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Scalar fields in the de Sitter spacetime

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Abstract. We examine long-wavelength correlation functions of massive scalar fields in de Sitter spacetime. For the theory with a quartic self-interaction, the two-point function is calculated up to two loops. Comparing our results with the Hartree-Fock approximation and with the stochastic approach shows that the former resums only the cactus type diagrams, whereas the latter contains the sunset diagram as well and produces the correct result. We compare our results with the preceding results obtained for the massless scalar field.

1. Introduction

The de Sitter universe is a spacetime with positive constant 4-curvature that is homogeneous and isotropic in both space and time. It is completely characterized by only one constant H and has as many symmetries as the flat (Minkowski) spacetime. De Sitter universe plays a central role in understanding the properties of cosmological inflation. Inflation is a stage of accelerated expansion of the early Universe. The expansion is quasi-exponential, and at lowest order it can be approximated by de Sitter space. The inflationary stage allows the growth of quantum fluctuations, which are necessary to explain observed large-scale structure of the Universe. So it is important to study quantum field theory in de Sitter background. Scalar fields in de Sitter space are of particular importance for understanding the period of inflation and the growth of quantum fluctuations. In the massive case if $m^2 \ll H^2$, the leading contribution to the renormalized correlator $\langle\phi^2\rangle_{\rm ren}$

$$\langle\phi^2(\vec{x},t)\rangle_{\rm ren} = \frac{3H^4}{8\pi^2 m^2} + \mathcal{O}\left(\left(m^2/H^2\right)^0\right), \tag{1}$$

derives entirely from the long-wavelength modes [1, 2, 3, 4]. When there is a self-interaction, each successive term in the weak coupling perturbative expansion contains higher and higher powers of H^2/m^2. The perturbation theory breaks down when the value of H^2/m^2 overwhelms the smallness of the coupling constant.

A non-perturbative method for calculating the expectation values of the coarse-grained theory, containing only the long-wavelength fluctuations of a scalar field, was proposed by

Starobinsky [5, 6]. In the framework of this method the expectation values can be determined by using a probability distribution function that is a solution to a simple Fokker-Planck equation. In the recent work [7] we considered a massive scalar field with a quartic self-interaction. We calculated a long-wavelength part of the two-point function up to two-loop order using the "in-in" formalism [8, 9]. We compare our results with the Hartree-Fock approximation and with the stochastic approach. We argue that the perturbative expression for the two-point function can be reorganized into a sum of exponential functions that depend on the two given points in a de Sitter invariant way.

2. De Sitter space in flat coordinates

We consider the de Sitter spacetime represented as an expanding spatially flat homogeneous and isotropic universe with the following metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)\ , \tag{2}$$

where the scale factor $a(t)$ is

$$a(t) = e^{Ht}\ , -\infty < t < \infty, \tag{3}$$

and H is the Hubble constant that characterizes the rate of expansion.

If we introduce a conformal time coordinate, given by $\eta(t) \equiv \int dt\, a^{-1}(t)$, then

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2), \tag{4}$$

$$a(\eta) = -\frac{1}{H\eta},\ -\infty < \eta < 0. \tag{5}$$

Physical distances are defined as: $\ell_{phys} = a(\eta)\ell = -\ell/(H\eta)$, while physical energy or momentum are given by : $k_{phys} = k/a(\eta) = -kH\eta$.

3. Massive scalar field in de Sitter space

We will study a massive scalar field with a quartic self-interaction

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4\right)\ . \tag{6}$$

When $\lambda = 0$, the equation of motion for the rescaled field $\chi \equiv a(\eta)\phi$ is

$$\chi'' - \nabla^2\chi - \frac{1}{\eta^2}\left(2 - \frac{m^2}{H^2}\right)\chi = 0\ . \tag{7}$$

We expand the field $\phi(\vec{x}, t)$ in terms of creation and annihilation operators

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3}\left\{\phi_k(\eta)e^{i\vec{k}\cdot\vec{x}}a_{\vec{k}} + \phi_k^*(\eta)e^{-i\vec{k}\cdot\vec{x}}a_{\vec{k}}^\dagger\right\}\ , \tag{8}$$

where the mode functions $\chi_k \equiv a(\eta)\phi_k$ obey the differential equation

$$\chi_k'' + k^2\left[1 - \frac{1}{k^2\eta^2}\left(2 - \frac{m^2}{H^2}\right)\right]\chi_k = 0\ , \tag{9}$$

where $k = |\vec{k}|$. The general solution of this equation can be expressed as a linear combination of Hankel functions:

$$\chi_k(\eta) = \sqrt{-k\eta}\left[A_k\, \mathcal{H}_\nu^{(1)}(-k\eta) + B_k\, \mathcal{H}_\nu^{(2)}(-k\eta)\right]\ , \tag{10}$$

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \ . \tag{11}$$

The choice of the coefficients A_k and B_k defines a vacuum state $|0\rangle$ annihilated by $a_{\vec{k}}$.

$$a_{\vec{k}}|0\rangle = 0 \ \text{ for any } \vec{k} \ . \tag{12}$$

If one wants to have a vacuum that in the remote past $\eta \to -\infty$ (or, equivalently, for modes with very short physical wavelength, $-kH\eta \gg H$) behaves like the vacuum in Minkowski spacetime,

$$\chi_k(\eta) \to \frac{e^{-ik\eta}}{\sqrt{2k}} \ , \tag{13}$$

one should choose

$$\chi_k(\eta) = -\frac{\sqrt{\pi}}{2}\sqrt{-\eta}\,\mathcal{H}_\nu^{(1)}\left(-k\eta\right) \ . \tag{14}$$

Such a choice is called the Bunch-Davies vacuum [10]. As long as $m \neq 0$, this state is de Sitter invariant. If $m^2 \ll H^2$, then

$$\nu \approx \frac{3}{2} - u, \text{with } u \equiv \frac{m^2}{3H^2} \ll 1 \ . \tag{15}$$

4. Perturbative calculation of the two-point correlation function

We present a perturbative calculation of the long-wavelength part of the two-point function. There are two reasons why it is meaningful to consider exclusively the long-wavelength modes The first reason is physical. The fluctuations relevant for the formation of the observed large-scale structure of the universe are those whose wavelength, by the end of inflation, has been stretched to a size much larger than the Hubble horizon. The second reason is mathematical. Calculations are much simpler if instead of the exact modes, one uses their long-wavelength limit. At the same time, in many cases the results can reflect the behavior of the untruncated theory. In the small mass limit, the long-wavelength two-point function matches with the untruncated one for large separations and for coinciding spacetime points. The free two-point function in a vacuum state $|0\rangle$ is

$$\begin{aligned} \langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda^0} &\equiv \langle 0|\phi(\vec{x},t_1)\phi(\vec{y},t_2)|0\rangle \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}\phi_k(\eta_1)\phi_k^*(\eta_2)e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2\frac{\sin kr}{kr}\phi_k(\eta_1)\phi_k^*(\eta_2) \ . \end{aligned} \tag{16}$$

Its long-wavelength part consists of modes with physical momenta much less than H:

$$\frac{k}{a(\eta_1)H} = -k\eta_1 < \epsilon \ , \ \ \frac{k}{a(\eta_2)H} = -k\eta_2 < \epsilon \ , \tag{17}$$

where $\epsilon \ll 1$. In this limit

$$\phi_k(\eta) \ \approx \ \frac{iH}{\sqrt{2}}(-\eta)^{3/2}(-k\eta)^{-\nu} = \frac{iH}{\sqrt{2k^3}}(-k\eta)^u. \tag{18}$$

The long-wavelength part of the two-point function is

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda^0,L} = \frac{H^2(\eta_1\eta_2)^u}{4\pi^2}\int_0^{-\epsilon/\eta_{\rm m}}\frac{dk}{k}\frac{k^{2u}\sin(kr)}{kr} \ , \tag{19}$$

where $\eta_{\rm m}$ is the earliest time that accompanies the momentum $\vec{k}$: $\eta_{\rm m} \equiv \min(\eta_1, \eta_2)$. In the case of coinciding spacetime points

$$\langle \phi^2(\vec{x}, t) \rangle_{\lambda^0, L} = \frac{H^2(-\eta)^{2u}}{4\pi^2} \int_0^{-\epsilon/\eta} \frac{dk}{k^{1-2u}} = \frac{H^2}{8\pi^2} \frac{\epsilon^{2u}}{u}. \tag{20}$$

If

$$\exp\left(-u^{-1}\right) \ll \epsilon \ll 1, \tag{21}$$

then ϵ^{2u} may be replaced by 1 and

$$\langle \phi^2(\vec{x}, t) \rangle_{\lambda^0, L} = \frac{H^2}{8\pi^2 u} = \frac{3H^4}{8\pi^2 m^2} . \tag{22}$$

Given two points in de Sitter space, there is a de Sitter invariant function associated with them [11, 4]

$$Z(X, Y) = -H^2 \eta_{\mu\nu} X^\mu Y^\nu, \tag{23}$$

where X and Y represent coordinates in five-dimensional Minkowski embedding space with the metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1, -1)$. In spatially flat coordinates

$$Z(\vec{x}_1, \eta_1; \vec{y}, \eta_2) = \frac{\eta_1^2 + \eta_2^2 - |\vec{x} - \vec{y}|^2}{2\eta_1\eta_2} . \tag{24}$$

If points are timelike separated, then $Z > 1$; if points are lightlike separated, then $Z = 1$, and if points are spacelike separated, then $Z < 1$. If $(\vec{x}, t_1)$ and $(\vec{y}, t_2)$ are related in such a way that

$$Z > 1 - \frac{1}{2\epsilon^2} , \tag{25}$$

then

$$-r/\eta_{\rm m} < 1/\epsilon$$

and we have

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2) \rangle_{\lambda^0, L} \quad = \quad \frac{H^2}{8\pi^2 u} e^{-uH|t_1 - t_2|} . \tag{26}$$

There are two important subcases for which the two-point function is given by the above expression. One is when points $(\vec{x}, t_1)$ and $(\vec{y}, t_2)$ are timelike or lightlike separated; the other is when these points have coinciding time coordinates and the physical spatial distance between them satisfies $a(t)r < (\epsilon H)^{-1}$. In the latter case, we obtain the same result as that for the coinciding spacetime points. This means that as far as the long-wavelength correlation function is concerned, there is no difference between coinciding spacetime points and points on a constant time hypersurface that are separated by a proper distance less than $(\epsilon H)^{-1}$.

Let us consider the case when

$$(-r/\eta_{\rm m}) > 1/\epsilon , \tag{27}$$

or

$$Z < 1 - \frac{1}{2\epsilon^2} \ll -1 .$$

It corresponds to the regime of large spacelike separation between points and

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda^0,L} = \frac{H^2}{8\pi^2 u}\left(\frac{\eta_1\eta_2}{r^2}\right)^u = \frac{H^2}{8\pi^2 u}e^{-uH(t_1+t_2)}(rH)^{-2u} \ . \tag{28}$$

In this regime the equal-time two-point function,

$$\langle\phi(\vec{x},t)\phi(\vec{y},t)\rangle_{\lambda^0,L} = \frac{3H^4}{8\pi^2 m^2}(RH)^{-\frac{2m^2}{3H^2}} \ , \tag{29}$$

depends only on the physical spatial distance $R \equiv re^{Ht}$.

The exact (untruncated) two-point correlator function is known, it is expressed through a hypergeometrical functions and its leading in the parameter u terms give the results coinciding with the long-wavelength correlators, presented above for the cases of coinciding spacetime points $Z = 1$ and the points separated by large timelike or spacelike intervals $|Z| \gg 1$.

Just as in flat spacetime, the expectation value of the commutator of two fields vanishes for spacelike separated points and is nonzero for timelike separated points. However, for the long-wavelength fields

$$\langle[\phi(\vec{x},t_1),\phi(\vec{y},t_2)]\rangle_{\lambda^0,L} = 0 \tag{30}$$

both for timelike and spacelike related points. The vanishing of this commutator indicates that the long-wavelength part of the field in a sense behaves like a classical quantity.

5. Schwinger–Keldysh technique

Schwinger-Keldysh or "in-in" or "closed time path" formalism [8, 9] serves for the calculations of expectation values of operators when only the initial state of the system is given. In contrast to the "in-out" formalism there are four types of the propagators and two types of vertices, characterizing the quantum fields on the way forward in time and "back in time". After some calculations one remains with the integrals including Wightman functions and theta-functions. Diagrams that correspond to these integrals look similar to Feynman diagrams. The one-loop correction to the two-point function is given by the following diagram:

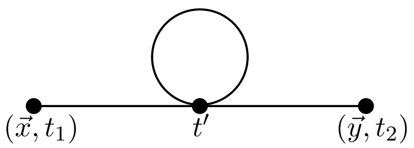

Fig.1 One-loop "seagull" diagram.

In the case of the timelike, lightlike or small spacelike separation between the points we have

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda,L} = -\frac{\lambda H^2}{64\pi^4 u^3}\Big(1 + uH|t_1 - t_2|\Big)e^{-uH|t_1-t_2|} \ . \tag{31}$$

In the case of coinciding spacetime points one finds

$$\langle\phi^2(\vec{x},t)\rangle_{\lambda,L} = -\frac{27\lambda H^8}{64\pi^4 m^6} \ . \tag{32}$$

The long-wavelength correlation function at large spacelike separations is

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_{\lambda,L} \tag{33}$$
$$= -\frac{\lambda H^2}{64\pi^4 u^3}\left\{1+u\ln\left(r^2H^2e^{H(t_1+t_2)}\right)\right\}e^{-uH(t_1+t_2)}(rH)^{-2u}\ . \tag{34}$$

The effective parameter of the perturbative expansion is not λ but λ/u^2, so the perturbation theory is valid as long as $\lambda \ll m^4/H^4$. To calculate the two-loop contribution to the two-point correlator, we should consider three diagrams. First of them is the diagram with two independent loops.

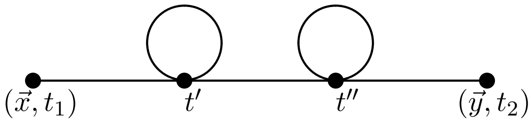

Fig.2 The “double seagull” diagram with two independent loops.

When $Z > 1-(2\epsilon^2)^{-1}$, we obtain

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle^{(1)}_{\lambda^2,L} \tag{35}$$
$$= \frac{\lambda^2H^2}{512\pi^6u^5}\left(1+uH|t_1-t_2|+\frac{1}{2}u^2H^2|t_1-t_2|^2\right)e^{-uH|t_1-t_2|}\ , \tag{36}$$

which for coinciding spacetime points becomes

$$\langle\phi^2(\vec{x},t)\rangle^{(1)}_{\lambda^2,L} = \frac{243\lambda^2H^{12}}{512\pi^6m^{10}}\ . \tag{37}$$

In $Z < 1-(2\epsilon^2)^{-1}$ regime, we have

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle^{(1)}_{\lambda^2,L} \tag{38}$$
$$= \frac{\lambda^2H^2}{512\pi^6u^5}\left\{1+u\ln\left(r^2H^2e^{H(t_1+t_2)}\right)\right. \tag{39}$$
$$\left.+\frac{1}{2}u^2\ln^2\left(r^2H^2e^{H(t_1+t_2)}\right)\right\}e^{-uH(t_1+t_2)}(rH)^{-2u}\ . \tag{40}$$

The second diagram can be called "snowman".

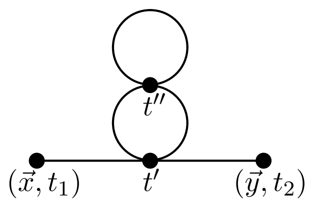

Fig. 3 The "snowman" diagram.

In the case $Z > 1 - (2\epsilon^2)^{-1}$ it gives

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle^{(2)}_{\lambda^2,L} = \frac{\lambda^2 H^2}{512\pi^6 u^5}\Big(1 + uH|t_1 - t_2|\Big)e^{-uH|t_1-t_2|}\ , \tag{41}$$

which for coinciding spacetime points reduces to

$$\langle\phi^2(\vec{x},t)\rangle^{(2)}_{\lambda^2,L} = \frac{243\lambda^2 H^{12}}{512\pi^6 m^{10}}\ . \tag{42}$$

When $Z < 1 - (2\epsilon^2)^{-1}$, we obtain

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle^{(2)}_{\lambda^2,L} \tag{43}$$

$$= \frac{\lambda^2 H^2}{512\pi^6 u^5}\left\{1 + u\ln\Big(r^2 H^2 e^{H(t_1+t_2)}\Big)\right\} e^{-uH(t_1+t_2)}(rH)^{-2u}\ . \tag{44}$$

The last two-loop diagram is "sunset".

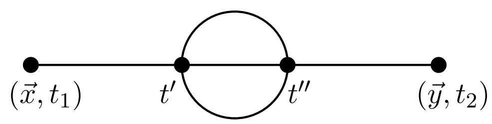

Fig. 4 The "sunset" diagram.

For $Z > 1 - (2\epsilon^2)^{-1}$, it gives

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle^{(3)}_{\lambda^2,L} = \frac{\lambda^2 H^2}{1024\pi^6 u^5}\Big(1 + 2uH|t_1 - t_2|\Big)e^{-uH|t_1-t_2|} \tag{45}$$

$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5}e^{-3uH|t_1-t_2|}\ . \tag{46}$$

For coinciding spacetime points, it reduces to

$$\langle\phi^2(\vec{x},t)\rangle^{(3)}_{\lambda^2,L} = \frac{81\lambda^2 H^{12}}{256\pi^6 m^{10}}\ . \tag{47}$$

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In the opposite regime, $Z < 1 - (2\epsilon^2)^{-1}$, we obtain

$$\langle \phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle^{(3)}_{\lambda^2, L} \tag{48}$$

$$= \frac{\lambda^2 H^2}{1024\pi^6 u^5}\left\{1 + 2u\ln\left(r^2 H^2 e^{H(t_1+t_2)}\right)\right\} e^{-uH(t_1+t_2)}(rH)^{-2u}$$
$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5} e^{-3uH(t_1+t_2)}(rH)^{-6u} \; . \tag{49}$$

6. Comparison with the Hartree-Fock approximation and with the stochastic approach

Starting with the Klein-Gordon equation and using the Hartree-Fock (Gaussian) approximation

$$\langle \phi^4\rangle = 3\langle\phi^2\rangle^2, \tag{50}$$

we arrive to the following equation for the two-point correlator:

$$\frac{\partial}{\partial t}\langle\phi^2\rangle_L = \frac{H^3}{4\pi^2} - \frac{2m^2}{3H}\langle\phi^2\rangle_L - \frac{2\lambda}{H}\langle\phi^2\rangle_L^2 \; . \tag{51}$$

As $t \to \infty$, all the solutions to this equation approach an equilibrium value that satisfies

$$\frac{H^3}{4\pi^2} - \frac{2m^2}{3H}\langle\phi^2\rangle_L - \frac{2\lambda}{H}\langle\phi^2\rangle_L^2 = 0 \; . \tag{52}$$

For $\lambda = 0$, we have

$$\langle\phi^2\rangle_L = \frac{3H^4}{8\pi^2 m^2} \; . \tag{53}$$

When $\lambda \neq 0$, we have

$$\langle\phi^2\rangle_L = \frac{m^2}{6\lambda}\left(\sqrt{1 + \frac{9\lambda H^4}{2\pi^2 m^4}} - 1\right) ; \tag{54}$$

we chose the root that coincides with the preceding expression in the limit $\lambda \to 0$. Assuming that $\lambda H^4/m^4 \ll 1$, and expanding the preceding expression yields

$$\langle\phi^2\rangle_L = \frac{3H^4}{8\pi^2 m^2} - \frac{27\lambda H^8}{64\pi^4 m^6} + \frac{243\lambda^2 H^{12}}{256\pi^6 m^{10}} + \mathcal{O}(\lambda^3) \; . \tag{55}$$

Comparing this expansion with the results obtained by the field-theoretical methods, we see that they match at zeroth- and first-order in λ, but there is a mismatch at second order: the λ^2-term omits the contribution of the sunset diagram and is equal to the sum of other two diagrams. Hence, it can be concluded that the Hartree-Fock approximation resums all cactus type diagrams of the perturbation theory

The stochastic approach [5, 6] argues that the behavior of the long-wavelength part of the quantum field $\phi(\vec{x}, t)$ in de Sitter space can be modelled by an auxiliary classical stochastic variable φ with a probability distribution $\rho(\varphi, t)$ that satisfies the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = \frac{H^3}{8\pi^2}\frac{\partial^2\rho}{\partial\varphi^2} + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\frac{\partial V}{\partial\varphi}\rho(t,\varphi)\right). \tag{56}$$

In our case the potential has the form

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\varphi^4 \; . \tag{57}$$

At late times any solution of the Fokker-Planck equation approaches the static equilibrium solution

$$\rho_{\rm eq}(\varphi) = N^{-1} \exp\left(-\frac{8\pi^2}{3H^4}V(\varphi)\right), \tag{58}$$

where N is the normalization constant fixed by the condition

$$\int_{-\infty}^{\infty} \rho_{\rm eq}(\varphi)\, d\varphi = 1 \ . \tag{59}$$

In our case we can calculate this normalization explicitly

$$\begin{aligned} N &= \int_{-\infty}^{\infty} \exp\left[-\frac{8\pi^2}{3H^4}\left(\frac{\lambda\varphi^4}{4} + \frac{m^2\varphi^2}{2}\right)\right] d\varphi \\ &= \frac{m}{\sqrt{2\lambda}} \exp(z)\mathcal{K}_{\frac{1}{4}}(z) \ , \end{aligned} \tag{60}$$

where $\mathcal{K}_{\frac{1}{4}}(z)$ is a modified Bessel function of the second kind, and $z \equiv \frac{\pi^2 m^4}{3\lambda H^4}$.

Using this equilibrium distribution, we obtain

$$\langle\varphi^2\rangle = \frac{m^2}{2\lambda}\frac{\mathcal{K}_{\frac{3}{4}}(z)}{\mathcal{K}_{\frac{1}{4}}(z)} - \frac{m^2}{2\lambda}. \tag{61}$$

Expanding this in the limit $\lambda H^4/m^4 \ll 1$ (which corresponds to $z \gg 1$) gives

$$\langle\varphi^2\rangle = \frac{3H^4}{8\pi^2 m^2} - \frac{27\lambda H^8}{64\pi^4 m^6} + \frac{81\lambda^2 H^{12}}{64\pi^6 m^{10}} + \mathcal{O}(\lambda^3) \ . \tag{62}$$

This result is in agreement with the result of the quantum field theory calculations, and unlike the Hartree-Fock approximation, it includes the contribution of the sunset diagram. The long-wavelength two-point function of $\phi(\vec{x}, t)$ too can be calculated by using the classical stochastic variable φ: if the points $(\vec{x}, t_1)$ and $(\vec{y}, t_2)$ are timelike or lightlike related, this correlation function is given by

$$\langle\phi(\vec{x}, t_1)\phi(\vec{y}, t_2)\rangle_L = \langle\varphi(t_1)\varphi(t_2)\rangle \ . \tag{63}$$

If the correlation function $\langle\varphi(t_1)\varphi(t_2)\rangle$ depends only on the absolute value of the time difference $T \equiv |t_1 - t_2|$, it can be expressed as

$$\langle\varphi(t_1)\varphi(t_2)\rangle = \int_{-\infty}^{\infty} \varphi\, \Xi(\varphi, T) d\varphi \ , \tag{64}$$

where the function $\Xi(\varphi, T)$ satisfies the Fokker-Planck equation,

$$\frac{\partial\Xi}{\partial T} = \frac{H^3}{8\pi^2}\frac{\partial^2\Xi}{\partial\varphi^2} + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\frac{\partial V}{\partial\varphi}\Xi(\varphi, T)\right) , \tag{65}$$

with the initial condition

$$\Xi(\varphi, 0) = \varphi\rho_{\rm eq}(\varphi) \ . \tag{66}$$

Derivatives of $\langle\varphi(t_1)\varphi(t_2)\rangle$ at $T = 0$ can be computed by using the equations above:

$$\frac{\partial}{\partial T}\langle\varphi(t_1)\varphi(t_2)\rangle\Big|_{T=0} = -\frac{H^3}{8\pi^2} \ , \tag{67}$$

$$\frac{\partial^2}{\partial T^2}\langle\varphi(t_1)\varphi(t_2)\rangle\Big|_{T=0} = \frac{H^2}{24\pi^2}\Big(3\lambda\langle\varphi^2\rangle + m^2\Big) \ , \tag{68}$$

and so on. It is easy to confirm that the T-derivatives of the two-point correlation function presented earlier (for $Z > 1 - \frac{1}{2\epsilon^2}$ case) satisfy these equalities as well.

7. Exponentiation of the perturbative series

The expression for the two-point correlation function can be presented in the following way: (the case $Z > 1 - \frac{1}{2\epsilon^2}$):

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_L = \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4}\right)e^{-uHT} \tag{69}$$

$$-\frac{\lambda H^3 T}{64\pi^4 u^2}\left(1 - \frac{3\lambda}{8\pi^2 u^2}\right)e^{-uHT}$$

$$+\frac{\lambda^2 H^4 T^2}{1024\pi^6 u^3}e^{-uHT} + \frac{\lambda^2 H^2}{3072\pi^6 u^5}e^{-3uHT} + \mathcal{O}(\lambda^3) \tag{70}$$

$$= \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4} + \mathcal{O}(\lambda^3)\right) \tag{71}$$

$$\times\left[1 - \frac{\lambda HT}{8\pi^2 u} + \frac{\lambda^2 HT}{32\pi^4 u^3} + \frac{1}{2}\left(\frac{\lambda HT}{8\pi^2 u}\right)^2 + \mathcal{O}(\lambda^3)\right]e^{-uHT}$$

$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5}e^{-3uHT} + \dots\,, \tag{72}$$

To second order in λ, the expression in squared brackets matches with the first three terms in the Taylor series of the exponential function

$$\exp\left[-\frac{\lambda HT}{8\pi^2 u}\left(1 - \frac{\lambda}{4\pi^2 u^2}\right)\right], \tag{73}$$

so it is plausible that an infinite series of diagrams may be resummed into this exponent. With this assumption, we arrive at

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_L = \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4} + \mathcal{O}(\lambda^3)\right) \tag{74}$$

$$\times\exp\left[-uHT\left(1 + \frac{\lambda}{8\pi^2 u^2} - \frac{\lambda^2}{32\pi^4 u^4} + \mathcal{O}(\lambda^3)\right)\right]$$

$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5}\exp\Big[-3uHT\Big] + \dots\,. \tag{75}$$

Analogously for $Z < 1 - \frac{1}{2\epsilon^2}$:

$$\langle\phi(\vec{x},t_1)\phi(\vec{y},t_2)\rangle_L = \frac{H^2}{8\pi^2 u}\left(1 - \frac{\lambda}{8\pi^2 u^2} + \frac{5\lambda^2}{128\pi^4 u^4} + \mathcal{O}(\lambda^3)\right) \tag{76}$$

$$\times\left(r^2 H^2 e^{H(t_1+t_2)}\right)^{-u\left(1+\frac{\lambda}{8\pi^2 u^2}-\frac{\lambda^2}{32\pi^4 u^4}+\mathcal{O}(\lambda^3)\right)}$$

$$+\frac{\lambda^2 H^2}{3072\pi^6 u^5}\left(r^2 H^2 e^{H(t_1+t_2)}\right)^{-3u} + \dots\,, \tag{77}$$

In this regime the equal-time correlation function depends only on the physical spatial distance $R \equiv re^{Ht}$.

We see that the perturbative corrections don't change the long-wavelength part of the commutator: just as in the free theory case, it is equal to zero both for timelike and spacelike related points.

As $T \to \infty$, the two-point function decays with the characteristic correlation time

$$T_c \sim \frac{1}{uH} = \frac{3H}{m^2} \gg \frac{1}{H} \; . \tag{78}$$

Similarly, as $R \to \infty$, the equal-time correlation function decays with the characteristic correlation length

$$R_c \sim \frac{1}{H} \exp\left(\frac{3H^2}{2m^2}\right) \; . \tag{79}$$

This behavior differs from a much faster exponential decay of the equal-time correlation function in flat spacetime: $\langle\phi(\vec{x},t)\phi(\vec{y},t)\rangle_{\text{flat}} \sim \sqrt{m/r^3}e^{-mr}$ as $r \to \infty$.

8. Relations with other works

The results of our paper coincide with those, obtained by other methods in papers [12, 13], where the p-representation for the correlators of the scalar fields on the flat de Sitter background [14] was used.

In our preceding paper [15] we considered a massless scalar field with quartic self-interaction in de Sitter spacetime and developed a semi-heuristic method for taking the late-time limit of a series of secularly growing terms obtained from quantum perturbative calculations. We compared our results with those coming from the stochastic approach and have found an astonishing agreement.

Our proposal was inspired by the renormalization group: to construct a simple autonomous first-order differential equation

$$\frac{df}{dt} = F[f(t)]$$

that produces the first terms of the expression for correlators, obtained by perturbative methods. Then, we can solve approximately this autonomous equation, obtaining the expressions without secular growth, while the initial terms of the perturbative expansion were proportional to the powers of the cosmic time parameter t.

9. Conclusions

We have calculated—up to two loops—the long-wavelength two-point function for a scalar theory with a small mass and a quartic interaction.

It has been shown that it is de Sitter invariant for coinciding points as well as at large spacelike and large timelike separations. We have demonstrated that the commutator of the long-wavelength part of the field is equal to zero both at the free theory level and with the perturbative corrections.

Our results are in agreement with Starobinsky's stochastic approach in which the coarse-grained quantum field is equivalent to a classical stochastic quantity. It would be interesting but more difficult to consider similar problems on more general backgrounds.

References

[1] Linde A D 1982 *Phys. Lett.* B **116** 335
[2] Starobinsky A A 1982 *Phys. Lett.* B **117** 175
[3] Vilenkin A and Ford L H 1982 *Phys. Rev.* D **26** 1231
[4] Allen B and Folacci A 1987 *Phys. Rev.* D **35** 3771
[5] Starobinsky A A 1986 Lect. Notes Phys. **246** 107
[6] Starobinsky A A and Yokoyama J 1994 *Phys. Rev.* D **50** 6357
[7] Kamenshchik A Y, Starobinsky A A and Vardanyan T 2022 *Eur. Phys. J* C **82** 345
[8] Schwinger J S 1961 *J. Math. Phys.* **2** 407
[9] L. V. Keldysh L V 1965 *Sov. Phys. JETP* **20** 1018

[10] Bunch T S and Davies P C W 1978 *Proc. Roy. Soc. Lond.* A **360** 117
[11] Allen B 1985 *Phys. Rev.* D **32** 3136
[12] Gautier F and Serreau J 2013 *Phys. Lett.* B **727** 541
[13] Gautier F and Serreau J 2015 *Phys. Rev.* D **92** 105035
[14] Parentani R and Serreau J 2013 *Phys. Rev.* D **87** 045020
[15] Kamenshchik A Y and Vardanyan T 2020 *Phys. Rev.* D **102** 065010