Stability analysis of dual viscous flows saturating a vertical porous pipe

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Stability analysis of dual viscous flows saturating a vertical porous pipe

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Abstract. The linear stability analysis of a mixed convection viscous flow in a vertical porous pipe is here investigated. The contribution of viscous heating is assumed to be non negligible. A fully developed flow regime is assumed for the basic state. The local balance equations for this state display dual stationary solutions. The dual branches of stationary solutions are determined numerically. Since the pipe is characterised by an isothermal lateral surface, the viscous heating is the sole cause of the buoyancy force. In order to investigate the stability of the basic dual solutions, small amplitude disturbances with the form of normal modes are superposed to the basic state. The solution of the eigenvalue problem obtained allows one to determine the growth rate associated to both the basic solution branches. The sign of the growth rate determines whether the particular basic solution is stable or unstable.

1. Introduction

In the last decades, several papers have been published reporting stability analyses of buoyant flows in porous media [1]. Most of these studies deal with the Rayleigh–Bénard type of instabilities occurring in porous layers saturated by a fluid and heated from below. The pioneering papers on this topic were authored by Horton, Rogers [2] and Lapwood [3]. Prats [4] expanded these earlier analyses by including a stationary horizontal throughflow across the layer. Barletta et al. [5] further extended this problem by considering the contribution of the viscous heating. The viscous heating becomes relevant when the throughflow is sufficiently intense and/or the fluid is characterised by high Prandtl numbers, namely it is both highly viscous and poorly conductive. When viscous heating is considered, the basic stationary solution of the governing equations may lose its uniqueness. More effort has been devoted to the investigation of dual solutions for clear fluids [6, 7, 8] compared to the convection of viscous fluids saturating porous layers [9]. Very recently, a few papers containing the stability analysis of dual stationary solutions have been published both for clear fluids [10, 11] and fluid saturated porous media [12]. The last paper deals with a two–dimensional flow in a plane vertical channel where the viscous dissipation effect is taken into account.

The paper is focussed on the three–dimensional linear stability analysis of a mixed convection flow in a vertical porous channel with circular cross–section when the contribution of viscous heating is non–negligible. The basic state whose stability is investigated feature dual stationary

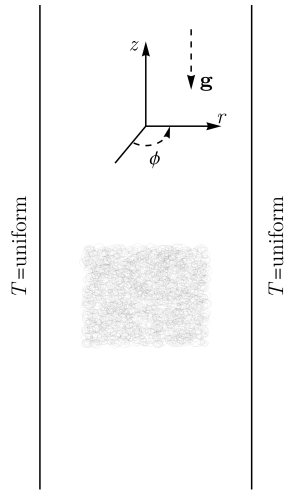

Figure 1. Sketch of the vertical porous layer.

solutions. The analysis aims to obtain the threshold values of the governing parameters for the onset of modal instability.

2. Mathematical model

A mixed convection in a vertical porous pipe characterised by a non–vanishing mass flow rate is assumed. The fluid saturating the porous pipe is in a fully developed flow regime and the mass flow rate depends on the imposed pressure gradient. The pipe has radius r_0 characterised by a uniform and constant temperature. Figure 1 displays a sketch of the porous duct.

The mass, momentum (Darcy's law) and the energy local balance equations employed to investigate the system just described are the following:

$$\begin{aligned}
&\frac{1}{r}\frac{\partial(r\,u)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0,\\
&\frac{\mu}{K}u = -\frac{\partial p}{\partial r},\\
&\frac{\mu}{K}v = -\frac{1}{r}\frac{\partial p}{\partial \phi},\\
&\frac{\mu}{K}w = -\frac{\partial p}{\partial z} + \rho\, g\, \beta\,(T - T_0),\\
&\sigma\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \phi} + w\frac{\partial T}{\partial z}\\
&\qquad = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\mu}{\rho\, c\, K}(u^2 + v^2 + w^2),\\
&r = r_0: \quad u = 0,\\
&r = r_0: \quad \frac{\partial T}{\partial z} = 0, \quad \frac{\partial T}{\partial \phi} = 0,
\end{aligned} \tag{1}$$

where (r, ϕ, z) are the cylindrical coordinates, (u, v, w) are the cylindrical components of the velocity vector, μ is the dynamic viscosity of the saturating fluid, K is the permeability of the porous medium, ρ is the density of the fluid evaluated at the reference temperature T_0, g is the modulus of the gravity acceleration vector $\mathbf{g}$, β is the thermal expansion coefficient of the fluid, c the specific heat of the fluid, σ is the ratio between the average volumetric heat capacity of the porous medium and the volumetric heat capacity of the fluid and α is the thermal diffusivity relative to the fluid saturated porous medium. The reference temperature T_0 is the average temperature on the pipe cross–section,

$$T_0 = \frac{1}{\pi\, r_0^2} \int_0^{r_0} \int_0^{2\pi} T\, r\, \mathrm{d}r \mathrm{d}\phi. \tag{2}$$

The last term of the local energy balance in Eq. (1) identifies the contribution of the viscous dissipation when Darcy's law is considered [1]. By employing the scaling

$$\frac{(r,z)}{r_0} \to (r,z), \qquad \frac{\alpha}{\sigma\, r_0^2} t \to t, \qquad \frac{Ge\, r_0}{\alpha}(u,v,w) \to (u,v,w), \qquad \frac{T - T_0}{\Delta T} \to T, \qquad \frac{Ge\, K}{\mu\, \alpha} p \to p, \tag{3}$$

the dimensionless form of Eq. (1) is obtained,

$$\begin{aligned}
&\frac{1}{r}\frac{\partial (r\,u)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0,\\
&u = -\frac{\partial p}{\partial r},\\
&v = -\frac{1}{r}\frac{\partial p}{\partial \phi},\\
&w = -\frac{\partial p}{\partial z} + T,\\
&\frac{\partial T}{\partial t} + \frac{1}{Ge}\left(u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \phi} + w\frac{\partial T}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + u^2 + v^2 + w^2,\\
&r = 1: \quad u = 0,\\
&r = 1: \quad \frac{\partial T}{\partial z} = 0, \quad \frac{\partial T}{\partial \phi} = 0,
\end{aligned} \tag{4}$$

where the Gebhart number Ge and ΔT are defined as

$$Ge = \frac{g\,\beta\, r_0}{c}, \quad \Delta T = \frac{\nu\, \alpha}{Ge^2\, K\, c}. \tag{5}$$

As a consequence of Eq. (2), the dimensionless temperature satisfies the constraint

$$\int_0^1 \int_0^{2\pi} T\, r\, \mathrm{d}r \mathrm{d}\phi = 0. \tag{6}$$

3. Stationary dual solutions

The basic stationary solutions of Eq. (4) are assumed to be fully developed flows. The basic solutions are characterised by zero radial velocity $u = 0$. Among all the possible fully developed flows, stationary axisymmetric solutions with zero angular velocity $v = 0$ are chosen.

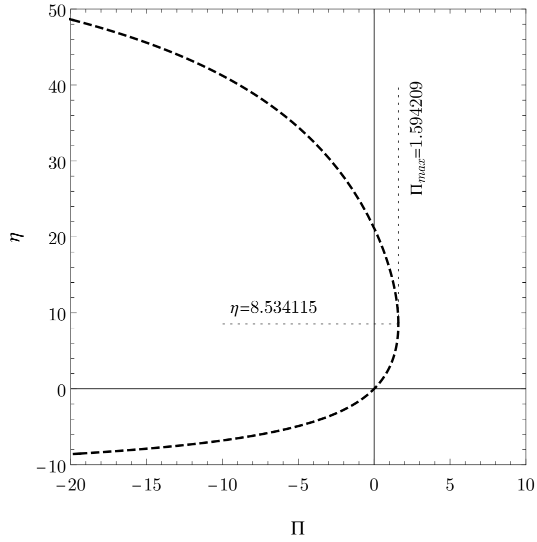

Figure 2. Dual basic solutions: for a given Π we obtains two possible velocity η on the pipe axis, $r \to 0$.

Equations (4) and (6) simplify to

$$\begin{aligned} &u_b = 0, \\ &v_b = 0, \\ &w_b = T_b + \Pi, \\ &\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_b}{\mathrm{d}r}\right) + w_b^2 = 0, \\ &r = 0: \qquad \frac{\mathrm{d}w_b}{\mathrm{d}r} = 0, \\ &\int_0^1 w_b\, r\mathrm{d}r = \Pi, \end{aligned} \tag{7}$$

where the subscript b denotes the basic state and the boundary condition $\mathrm{d}w_b/\mathrm{d}r = 0$ at $r = 0$ can be employed because the flow is axisymmetric. One can note that the assumption $u = v = 0$ yields, from Eq. (4), that p depends only on z. From Eqs. (6) and (7), it can be inferred that $\mathrm{d}^2p/\mathrm{d}z^2$ is zero, so that one can define the dimensionless constant $\Pi = -\mathrm{d}p/\mathrm{d}z$.
The basic state is obtained by solving the initial value problem

$$\begin{aligned} &\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_b}{\mathrm{d}r}\right) + w_b^2 = 0, \\ &r = \varepsilon: \qquad w_b = \eta, \qquad \frac{\mathrm{d}w_b}{\mathrm{d}r} = 0, \end{aligned} \tag{8}$$

where η is the unknown velocity at $r = 0$ and ε is a small cutoff $\varepsilon \ll 1$, which is employed to overcome the numerical issue introduced by the singular behaviour of the ordinary differential equation (11) at $r = 0$. For any given value of the axial velocity η one may obtain a single value of Π by employing the last of equations (7), namely

$$\int_0^1 w_b\, r\mathrm{d}r = \Pi. \tag{9}$$

Table 1. Values of $\eta(\Pi_{max})$ and Π_{max} as functions of ε.

ε	$\eta(\Pi_{max})$	Π_{max}
10^{-1}	8.761257	1.672008
10^{-2}	8.529247	1.595070
10^{-3}	8.533982	1.594218
10^{-4}	8.534112	1.594209
10^{-6}	8.534115	1.594209
10^{-8}	8.534115	1.594209

The basic stationary solutions are displayed in Fig. 2: by fixing the basic imposed pressure gradient Π, two possible values of velocity at the pipe axis, $r = 0$, are obtained. One can conclude that a single value of the pressure gradient may produce more than one velocity profile. The curve in Fig. 2 displays a maximum value of Π,

$$\Pi_{max} = 1.594209, \qquad \eta(\Pi_{max}) = 8.534115. \tag{10}$$

The basic state solutions are sensitive to the values of ε employed. In Table 1 the values of $\eta(\Pi_{max})$ and Π_{max} as functions on ε are reported. The values are obtained by imposing 7 effective digits of accuracy. One can conclude that a reasonable value of ε to be employed for the computations is 10^{-4}.

4. Linear stability analysis

The stability of the dual basic solutions are here tested by superposing small amplitude disturbances to the basic state, namely

$$\mathbf{u} = \mathbf{u}_b(r) + \varepsilon \mathbf{U}(r,\phi,z,t), \quad p = -\Pi\, z + \varepsilon P(r,\phi,z,t), \quad T = T_b(r) + \varepsilon \Theta(r,\phi,z,t). \tag{11}$$

One may substitute Eq. (11) in Eq. (4) and then retain only terms $O(\varepsilon)$ to linearise the equations obtained, namely

$$\begin{aligned}
&\frac{1}{r}\frac{\partial(r\,U)}{\partial r} + \frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{\partial W}{\partial z} = 0,\\
&U = -\frac{\partial P}{\partial r},\\
&V = -\frac{1}{r}\frac{\partial P}{\partial \phi},\\
&W = -\frac{\partial P}{\partial z} + \Theta,\\
&\frac{\partial \Theta}{\partial t} + \frac{1}{Ge}\left(U\frac{\mathrm{d}w_b}{\mathrm{d}r} + w_b\frac{\partial \Theta}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \phi^2} + \frac{\partial^2 \Theta}{\partial z^2} + 2w_b\,W,\\
&r = 1: \quad U = 0,\\
&r = 1: \quad \frac{\partial \Theta}{\partial z} = 0, \quad \frac{\partial \Theta}{\partial \phi} = 0.
\end{aligned} \tag{12}$$

By employing the scaling

$$z\,Ge \to z, \quad \frac{1}{Ge^2}P \to P, \tag{13}$$

Eq. (12) can be rewritten as follows:

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial(r\,U)}{\partial r}+\frac{1}{r}\frac{\partial V}{\partial \phi}+Ge\frac{\partial W}{\partial z}=0,\\
&U=-Ge^2\frac{\partial P}{\partial r},\\
&V=-\frac{Ge^2}{r}\frac{\partial P}{\partial \phi},\\
&W=-Ge^3\frac{\partial P}{\partial z}+\Theta,\\
&\frac{\partial \Theta}{\partial t}+\frac{1}{Ge}\left(U\frac{\mathrm{d}w_b}{\mathrm{d}r}+Ge\,w_b\frac{\partial \Theta}{\partial z}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Theta}{\partial \phi^2}+Ge^2\frac{\partial^2\Theta}{\partial z^2}+2w_b\,W,\\
&r=1:\quad U=0,\\
&r=1:\quad \frac{\partial \Theta}{\partial z}=0,\quad \frac{\partial \Theta}{\partial \phi}=0.
\end{aligned}
\tag{14}
$$

The value of the Gebhart number is typically much less than 1 (for instance water at 20°C flowing inside a 10 cm radius pipe yields $Ge \approx 10^{-7}$). For the limiting case $Ge \to 0$, the governing equations (14) can be drastically simplified to

$$
\begin{aligned}
&U=V=0,\\
&W=\Theta,\\
&\frac{\partial \Theta}{\partial t}+w_b\frac{\partial \Theta}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Theta}{\partial \phi^2}+2w_b\,W,\\
&r=1:\quad \frac{\partial \Theta}{\partial z}=0,\quad \frac{\partial \Theta}{\partial \phi}=0,
\end{aligned}
\tag{15}
$$

where the mass balance equation is not reported since it is identically satisfied. Equation (15) can be simplified by eliminating the presence of the temperature to obtain

$$
\begin{aligned}
&\frac{\partial W}{\partial t}+w_b\frac{\partial W}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 W}{\partial \phi^2}+2w_b\,W,\\
&r=1:\quad \frac{\partial W}{\partial z}=0,\quad \frac{\partial W}{\partial \phi}=0.
\end{aligned}
\tag{16}
$$

By assuming the disturbances to have the form of normal modes,

$$
W(r,t)=h(r)\cos(m\,\phi)e^{\lambda\,t}e^{i\,k\,z},
\tag{17}
$$

one obtains

$$
\begin{aligned}
&\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}h}{\mathrm{d}r}\right)+\left(2\,w_b-\lambda-\frac{m^2}{r^2}-i\,k\,w_b\right)\,h=0,\\
&r=1:\quad h=0.
\end{aligned}
\tag{18}
$$

In Eq. (17) h is a complex eigenfunction, m is a non negative integer while $\lambda=\lambda_r-i\omega$, λ_r is the growth/damping rate, ω is the angular frequency and k is the wavenumber.

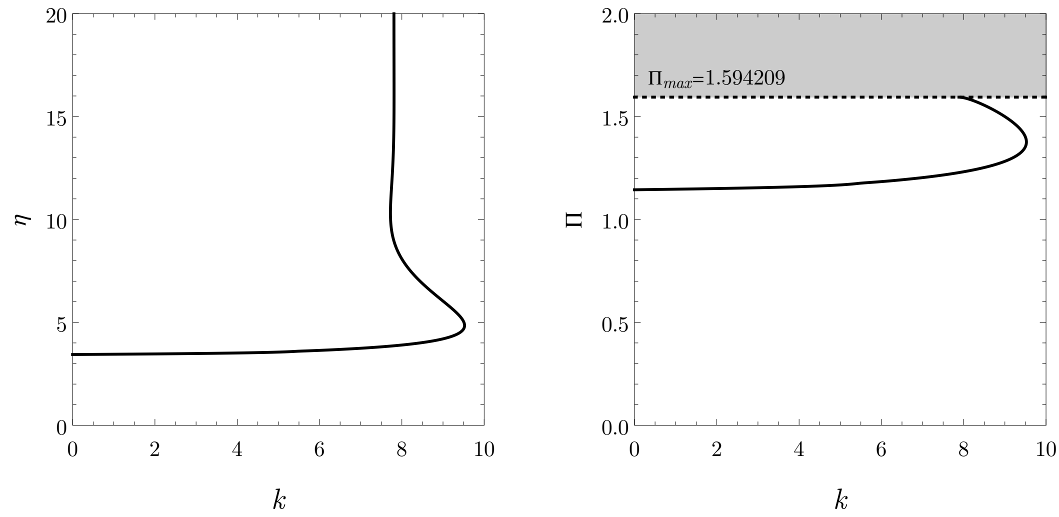

Figure 3. Neutral stability curve for $m = 0$: $\eta(k)$ on the left hand frame and $\Pi(k)$ on the right hand frame.

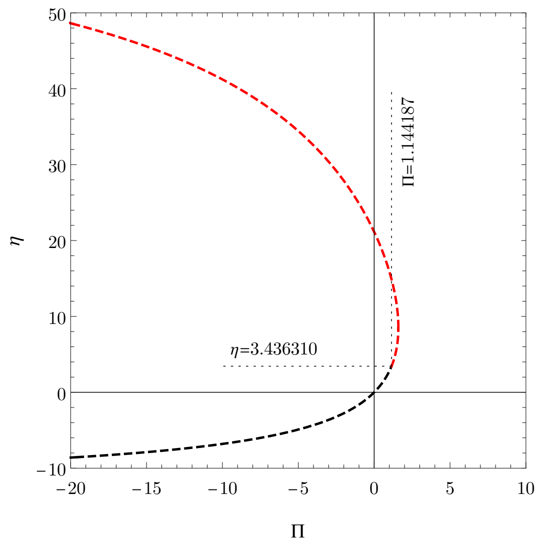

Figure 4. Possibly unstable basic solutions, red dashed curve, against the stable basic solutions, black dashed curve.

5. Discussion of the Results

The solutions of Eq. (18) are reported in Fig. 3. This figure displays the curves $\eta(k)$ and $\Pi(k)$ obtained by fixing the growth rates $\lambda_r = 0$ and for the modes defined by $m = 0$. For the case $m = 1$ and $m = 2$, no solutions charaterised by $\lambda_r = 0$ are obtained and the growth rates λ_r are always negative. One can conclude that, except for the axisymmetric modes characterised by $m = 0$, the other modes are stable.

The left and the right frame of Fig. 3 report the same neutral stability; they are reported by employing either η or Π. For the interpretation of Fig. 3 is necessary to look at Fig. 4 where the basic solutions that can possibly become unstable are identified by a red dashed line.

By comparing Fig. 3 and Fig. 4 one can conclude that the basic solutions characterised by $\eta < 3.436310$ cannot become unstable. These solutions are, in fact, endowed with negative growth rates. It is worth noting that right–hand frame of Fig. 3 displays a gray area for $\Pi > \Pi_{max}$. One has to disregard this area since the parameter Π shows a maximum, $\Pi_{max} = 1.594209$, in its range of variation.

6. Conclusions

The stability analysis presented in this paper aims to investigate the role of viscous dissipation for mixed convection flows in a fluid saturated vertical porous pipe. The problem presents dual stationary solutions whose linear stability is studied. The main conclusions one can draw from this analysis are the following:

- The basic stationary solutions are dual: for a given value of the imposed pressure gradient, two different possible velocity profiles exist;
- There exists a maximum value of basic pressure gradient beyond which no basic stationary solution is obtained;
- Part of the basic stationary solutions are always linearly stable, which belong to the lower branch of the dual flows;
- Only the axisymmetric disturbances can trigger the instability.

Acknowledgments

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