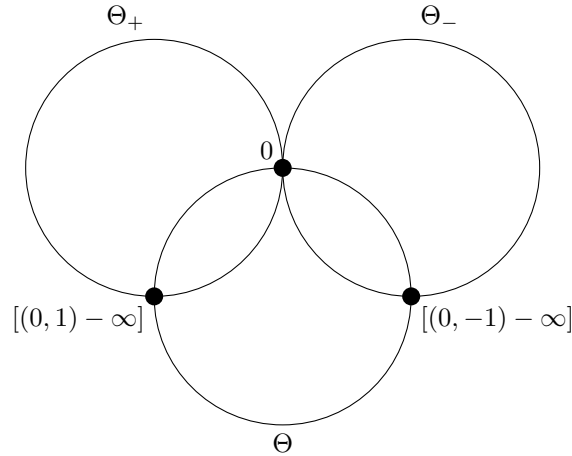

Figure 2: When $g = 2$ the divisor at infinity of $\mu^{-1}(f)$ consist of three copies of the curve $y^2 = f(x)$, intersecting according to the indicated pattern.

To do this, we first recall the general fact that two translates of the theta divisor (also called *theta curves*) intersect in two points which coincide if and only if the curves are tangent. Consider first a point (divisor class) in $\Theta_+ \cap \Theta_-$. It must be of the form $[P + (0, 1) - 2\infty]$ and of the form $[Q + (0, -1) - 2\infty]$, for some $P, Q \in \bar{\Gamma}_f$. In particular, the points P and Q must be such that $P + (0, 1) \sim Q + (0, -1)$; as we already recalled, a linear equivalence of such divisors amounts to equality, so that $P = (0, -1)$ and $Q = (0, 1)$ and there is a unique intersection point $[(0, 1) + (0, -1) - 2\infty]$ which is the origin O of $\mathrm{Jac}(\bar{\Gamma}_f)$, since $(0, 1) + (0, -1) \sim 2\infty$. Consider next a point in $\Theta \cap \Theta_\pm$. It must be both of the form $[P - \infty]$ and $[Q + (0, \pm 1) - 2\infty]$, for some $P, Q \in \bar{\Gamma}_f$. This leads us now to the linear equivalence $P + \infty \sim Q + (0, \pm 1)$, whose only solutions are $P = \infty$, $Q = (0, \mp 1)$ and $P = (0, \pm 1)$, $Q = \infty$; the first solution corresponds again to the origin O while the other intersection point is the point $[(0, \pm 1) - \infty]$ (see Figure 2).

4.3 Discrete Liouville and algebraic integrability

We are now ready to show that the Volterra map $\mathcal{V}_g$ is Liouville integrable on $(M_g, \{\cdot, \cdot\}^\phi)$ when $\phi \neq 0$. Recall that a birational map R on an algebraic Poisson manifold $(M, \{\cdot, \cdot\})$ of dimension n and (Poisson) rank $2r$ is said to be *Liouville integrable* when the following conditions are satisfied:

(1) R is a Poisson map;

(2) R has $n - r$ functionally independent invariants in involution.

In the case at hand, $M = M_g$ so that $n = 3g + 1$, and $R = \mathcal{V}_g$, which we already know to be a Poisson map (Proposition 4.1). Also, we already have $2g + 1$ invariants in involution (Proposition 4.2). So it will be sufficient to show, in the proof which follows, that the rank of $\{\cdot, \cdot\}^\phi$ is $2g$ and that the invariants $H_1, \ldots, H_{2g+1}$ are functionally independent.

Proposition 4.9. *Let $\phi \in \mathbb{C}[x]$ be any non-zero polynomial of degree at most $g + 1$, vanishing at 0. Then the Volterra map $\mathcal{V}_g$ is Liouville integrable on $(M_g, \{\cdot, \cdot\}^\phi)$.*

Proof. We first show that the components of μ, which are the $2g + 1$ polynomial functions H_i, defined by $\mu(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = 1 + \sum_{i=1}^{2g+1} H_i x^i$, are independent. According to Proposition 4.6, the generic fiber of μ (which is the generic fiber of $H_1, \ldots, H_{2g+1}$) is an open subset of the Jacobian of a curve of genus g, hence has dimension g. The dimension of the generic fiber of μ is given by $\dim M_g - s = 3g + 1 - s$, where s denotes the number of independent functions in $H_1, \ldots, H_{2g+1}$. Therefore, $s = 2g + 1$ and the components of μ are independent.

It remains to be shown that the rank of $\{\cdot, \cdot\}^\phi$ is $2g$. Since $\dim M = 3g + 1$ and $\{\cdot, \cdot\}^\phi$ admits $2g + 1$ independent functions in involution, the rank of $\{\cdot, \cdot\}^\phi$ is at most $2g$ (see [53, Prop. 3.4]); to show equality, in a neighborhood of a generic point $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) \in M_g$ we take the functions $x_1, \ldots, x_n, y_1, \ldots, y_n$, which we constructed in the proof of Proposition 4.6. Then $\{x_i, x_j\}^\phi = \{y_i, y_j\}^\phi = 0$, since $\{\mathcal{P}(x), \mathcal{P}(y)\}^\phi = \{\mathcal{Q}(x), \mathcal{Q}(y)\}^\phi = 0$. We show that the brackets $\{x_i, y_j\}^\phi$ are non-zero if and only if $i = j$, from which it follows that the rank of $\{\cdot, \cdot\}^\phi$ is indeed $2g$. To do this, we compute for $1 \leqslant i \leqslant g$ the Poisson bracket $\{y_i, \ln \mathcal{Q}(y)\}^\phi$ in two different ways. First, using (4.2),

$$\{y_i, \ln \mathcal{Q}(y)\}^\phi = \{\mathcal{P}(x_i), \ln \mathcal{Q}(y)\}^\phi = \frac{\{\mathcal{P}(x_i), \mathcal{Q}(y)\}^\phi}{\mathcal{Q}(y)} = \frac{y\phi(x_i) - x_i\phi(y)\mathcal{Q}(x_i)/\mathcal{Q}(y)}{x_i - y} = y\frac{\phi(x_i)}{x_i - y} , \quad (4.19)$$

and next, using (4.15),

$$\{y_i, \ln \mathcal{Q}(y)\}^\phi = \sum_{j=1}^{g} \{y_i, \ln(1 - y/x_j)\}^\phi = y\sum_{j=1}^{g} \frac{\{y_i, x_j\}^\phi}{x_j(x_j - y)} , \quad (4.20)$$

so that

$$\frac{\phi(x_i)}{x_i - y} = \sum_{j=1}^{g} \frac{\{y_i, x_j\}^\phi}{x_j(x_j - y)} .$$

Since (generically) all x_j are different, $\{x_j, y_i\}^\phi = 0$ when $j \neq i$, while $\{x_i, y_i\}^\phi = -x_i\phi(x_i)$, so that

$$\{x_i, y_j\}^\phi = -x_i\phi(x_i)\delta_{ij} . \quad (4.21)$$

Since $\phi \neq 0$, this shows that the rank of $\{\cdot, \cdot\}^\phi$ is $2g$. $\qquad\square$

We now conclude with the main result of this section, namely that the Volterra map is a *discrete a.c.i. system*. By this we mean that, besides being Liouville integrable, the generic level sets defined by the invariants are affine parts of Abelian varieties (complex algebraic tori) and the restriction of the map to any of these Abelian varieties is a translation.

Theorem 4.10. *The Volterra map $\mathcal{V}_g$ is a discrete a.c.i. system on $(M_g, \{\cdot, \cdot\}^\phi)$.*

Proof. Liouville integrability was already shown in Proposition 4.9. Let $f \in \mathbb{C}[x]$ be a polynomial of degree $2g + 1$, without repeated roots and with $f(0) = 1$. Writing $f(x) = 1 + \sum_{i=0}^{2g+1} c_i x^i$, the common level set defined by $H_i = c_i$, $i = 1, 2, \ldots, 2g + 1$ is the fiber $\mu^{-1}(f)$ which was shown in Proposition 4.6 to be an affine part of the Jacobian of $\bar{\Gamma}_f$, and this is indeed an Abelian variety. It remains to be shown that the restriction of the Volterra map $\mathcal{V}_g$ to $\overline{\mu^{-1}(f)} \cong \mathrm{Jac}(\bar{\Gamma}_f)$ is a translation; more precisely we will show that it is

a translation over $[(0, -1) - \infty] = [\infty - (0, 1)]$. Let $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) \in \mu^{-1}(f)$ be a generic point (a regular triplet) so that $(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{R}}) := \mathcal{V}_g(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ also belongs to $\mu^{-1}(f)$. The degree g divisors on $\bar{\Gamma}_f$ corresponding to these two triplets are respectively denoted by $\mathcal{D} = \sum_{i=1}^{g}(x_i, y_i)$ and $\tilde{\mathcal{D}} = \sum_{i=1}^{g}(\tilde{x}_i, \tilde{y}_i)$ (so that the corresponding divisor classes in $\mathrm{Jac}(\bar{\Gamma}_f)$ are $[\mathcal{D} - g\infty]$ and $[\tilde{\mathcal{D}} - g\infty]$, respectively). Consider the rational function $\tilde{F}$ on $\bar{\Gamma}_f$ given by the action of $\mathcal{V}_g$ on (3.6), which in view of (4.3) and the definition (4.14) of f, can be written in a few different ways:

$$\tilde{F} := \frac{y + \tilde{\mathcal{P}}(x)}{\tilde{\mathcal{Q}}(x)} = \frac{\tilde{\mathcal{R}}(x)}{y - \tilde{\mathcal{P}}(x)} = \frac{-wx\mathcal{Q}(x)}{y - \mathcal{Q}(x) + \mathcal{P}(x)} \ . \tag{4.22}$$

It is clear from the last, respectively first, expression that $\tilde{F}$ has a simple zero at the each of the points (x_i, y_i) and $(0, -1)$, and a simple pole at each of the points $(\tilde{x}_i, \tilde{y}_i)$ and ∞. To verify the behaviour at ∞, one should introduce a local parameter z such that $x = 1/z^2$, $y = \sqrt{c_{2g+1}}z^{-(2g+1)}(1 + \mathcal{O}(z))$ there, which gives $\tilde{F} = \sqrt{c_{2g+1}}q_g^{-1}/z + \mathcal{O}(1)$. For the other zero or pole candidates in Γ_f, coming from places where the numerators or denominators in (4.22) vanish, one checks using one of the alternative formulae that $\tilde{F}$ is finite and non-zero at these points. Thus $\tilde{F}$ has precisely $g + 1$ zeros and $g + 1$ poles, in accord with the fact that the degree of the divisor of a rational function is zero. The upshot is that the divisor of zeros and poles of $\tilde{F}$ is given by

$$(\tilde{F}) = \sum_{i=1}^{g}(x_i, y_i) + (0, -1) - \sum_{i=1}^{g}(\tilde{x}_i, \tilde{y}_i) - \infty = \mathcal{D} - \tilde{\mathcal{D}} + (0, -1) - \infty \ ,$$

which leads to the linear equivalence $\mathcal{D} + (0, -1) \sim \tilde{\mathcal{D}} + \infty$, and hence to

$$[\tilde{\mathcal{D}} - g\infty] = [\mathcal{D} - g\infty] + [(0, -1) - \infty] \ , \tag{4.23}$$

as was to be shown. □

According to Proposition 4.3, when $\phi = \phi_g x^g + \phi_{g+1} x^{g+1}$ we can restrict the Volterra map and its Poisson structure to $M_g^c = \cap_{i=1}^{g}(H_i = c_i)$, and so by the above theorem the Volterra map is a discrete a.c.i. system on $(M_g^c, \{\cdot, \cdot\}^{\phi})$. In particular, the recursion relations obtained by fixing the invariants $H_1, \ldots, H_g$ to generic values c_i (and possibly also fixing the other Casimir C_{ϕ} to some generic value) are discrete a.c.i. systems.

Example 4.11. In the genus 1 case, it follows that (3.34), (3.37) and (3.39), equipped respectively with the Poisson structures $\{\cdot, \cdot\}^{\phi}$, $\{\cdot, \cdot\}^x$ and $\{\cdot, \cdot\}^{x^2}$, are discrete a.c.i. systems. The same holds for (3.45), which is a discrete a.c.i. system with respect to the quadratic Poisson structure (4.12).

Example 4.12. In the genus 2 case, we have that for generic a, b, ν the map (P.iv) is a discrete a.c.i. system.

5 Continuous flows and the infinite Volterra and Toda lattices

The discrete integrable systems that we have discussed so far are naturally associated with continuous systems which are equally integrable. More precisely, Liouville integrability of the Hamiltonian systems associated with the Volterra maps comes for free, and with some extra work we show that these continuous systems are also algebraically integrable. We further show that in the genus g case any solution $w_i(t)$ of one of the integrable Hamiltonian vector fields extends, under the action of the Volterra map, to a sequence $(w_n(t))_{n \in \mathbb{Z}}$ that is a solution to the infinite Volterra lattice; notice that in particular, as discussed in the introduction, this applies to the map (P.iv). We also discuss the relation between the infinite Toda and Volterra lattices, which explains in part how some of the results in this paper are related to the results in [28], and what motivated us to introduce S-fractions and the corresponding Mumford-like systems to study the map (P.iv) and its higher genus analogues.

5.1 Liouville and algebraic integrability

Recall that on M_g we have a family of compatible Poisson brackets $\{\cdot,\cdot\}^\phi$ of rank $2g$, as well as a family of polynomial functions $H_1,\ldots,H_{2g+1}$, where H_i is the coefficient in x^i of $\mathcal{P}^2(x) + \mathcal{Q}(x)\mathcal{R}(x)$; said differently, $H_1,\ldots,H_{2g+1}$ are the components of the momentum map $\mu : M_g \to \mathbb{C}[x]$. For the sake of clarity, and since the choice of Poisson structure is not important for what follows, we will only consider $\phi = x^{g+1}$ here, and henceforth write $\{\cdot,\cdot\}$ for $\{\cdot,\cdot\}^\phi$. With this choice of ϕ, (4.6) – (4.8) become

$$\left\{\mathcal{P}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\right\} = y^{g+1} \frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{x - y} ,$$

$$\left\{\mathcal{Q}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\right\} = 2y^{g+1} \frac{x\mathcal{P}(y)\mathcal{Q}(x) - y\mathcal{P}(x)\mathcal{Q}(y)}{x - y} - y^{g+1}\mathcal{Q}(x)\mathcal{Q}(y) ,$$

$$\left\{\mathcal{R}(x), \mathcal{P}(y)^2 + \mathcal{Q}(y)\mathcal{R}(y)\right\} = 2xy^{g+1} \frac{\mathcal{P}(x)\mathcal{R}(y) - \mathcal{P}(y)\mathcal{R}(x)}{x - y} + y^{g+1}\mathcal{Q}(y)\mathcal{R}(x) .$$

As we have seen, $H_1,\ldots,H_g$ are Casimir functions of the Poisson bracket, as well as $H_{g+1} = C_\phi$ (see (4.9)). The vector fields $\frac{1}{2}X_{H_{g+i+1}}$ are denoted by X_i; we will mainly be interested in $X_1 = \frac{1}{2}X_{H_{g+2}}$, which we can compute by dividing the above equations by $2y^{g+2}$ and taking the limit for $y \to 0$, so that

$$\dot{\mathcal{P}}(x) = \lim_{y \to 0} \frac{y\mathcal{Q}(y)\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}(y)}{2y(x - y)} = \frac{2\mathcal{R}(x) - x\mathcal{Q}(x)\mathcal{R}'(0)}{2x} = \frac{\mathcal{R}(x)}{x} - \frac{r_1}{2}\mathcal{Q}(x) , \qquad (5.1)$$

where the dot denotes the derivative $\frac{\mathrm{d}}{\mathrm{d}t}$, and similarly

$$\dot{\mathcal{Q}}(x) = \frac{2p_1 - q_1}{2}\mathcal{Q}(x) - \frac{2\mathcal{P}(x) - \mathcal{Q}(x)}{x} ,$$

$$\dot{\mathcal{R}}(x) = r_1\mathcal{P}(x) + \frac{q_1 - 2p_1}{2}\mathcal{R}(x) - \frac{\mathcal{R}(x)}{x} . \qquad (5.2)$$

Notice that the vector field X_1 is (non-homogeneous) quadratic. From Proposition 4.2, the functions H_j are in involution with one another with respect to $\{\cdot,\cdot\}$, which means that the vector fields X_i all commute.

Note that the Liouville integrability of this continuous system is incorporated into the Liouville integrability of the discrete system, so the following statement is an automatic consequence of Proposition 4.9.

Proposition 5.1. *The Hamiltonian system $(M_g, \{\cdot,\cdot\}, \mu)$ is a Liouville integrable system.*

We now turn to the algebraic integrability of the Mumford-like system, which is slightly more involved in the continuous case. Recall (for example from [1, Ch. 6]) that $(M_g, \{\cdot,\cdot\}, \mu)$ being an a.c.i. system means that

(1) $(M_g, \{\cdot,\cdot\}, \mu)$ is a (complex) Liouville integrable system;

(2) The generic fiber of μ is (isomorphic to) an affine part of an Abelian variety;

(3) The integrable vector fields are holomorphic (hence constant) on these Abelian varieties.

Items (1) and (2) have been shown already, in Propositions 5.1 and 4.6, respectively, so it only remains to address item (3).

Proposition 5.2. *The Hamiltonian system $(M_g, \{\cdot,\cdot\}, \mu)$ is an algebraic completely integrable system (a.c.i. system).*

Proof. We show (3) for one of the integrable vector fields; then it also holds for the other integrable vector fields, since the latter are holomorphic on the fiber and commute with a constant vector field.

The vector field which we consider is the Hamiltonian vector field X_1, given by (5.1) and (5.2). Let $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$ be a generic point of M_g and consider for small $|t|$ the integral curve $t \mapsto (\mathcal{P}_t, \mathcal{Q}_t, \mathcal{R}_t)$ of X_1,

starting at $(\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0)$. Let $\mathcal{D}_t = \sum_{i=1}^{g}(x_i(t), y_i(t)) - g\infty$ denote the associated divisor on the algebraic curve Γ_f, defined by it; recall that Γ_f is given by $y^2 = f(x)$ where $f = \mathcal{P}_0^2 + \mathcal{Q}_0\mathcal{R}_0 = \mathcal{P}_t^2 + \mathcal{Q}_t\mathcal{R}_t$. Since the $x_i(t)$ are the roots of $\mathcal{Q}_t(x)$, upon substituting $x = x_i(t)$ in the equation (5.2) for $\dot{\mathcal{Q}}(x)$ we get

$$\dot{\mathcal{Q}}(x_i(t)) = -2\frac{\mathcal{P}(x_i(t))}{x_i(t)} = -2\frac{y_i(t)}{x_i(t)} .$$

However, we can also compute $\dot{\mathcal{Q}}(x_i(t))$ directly from the explicit formula (4.15) for $\mathcal{Q}(x)$, to wit

$$\dot{\mathcal{Q}}(x_i(t)) = \frac{2\dot{x}_i(t)}{x_i(t)} \prod_{j \neq i}\left(1 - \frac{x_i(t)}{x_j(t)}\right) .$$

Comparing these two expressions gives

$$y_i(t) = -\dot{x}_i(t) \prod_{j \neq i}\left(1 - \frac{x_i(t)}{x_j(t)}\right) . \tag{5.3}$$

It follows that for $k = 0, \ldots, g-1$,

$$\sum_{i=1}^{g} \frac{x_i^k(t)\mathrm{d}x_i(t)}{y_i(t)} = -\sum_{i=1}^{g} x_i^k(t) \prod_{j \neq i} \frac{x_j(t)}{x_j(t) - x_i(t)}\mathrm{d}t = -\delta_{k,0}\,\mathrm{d}t . \tag{5.4}$$

Above we have used the following identity which is well-known in the theory of symmetric functions:

$$\sum_{i=1}^{g} x_i^k \prod_{j \neq i} \frac{x_j}{x_j - x_i} = \delta_{k,0} , \qquad k = 0, \ldots, g-1; \tag{5.5}$$

the proof of the latter follows easily from the fact that any lowest degree antisymmetric polynomial in g variables is, up to a factor, the Vandermonde determinant. Integrating (5.4) from 0 to t gives

$$\int_{\mathcal{D}_0}^{\mathcal{D}_t} \frac{x^k\mathrm{d}x}{y} = -t\,\delta_{k,0} . \tag{5.6}$$

The left-hand side of (5.6) contains the differentials $x^k\mathrm{d}x/y$ for $k = 0, \ldots, g-1$, which form a basis for the holomorphic differentials on $\bar{\Gamma}_f$. Thus the left-hand side of (5.6) is the image of the divisor $\mathcal{D}_t - \mathcal{D}_0$ under the Abel map, which is (by Abel's Theorem) an isomorphism between the algebraic Jacobian of $\bar{\Gamma}_f$, consisting of degree zero divisor classes on $\bar{\Gamma}_f$, and the analytic Jacobian of $\bar{\Gamma}_f$, which is a complex torus, that is

$$\mathrm{Jac}(\bar{\Gamma}_f) \cong H^0\left(\Omega_{\bar{\Gamma}_f}^1\right)^* / H_1(\bar{\Gamma}_f).$$

Formula (5.6) then says that the integral curve of X_1 is a straight line in this complex torus, as was to be shown. $\qquad\square$

It follows that for, generic initial conditions, the solutions to X_1 are meromorphic functions in t.

5.2 Genus g solutions to the infinite Volterra and Toda lattices

The infinite Volterra lattice is given by the set of equations

$$\dot{w}_n = w_n(w_{n+1} - w_{n-1}) , \qquad n \in \mathbb{Z} . \tag{5.7}$$

It was first considered by Kac and van Moerbeke [31], who also studied the N-periodic case ($w_{N+n} = w_n$ for all n). We now show that the Volterra map allows us to define, for any g, infinite sequences of meromorphic

functions $(w_n(t))_{n \in \mathbb{Z}}$ which satisfy (5.7). Since these sequences are defined from solutions of the genus g Mumford-like system, and hence can be written in terms of genus g theta functions, we will refer to these solutions to the Volterra lattice as *genus g solutions*.

Let $g > 0$ be fixed and consider the vector field on $\mathbb{C}_w^{3g+1}$, corresponding to the vector field $\frac{1}{2}X_1$ on the genus g Mumford-like system, via the birational transformation constructed in Section 3.5. For the sake of brevity, let us call this the w-system (in genus g). By algebraic integrability, the vector field $\frac{1}{2}X_1$ has globally defined meromorphic solutions $w_1(t), \ldots, w_{3g+1}(t)$, corresponding to generic initial conditions. Using the recursion relation induced by the action of the Volterra map on $\mathbb{C}_w^{3g+1}$, we get also globally defined meromorphic functions $w_n(t)$ for all $n > 3g + 1$ and all $n \leqslant 0$. Algebraic integrability further implies that the recursion and the flow of the vector field must commute, as they both correspond to translations on the fibers of the momentum map, which are affine parts of g-dimensional tori. (The fact that the map and the flow commute is already a consequence of Liouville integrability.) It follows that all formulae only involving the variables w_n remain valid when all indices are shifted by the same integer. In the proof of the theorem which follows we will make extensive use of the birational transformation between the w-system and the Mumford-like system. The triplet corresponding to a meromorphic solution $(w_1(t), \ldots, w_{3g+1}(t))$ of the w-system in genus g will be denoted $(\mathcal{P}_0(x; t), \mathcal{Q}_0(x; t), \mathcal{R}_0(x; t))$, the index 0 being added because we will also use the triplets $(\mathcal{P}_n(x; t), \mathcal{Q}_n(x; t), \mathcal{R}_n(x; t))$, obtained from it by applying the Volterra map or its inverse several times. Again, all formulae involving only the polynomials $\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n$, $n \in \mathbb{Z}$, remain valid when all indices are shifted by the same integer, and for any $n \in \mathbb{Z}$, $(\mathcal{P}_n(x; t), \mathcal{Q}_n(x; t), \mathcal{R}_n(x; t))$ corresponds to $(w_{n+1}(t), \ldots, w_{3g+n+1}(t))$ under the birational map.

Theorem 5.3. *The sequence of meromorphic functions $(w_n(t))_{n \in \mathbb{Z}}$ is a solution to the infinite Volterra lattice* (5.7).

Proof. We first recall the recursion relations (3.9) for the triplets $(\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n)$, which we evaluate at any meromorphic solution to $X_1 = \frac{1}{2}X_{H_{g+2}}$:

$$\mathcal{P}_{n+1}(x; t) = \mathcal{Q}_n(x; t) - \mathcal{P}_n(x; t) \;, \quad \mathcal{Q}_{n+1}(x; t) = \frac{2\mathcal{P}_n(x; t) - \mathcal{Q}_n(x; t) + \mathcal{R}_n(x; t)}{-w_{n+1}(t)x} \;, \tag{5.8}$$

$$\mathcal{R}_{n+1}(x; t) = -w_{n+1}(t)x\mathcal{Q}_n(x; t) \;. \tag{5.9}$$

From (5.9), since $H_1 = 2(p_1 + r_1)$ is a first integral, we have

$$w_n(t) = -\frac{1}{2}r_{n,1}(t) = \frac{1}{2}p_{n,1}(t) - \frac{c_1}{4} \;, \quad \text{and} \quad r_{n,2}(t) = -w_n(t)q_{n-1,1}(t) \;, \tag{5.10}$$

where c_1 is a constant. It follows that

$$\dot{w}_n(t) \overset{(5.10)}{=} \frac{1}{2}\dot{p}_{n,1}(t) \overset{(5.1)}{=} \frac{1}{2}r_{n,2}(t) - \frac{1}{4}r_{n,1}(t)q_{n,1}(t) \overset{(5.10)}{=} \frac{w_n(t)}{2}(q_{n,1}(t) - q_{n-1,1}(t))$$

$$\overset{(5.8)}{=} \frac{w_n(t)}{2}(p_{n+1,1}(t) - p_{n-1,1}(t)) \overset{(5.10)}{=} w_n(t)(w_{n+1}(t) - w_{n-1}(t)) \;,$$

as was to be shown. $\qquad \square$

Remark 5.4. It is fairly straightforward to modify the proof of Proposition 5.2, and the preceding result, to all of the Hamiltonian vector fields X_i, associated with times t_i, $1 \leqslant i \leqslant g$, which correspond to the first g flows in the Volterra lattice hierarchy. This replaces t by t_i and modifies the Kronecker delta on the right-hand side of (5.6) to $\delta_{k,i-1}$, hence producing solutions that are meromorphic in $t = t_1, t_2, \ldots, t_g$.

We now apply a standard Miura-like formula, to show how a genus g solution of the Volterra lattice, given by an infinite sequence of meromorphic functions $w_n(t)$, also leads to a corresponding solution to the infinite Toda lattice, given by

$$\frac{da_n}{dt} = a_n(b_{n-1} - b_n) \;, \qquad \frac{db_n}{dt} = a_n - a_{n+1} \;. \tag{5.11}$$

(These are almost the same as the Flaschka variables for the Toda lattice, except that traditionally $\sqrt{a_n}$ is used in place of a_n; and similarly, the quantities $\sqrt{w_n}$ are used in [37].)

Corollary 5.5. *Let* $w_n(t)$, $n \in \mathbb{Z}$ *be a genus g meromorphic solution to the infinite Volterra lattice* (5.7). *Upon setting, for $j \in \mathbb{Z}$,*

$$a_{j+1} := w_{2j-1} w_{2j} , \qquad b_{j+1} := -w_{2j} - w_{2j+1} , \tag{5.12}$$

the sequence of meromorphic functions $a_j(t), b_j(t)$ *is a solution to the infinite Toda lattice, while another sequence of meromorphic solutions to* (5.11) *is given for $j \in \mathbb{Z}$ by*

$$a_{j+1}^* := w_{2j} w_{2j+1} , \qquad b_{j+1}^* := -w_{2j+1} - w_{2j+2} . \tag{5.13}$$

Proof. Differentiating (5.12) and using (5.7) one gets immediately (5.11), and similarly for (5.13). $\square$

Example 5.6. Theorem 5.3 and Corollary 5.5 imply that we can obtain elliptic (genus 1) solutions to the infinite Volterra and Toda lattices, by starting from a generic solution to (3.34). On a fixed orbit of the latter, any such solution can be identifed with an orbit of the QRT map (3.45) associated with Somos-5. Hence this means that the analytic results of [24] can be applied, to write the tau function explicitly as

$$\tau_n = A_{\pm} B^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}},$$

where A_+, A_-, B are non-zero constants (with $A_{\pm}$ chosen according to the parity of n), and $\sigma(z) = \sigma(z; g_2, g_3)$ denotes the Weierstrass sigma function associated with an elliptic curve $y^2 = 4x^3 - g_2 x - g_3$, isomorphic to (3.41). The parameters z, g_2, g_3 all depend on c_1, c_2, c_3, while z_0 also depends on the initial point on the orbit. Then we can write the solution of the map explicitly in terms of the Weierstrass zeta function, as

$$w_n = \frac{\sigma(z_0 + nz)\sigma(z_0 + (n+3)z)}{\sigma(z)^4 \sigma(z_0 + (n+1)z)\sigma(z_0 + (n+2)z)} = \hat{c}\Big(\zeta(z_0 + (n+2)z) - \zeta(z_0 + (n+1)z) + C\Big), \tag{5.14}$$

where $\hat{c} = \sigma(2z)/\sigma(z)^4$, $C = \zeta(z) - \zeta(2z)$. Now extending this by the flow of the vector field X_1, with parameter t, we find that only z_0 changes, being replaced by $z_0 + \hat{c}t$ (giving a linear flow on the Jacobian of the elliptic curve). Hence we arrive at the genus 1 solution of the Volterra lattice, given for $n \in \mathbb{Z}$ by

$$w_n(t) = \hat{c}\Big(\zeta(z_0 + \hat{c}t + (n+2)z) - \zeta(z_0 + \hat{c}t + (n+1)z) + C\Big)$$

(equivalent to the travelling waves found in [55]), and from (5.12) we get a corresponding genus 1 solution of the Toda lattice, that is

$$a_n(t) = \hat{c}^4 \Big(\wp(2z) - \wp(z_0 + \hat{c}t + (2n-1)z)\Big) , \tag{5.15}$$

$$b_n(t) = \hat{c}\Big(\zeta(z_0 + \hat{c}t + (2n-1)z) - \zeta(z_0 + \hat{c}t + (2n+1)z) - 2C\Big) ,$$

written in terms of the Weierstrass $\wp$ function, with the constants $\hat{c}$, C as above.

Example 5.7. Similarly, we can produce genus 2 solutions to the infinite Volterra and Toda lattices, by starting from a generic solution to the map (P.iv), extended to meromorphic functions $w_n(t)$ by the flow of the vector field X_1.

In [37], the transformation (5.12) was used to connect the finite Volterra and Toda lattices by Moser, who attributed it to Hénon. The same transformation has further been applied to connect real-valued solutions of the infinite lattices, subject to suitable (smoothness/boundedness) conditions [19]. Moser also employed finite continued fractions in [37]. However, it turns out that the map (5.12) has a much earlier origin in the classical theory of continued fractions, where it arises from the method of contraction for S-fractions (see J.3 in [48], and [46]), a fact that has perhaps been overlooked in the integrable systems literature. In the next subsection, we show how the Volterra maps, as presented in this paper, are related to the integrable maps recently constructed by one of us [28]; the key is to apply contraction to the S-fraction (3.15), which produces a J-fraction, and thereby yields the Miura-type formula (5.12).

5.3   Contraction of continued fractions

We take a fixed set of coefficients c_i, which are arbitrary except that, as usual, we assume that the polynomial

$$f(x) = 1 + \sum_{i=1}^{2g+1} c_i x^i \tag{5.16}$$

is square-free with $c_{2g+1} \neq 0$, so that the hyperelliptic curve $\Gamma_f : y^2 = f(x)$ is smooth and has genus g. In order to simplify the presentation below, initially we make the further assumption that $c_1 = 0$. Then setting

$$X = \frac{1}{x}, \qquad Y = \frac{y}{x^{g+1}} \tag{5.17}$$

establishes a birational isomorphism between Γ_f and an algebraic curve $\mathcal{C}$ which (by completing the square) can be written in the form

$$\mathcal{C} : \ Y^2 = \hat{\mathrm{F}}(X), \qquad \hat{\mathrm{F}}(X) = A(X)^2 + 4R(X), \tag{5.18}$$

where $A(X)$ is a monic polynomial in X of degree $g + 1$ with no term of degree g (so that the right-hand side of (5.18) has no degree $2g + 1$ term), and R is a polynomial of degree at most g in X, not identically zero but otherwise arbitrary; such curves are exactly the ones which were considered in [28].

Let $(\mathcal{P}, \mathcal{Q}, \mathcal{R}) = (\mathcal{P}_0, \mathcal{Q}_0, \mathcal{R}_0) \in M_g$ be a generic triplet satisfying $\mathcal{P}^2 + \mathcal{Q}\mathcal{R} = f$, in the sense discussed above (3.15). As we have seen in (3.15), the associated rational function on Γ_f, denoted F_0, admits the following expansion as an S-fraction:

$$F_0 = \frac{y + \mathcal{P}_0(x)}{\mathcal{Q}_0(x)} = 1 - \cfrac{w_1 x}{1 - \cfrac{w_2 x}{1 - \cfrac{w_3 x}{1 - \cdots}}}. \tag{5.19}$$

Now, upon multiplying (5.19) by $x^{-1} = X$, rewriting the S-fraction in terms of the new spectral parameter X, and then applying contraction, we find

$$x^{-1} F_0 = X - \cfrac{w_1}{1 - \cfrac{w_2}{X - \cfrac{w_3}{1 - \cfrac{w_4}{X - \cdots}}}} = X + b_1 - \cfrac{a_2}{X + b_2 - \cfrac{a_3}{X + b_3 - \cfrac{a_4}{X + b_4 - \cdots}}}, \tag{5.20}$$

where the J-fraction on the right above is the *contraction* of the S-fraction. More precisely, the second equality in (5.20) is an identity of continued fractions, obtained by combining successive pairs of adjacent lines in the S-fraction into a single sequence of lines in the J-fraction, with coefficients a_j, b_j related to w_j by

$$b_1 = -w_1, \qquad a_{j+1} = w_{2j-1} w_{2j}, \qquad b_{j+1} = -w_{2j} - w_{2j+1}, \qquad \text{for} \quad j \geqslant 1. \tag{5.21}$$

In particular, we observe that this reproduces the transformation (5.12) between the Volterra and Toda lattices, but for indices $j \geqslant 1$ only.

We now briefly recall the construction of integrable maps associated with J-fractions, as presented in [28]. The starting point is a rational function Y_0 on an even hyperelliptic curve $\mathcal{C}$ of the form (5.18), whose completion $\bar{\mathcal{C}}$ includes two points ∞_1, ∞_2 at infinity. This function has $g + 1$ simple poles and $g + 1$ simple zeros, with one pole being at the point ∞_1 (where $Y \sim X^{g+1} \sim A(X)$ as $X \to \infty$), and one zero being at ∞_2 (where $Y \sim -X^{g+1} \sim -A(X)$), taking the form

$$Y_0 = \frac{Y + P_0(X)}{Q_0(X)}, \tag{5.22}$$

for polynomials P_0, of degree $g+1$ with no term at $\mathcal{O}(X^g)$, and Q_0, of degree g; and there exists another polynomial Q_{-1}, of degree g, satisfying $Y^2 = P_0^2 + Q_0 Q_{-1} = \hat{\mathrm{F}}$. The expansion of Y_0 around the point ∞_1, with X^{-1} as a local parameter, can be considered as an element of $\mathbb{C}((X^{-1}))$, and it was shown by van der Poorten (see [40, 43]) that this power series admits a J-fraction expansion of the form

$$Y_0 = \alpha_0 + \cfrac{1}{Y_1} = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{Y_2}} = \cdots = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots}}} \tag{5.23}$$

with $\alpha_n := \lfloor Y_n \rfloor$, the polynomial part of Y_n. Furthermore, for a generic choice of such P_0, Q_0 in (5.22), the polynomial parts α_n are of degree 1 in X for any n, and the recursion $Y_n = \alpha_n + \frac{1}{Y_{n+1}}$ leads to a sequence of polynomials P_n, Q_n satisfying the same degree constraints as above, such that

$$Y_n = \frac{Y + P_n(X)}{Q_n(X)} = \frac{Q_{n-1}(X)}{Y - P_n(X)} \implies Y^2 = \hat{\mathrm{F}}(X) = P_n(X)^2 + Q_n(X) Q_{n-1}(X), \tag{5.24}$$

where the above relations extend to all $n \in \mathbb{Z}$, not just $n \geqslant 0$, by reversing (5.23) to find Y_{-1} from Y_0, etc.

The situation for the J-fraction expansion (5.23) is very similar to that for the expansion (5.19), and allows the construction of a birational dynamical system that is defined by a recursion for the polynomials P_n, Q_n, analogous to the derivation of the Volterra map from the S-fraction in Section 3. Here we refer to the dynamical system (in dimension $3g+1$) defined by (5.23) as the *J-fraction map* in genus g, denoted $\mathcal{J}_g$. In order to explain the very close connection between $\mathcal{V}_g$ and $\mathcal{J}_g$, and prove Theorem 5.10, we further summarize some features of the latter, while referring the reader to [28] for a complete description.

To describe the dynamics defined by P_n, Q_n, new variables u_n, d_n and v_n are introduced from

$$P_n(X) = A(X) + 2d_n X^{g-1} + \mathcal{O}(X^{g-2}), \quad Q_n(X) = u_n\big(X^g - v_n X^{g-1} + \mathcal{O}(X^{g-2})\big), \tag{5.25}$$

so that from the terms of $\mathcal{O}(X^{2g})$ in the equation for $\hat{\mathrm{F}}$ on the right-hand side of (5.24), the relation

$$u_n u_{n-1} = -4d_n \neq 0 \tag{5.26}$$

must hold, while calculating the (degree 1) polynomial parts in each line of (5.23) shows that, for any n, we have $\alpha_n = (X + b_n)/u_n$. Thus, upon substituting for α_n and rescaling each line of the continued fraction using (5.26), we may rewrite the J-fraction (5.23) more explicitly as

$$Y_0 = \alpha_0 + \cfrac{1}{Y_1} = \cfrac{2(X + v_0)}{u_0} + \cfrac{1}{\cfrac{2(X+v_1)}{u_1} + \cfrac{1}{\cfrac{2(X+v_2)}{u_2} - \cdots}},$$

where, by setting $\hat{s}_0 = \frac{u_1}{2}$, we have

$$Y_1 = \frac{1}{\hat{s}_0}\left(X + v_1 - \cfrac{d_2}{X + v_2 - \cfrac{d_3}{X + v_3 - \cdots}}\right). \tag{5.27}$$

Then the J-fraction map $\mathcal{J}_g$ is a dynamical system on an affine phase space $\hat{M}_g^{\hat{c}_1=0}$ of dimension $3g+1$, which fibers over the space of curves $\mathcal{C}$ of the form (5.18), with each (generic) fiber being an affine part of the corresponding Jacobian variety $\mathrm{Jac}(\bar{\mathcal{C}})$; and on each such fiber, the map corresponds to a translation by the class of the divisor $\infty_2 - \infty_1$. It is defined recursively by (5.24), in terms of the coefficients of the polynomials P_n, Q_n, except that the prefactors u_n in front of each Q_n, as in (5.25), are completely decoupled

from the dynamics. Indeed, the constant $\hat{s}_0$ in (5.27) is arbitrary: it determines the first coefficient in the series expansion of the moment generating function $1/Y_1 = \sum_{j \geqslant 0} \hat{s}_j X^{-j-1}$, whose coefficients allow the solutions of $\mathcal{J}_g$ to be written in terms of tau functions given by Hankel determinants; but $\hat{s}_0$ can be removed by a gauge transformation on the tau functions. Moreover, once $\hat{s}_0$ is fixed, then u_1 and all the other prefactors u_n are determined from $\hat{s}_0$ and d_n, due to (5.26); and the phase space $\hat{M}_g^{\hat{c}_1=0}$ (which is an affine space of Lax matrices) does not include the parameter $\hat{s}_0$. After decoupling from u_n, the map $\mathcal{J}_g$ can be written equivalently as a recursion for the remaining coefficients in P_n, Q_n, or as coupled recurrences for the quantities d_n, v_n. (See (5.33) and (5.35) in the examples below for the cases $g = 1$ and $g = 2$, respectively.)

We would now like to identify (5.27) with the J-fraction appearing on the right in (5.20), but there are two problems: firstly, the relation (5.21) is valid only for $j \geqslant 1$, and gives a different formula for b_1 when $j = 0$; and secondly, we initially made the assumption that $c_1 = 0$, which does not hold in general. To relax the latter assumption, we must shift the spectral parameter X, as in (5.30), make a compensating shift in b_j, and allow a linear relation between $x^{-1}F_0$ and Y_1, yielding a modification of (5.12), valid for all $j \in \mathbb{Z}$.

Proposition 5.8. *The odd, genus g spectral curve $\Gamma_f : y^2 = f(x)$ with $f(x) = 1 + \sum_{i=1}^{2g+1} c_i x^i$, associated with a generic orbit of the Volterra map $\mathcal{V}_g$, is isomorphic to $\mathcal{C}$, the even spectral curve (5.18) for a corresponding orbit of the J-fraction map $\mathcal{J}_g$, via the birational equivalence $X = x^{-1} + c_1/(2g+2)$, $Y = y/x^{g+1}$. Under this isomorphism of curves, the function F_0 on Γ_f and the function Y_1 on $\mathcal{C}$ are related by*

$$x^{-1}F_0 - w_0 = \hat{s}_0 Y_1, \tag{5.28}$$

and the quantities d_n, v_n satisfying the map $\mathcal{J}_g$ are given in terms of the solution of $\mathcal{V}_g$ by

$$d_{j+1} = w_{2j-1}w_{2j}, \qquad v_{j+1} = -w_{2j} - w_{2j+1} - \frac{c_1}{2(g+1)}, \qquad \text{for} \quad j \in \mathbb{Z}. \tag{5.29}$$

Hence each iteration on the orbit of $\mathcal{J}_g$ corresponds to two iterations on the corresponding orbit of $\mathcal{V}_g$.

Proof. The shift in X in the birational transformation, as in (5.30), ensures that the equation $Y^2 = \hat{F}(X) = A(X)^2 + 4R(X)$ for $\mathcal{C}$ has no term at $\mathcal{O}(X^{2g+1})$ in $\hat{F}(X)$, so that $A(X) = X^{g+1} + \mathcal{O}(X^{g-1})$, as required for a spectral curve of the J-fraction map. Also, from the explicit form of the function F_0 in (5.19), we may rewrite the left-hand side of (5.28) in terms of Y and $\tilde{X} = X - c_1/(2g+2)$, as

$$\frac{\tilde{X}\left(\tilde{X}^{-(g+1)}Y + \mathcal{P}_0(\tilde{X}^{-1})\right) - w_0 \mathcal{Q}_0(\tilde{X}^{-1})}{\mathcal{Q}_0(\tilde{X}^{-1})} = \frac{Y + P_1(X)}{Q_1(X)/\hat{s}_0}$$

where we calculate $P_1(X) = \tilde{X}^{g+1}\mathcal{P}_0(\tilde{X}^{-1}) - w_0\tilde{X}^g\mathcal{Q}_0(\tilde{X}^{-1}) = \tilde{X}^{g+1} + (p_{0,1} - 2w_0)\tilde{X}^g + \mathcal{O}(\tilde{X}^{g-1})$, and then in view of (3.25) we see that $P_1(X) = X^{g+1} + \mathcal{O}(X^{g-1})$, while $Q_1(X) = \hat{s}_0 \tilde{X}^g \mathcal{Q}_0(\tilde{X}^{-1}) = \frac{u_1}{2}\left(2\tilde{X}^g + \mathcal{O}(\tilde{X}^{g-1})\right) = u_1\left(X^g + \mathcal{O}(X^{g-1})\right)$, so both of P_1 and Q_1 are polynomials in X of the required form for the J-fraction map. Now from the S-fraction in (5.28), we find that combining contraction with the shift of spectral parameter modifies (5.20), so that, in terms of fractions in X, $x^{-1}F_0 - w_0$ is equal to

$$X - \frac{c_1}{2g+2} - w_0 - \cfrac{w_1}{1 - \cfrac{w_2}{X - \frac{c_1}{2g+2} - \cfrac{w_3}{1 - \cfrac{w_4}{X - \frac{c_1}{2g+2} - \cdots}}}} = X + v_1 - \cfrac{d_2}{X + v_2 - \cfrac{d_3}{X + v_3 - \cfrac{d_4}{X + v_4 - \cdots}}},$$

where we have inserted the formula for Y_1 from (5.27), and cancelled the arbitrary constant $\hat{s}_0 = \frac{u_1}{2}$. Comparing the first line of the above fractions on each side, we see that $v_1 = -w_0 - w_1 - c_1/(2g+2)$, which is the correct form of the relation for v_1 in (5.29) when $j = 0$, and contraction of the subsequent lines on the left give these expressions for d_{j+1}, v_{j+1} for all $j \geqslant 1$. One can also shift the fraction on the left down by

two lines, to get a relation between F_2 and Y_2, and the fraction can be inverted to find a relation between F_{-2} and Y_0; so, by continuing down or up in this way, we find that $x^{-1}F_{2j} - w_{2j} = \frac{u_{j+1}}{2}Y_{j+1}$ holds for all $j \in \mathbb{Z}$, extending (5.29) to negative j as well. In all expressions, a shift of indices $j \to j+1$ gives a single iteration of $\mathcal{J}_g$, but all indices of the Volterra variables increase by 2, giving two iterations of $\mathcal{V}_g$.   □

Corollary 5.9. *Under the action of the Hamiltonian vector field* X_1, *each generic solution of* $\mathcal{V}_g$ *produces a genus g solution of the Toda lattice equation* (1.11) *which also satisfies the map* $\mathcal{J}_g$, *via the transformation* (5.29).

Proof. This follows immediately from the preceding result, by applying Theorem 5.3, and noting that the result of Corollary 5.5 still stands if we set $a_n = d_n$, $b_n = v_n + c_1/(2g+2)$ for all n.   □

The main results of this subsection are collected in the following statement.

Theorem 5.10. *By contraction of the S-fraction* (5.19) *for the associated rational function* F_0, *a generic orbit of the Volterra map* $\mathcal{V}_g$ *corresponding to a fixed odd spectral curve* $\Gamma_f : y^2 = f(x)$ *of genus g, for square-free* $f(x)$ *as in* (5.16), *is transformed to an orbit of the integrable map* $\mathcal{J}_g$ *constructed in [28] from the J-fraction* (1.10), *with the even spectral curve* $\mathcal{C}$ *given by* (5.18) *with* $\hat{\mathrm{F}}(X) = X^{2g+2} + \sum_{j=2}^{2g+2} \hat{c}_j X^{2g+2-j}$. *For coefficients* $\hat{c}_j$ *given suitably in terms of* c_i, *there is a birational equivalence between* Γ_f *and* $\mathcal{C}$, *given by*

$$X = x^{-1} + \frac{c_1}{2(g+1)}, \qquad Y = \frac{y}{x^{g+1}}. \tag{5.30}$$

Moreover, the translation on $\mathrm{Jac}(\bar{\mathcal{C}})$ *associated with a single iteration of the J-fraction map corresponds to twice the translation on* $\mathrm{Jac}(\bar{\Gamma}_f)$ *associated with each iteration of* $\mathcal{V}_g$. *In fact, each generic orbit of* $\mathcal{V}_g$ *is related to two different orbits of* $\mathcal{J}_g$ *in this way.*

Proof. The main statements in the theorem were already proved in Proposition 5.8. For the relation between shifts on complex tori, note that in M_g, we have that $\mu^{-1}(f)$, the fiber over a generic curve Γ_f, is an affine part of $\mathrm{Jac}(\bar{\Gamma}_f)$, while in $\hat{M}_g^{\hat{c}_1=0}$ the fiber over $\mathcal{C}$ is an affine part of $\mathrm{Jac}(\bar{\mathcal{C}})$; but then the isomorphism (5.30) between these two spectral curves means that $\mathrm{Jac}(\bar{\Gamma}_f) \cong \mathrm{Jac}(\bar{\mathcal{C}})$. It was shown in [28] that each iteration of the J-fraction map $\mathcal{J}_g$ corresponds to a translation by the class of the divisor $\infty_2 - \infty_1$ on $\mathrm{Jac}\,\bar{\mathcal{C}}$, where $\infty_{1,2}$ are the two points at infinity on $\bar{\mathcal{C}}$, and these are equivalent to the points $(0, \pm 1)$ on (3.5). So in terms of $\mathrm{Jac}(\bar{\Gamma}_f)$, this is a translation by the class of the divisor $(0, -1) - (0, 1) = 2(0, -1) - (0, -1) - (0, 1) \sim 2\big((0, -1) - \infty\big)$, that is by $2[(0, -1) - \infty]$, twice the shift corresponding to the Volterra map $\mathcal{V}_g$ (as found in the proof of Theorem 4.10). Finally, notice that in Corollary 5.5 there is the second, alternative formula (5.13), with the indices on all w_i shifted one step forwards. This corresponds to the freedom to start the contraction procedure one line lower in the S-fraction (5.20), beginning with F_1 rather than F_0; so the indices on all Volterra variables must be shifted by the same amount, and the relation (5.28) with the corresponding rational function on $\mathcal{C}$ is modified to $x^{-1}F_1 - w_1 = \hat{s}_0 Y_1$. Then, in terms of the J-fraction coefficients, this produces

$$d_{j+1} = w_{2j}w_{2j+1}, \qquad v_{j+1} = -w_{2j+1} - w_{2j+2} - \frac{c_1}{2(g+1)}, \qquad \text{for} \quad j \in \mathbb{Z}. \tag{5.31}$$

Thus each orbit of $\mathcal{V}_g$ is transformed to two different orbits of $\mathcal{J}_g$, since the resulting orbit of the latter map remains the same when the index on the w_i in (5.31) is shifted by a multiple of 2.   □

Remark 5.11. In [28], the phase space $\hat{M}_g^{\hat{c}_1=0}$ for the J-fraction map $\mathcal{J}_g$ is obtained from $\hat{M}_g$, an affine space of dimension $3g+2$ with a specific Poisson structure $\{\cdot, \cdot\}$, by restricting to a subvariety defined by setting the value of one of the Casimirs to zero. (This is analogous to the situation described in the appendix, Section 7.) Although Theorem 5.10 has been stated in terms of a correspondence between specific orbits of $\mathcal{V}_g$ and $\mathcal{J}_g$, the considerations in the proof make it clear that, since the coresponding generic fibers are birationally equivalent, the restriction of $\mathcal{J}_g$ to each fiber is (conjugate to) the square $(\mathcal{V}_g)^2 = \mathcal{V}_g \circ \mathcal{V}_g$. This gives a strong hint that $\hat{M}_g^{\hat{c}_1=0}$ and M_g should also be birationally equivalent, making this into a global

statement about the two maps. While this global statement is by no means obvious, the first example below shows that it is correct when $g = 1$; but it is not true as a Poisson isomorphism, at least for the specific Poisson structure introduced in [28]. The problem of finding alternative Poisson structures on $\hat{M}_g$, and making all these statements precise, is best left for future work.

Example 5.12. When $g = 1$, with the cubic $\Gamma_f : y^2 = 1 + c_1 x + c_2 x^2 + c_3 x^3$, the transformation (5.30) is

$$X = x^{-1} + \frac{c_1}{4}, \qquad Y = \frac{y}{x^2} \qquad \Longrightarrow \qquad \mathcal{C}: \qquad Y^2 = (X^2 + \hat{f})^2 + 4\hat{u}X + 4\hat{h},$$

where the quartic curve $\mathcal{C}$ is written in terms of the parameters

$$\hat{f} = \frac{1}{2}c_2 - \frac{3}{16}c_1^2, \quad \hat{u} = \frac{1}{4}c_3 - \frac{1}{8}c_1 c_2 + \frac{1}{32}c_1^3, \quad \hat{h} = \frac{1}{16}(c_1^2 c_2 - c_1 c_3 - c_2^2) - \frac{3}{256}c_1^4. \tag{5.32}$$

Under the transformation (5.29), solutions of the $g = 1$ Volterra map $\mathcal{V}_1$, given by (3.34), or equivalently by (3.37) (with fixed c_3), or by (3.39) (with fixed c_2), are mapped to solutions of the corresponding J-fraction map, which (according to the results in Example 3.2 in [28]) can be written as a 2D map defined by

$$d_{n+1} = -d_n - v_n^2 - \hat{f}, \tag{5.33}$$

$$v_{n+1} = -v_n + \frac{\hat{u}}{d_{n+1}},$$

on a reduced phase space with fixed parameters $\hat{f}, \hat{u}$, which are determined from (5.32) in terms of the values of the constants c_1, c_2, c_3 for the solution of the map $\mathcal{V}_1$. The map (5.33) has the conserved quantity

$$\hat{H} = d_n(v_n^2 + d_n + \hat{f}) - \hat{u}v_n,$$

which, on the orbit corresponding to a fixed solution of $\mathcal{V}_1$, takes the value $\hat{H} = \hat{h}$ given in (5.32). The Poisson bracket presented for the J-fraction maps in [28] becomes the canonical bracket

$$\{v_n, d_n\} = 1 \tag{5.34}$$

on the 2D phase space of the map (5.33), with coordinates (d_n, v_n), and it can be verified directly that the vector field $\{\cdot, \hat{H}\}$ extends to the Toda lattice flow (1.11) for all $n \in \mathbb{Z}$ under the action of this J-fraction map. Upon comparing with Example 5.6, it is clear that the analytic expressions for a_n, b_n in (5.15) provide explicit formulae for the solutions of both the map (5.33) and the Toda lattice, by setting $d_n(t) = a_n(t)$ and $v_n(t) = b_n(t) - \frac{1}{4}c_1$. Upon comparison of (5.14) with (5.15), it can be seen that each iteration of $\mathcal{V}_1$ gives a shift by z on the Jacobian of the elliptic curve, while each iteration of (5.33) produces a shift by $2z$.

However, the bracket (5.34) cannot be a reduction of any of the Poisson structures in Example 4.4, because the parameters $\hat{f}, \hat{u}$ do not correspond to Casimirs of any of these brackets on the phase space M_1. (This is immediately obvious for the pencil of brackets generated by $\{\cdot, \cdot\}^x$ and $\{\cdot, \cdot\}^{x^2}$, while a short calculation shows this to be the case for (4.12) as well.) Nevertheless, it is still possible to interpret (5.29) as a Poisson map in terms of members of the pencil $\{\cdot, \cdot\}^{\phi}$, with c_1 being the fixed value of a Casimir. For instance, using the Nambu-Poisson structure, we can construct a birational transformation between the symplectic leaves of $\{\cdot, \cdot\}^{x^2}$, that is $M_1^c \cap (H_2 = c_2)$, and a 2D phase space for the J-fraction map with coordinates (d_n, v_n), with c_1, c_2 viewed as fixed parameters, on which the bracket reduces to

$$\{v_n, d_n\}^{x^2} = v_n - \frac{1}{4}c_1.$$

Now we take a particular numerical example, with the elliptic curve $y^2 = 1 - 4x + 4x^3$ ($c_1 = -4$, $c_2 = 0$, $c_3 = 4$), and $w_1 = 1$, $w_2 = 2$. The sequence (w_n) extends backwards to $n \leqslant 0$ to give a singular orbit of (3.34), with the same singularity pattern appearing as was found for (P.iv) in Section 2:

$$\ldots, \frac{7}{15}, \frac{10}{3}, -\frac{3}{2}, -\frac{1}{2}, 2, 1, 0, \infty, \infty, 0, 1, 2, -\frac{1}{2}, -\frac{3}{2}, \frac{10}{3}, \frac{7}{15}, \ldots.$$

This orbit is symmetrical, in the sense that $w_{-n} = w_{n-3}$ for all $n \in \mathbb{Z}$. By applying the formulae in Example 3.9, the solution is expressed in terms of Hankel determinants Δ_n constructed from the moment sequence determined from $s_1 = 1$, $s_2 = 2$, $s_j = \sum_{i=1}^{j-1} s_i s_{j-i} - \sum_{i=1}^{j-2} s_i s_{j-i-1}$ $(j \geqslant 3)$, that is $(s_j)_{j\geqslant 1}$: $1, 2, 3, 6, 14, 37, 105, 312, 956, 2996, 9554, \ldots$, which gives

$$(\Delta_n)_{n\geqslant -2}: \quad 1, 1, 1, 1, 2, -1, 3, -5, 7, -4, 23, 29, 59, 129, 314, \ldots.$$

With $\Delta_{-3} = 0$, this extends backwards to a sequence of tau functions $\tau_n = \Delta_{n-3} = (-1)^{n+1}\tau_{-n}$ for $n \in \mathbb{Z}$; hence from (3.42) and (3.48) we find $c_2' = -1$, $c_3' = 1$, so for all n they satisfy the Somos-5 relation

$$\tau_{n+5}\tau_n = -\tau_{n+4}\tau_{n+1} + \tau_{n+3}\tau_{n+2}.$$

Applying (5.30) in this case produces the quartic curve $Y^2 = (X^2 - 3)^2 - 4(X + 2)$, the same one as in Example 4.2 from [28], and the contraction formulae (5.29) and (5.31) with $j = 1$ give initial points on two different orbits of the map (5.33) with parameters $\hat{f} = -3$, $\hat{u} = -1$, namely $(d_2, v_2) = (2, -\frac{1}{2})$ and $(-1, 3)$, respectively, which both correspond to the value $\hat{h} = -2$ for the conserved quantity of this map. For these two different orbits, we find that

$$d_n = \frac{\hat{\tau}_{n-1}\hat{\tau}_{n+1}}{\hat{\tau}_n^2},$$

where $\hat{\tau}_n = \Delta_{2n-4}$ (even index Hankel determinants) for the first orbit, and $\hat{\tau}_n = \Delta_{2n-3}$ (odd index Hankel determinants) for the second one. It follows from Proposition 5.1 in [28] that either of these even/odd index subsequences $(1, 1, 2, 3, 7, 23, 59, 314, \ldots$ and $1, 1, -1, -5, -4, 29, 129, \ldots$, respectively) must satisfy the same Somos-4 relation, in this case the original one introduced by Somos [47]:

$$\hat{\tau}_{n+4}\hat{\tau}_n = \hat{\tau}_{n+3}\hat{\tau}_{n+1} + \hat{\tau}_{n+2}^2.$$

This connection between Somos-5 and Somos-4 has already been exploited elsewhere (in [24], and in [8], for example); however, the specific Hankel determinants for Somos-4 obtained here via contraction differ from the ones found in [8], and also from the ones derived directly from the J-fraction in [28].

Example 5.13. An orbit of the map (P.iv), with the spectral curve (1.8), can be transformed to an orbit of the $g = 2$ J-fraction map, as discussed in Example 3.3 of [28], which has an associated sextic curve, related to it via $X = x^{-1} + \nu/3$, $Y = y/x^3$, that can be taken in the form

$$\mathcal{C}: \quad Y^2 = (X^3 + \hat{f}X + \hat{g})^2 + 4(\hat{u}X^2 + \hat{h}_1 X + \hat{h}_2).$$

Defined on a 4D phase space with coordinates $(d_{n-1}, d_n, v_{n-1}, v_n)$ and depending on the 3 parameters $\hat{f}, \hat{g}, \hat{u}$, the map in [28] is given by

$$d_{n+1} + d_n + d_{n-1} + \hat{u}/d_n + v_n^2 + v_n v_{n-1} + v_{n-1}^2 + \hat{f} = 0, \tag{5.35}$$

$$(2v_n + v_{n-1})d_n + (2v_n + v_{n+1})d_{n+1} + v_n^3 + \hat{f}v_n + \hat{g} = 0.$$

More precisely, there are two different orbits of (5.35) obtained from each orbit of (P.iv), depending on whether the formula (5.29) or (5.31) is applied. Then, upon writing each term w_j satisfying (P.iv) as a ratio of the Hankel determinants Δ_j defined in Example 3.11, we see that the quantities d_n that appear in the solution of the J-fraction map, as above, are given as a ratio of tau functions, in two different ways:

$$d_n = \frac{\hat{\tau}_{n-1}\hat{\tau}_{n+1}}{\hat{\tau}_n^2}, \qquad \hat{\tau}_n = \Delta_{2n-4} \quad \text{or} \quad \Delta_{2n-3},$$

for $n \in \mathbb{Z}$, where the orbit is determined by the choice of parity of the index on Δ_j. It follows from the proof of Theorem 5.5 in [28] that (regardless of which choice is made), the tau functions $\hat{\tau}_n$ satisfy a Somos-8 relation, which explains why the relation (2.9) appears in Example 2.2.

6 Conclusions and outlook

We have seen how the map (P.iv) obtained by Gubbiotti et al. can naturally be viewed as the $g = 2$ member of a family of algebraically integrable maps, defined for each g, that are naturally related to the infinite Volterra lattice equation, leading to genus g solutions of the latter. This begs the question as to what can be said about the other Liouville integrable maps in 4D found in [21], namely (P.v) and (P.vi), which take the same form (1.1) but for a different rational function F. Note that (P.vi) depends on an extra parameter compared with (P.v), which we denote here by δ (instead of δ^2 in [21]). In fact, (P.v) arises from (P.vi) in the limit $\delta \to 0$.

So far we have made the following observations:

- Each solution $\hat{w}_n$ of (P.v) is mapped to a solution w_n of (P.iv) via the transformation

$$w_n = \hat{w}_{n+1}\hat{w}_n, \tag{6.1}$$

  by suitably identifying the parameters ν, a, b for (P.iv) in terms of the parameters and first integrals for (P.v).

- Under the flow of the Hamiltonian vector field $\frac{d}{dt}$ associated with one of its first integrals, the sequence $(\hat{w}_n)_{n \in \mathbb{Z}}$ generated by iteration of the map (P.v) extends to a solution of the modified Volterra lattice in the form

$$\frac{d\hat{w}_n}{dt} = \hat{w}_n^2(\hat{w}_{n+1} - \hat{w}_{n-1}). \tag{6.2}$$

- Each solution $\hat{w}_n$ of (P.vi) is mapped to a pair of solutions $w_n^{(+)}$, $w_n^{(-)}$ of (P.iv), via the transformations

$$w_n^{(\pm)} = (\hat{w}_{n+1} \pm \delta)(\hat{w}_n \mp \delta), \tag{6.3}$$

  by suitably identifying the parameters ν, a, b for (P.iv) in terms of the parameters and first integrals for (P.vi).

- Under the flow of the Hamiltonian vector field $\frac{d}{dt}$ associated with one of its first integrals, the sequence $(\hat{w}_n)_{n \in \mathbb{Z}}$ generated by iteration of the map (P.vi) extends to a solution of the modified Volterra lattice in the form

$$\frac{d\hat{w}_n}{dt} = (\hat{w}_n^2 - \delta^2)(\hat{w}_{n+1} - \hat{w}_{n-1}). \tag{6.4}$$

The formulae (6.1) and (6.3) are the well-known expressions for the Miura transformation connecting the two forms of the modified Volterra lattice equation, given by (6.2) and (6.4), respectively, to the Volterra lattice (5.7). Thus the above statements about the connections between the maps can be viewed as restrictions of a Miura transformation to a finite-dimensional phase space. Preliminary calculations, and the results of [55] on elliptic solutions, indicate that this picture should extend to arbitrary genus g. We propose that the full description of the above observations, and their extension to genus g analogues of the maps (P.v) and (P.vi), should be left as the subject of future work.

It is also worth pointing out that part of the original motivation for the work in [21] was to consider autonomous versions of the higher order discrete Painlevé equations from [11], and new applications of the latter have been found very recently. Non-autonomous analogues of the Volterra maps $\mathcal{V}_g$ have been considered in the context of Hermitian matrix models [4], where they arise as string equations, and they also appear as recursion relations for orthogonal polynomials associated with generalised Freud weights of higher order [10]. In these applications, the algebro-geometric structure of the Volterra maps should be relevant to the asymptotic description of the oscillatory behaviour that is observed in specific parameter regimes.

7 Appendix: the relation between the Mumford-like and the even Mumford system

Here we now show that the Hamiltonian system $(M_g, \{\cdot, \cdot\}^\phi, \mu)$ associated with the Volterra map is birationally isomorphic (as a Poisson isomorphism) to the even Mumford system, or more precisely to a subsystem thereof, obtained simply by fixing the value of one of the Casimirs. Recall from [53, Ch. 6] that the even Mumford system (of genus g) is the Hamiltonian system $(M_g', \{\cdot, \cdot\}'^{\,\psi}, \mu')$, whose phase space M_g' is the $(3g+2)$-dimensional affine space

$$M_g' := \left\{ (U(\xi), V(\xi), W(\xi)) \in \mathbb{C}[\xi]^3 \;\middle|\; \begin{array}{l} \deg U(\xi) = g\,, \qquad U \text{ monic} \\ \deg V(\xi) < g\,, \\ \deg W(\xi) = g+1\,, \quad W \text{ monic} \end{array} \right\}\,. \tag{7.1}$$

Elements $(U(\xi), V(\xi), W(\xi))$ of M_g' are written as Lax matrices

$$\mathbf{L}'(\xi) := \left(\begin{array}{cc} V(\xi) & U(\xi) \\ W(\xi) & -V(\xi) \end{array} \right)\,,$$

whose polynomial entries have the form

$$U(\xi) = x^g + \sum_{i=0}^{g-1} U_i \xi^i\,, \qquad V(\xi) = \sum_{i=0}^{g-1} V_i \xi^i\,, \qquad W(\xi) = x^{g+2} + \sum_{i=0}^{g+1} W_i \xi^i\,.$$

The $3g+2$ coefficients W_{g+1}, W_g and U_i, V_i, W_i with $0 \leqslant i < g$ are used as linear coordinates on M_g'. The momentum map μ' is given by

$$\begin{array}{rcl} \mu' \;:\; & M_g' & \to \quad \mathbb{C}[\xi] \\ \mathbf{L}'(\xi) = \left(\begin{array}{cc} V(\xi) & U(\xi) \\ W(\xi) & -V(\xi) \end{array} \right) & \mapsto & -\det \mathbf{L}'(\xi) = V(\xi)^2 + U(\xi)W(\xi)\,. \end{array}$$

It is clear that $-\det \mathbf{L}'(\xi)$ is monic of degree $2g+2$, so $2g+2$ polynomial functions $H_0', H_1', \ldots, H_{2g+1}'$ on M_g' are defined by

$$V(\xi)^2 + U(\xi)W(\xi) = \xi^{2g+2} + \sum_{i=0}^{2g+1} H_i' \xi^i\,,$$

For any non-zero polynomial ψ of degree at most $g+1$, a Poisson structure of rank $2g$ on M_g' is given by

$$\{U(\xi), U(\eta)\}'^{\,\psi} = \{V(\xi), V(\eta)\}'^{\,\psi} = 0\,,$$

$$\{U(\xi), V(\eta)\}'^{\,\psi} = \frac{U(\xi)\psi(\eta) - U(\eta)\psi(\xi)}{\xi - \eta}\,,$$

$$\{U(\xi), W(\eta)\}'^{\,\psi} = -2\frac{V(\xi)\psi(\eta) - V(\eta)\psi(\xi)}{\xi - \eta}\,,$$

$$\{V(\xi), W(\eta)\}'^{\,\psi} = \frac{W(\xi)\psi(\eta) - W(\eta)\psi(\xi)}{\xi - \eta} - (\xi + \eta + W_{g+1} - U_{g-1})U(\xi)\psi(\eta)\,,$$

$$\{W(\xi), W(\eta)\}'^{\,\psi} = 2(\xi + \eta + W_{g+1} - U_{g-1})\left(V(\xi)\psi(\eta) - V(\eta)\psi(\xi)\right)\,.$$

The polynomial functions H_i' are independent and in involution, which accounts for the Liouville integrability of the even Mumford system. It is algebraically integrable, with the fiber of μ' over any monic polynomial $f'(x)$ of degree $2g+2$ and without multiple roots being an affine part of the Jacobian of the smooth hyperelliptic curve defined by $\eta^2 = f'(\xi)$.

For the isomorphism, we consider only the Poisson structures $\{\cdot,\cdot\}'^{\,\psi}$ for which $\psi(0)=0$. They admit H_0' as Casimir function, hence we can restrict $(M_g', \{\cdot,\cdot\}'^{\,\psi}, \mu')$ to the subvariety, defined by $H_0'=0$; the resulting system is denoted by $(M_g^0, \{\cdot,\cdot\}^{0,\psi}, \mu^0)$. We show that this system is birationally equivalent to the Mumford-like system $(M_g, \{\cdot,\cdot\}^{\phi}, \mu)$, where the relation between ψ and ϕ will be spelled out below. To do this, we construct a biregular map $\tilde{\Psi}$ and a birational map Ψ making the following diagram commutative:

$$
\begin{array}{ccc}
M_g^0 & \xrightarrow{\ \Psi\ } & M_g \\
{\scriptstyle \mu^0}\downarrow & & \downarrow{\scriptstyle \mu} \\
B_g^0 & \xrightarrow[\ \tilde{\Psi}\ ]{} & B_g
\end{array}
$$

In this diagram, B_g and B_g^0 are the images of μ and μ^0, which we view as spaces of curves: B_g consists of the hyperelliptic curves of the form $y^2=f(x)$, with $f(0)=1$ and $\deg f \leqslant 2g+1$, while B_g^0 consists of the hyperelliptic curves of the form $\eta^2=f'(\xi)$, with f' monic of degree $2g+2$, vanishing at 0. The curves of B_g^0 have two points at infinity, which we denote by ∞_1 and ∞_2.

We first establish a natural correspondence between the curves of B_g^0 and the curves of B_g. Let $y^2=f(x)$ be a curve of B_g and substitute $x=\xi^{-1}$ and $y=\eta\xi^{-g-1}$, to get $\eta^2=f'(\xi)=\xi^{2g+2}f(\xi^{-1})$, where $f'(0)=0$ and f' is monic of degree $2g+2$, so $\eta^2=f'(\xi)$ is a curve of B_g^0. From the latter, one gets back $y^2=f(x)$ by setting $\xi=x^{-1}$ and $\eta=yx^{-g-1}$. Notice that when $f(x)=1+\sum_{i=1}^{2g+1}c_ix^i$ then $\eta^2=\xi^{2g+2}+\sum_{i=1}^{2g+1}c_{2g+2-i}\xi^i$, which yields the biregular map $\tilde{\Psi}$.

For the construction of Ψ, the biregular map $\tilde{\Psi}$ between the spaces of curves $y^2=f(x)$ and $\eta^2=f'(\xi)$ is extended to divisors on these curves. To do this, we compare the description of points $(\mathcal{P}(x), \mathcal{Q}(x), \mathcal{R}(x))$ on a generic fiber of μ in terms of divisors on the corresponding curve $y^2=f(x)$ with the description of points $(U(\xi), V(\xi), W(\xi))$ on a generic fiber of μ' in terms of divisors on the corresponding curve $\eta^2=f'(\xi)$. The first description was given in the proof of Proposition 4.6, while the second description, which we quickly recall, can be found in [53, Ch. 6]. Let $\xi_1,\ldots,\xi_g$ denote the roots of $U(\xi)$ and let $\eta_i:=V(\xi_i)$, for $i=1,\ldots,g$. Then the points (ξ_i,η_i) belong to the curve $\eta^2=\phi(\xi)$ and so the divisor class $[\sum_{i=1}^{g}(\xi_i,\eta_i)-g\infty_1]$ is a point of its Jacobian. The relation between the polynomials $\mathcal{Q}(x)$ and $U(\xi)$ is clearly given by

$$
\frac{\xi^g}{2}\mathcal{Q}(\xi^{-1})=U(\xi)\ .
$$

Indeed, both sides of this equality are polynomials of degree $2g+2$ which vanish for $\xi=\xi_i=x_i^{-1}$ (which we may assume to be different). Similarly, $\mathcal{P}(x)$ is related to $U(\xi)$ and $V(\xi)$ by

$$
\xi^g\mathcal{P}(\xi^{-1})-U(\xi)=\frac{U_0V(\xi)-V_0U(\xi)}{\xi U_0}\ ,
$$

because both sides of this equation are the unique polynomial in ξ of degree less than g which takes for $\xi=\xi_i$ the value η_i/ξ_i. A formula for $\mathcal{R}$ follows from the equations of the curves, namely

$$
\mathcal{P}^2(\xi^{-1})+\mathcal{Q}(\xi^{-1})\mathcal{R}(\xi^{-1})=f(\xi^{-1})=\xi^{-2g-2}f'(\xi)=\xi^{-2g-2}(V^2(\xi)+U(\xi)W(\xi))\ .
$$

It is clear that this defines a birational map between M_g and M_g^0. Under this map, H_i corresponds to H_{2g+2-i}', for $i=1,\ldots,2g+1$. To see that Ψ is a Poisson map, hence a birational Poisson isomorphism, we recall that the Poisson bracket $\{\cdot,\cdot\}'^{\,\psi}$ is given in terms of the ξ_i and η_j by $\{\xi_i,\eta_j\}'^{\,\psi}=\psi(\xi_i)\delta_{ij}$, and hence

$$
\{x_i,y_j\}'^{\,\psi}=\left\{\xi_i^{-1},\eta_j\xi_j^{-g-1}\right\}'^{\,\psi}=-\xi_i^{-2}\xi_j^{-g-1}\{\xi_i,\eta_j\}'^{\,\psi}=-\xi_i^{-g-3}\psi(\xi_i)\delta_{ij}=-x_i^{g+3}\psi(x_i^{-1})\delta_{ij}\ .
$$

Compared with (4.21), this shows that $\Psi:(M_g^0,\{\cdot,\cdot\}'^{\,\psi})\to(M_g,\{\cdot,\cdot\}^{\phi})$ is a birational Poisson isomorphism when taking $\phi(x)=\psi(x^{-1})x^{g+2}$; notice that ϕ, defined by this formula, is indeed a polynomial of degree

at most $g + 1$, vanishing at 0 and that we get all such polynomials ϕ for some appropriate polynomial ψ of degree at most $g + 1$, vanishing at 0.

References

[1] M. Adler, P. van Moerbeke, and P. Vanhaecke. *Algebraic integrability, Painlevé geometry and Lie algebras*, vol. **47** of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 2004.

[2] N.I. Akhiezer, Elements of the Theory of Elliptic Functions, Translations of Mathematical Monographs, vol. **79**, American Mathematical Society, 1990.

[3] A.I. Aptekarev, Asymptotic properties of polynomials orthogonal on a system of contours and periodic motions of Toda lattices, Mat. Sb. **125** (1984) 231–258.

[4] C. Benassi and A. Moro, Thermodynamic limit and dispersive regularization in matrix models, Phys. Rev. E **101** (2020) 052118.

[5] M. Bertola, Padé approximants on Riemann surfaces and KP tau functions, Analysis Math. Phys. **11** (2021) 149.

[6] M. Bellon and C.M. Viallet, Algebraic entropy, Comm. Math. Phys. **204** (1999) 425–437.

[7] M. Bruschi, O. Ragnisco, P.M. Santini and T. Gui-Zhang, Integrable symplectic maps, Physica D **49** (1991) 273–294.

[8] X.-K. Chang, X.-B. Hu and G. Xin, Hankel Determinant Solutions to Several Discrete Integrable Systems and the Laurent Property, SIAM J. Discrete Math. **29** (2015) 667–682.

[9] X.-M. Chen, X.-B. Hu and F. Müller-Hoissen, Non-isospectral extension of the Volterra lattice hierarchy, and Hankel determinants, Nonlinearity **31** (2018) 4393–4422.

[10] P.A. Clarkson, K. Jordaan and A. Loureiro, Generalized higher-order Freud weights, Proc. R. Soc. A **479** (2023) 20220788.

[11] C. Cresswell and N. Joshi, The discrete first, second and thirty-fourth Painlevé hierarchies, J. Phys. A: Math. Gen. **32** (1999) 655–69.

[12] D.K. Demskoi, D.T. Tran, P.H. van der Kamp and G.R.W. Quispel, A novel nth order difference equation that may be integrable, J. Phys. A: Math. Theor. **45** (2012) 135202.

[13] J. Diller and C. Favre, Dynamics of bimeromorphic maps of surfaces, Amer. J. of Math. **123** (2001) 1135–1169.

[14] A. Doliwa and A. Siemaszko, Hermite-Padé approximation and integrability, J. Approximation Theory **292** (2023); `arXiv:2201.06829`

[15] J.J. Duistermaat, Discrete Integrable Systems: QRT Maps and Elliptic Surfaces, Springer, 2010.

[16] D. Duverney and I. Shiokawa, Continuants and convergence of certain continued fractions, `arXiv:2210.09862v1`

[17] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, J. Amer. Math. Soc. **15** (2002) 497–529.

[18] A.P. Fordy and A.N.W Hone, Discrete integrable systems and Poisson algebras from cluster maps, Comm. Math. Phys. 325 (2014) 527–584.

[19] F. Gesztesy, H. Holden, B. Simon and Z. Zhao, On the Toda and Kac-van Moerbeke Systems, Trans. Amer. Math. Soc. **339** (1993) 849–868.

[20] G. Gubbiotti, N. Joshi, D.T. Tran and C.-M. Viallet, Complexity and integrability in 4D bi-rational maps with two invariants. In: Nijhoff F., Shi Y., Zhang D. (eds), Asymptotic, Algebraic and Geometric Aspects of Integrable Systems. Springer Proceedings in Mathematics & Statistics, vol. **338**, Springer, Cham, pp.17–36, 2020.

[21] G. Gubbiotti, N. Joshi, D.T. Tran and C.-M. Viallet, Bi-rational maps in four dimensions with two invariants, J. Phys. A: Math. Theor. **53** (2020) 115201.

[22] G. Gubbiotti, Lagrangians and integrability for additive fourth-order difference equations, Eur. Phys. J. Plus **135** (2020) 853.

[23] K. Hamad, A.N.W. Hone, P. van der Kamp and G.R.W. Quispel, QRT maps and related Laurent systems, Adv. Appl. Math. **96** (2018) 216–248.

[24] A.N.W. Hone, Sigma function solution of the initial value problem for Somos 5 sequences, Trans. Amer. Math. Soc. **359** (2007) 5019–5034.

[25] A. N. W. Hone, Singularity confinement for maps with the Laurent property, Phys. Lett. A **361** (2007) 341–345.

[26] A.N.W. Hone, Analytic solution and integrability for a bilinear recurrence of order six, Applicable Analysis **89** (2010) 473–492.

[27] A.N.W. Hone, T.E. Kouloukas and G.R.W. Quispel, Some integrable maps and their Hirota bilinear forms, J. Phys. A: Math. Theor. **51** (2018) 044004.

[28] A.N.W. Hone, Continued fractions and Hankel determinants from hyperelliptic curves, Comm. Pure Appl. Math. **73** (2020) 2310–2347.

[29] A. Iatrou and J.A.G. Roberts, Integrable mappings of the plane preserving biquadratic invariant curves II, Nonlinearity **15** (2002) 459–489.

[30] N. Joshi and C.-M. Viallet, Rational maps with invariant surfaces, J. Int. Syst. **3** (2018) xyy017.

[31] M. Kac and P. van Moerbeke, On an Explicitly Soluble System of Nonlinear Differential Equations Related to Certain Toda Lattices, Adv. Math. **16** (1975) 160–169.

[32] M. Kanki, J. Mada, K.M. Tamizhmani, T. Tokihiro, Discrete Painlevé II equation over finite fields, J. Phys. A: Math. Theor. **45** (2012) 342001.

[33] T. Lam and P. Pylyavskyy, Laurent phenomenon algebras, Cam. J. Math. **4** (2012) 121–162.

[34] S. Maeda, Completely integrable symplectic mapping, Proc. Japan Acad. Ser. A Math. Sci. **63** (1987) 198–200.

[35] T. Mase, Investigation into the role of the Laurent property in integrability, J. Math. Phys. **57** (2016) 022703.

[36] T. Mase, R. Willox, A. Ramani, B. Grammaticos, Singularity confinement as an integrability criterion, J. Phys. A: Math. Theor. **52** (2019) 205201.

[37] J. Moser, Three Integrable Systems Connected with Isospectral Deformations, Adv. Math. **16** (1975) 197–220.

[38] D. Mumford, Tata Lectures on Theta II, Birkhäuser, Boston, 1984.

[39] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A006721`

[40] A.J. van der Poorten, Non-periodic continued fractions in hyperelliptic function fields. Bull. Austral. Math. Soc. **64** (2001) 331–343.

[41] A.J. van der Poorten, Elliptic curves and continued fractions. J. Integer Seq. **8** (2005), no. 2, Article 05.2.5, 19 pp.

[42] A.J. van der Poorten, Curves of genus 2, continued fractions, and Somos sequences, J. Integer Seq. **8** (2005), no. 3, Article 05.3.4, 9 pp.

[43] A.J. van der Poorten, Hyperelliptic curves, continued fractions, and Somos sequences, Dynamics & stochastics, 212–224. IMS Lecture Notes Monogr. Ser. **48**, Inst. Math. Statist., Beachwood, OH, 2006.

[44] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, Integrable mappings and soliton equations, Phys. Lett. A **126** (1988) 419–421.

[45] G.R.W. Quispel, H.W. Capel and J.A.G. Roberts, Duality for discrete integrable systems, J. Phys. A: Math. Gen. **38** (2005) 3965–3980.

[46] J. Shohat, On Stieltjes Continued Fractions, Amer. J. Math. **54** (1932) 79–84.

[47] M. Somos, Problem 1470, Crux Mathematicorum **15** (1989) 208.

[48] T.-J. Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci. Toulouse Math. **8** (1894) J1-J122.

[49] A.K. Svinin, On some classes of discrete polynomials and ordinary difference equations, J. Phys. A: Math. Theor. **47** (2014) 155201.

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

[50] A.K. Svinin, On integrals for some class of ordinary difference equations admitting a Lax representation, J. Phys. A: Math. Theor. **49** (2016) 095201.

[51] A.K. Svinin, On solutions for some class of integrable discrete difference equations, `arXiv:2106.00539`

[52] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys. **160** (1994) 295–315.

[53] P. Vanhaecke, Integrable systems in the realm of algebraic geometry, 2nd edition, Springer, 2001.

[54] A.P. Veselov, Integrable Maps. Russ. Math. Surv. **46** (1991) 1–51.

[55] Z. Yan, Discrete exact solutions of modified Volterra and Volterra lattice equations via the new discrete sine-Gordon expansion algorithm, Nonlinear Analysis **64** (2006) 1798–1811.

[56] U. Zannier, Hyperelliptic continued fractions and generalized Jacobians, Amer. J. Math. **141** (2019) 1–40.