

REPRESENTATIONS FROM GROUP ACTIONS ON WORDS AND MATRICES

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ABSTRACT

Representations from Group Actions on Words and Matrices

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We provide a combinatorial interpretation of the frequency of any irreducible representation of S_n in representations of S_n arising from group actions on words. Recognizing that representations arising from group actions naturally split across orbits yields combinatorial interpretations of the irreducible decompositions of representations from similar group actions. The generalization from group actions on words to group actions on matrices gives rise to representations that prove to be much less transparent. We share the progress made thus far on the open problem of determining the irreducible decomposition of certain representations of $S_m \times S_n$ arising from group actions on matrices.

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TABLE OF CONTENTS

	Page
LIST OF TABLES	viii
CHAPTER	
1 Preliminaries	1
1.1 Permutations	1
1.2 Partitions and Young Diagrams	2
1.3 The Symmetric Group	4
1.4 Group Actions	7
2 Introduction to Representation Theory of Finite Groups	12
2.1 Matrix Representations	12
2.2 Combining Representations	16
2.3 Irreducible Representations	20
2.4 Characters	27
2.5 Inner Products	31
2.6 Modules and Group Algebras	39
2.7 Character Tables	46
3 Irreducible Representations of the Symmetric Group	50
3.1 Tableau and Tabloids	50
3.2 Specht Modules	53
3.3 Irreducibility of Specht Modules	55
3.4 Dimensions of the Irreducible Representations	59
4 Symmetric Functions and the Murnaghan-Nakayama Rule	65
4.1 Standard Bases for Symmetric Functions	65

4.2	Rim-Hook Tableaux	68
4.3	Relationships Between Bases	71
4.4	The Frobenius Map	80
4.5	Character Tables of the Symmetric Group	81
5	Representations from an Action on Words	85
5.1	A Combinatorial Interpretation	85
5.2	Generalizations from Restricting the Action to an Orbit	87
5.3	The Robinson-Schensted-Knuth Correspondence	94
6	Representations from an Action on Matrices	99
6.1	Character Tables of $S_m \times S_n$	99
6.2	An Open Problem	101
6.3	Current Progress	104
	BIBLIOGRAPHY	110

LIST OF TABLES

Table		Page
2.1	Character Table of S_3	48
4.1	Character Table of S_4	83
4.2	Character Table of S_5	84
4.3	Character Table of S_6	84
6.1	Example Irreducible Decomposition of X_A	108

Chapter 1

PRELIMINARIES

In this chapter, we present background information necessary to entering the realm of representation theory. We assume the reader has an understand of linear algebra and an understanding of the fundamentals of group theory. We specifically introduce permutations, partitions, the symmetric group, and group actions.

1.1 Permutations

Definition 1.1.1. A *permutation* of a positive integer n is a bijection $\sigma : [n] \rightarrow [n]$ where $[n]$ denotes the set of integers from 1 to n .

Example 1.1.2. We can represent permutations in various notation. Consider the permutation σ of 6 defined by the following mappings:

$$\begin{array}{lll} 1 \mapsto 3 & 2 \mapsto 2 & 3 \mapsto 6 \\ 4 \mapsto 5 & 5 \mapsto 4 & 6 \mapsto 1 \end{array}$$

In *two line notation*, we write the integers 1 through n in one line and we write their image under σ directly below to form another line:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 5 & 4 & 1 \end{array}$$

In *one line notation*, we omit top line of two line notation since it provides no new information and we write:

$$3 \ 2 \ 6 \ 5 \ 4 \ 1$$

Since 4 maps to 5 which maps to 4, we say that $(4 \ 5)$ is a cycle of σ . Since 3 maps to 6 which maps to 1 which maps to 3, $(3 \ 6 \ 1)$ is a cycle of σ . Since 2 maps to itself, it is in a 1-cycle: (2) . Listing these cycles side by side,

$$\sigma = (45)(361)(2)$$

is a way of writing our permutation in *cycle notation*.

At the moment however, this is not unique, the choice to begin our cycles at 4, 3, and 2 was arbitrary, as was the choice to arrange the cycles in this particular order. Standard convention is to write cycles with their smallest element first in order of increasing smallest element,

$$\sigma = (136)(2)(45)$$

called *canonical cycle notation*. We often omit 1-cycles for brevity and simply write

$$\sigma = (136)(45).$$

1.2 Partitions and Young Diagrams

Definition 1.2.1. A *partition* of a positive integer n is a multiset of positive integers that sum to n .

We write $\lambda \vdash n$ to indicate that λ is a partition of n .

Example 1.2.2. By convention we write partitions as a list ordered from largest part to smallest part. For example,

$$\lambda_1 = 4 \quad \lambda_2 = 3 \ 1 \quad \lambda_3 = 2 \ 2 \quad \lambda_4 = 2 \ 1 \ 1 \quad \lambda_5 = 1 \ 1 \ 1 \ 1$$

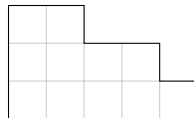
are the five partitions of 4. As a shorthand, the partition $\lambda = 4 \ 4 \ 4 \ 3 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \vdash 20$ can be written as $4^3 \ 3 \ 1^5 \vdash 20$.

We can represent partitions visually with Young diagrams.

Definition 1.2.3. A *Young diagram* for a partition λ is a grid of cells where each row represents one part of λ by setting the length of that row equal to that part of λ .

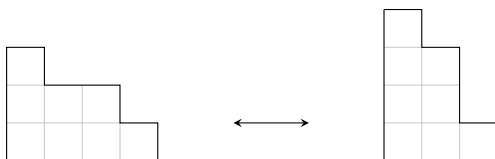
The two conventions for drawing these diagrams are English notation and French notation, putting the largest part in the top row or in the bottom row respectively.

We will adopt the French notation. For example,



is the Young diagram for the partition $5 \ 4 \ 2$.

Example 1.2.4. Young diagrams are useful in proving facts about partitions. For example, the number of partitions of n with largest part k is equal to the number of partitions of n into at most k parts. We can prove this by describing a bijection between the set of partitions of n with largest part k and the set of partitions of n into at most k parts. An elegant visualization of such a bijection is recognized by representing these partitions with their Young diagrams



and reflecting across the diagonal through the bottom left corner. This pairs every partition that had largest part k with a partition into at most k parts.

Describing bijections by their effect on Young Diagrams is effective in proving a host of other partition identities. Furthermore, this strategy of visually representing something combinatorial, like our partitions, as an object or diagram will prove itself to be tremendously fruitful. It will allow us to prove algebraic statements combinatorially through bijections between particular sets of objects.

1.3 The Symmetric Group

Definition 1.3.1. For a positive integer n , the *symmetric group* S_n is the set of all permutations of n equipped with the operation of function composition.

Example 1.3.2. Using $*$ to denote our group operation, we observe that

$$(134)(26) * (24)(35) = (135462)$$

by first applying $(24)(35)$ and then applying $(134)(26)$ to each number in $[n]$. It is not difficult to verify that this is in fact a group.

Definition 1.3.3. A *transposition* is a permutation that consists of a single 2-cycle.

Definition 1.3.4. If a transposition has the form $(i, i + 1)$ for some positive integer i , we say that it is an *adjacent transposition*.

Proposition 1.3.5. *Every permutation can be written as a product of adjacent transpositions.*

Definition 1.3.6. We say that a permutation is *even* if it can be written as a product of an even number of transpositions. We say that a permutation is *odd* if it can be written as a product of an odd number of transpositions.

There are multiple ways to write a given permutation as a product of transpositions, but parity of permutations is nevertheless well defined. Every permutation is either even or odd.

Example 1.3.7. We could rewrite the permutation $\sigma = (136)(45)$ as

$$\sigma = (13) * (36) * (45),$$

a product of transpositions. Splitting up each of these transpositions, we could write σ as

$$\sigma = (12) * (23) * (12) * (34) * (45) * (56) * (45) * (34) * (45),$$

a product of adjacent transpositions. Since σ can be expressed as a product of 3 or 9 transpositions, σ is an odd permutation.

If two even permutations are multiplied, or if two odd permutations are multiplied, the result will be even. If an even permutation is multiplied with an odd permutation, the result will be odd.

Definition 1.3.8. The *cycle type* of a permutation is the partition consisting of the length of each of its cycles.

For example, if $n = 7$ and $\sigma = (134)(26)$, then the cycle type of σ is the partition

3 2 1 1

because σ has one cycle of length 3, one cycle of length 2, and 2 cycles of length 1: (5) and (7).

Definition 1.3.9. Two group elements $a, b \in G$ are *conjugate* if there exists another $g \in G$ such that $gbg^{-1} = a$.

Conjugacy defines an equivalence relation on the group. The equivalence classes of this relation are called *conjugacy classes*. It turns out that the conjugacy classes of the symmetric group are determined by cycle type.

Theorem 1.3.10. *Let $\sigma, \tau \in S_n$. We have that σ is in the same conjugacy class as τ if and only if σ and τ have the same cycle type.*

Proof. Let $\sigma, \tau \in S_n$. If $\tau(i) = j$, then $(\sigma\tau\sigma^{-1})(\sigma(i)) = \sigma(\tau\sigma^{-1}\sigma(i)) = \sigma(\tau(i)) = \sigma(j)$. Since σ is a permutation, every number in $[n]$ is equal to $\sigma(i)$ for some i . When we write the cycle notation for τ , $\tau(i) = j$ means that j follows i in a cycle. Since this implies $(\sigma\tau\sigma^{-1})(\sigma(i)) = \sigma(j)$, the cycle notation for $\sigma\tau\sigma^{-1}$ will have $\sigma(j)$ follow $\sigma(i)$ in a cycle. Thus, we can write $\sigma\tau\sigma^{-1}$ in cycle notation by substituting $\sigma(i)$ in for i in the cycle expression of τ . As a consequence, $\sigma\tau\sigma^{-1}$ has the same cycle type as σ .

Furthermore, for any two permutations σ and τ of the same cycle type, there exists a third permutation π by which we can conjugate to send one to the other. We can describe π explicitly. List σ in cycle notation. Then list τ in cycle notation directly below σ , ordering the cycles in a way that matches the lengths of the cycles of σ above it. This may not be unique. Define π to be the permutation that maps any number in σ to the number immediately below it in τ .

For example if $n = 7$, $\sigma = (236)(45)$, and $\tau = (17)(465)$, we write

$$\sigma = (1) (2\ 3\ 6) (4\ 5) (7)$$

$$\tau = (2) (4\ 6\ 5) (1\ 7) (3)$$

rearranging τ to match the cycle structure of σ . We read off π in two line notation,

$$\pi = \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ 2 & 4 & 6 & 1 & 7 & 5 & 3 & \end{array} \quad (1.1)$$

by removing the parenthesis and rearranging the columns so the top row is increasing. Observe, $\pi\sigma\pi^{-1} = \tau$. Therefore, having the same cycle type is equivalent to being conjugate. \square

Since cycle types are partitions, we can index the conjugacy classes of S_n by partitions.

1.4 Group Actions

Definition 1.4.1. Let C be a set and G be a group. An *action* of G on C is a function $\phi : G \times C \mapsto C$ that satisfies

1. $\phi(e, c) = c$
2. $\phi(g, \phi(h, c)) = \phi(gh, c)$ for all $g, h \in G$

for any $c \in C$ where e represents the identity in G .

When there is only one action defined, we use the shorthand notation

$$g \cdot c := \phi(g, c). \quad (1.2)$$

An action is a way for a group to operate on a set. In other words, it can also be viewed as a group homomorphism from G to the set of bijections of C .

Definition 1.4.2. If C is a set along with an action of a group G on C , we say that C is a G -set.

Definition 1.4.3. For a finite group G with a finite G -set C , we define

1. the *stabilizer* of $c \in C$ to be the subgroup $G_c := \{g \in G : g \cdot c = c\}$.
2. the *fixed point set* of $g \in G$ to be the subset $C^g := \{c \in C : g \cdot c = c\}$.
3. the *orbit* of $c \in C$ to be the set $G \cdot c = \{g \cdot c : g \in G\}$.

Example 1.4.4. Consider $C = \{112, 113, 121, 123, 131, 132, 211, 213, 231, 311, 312, 321\}$, the set of all words whose letters come from the multiset $\mathcal{A} = \{1, 1, 2, 3\}$. The multiplication

$$\sigma \cdot (c_1 c_2 c_3) = c_{\sigma(1)} c_{\sigma(2)} c_{\sigma(3)} \tag{1.3}$$

has $\sigma \cdot c \in C$ for all $\sigma \in S_n$ and $c \in C$ and satisfies the two properties in Definition 1.4.1. So it defines a group action of S_3 on C , allowing us to view C as an S_3 -set. Lets compute the stabilizer, orbit, and fixed point sets for some specific elements.

Consider $c = 131 \in C$. We say that the permutations e and (12) *fix* c because $e \cdot c = c$ and $(12) \cdot c = c$. Since these are the only permutations that fix c , the stabilizer of c is

$$(S_3)_c = \{e, (12)\}.$$

The orbit of c is the subset of C that is reached by acting on c , which in this case is the subset consisting of the permutations of c . So the orbit of c is

$$S_3 \cdot c = \{113, 131, 311\}.$$

Consider $\sigma = (12) \in S_3$. The fixed point set of σ consists of the elements of c that are unchanged by switching the first two letters of the word. So the fixed point set

of σ is

$$C^\sigma = \{112, 113\}.$$

Theorem 1.4.5. *For a finite G -set C , the set of all orbits of elements of C partitions the set C .*

Proof. We will prove that the property of two elements sharing an orbit is an equivalence relation. We have reflexivity because $e \cdot c = c$. We also have symmetry because if $g \cdot c_1 = c_2$, then $g^{-1} \cdot c_2 = c_1$.

Assume c_1 shares an orbit with c_2 and c_2 shares an orbit with c_3 . Then there exist $g_1, g_2 \in G$ such that $c_1 = g_1 \cdot c_2$ and $c_2 = g_2 \cdot c_3$. Substituting, $c_1 = g_1 \cdot (g_2 \cdot c_3)$. By the definition of action, $c_1 = g_1 g_2 \cdot c_3$. So c_1 and c_3 share an orbit, proving transitivity. Since the orbits of C are the equivalence classes of this relation, they partition C . \square

Definition 1.4.6. For a finite G -set C , the *orbit space* C/G is the set of all orbits of elements of C .

Theorem 1.4.7. *Let C be a finite G -set. For any $c \in C$, there is a bijective correspondence between $G \cdot c$ and the quotient group G/G_c .*

Proof. Define the map $\phi : G \mapsto G \cdot c$ by $\phi(g) = g \cdot c$. Since c is acted on by every group element, this map is surjective. Observe that

$$\begin{aligned} \phi(g_1) = \phi(g_2) &\iff g_1 \cdot c = g_2 \cdot c \\ &\iff g_2^{-1} g_1 \cdot c = c \\ &\iff g_2^{-1} g_1 \in G_c \\ &\iff g_1 \in g_2 G_c \end{aligned}$$

where g_2G_c denotes the coset of G_c in G/G_c containing g_2 . It follows that the map $f : G/G_c \mapsto G \cdot c$ defined by $f(gG_c) = g \cdot c$ is a well defined bijection. \square

Corollary 1.4.8. *Let C be a finite G -set. For any $c \in C$, the order of the stabilizer of c times the order of the orbit of c equals the order of the group: $|G_c||G \cdot c| = |G|$.*

Proof. Equating the cardinality of two sets in the aforementioned bijection, $|G/G_c| = |G \cdot c|$. It follows that

$$\frac{|G|}{|G_c|} = |G \cdot c|$$

and multiplying both sides by $|G_c|$ completes the proof \square

Theorem 1.4.9 (Burnside). *For a finite G -set C , we have*

$$|C/G| = \frac{1}{|G|} \sum_{g \in G} |C^g|.$$

Proof. Since C/G is finite, we can index its elements as O_1, \dots, O_n . Observe that

$$|C/G| = \sum_{i=1}^n 1 = \sum_{i=1}^n \sum_{c \in O_i} \frac{1}{|O_i|}.$$

By Theorem 1.4.5, the orbits of C partition C so the double sum can be rewritten as a sum over all elements of C :

$$|C/G| = \sum_{c \in C} \frac{1}{|G \cdot c|}$$

By Corollary 1.4.8,

$$|C/G| = \frac{1}{|G|} \sum_{c \in C} |G_c|. \tag{1.4}$$

Noticing that

$$\sum_{c \in C} |G_c| = |\{(g, c) \in G \times C : g \cdot c = c\}| = \sum_{g \in G} |C^g| \quad (1.5)$$

completes the proof.

□

Chapter 2

INTRODUCTION TO REPRESENTATION THEORY OF FINITE GROUPS

In this chapter, we introduce and study matrix representations. At their core, matrix representations are a way in which we can utilize the powerful tools of linear algebra to understand more about the often non-linear structure of groups.

2.1 Matrix Representations

Definition 2.1.1. A *matrix representation* of a finite group G , of *dimension* $d > 0$, is a function X that assigns each element in G to a $d \times d$ matrix, satisfying

1. $X(e) = I_d$
2. $X(gh) = X(g)X(h)$ for all $g, h \in G$

where e represents the identity in G and I_d represents the $d \times d$ identity matrix.

In other words, a matrix representation is a homomorphism $X : G \rightarrow GL_d$, where GL_d denotes the general linear group of $d \times d$ invertible matrices. We will refer to matrix representations simply as representations.

Definition 2.1.2. The function $X(g) = [1]$ for all $g \in G$ is called the *trivial representation* for any group G .

Since $X(e) = [1]$ and

$$X(g)X(h) = [1][1] = [1] = X(gh) \tag{2.1}$$

for all $g, h \in G$, this is in fact a representation. Representations preserve all, none, or only some of the structure of the group they represent. The trivial representation preserves none—every group has a trivial representation and it does not reveal any information about the group.

Definition 2.1.3. The function $X(\sigma) = [1]$ if σ is even and $X(\sigma) = [-1]$ if σ is odd, for $\sigma \in S_n$ is called the *sign representation*.

This representation of S_n captures the structure in parity of permutations. Both the trivial and sign representations have dimension one because they map to GL_1 .

A common way to construct representations is from group actions. Every group action gives rise to a representation of that group in the following way.

Theorem 2.1.4. Consider an action of a group G on a finite set $C = \{c_1, \dots, c_d\}$. The function $X : S_n \mapsto GL_d$ defined by $X(g) = ||X_{ij}(g)||$ where

$$X_{ij}(g) = \begin{cases} 1 & \text{if } g \cdot c_j = c_i \\ 0 & \text{if } g \cdot c_j \neq c_i \end{cases}$$

is a representation of G .

Proof. Since $e \cdot c_i = c_i$ for all $i \in [d]$, $X(e) = I_d$. Let $g, h \in G$. Observe that $X(gh)$ and $X(g)X(h)$ both have exactly one 1 in each row and column and zeros elsewhere. Suppose the i, j entry of $X(g)X(h)$ is a 1 for some $i, j \in [d]$. This entry is given by the dot product of the i^{th} row of $X(g)$ with the j^{th} row of $X(h)$. So there exists $k \in [d]$ such that $X_{ik}(g) = 1$ and $X_{kj}(h) = 1$. This means that $g \cdot c_k = c_i$ and $h \cdot c_j = c_k$. Substituting and applying properties of group actions gives $gh \cdot c_j = c_i$. So $X(gh)_{ij} = 1$. Thus, $X(g)X(h) = X(gh)$. \square

When we construct representations as described above, we say that X is the representation arising from the action of G on C . Notice that $\dim(X) = |C|$ where $\dim(X)$ denotes the dimension of X . Let us see how some group actions actually define representations.

Definition 2.1.5. The *defining representation* of S_n is the representation arising from the natural action of S_n on $[n]$.

In other words, the defining representation of S_n is the function X that assigns each $\sigma \in S_n$ to an $n \times n$ matrix whose i, j entry is 1 if and only if $\sigma(j) = i$ and is 0 otherwise. This representation has dimension n because it maps to n by n matrices.

Example 2.1.6. Let $n = 4$ and consider the permutation $(142) \in S_n$. Since $\sigma(1) = 4$, $\sigma(2) = 1$, $\sigma(3) = 3$, and $\sigma(4) = 2$, the matrix

$$X((142)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

has ones in positions $(4,1)$, $(1,2)$, $(3,3)$, and $(2,4)$.

The defining representation is convenient because if we permute the indices of a vector (x_1, \dots, x_n) by σ , we get the same result as if we right multiply by $X(\sigma)$. When $\sigma = (142)$ as above, we observe

$$(x_1, x_2, x_3, x_4) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = (x_4, x_1, x_3, x_2) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}).$$

So we can apply permutations with matrix multiplication. The matrix $X(\sigma)$ is sometimes called the *permutation matrix* of σ for this reason.

Let us look at a second representation constructed in this way. For a group G , there is a natural action from $G \times G \mapsto G$ defined by $g \cdot h = gh$. In other words, G acts on itself by left multiplication.

Definition 2.1.7. For a group G , the representation X arising from the action of G on itself by left multiplication is called the *(left) regular representation* of G .

In other words, the regular representation of G is defined to be the function X that assigns each $h \in G$ to a matrix whose i, j entry is a 1 if and only if $hg_j = g_i$ and is 0 otherwise.

Example 2.1.8. Consider the group S_3 . It has order $3! = 6$ so the regular representation X of S_3 will have dimension 6. Let us compute the value of the regular representation at $\sigma = (12)$. Indexing the 6 group elements as

$$g_1 = e, \quad g_2 = (12), \quad g_3 = (13), \quad g_4 = (23), \quad g_5 = (123), \quad g_6 = (132),$$

we can compute what matrix will result. Since $\sigma g_1 = (12)e = (12) = g_2$, the first column will have a one in the second row and zeros elsewhere. Since $\sigma g_2 = (12)(12) = e = g_1$, the second column will have a one in the first row and zeros elsewhere. Since $\sigma g_3 = (12)(13) = (132) = g_6$, the third column will have a one in the sixth row and

zeros elsewhere. Computing the last three rows,

$$X(\sigma) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We only have a notion of the defining representation for S_n , but the regular representation is defined for any group G . We saw that every group has the trivial representation, preserving none of its structure. The regular representation, for any group, preserves all of its structure. The matrices that are formed can be viewed as recipes for how to compute the group operation. As it turns out the regular representation will prove to be quite interesting to explore.

2.2 Combining Representations

In this section, we introduce notions of addition and multiplication of representations.

Definition 2.2.1. Two representations X and Y of a group G are *equivalent*, denoted $X \cong Y$, if there exists a fixed change of basis matrix B such that $B^{-1}X(g)B = Y(g)$ for all $g \in G$. Otherwise, we say X and Y are *inequivalent*.

Theorem 2.2.2. *Representation equivalence as described above defines an equivalence relation.*

Proof. Let X , Y , and Z be representations of a group G and let $g \in G$. Since $I^{-1}X(g)I = X(g)$, we have reflexivity. Suppose $B^{-1}X(g)B = Y(g)$. Then

$$(B^{-1})^{-1}Y(g)B^{-1} = X(g)$$

which implies symmetry. Now also suppose that $C^{-1}Y(g)C = Z(g)$. Then

$$(BC)^{-1}X(g)BC = X(g)$$

which implies transitivity. □

To define the addition of two representations we first consider a slightly less common notion of addition of matrices.

Definition 2.2.3. Let A and B be matrices. The *direct sum* $A \oplus B$ is the block matrix

$$A \oplus B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right]$$

where 0 is a nonempty matrix of zeros.

Note that this direct sum is associative.

Theorem 2.2.4. Let A and B be $d \times d$ matrices and let C and D be $f \times f$ matrices.

We have

$$(A \oplus B)(C \oplus D) = AB \oplus CD.$$

Proof. This is theorem just acknowledges that multiplication of block diagonal matrices works nicely. We observe

$$(A \oplus B)(C \oplus D) = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right] \left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array} \right] = \left[\begin{array}{c|c} AC & 0 \\ \hline 0 & BD \end{array} \right] = AB \oplus CD.$$

□

Definition 2.2.5. Let X, Y be matrix representations of group G with dimension d and f respectively. We define the *direct sum representation* $X \oplus Y$ of G to be

$$(X \oplus Y)(g) = X(g) \oplus Y(g).$$

Verifying that

$$(X \oplus Y)(e) = \left[\begin{array}{c|c} I_d & 0 \\ \hline 0 & I_f \end{array} \right] = I_{d+f}$$

and by Theorem 2.2.4 that

$$\begin{aligned} (X \oplus Y)(g)(X \oplus Y)(h) &= (X(g) \oplus Y(g))(X(h) \oplus Y(h)) \\ &= (X(g)X(h) \oplus Y(g)Y(h)) \\ &= (X(gh) \oplus Y(gh)) \\ &= (X \oplus Y)(gh) \end{aligned}$$

confirms that $X \oplus Y$ is in fact a representation of G . Notice that this notion of direct sum of representations is associative and the dimension of $X \oplus Y$ is $d + f$. For positive integer m , we define

$$mX = \underbrace{X \oplus \dots \oplus X}_{m \text{ times}} \tag{2.2}$$

for a notion of scalar multiplication.

Theorem 2.2.6. *If $X = m_1X_1 \oplus \cdots \oplus m_kX_k$, then the dimension of X is*

$$\dim(X) = \sum_{i=1}^k m_i \dim(X_i)$$

Proof. Reindex, allowing repeats, such that $X = X_1 \oplus \cdots \oplus X_l$. We observed that the dimension of the direct sum of two representations is the sum of the dimensions. With this, induction on l proves the result. \square

Now we define a notion of multiplying representations together.

Definition 2.2.7. For matrices $A = \|a_{ij}\|$ and B , the *tensor product* $A \otimes B$ is the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 2.2.8. *Let A and B be $d \times d$ matrices and let C and D be $f \times f$ matrices. We have*

$$(A \otimes B)(C \otimes D) = AB \otimes CD.$$

Proof. Suppose $A = \|a_{ij}\|$ and $C = \|c_{ij}\|$. Then

$$\begin{aligned} (A \otimes B)(C \otimes D) &= \|a_{ij}B\| \cdot \|c_{ij}D\| \\ &= \left\| \sum_k a_{ik}Bc_{kj}D \right\| \\ &= \left\| \sum_k a_{ik}c_{kj}BD \right\| \\ &= AC \otimes CD. \end{aligned}$$

\square

Definition 2.2.9. Let X and Y be representations for groups G and H respectively. The *tensor product representation* $X \otimes Y$ of $G \times H$ is defined by

$$(X \otimes Y)(g) = X(g) \otimes Y(h) = \begin{bmatrix} x_{11}(g)Y(h) & x_{12}(g)Y(h) & \dots \\ x_{21}(g)Y(h) & x_{22}(g)Y(h) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where $x_{ij}(g)$ denotes the i, j entry of $X(g)$.

To verify that this is a representation, observe

$$(X \otimes Y)(e, e) = (X(e) \otimes Y(e)) = I \otimes I = \|\delta_{i,j}I\| = I.$$

Now let $(g_1, h_1), (g_2, h_2) \in G \times H$. By Theorem 2.2.8,

$$\begin{aligned} (X \otimes Y)(g_1, h_1) \cdot (X \otimes Y)(g_2, h_2) &= (X(g_1) \otimes Y(h_1)) \cdot (X(g_2) \otimes Y(h_2)) \\ &= X(g_1)X(g_2) \otimes Y(h_1)Y(h_2) \\ &= X(g_1g_2) \otimes Y(h_1h_2) \\ &= (X \otimes Y)(g_1g_2, h_1h_2), \end{aligned}$$

confirming that this is in fact a representation of $G \times H$.

Note that $\dim(X \otimes Y) = \dim(X) \dim(Y)$.

2.3 Irreducible Representations

Often, representations can be realized as combinations of other representations of smaller dimension. The ones that cannot be broken down in this way are called irreducible representations and they are the building blocks from which all other

representations emerge. There are infinitely many representations of any finite group, but as we will prove in Theorem 2.6.14, only a handful of them are irreducible.

Definition 2.3.1. Let X and Y be representations of a group G . If $X \cong Y \oplus Z$ for some representation Z of G , then Y is a *subrepresentation* of X .

Definition 2.3.2. Let X be a representation of a group G . Then X is *reducible* if it has a subrepresentation and X is *irreducible* otherwise.

Equivalently, a representation of a group G is reducible if there exists a basis with respect to which every $g \in G$ is assigned to a block matrix of the form

$$B^{-1}X(g)B = \left[\begin{array}{c|c} A(g) & 0 \\ \hline 0 & C(g) \end{array} \right] \quad (2.3)$$

where B is the change of basis matrix, 0 is a nonempty matrix of zeros, and $A(g)$ is a square matrix of the same size for each g .

Theorem 2.3.3. *If a representation X of a group G has dimension 1, then X is irreducible.*

Proof. Suppose that X has dimension 1 and is reducible. Then there exist representations Y and Z of G such that $X \cong Y \oplus Z$. By Theorem 2.2.6, taking the dimension of both sides gives $1 = \dim(Y) + \dim(Z)$. This is a contradiction because we require representations to have dimension at least 1. \square

Theorem 2.3.4 (Maschke). *Let X be a representation over \mathbb{C} of a finite group G with dimension $d > 0$. Then*

$$X \cong X_1 \oplus \cdots \oplus X_k$$

where each X_i is an irreducible representation of G .

Proof. We will use strong induction. Assume the result holds for any representation of dimension less than d . If X is irreducible, the result is immediate. Assume X is reducible. Then there exist representations Y , and Z of G such that $X \cong Y \oplus Z$. Since all representations have dimension at least 1, $\dim(Y) < n$ and $\dim(Z) < n$. By the inductive hypothesis, $Y \cong X_1 \oplus \cdots \oplus X_j$ and $Z \cong X_{j+1} \oplus \cdots \oplus X_k$ where X_1, \dots, X_k are irreducible. Thus, $X \cong X_1 \oplus \cdots \oplus X_j \oplus X_{j+1} \oplus \cdots \oplus X_k$ \square

It follows from definitions that Theorem 2.3.4 is equivalent to the existence of a change of basis matrix B such that

$$B^{-1}X(g)B = \begin{bmatrix} X_1(g) & 0 & \cdots & 0 \\ 0 & X_2(g) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_k(g) \end{bmatrix} \quad (2.4)$$

where each X_i is an irreducible representation of G . Note that this is still a block matrix even though we omitted the vertical and horizontal lines.

As it turns out, we can relax the requirements on what suffices to showing that a representation is reducible. We include the following theorem without proof.

Theorem 2.3.5. *A representation of a group G is reducible if there exists a basis B with respect to which every $g \in G$ is assigned to a block matrix of the form*

$$B^{-1}X(g)B = \left[\begin{array}{c|c} A(g) & D(g) \\ \hline 0 & C(g) \end{array} \right]$$

where B is the change of basis matrix, 0 is a nonempty matrix of zeros, and $A(g)$ is a square matrix of the same size for each g .

This is commonly taken to be the definition of irreducible and then it is proven that this agrees with our definition over \mathbb{C} .

Example 2.3.6. Consider the defining representation X of S_3 . Recall that this representation acts on \mathbb{C}^3 , by permuting the entries of a vector. So X acts invariantly on the subspace of \mathbb{C}^3 spanned by the vector $(1, 1, 1) \in \mathbb{C}^3$. Since the change of basis matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has $(1, 1, 1)$ as a column it will isolate the invariant subspace. This will show that X is reducible.

The defining representation for S_3 is the function X consisting of the following six mappings:

$$\begin{aligned} X(e) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & X((12)) &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ X((13)) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} & X((23)) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ X((123)) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} & X((132)) &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

We compute the matrices corresponding to each of the six group elements with respect to this new basis:

$$\begin{aligned}
 B^{-1}X(e)B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & B^{-1}X((12))B &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\
 B^{-1}X((13))B &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} & B^{-1}X((23))B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 B^{-1}X((123))B &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} & B^{-1}X((132))B &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}
 \end{aligned}$$

By Theorem 2.3.5, we conclude that the defining representation of S_3 is reducible because every matrix fits the required form with zeros in the first column after the first entry.

By the definition of reducible, we can find a different change of basis matrix B that completely breaks our representation into block diagonal form. We need the other two columns of our change of basis matrix to span the orthogonal complement of the space spanned by $(1, 1, 1)$. The new change of basis

$$B = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has the latter two columns orthogonal to the first. This can be verified by computing dot products. For example, since

$$(1, 1, 1) \cdot (-1, 1, 0) = 1(-1) + 1(1) + 1(0) = 0, \quad (2.5)$$

the first two columns are orthogonal.

We again compute the matrices corresponding to each of the six group elements with respect to this new basis:

$$\begin{aligned}
 B^{-1}X(e)B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & B^{-1}X((12))B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
 B^{-1}X((13))B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} & B^{-1}X((23))B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
 B^{-1}X((123))B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} & B^{-1}X((132))B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

As expected these matrices all take a block diagonal form. The upper left block is [1] for all of them, the trivial representation. The bottom right block, however is something new.

Peeling off the bottom right 2×2 block, we can verify by direct matrix multiplication that the function $Y : S_3 \mapsto GL_2$ defined by

$$\begin{aligned}
 Y(e) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & Y((12)) &= \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \\
 Y((13)) &= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} & Y((23)) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 Y((123)) &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & Y((132)) &= \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}
 \end{aligned}$$

is another representation of S_3 . Letting Z denote the trivial representation, we have observed $B^{-1}X(\sigma)B = Z(g) \oplus Y(\sigma)$ for all $\sigma \in S_3$. So we can say that $X \cong Z \oplus Y$. Thus both Y and Z are subrepresentations of S_3 .

The trivial representation, we know is irreducible by Theorem 2.3.3 because it has dimension 1. We will prove it later on, but it turns out this representation Y is irreducible as well. The representation Y is sometimes called the *standard representation* of S_3 .

The results in this section tell us that any representation of a finite group can be completely broken down into a combination of irreducible representations of the same group. This naturally gives rise to two big questions. Given any finite group, what are all of its irreducible representations up to equivalence? And, given a representation of a finite group, what is its irreducible decomposition? In other words, how many copies of each irreducible representation of the group are contained in our representation?

2.4 Characters

One downside of representations is that assigning each element to a matrix, especially larger matrices, requires storing a lot of information. To circumvent this issue, we introduce the character of a representation, which we will prove to be a numerical invariant under our notions of addition and multiplication.

Characters of representation are surprisingly powerful. As we will see in Section 2.5, for finite representations of a group G over \mathbb{C} , the character actually determines the representation up to equivalence.

Definition 2.4.1. Let X be a representation of a group G . The *character* of X is the function $\chi : G \rightarrow \mathbb{C}$ defined by

$$\chi(g) = \text{tr}(X(g))$$

where $\text{tr}(A)$ denotes the trace of the matrix A .

Example 2.4.2. Consider the defining representation X of S_3 defined. Let χ be the character of X . Computing the value of χ on specific elements of S_3 , we see that

$$\chi(e) = \text{tr}(X(e)) = \text{tr} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 + 1 + 1 = 3,$$

and

$$\chi((12)) = \text{tr}(X(g)) = \text{tr} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 + 0 + 1 = 1.$$

Continuing, we observe that $\chi((13)) = 1$, $\chi((23)) = 1$, $\chi((123)) = 0$, and $\chi((132)) = 0$. With the value of χ on all elements of the group, we could completely describe the function χ in a table:

$$\begin{array}{c|cccccc} & e & (12) & (13) & (23) & (123) & (132) \\ \hline \chi & 3 & 1 & 1 & 1 & 0 & 0 \end{array}.$$

In this example it appears that χ is constant on each of the three conjugacy classes of S_3 . As we will prove with Theorem 2.4.3, this holds in general for any representation of any group. So we still retain all the information of χ when we write the table more concisely as

$$\begin{array}{c|ccc} & K_1 & K_2 & K_3 \\ \hline \chi & 3 & 1 & 0 \end{array},$$

where the number below K_1 , K_2 , and K_3 indicates the value of χ on each of the three conjugacy classes of S_3 . This is called the *character row* of χ . Recall that the conjugacy classes of the symmetric group correspond precisely to cycle type. So, for a representation of the symmetric group, it is typical to write

$$\begin{array}{c|ccc} & K_{1,1,1} & K_{2,1} & K_3 \\ \hline \chi & 3 & 1 & 0 \end{array},$$

where K_λ denotes the conjugacy class corresponding to cycle type λ .

Theorem 2.4.3. *Let X be a representation of a group G with character χ . If g_1 and g_2 are in the same conjugacy class of G , $\chi(g_1) = \chi(g_2)$.*

Proof. Let X be a representation of a group G with character χ . Suppose $g_1, g_2, h \in G$ have $g_1 = h^{-1}g_2h$. Then

$$\begin{aligned}
 \chi(g_1) &= \chi(h^{-1}g_2h) \\
 &= \text{tr}(X(h^{-1}g_2h)) \\
 &= \text{tr}(X(h^{-1})X(g_2)X(h)) \\
 &= \text{tr}(X(h^{-1})X(h)X(g_2)) \\
 &= \text{tr}(X(g_2)) \\
 &= \chi(g_2).
 \end{aligned}$$

□

Definition 2.4.4. A *class function* for a group G is any function from a G to \mathbb{C} that is constant on conjugacy classes.

As we proved in the previous theorem, characters are examples of class functions.

Example 2.4.5. Now let X be the defining representation on S_4 with character χ . We will use the previous theorem to compute the character row. Summing the entries of the main diagonal of

$$X((142)) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

reveals that $\chi((124)) = 1$. We could repeat for the other 23 matrices and we would have a complete picture of χ , but we can be much more efficient than that. Since χ is constant on conjugacy classes, we know that $\chi((123)) = 1$ as well without computing $X((123))$, since (123) and (142) are in the same conjugacy class. In general we just compute the value on the character for one representative from each conjugacy class.

In this example we can reason through the other character values without computing the matrices. By definition of the defining representation, a 1 on the main diagonal corresponds to a fixed point of the permutation. So in this case the character function counts the number of fixed points for a given permutation. Our example, (142) had a single fixed point, 3, hence a character of 1. Since the conjugacy classes of S_n are indexed by partitions of n , choosing a representative element from each of the conjugacy classes and computing

$$\chi((1234)) = 0 \quad \chi((12)(34)) = 0 \quad \chi((123)) = 1 \quad \chi((12)) = 2 \quad \chi(e) = 4$$

allows us to fill out the character row,

$$\begin{array}{c|ccccc} & K_{1111} & K_{211} & K_{22} & K_{31} & K_4 \\ \hline \chi & 4 & 2 & 1 & 0 & 0 \end{array}.$$

Notice that the character of the identity reveals the dimension of the representation, because $\text{tr}(I_d) = d$. In this case we can read off from the left entry of the table that we have a 4-dimensional representation.

This method of counting fixed points holds more generally.

Theorem 2.4.6. *Let X be a representation of a group G arising from an action of G on a set C . For any $g \in G$, $\chi(g) = |C^g|$, where C^g is the fixed point set of g .*

Proof. Recall that $\chi(g) := \text{tr}(X(g))$, which equals the sum of the diagonal entries of $X(g)$. But each nonzero entry on the diagonal of $X(g)$ corresponds to an element of C that is fixed by g . Since each of these nonzero entries is a one, $\chi(g) = |C^g|$. \square

Theorem 2.4.7. *Let X be a representations of a group G with characters χ . Let*

$$X \cong X_1 \oplus \cdots \oplus X_k$$

where the X_i are irreducible representations of G with characters χ_i . Then $\chi = \chi_1 + \cdots + \chi_k$.

Proof. Let $g \in G$. Then

$$\begin{aligned}\chi(g) &= \text{tr}(X(g)) \\ &= \text{tr}(X_1(g) \oplus \cdots \oplus X_k(g)) \\ &= \text{tr}(X_1(g)) + \cdots + \text{tr}(X_k(g)) \\ &= \chi_1(g) + \cdots + \chi_k(g).\end{aligned}$$

□

Definition 2.4.8. If we have a character of an irreducible representation, we call it an *irreducible character*.

2.5 Inner Products

In this section, we define a way to multiply character functions with an inner product. We will prove that the inner product of two characters reveals critical information toward answering the question posed at the end of Section 2.3: How does a given representation of a group breakdown into irreducible components?

Definition 2.5.1. Let X and Y be representations of G with characters χ and ψ respectively. The *inner product* of χ and ψ is

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}).$$

Since characters are constant on conjugacy classes, we can rewrite the inner product of two characters as

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_K |K| \chi_K \overline{\psi_K} \quad (2.6)$$

where K iterates through all conjugacy classes of G and χ_K and ψ_K are the constant values of χ and ψ on the conjugacy class K respectively.

The inner product of characters reveals information about the relationship between the representations they correspond to. Specifically, as we will prove in Theorem 2.5.4, the inner product of two irreducible characters is 1 if they are the equivalent and 0 if they are inequivalent. We set the stage for the proof of this powerful result with a few lemmas.

Lemma 2.5.2 (Schur). *Let X and Y be inequivalent irreducible matrix representations of a group G . If $X(g)B = BY(g)$, then B is the zero matrix.*

Proof. If B was invertible, then $Y(g) = B^{-1}X(g)B$ for all $g \in G$. So X and Y would be equivalent, a contradiction. Since B is not invertible, there exists a nonzero vector u such that $Bu = 0$ or $uB = 0$.

Suppose $Bu = 0$. Then the kernel of B has dimension at least one. Let P be the projection matrix onto the kernel of B . For any vector v , $Pv \in \ker(B)$. So $BPv = 0$. Multiplying on the right by P ,

$$BY(g)P = X(g)BP = 0$$

because $BP = 0$. It follows that for any vector $v \in \ker(B)$, $BY(g)v = 0$, which implies that $Y(g)v \in \ker(B)$. In other words, $Y(g)$ acts invariantly on $\ker(B)$. Since Y is irreducible, $\ker(B)$ must be the whole space on which B acts. So $B = 0$.

If $uB = 0$, a similar argument shows $B = 0$. □

Lemma 2.5.3. *For an irreducible representation X of a group G , if $X(g)B = BX(g)$ for all $g \in G$, then $B = \lambda I$, a scalar multiple of the identity matrix.*

Proof. The square matrix B has at least one eigenvalue $c = \lambda$ because its characteristic equation $\det(B - cI) = 0$ has at least one root over \mathbb{C} , by the fundamental theorem of algebra. So the kernel of $B - \lambda I$ has dimension at least one. Since $(\lambda I)X(g) = \lambda X(g) = X(g)(\lambda I)$, we have

$$(B - \lambda I)X(g) = X(g)(B - \lambda I).$$

Let P be the projection matrix onto the kernel of $B - \lambda I$. For any vector v , $Pv \in \ker(B - \lambda I)$. So $BPv = 0$. Multiplying on the right by P ,

$$(B - \lambda I)Y(g)P = X(g)(B - \lambda I)P = 0$$

because $(B - \lambda I)P = 0$. It follows that for any vector $v \in \ker(B - \lambda I)$, $(B - \lambda I)Y(g)v = 0$, which implies that $Y(g)v \in \ker(B - \lambda I)$. In other words, $Y(g)$ acts invariantly on $\ker(B - \lambda I)$. Since Y is irreducible, $\ker(B - \lambda I)$ must be the whole space on which B acts. So $B - \lambda I = 0$. Thus, $B = \lambda I$. \square

Now we are ready to state and prove the powerful character relation.

Theorem 2.5.4. *The inner product of two irreducible characters χ and ψ of a group G is 1 if and only if $\chi = \psi$ and 0 otherwise. In symbols,*

$$\langle \chi, \psi \rangle = \delta_{\chi, \psi}$$

where

$$\delta_{\chi,\psi} = \begin{cases} 1 & \text{if } \psi = \chi \\ 0 & \text{if } \psi \neq \chi \end{cases}.$$

Proof. Let X and Y be representations of G with characters χ and ψ respectively. Let $A = \|A_{i,j}\|$ be a matrix of indeterminates. The matrix

$$B = \frac{1}{|G|} \sum_{g \in G} X(g)AY(g^{-1})$$

satisfies

$$\begin{aligned} X(h)BY(h^{-1}) &= \frac{1}{|G|} \sum_{g \in G} X(h)X(g)AY(g^{-1})Y(h^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} X(hg)AY(g^{-1}h^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} X(g)AY(g) \\ &= B \end{aligned}$$

which implies that $X(h)B = BY(h)$ for all $h \in G$. We chose the matrix B such that this would be the case because we can now apply Lemma 2.5.2 and Lemma 2.5.3.

Case 1: Suppose $\chi \neq \psi$. Then $X \not\cong Y$. By Lemma 2.5.2, $B = 0$. Examining the i, j entry of the matrix B ,

$$0 = B_{ij} = \frac{1}{|G|} \sum_{g \in G} \sum_{k,l} X_{ik}(g)A_{kl}Y_{lj}(g^{-1})$$

for all i and j . Since the A_{ij} are all indeterminate, equating coefficients of A_{ij} gives

$$0 = \frac{1}{|G|} \sum_{g \in G} X_{ik}(g)Y_{lj}(g^{-1}) \tag{2.7}$$

for all i, j, k and l . Since the character of a representation is the trace of the matrix, $\chi(g) = \sum_i X_{ii}(g)$ and $\psi(g^{-1}) = \sum_j Y_{jj}(g^{-1})$. Substituting these into the definition of inner product and reordering the sums,

$$\begin{aligned}
\langle \chi, \psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_i X_{ii}(g) \sum_j Y_{jj}(g^{-1}) \\
&= \sum_{i,j} \frac{1}{|G|} \sum_{g \in G} X_{ii}(g) Y_{jj}(g^{-1}) \\
&= 0
\end{aligned}$$

by Equation 2.7.

Case 2: Now suppose $\chi = \psi$. Since the result we are proving only involves the characters, we can assume without loss of generality that $X = Y$. By Lemma 2.5.3, $B = cI_d$ for some scalar c . Computing the trace of B ,

$$\begin{aligned}
cd = \text{tr}(B) &= \frac{1}{|G|} \sum_{g \in G} \text{tr}(X(g)AY(g^{-1})) \\
&= \frac{1}{|G|} \sum_{g \in G} \text{tr}(X(g)Y(g^{-1})A) \\
&= \frac{1}{|G|} \sum_{g \in G} \text{tr}(A) \\
&= \text{tr}(A).
\end{aligned}$$

So $B_{ii} = c = \frac{1}{d} \text{tr}(A)$. This can be written as

$$B_{ii} = \frac{1}{|G|} \sum_{g \in G} \sum_{k,l} X_{ik}(g) A_{kl} Y_{lj}(g^{-1}) = \frac{1}{d} \sum_i A_{ii}.$$

Equating coefficients of monomials,

$$\frac{1}{|G|} \sum_{g \in G} X_{ik}(g)Y_{lj}(g^{-1}) = \frac{1}{d} \delta_{ij}$$

for all i, j, k , and l . As in Case 1,

$$\langle \chi, \psi \rangle = \sum_{i,j} \frac{1}{|G|} \sum_{g \in G} X_{ii}(g)Y_{jj}(g^{-1}).$$

Since $B_{ij} = 0$ for $i \neq j$, these terms of the sum are zero for the same reason as before.

So

$$\begin{aligned} \langle \chi, \psi \rangle &= \sum_i \frac{1}{|G|} \sum_{g \in G} X_{ii}(g)Y_{ii}(g^{-1}) \\ &= \sum_i \frac{1}{d} \\ &= 1. \end{aligned}$$

□

Corollary 2.5.5. *Let X, Y be matrix representations of a group G with characters χ and ψ respectively. Let*

$$X \cong m_1 X_1 \oplus \cdots \oplus m_k X_k$$

where the X_i are pairwise inequivalent irreducible representations of G with characters χ_i . Then,

1. $\langle \chi, \chi_j \rangle = m_j$ for all j .
2. $\langle \chi, \chi \rangle = m_1^2 + \cdots + m_k^2$.
3. X is irreducible if and only if $\langle \chi, \chi \rangle = 1$.

4. $X \cong Y$ if and only if $\chi = \psi$.

Proof. 1. By the previous theorem,

$$\langle \chi, \chi_j \rangle = \left\langle \sum_i m_i \chi_i, \chi_j \right\rangle = \sum_i m_i \langle \chi_i, \chi_j \rangle = \sum_i m_i \delta_{ij} = m_j.$$

2. Again by Theorem 2.5.4,

$$\langle \chi, \chi \rangle = \left\langle \sum_i m_i \chi_i, \sum_j m_j \chi_j \right\rangle = \sum_{i,j} m_i m_j \langle \chi_i, \chi_j \rangle = \sum_{ij} m_i m_j \delta_{ij} = \sum_i m_i^2.$$

3. Theorem 2.5.4 gives the forward direction. For the reverse, assume $\langle \chi, \chi \rangle = 1$. Then by 2, $1 = m_1^2 + \cdots + m_k^2$. Since the m_i are positive integers, one of them must be one and the rest zero. So X is irreducible.

4. The forward direction is true because trace is invariant under change of basis. For the reverse, assume $\chi = \psi$. Since $\chi = \psi$, $\langle \chi, \chi_i \rangle = \langle \psi, \chi_i \rangle$ for all i . So the irreducible decomposition of contains exactly m_i copies of X_i for each X_i . \square

As a consequence of this corollary, the irreducible decomposition given by Theorem 2.3.4 is unique.

Theorem 2.5.6. *Let X be the regular representation of a group G with character χ . If Y is an irreducible representation of G with character ψ , then $\langle \chi, \psi \rangle = \dim(Y)$.*

Proof. Let Y be an irreducible representation of G with character ψ . Since $\chi(g)$ counts the number of ones on the diagonal of $X(g)$, $\chi(g)$ is the number of group elements fixed by g . Since the identity fixes every element of the group, $\chi(e) = |G|$. On the other hand, $gh = h$ forces g to be the identity. So, if g is not the identity,

g fixes no elements of the group. Thus, χ is constantly 0 on all non-identity group elements.

Since χ is 0 on all non-identity group elements,

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \chi(e) \psi(e) = \dim(Y)$$

because $\psi(e) = \dim(Y)$. □

This implies that every irreducible representation of G shows up exactly a number of times equal to its dimension in the irreducible decomposition of the regular representation of G . Specifically, if we can decompose the regular representation of a group, we will have a complete list of irreducible representations of that group.

Corollary 2.5.7. *Let X_1, \dots, X_k be a complete list of pairwise inequivalent representations of a group G . Let $d_i = \dim(X_i)$ for each $i \in [k]$. Then*

$$|G| = \sum_{i=1}^k d_i^2.$$

Proof. Let X be the regular representation of G with character χ . By Theorem 2.5.6,

$$X \cong d_1 X_1 \oplus \cdots \oplus d_k X_k. \tag{2.8}$$

Since $\dim(X) = |G|$, taking the dimension of both sides by Theorem 2.2.6 proves the result. □

2.6 Modules and Group Algebras

With this powerful inner product now in our toolkit, we need a complete list of irreducible representations for whatever group we want to study. By Theorem 2.5.6, we now know that any finite group is guaranteed to have finitely many irreducible representations. However, we do not know in general how many irreducible representations a given group has.

In this section, we define the vector space analog of matrix representations: G -modules. We also define the group algebra and use inner products to completely break it down into its irreducible decomposition. This will be used to prove that the number of irreducible representations of a finite group is equal to the number of conjugacy classes of that group.

Definition 2.6.1. Let G be a group. A G -module V is a vector space equipped with a homomorphism $\phi : G \mapsto GL(V)$ where $GL(V)$ is the set of linear transformations from V to itself. If the group is obvious, we just call V a module.

For a G -module, as a short hand for $(\phi(g))(v)$, we write gv , for any $g \in G$ and $v \in V$, thinking of the maps to linear transformations as notion of multiplication of group elements times elements of the vector space. So an equivalent definition of G -module is a vector space with a notion of multiplication of group elements times vectors such that

1. $gv \in V$,
2. $g(cv + dw) = cv + dgw$,
3. $gh(v) = g(hv)$, and
4. $ev = v$

for all $g, h \in G$, $v, w \in V$, and $c, d \in \mathbb{C}$.

If we fix a basis, the linear transformations can all be written as matrices with respect to that basis which defines a representation of G . So every G -module corresponds to an equivalence class of representations of G .

Definition 2.6.2. An *algebra* A is a vector space, equipped with an additional notion of multiplication of vectors, $\cdot : A \times A \rightarrow A$ that respects the following properties:

1. $x \cdot (y + z) = xy + xz$
2. $(x + y) \cdot z = xz + yz$
3. $(cx) \cdot (dy) = (cd)(x \cdot y)$

where $x, y, z \in A$ and c and d are scalars from the field.

Theorem 2.6.3. *If a G -module V contains no nontrivial subspace that is invariant under the action of G , V corresponds to an irreducible representation of G .*

Proof. Suppose V contains a nontrivial subspace W invariant under the action of G . Then there exists an orthogonal basis with respect to which for all g , the matrix corresponding to the linear transformation of multiplication by g has a block diagonal form. The two blocks apply the transformation individually to W and its orthogonal complement. But then the corresponding representation is reducible, a contradiction.

□

Definition 2.6.4. For a G -set S , we define

$$\mathbb{C}S = \{c_1s_1 + \cdots + c_k s_k : c_i \in \mathbb{C}, s_i \in S\}$$

to be the G -module of formal linear combinations of elements of S .

The module $\mathbb{C}S$ corresponds to the representation arising from the action of G on S .

Definition 2.6.5. The *group algebra* $\mathbb{C}[G]$ of a group $G = \{g_1, \dots, g_k\}$ is the vector space of formal linear combinations

$$\mathbb{C}[G] = \{c_1g_1 + \dots + c_kg_k : c_i \in \mathbb{C}\}$$

along with multiplication defined by $\sum_i c_i g_i \cdot \sum_j c_j g_j = \sum_{i,j} c_i c_j (g_i g_j)$, an extension of the group multiplication.

The group algebra $\mathbb{C}[G]$ is a module where both the vector space and the group are G . The corresponding representation is the regular representation, where the group acts on itself.

Proposition 2.6.6. *The set of all $d \times d$ matrices, Mat_d , is an algebra.*

Definition 2.6.7. Let X be a representation of a group G . The *commutant algebra* of X is

$$\text{Com}(X) = \{T \in \text{Mat}_d : TX(g) = X(g)T \text{ for all } g \in G\}.$$

Noticing that $\text{Com}(X)$ is closed under addition and multiplication is sufficient to confirm that $\text{Com}(X)$ is in fact an algebra because the other requirements are inherited from Mat_d .

Definition 2.6.8. Let V be a G -module. The *endomorphism algebra* of V is

$$\text{End}(V) = \{\phi : V \mapsto V : \phi \text{ is a homomorphism}\}.$$

Proposition 2.6.9. *If X is a representation of G with corresponding module V , then $\text{Com}(X) \cong \text{End}(V)$.*

Definition 2.6.10. The *center* Z_G of a group G is the set of all elements of G that commute with everything in G .

Proposition 2.6.11. *The center of Mat_d only contains scalar multiples of the identity.*

Theorem 2.6.12. *Let X be a representation of a finite group G with dimension $d > 0$ such that*

$$X = m_1 X_1 \oplus \cdots \oplus m_k X_k$$

where the X_i are pairwise inequivalent irreducible representations of G with dimension d_i . Then $A \in \text{Com}(X)$ if and only if there exist $M_i \in \text{Mat}_{m_i}$ such that

$$A = \bigoplus_{i=1}^k M_i \otimes I_{d_i}.$$

Proof. Expanding each scalar multiplication, rewrite

$$X = \underbrace{(X_1 \oplus \cdots \oplus X_1)}_{m_1 \text{ times}} \oplus \cdots \oplus \underbrace{(X_k \oplus \cdots \oplus X_k)}_{m_k \text{ times}} = Y_1 \oplus \cdots \oplus Y_l$$

where $Y_i = X_1$ for $i \in [m_1]$, $Y_i = X_2$ for $i \in [m_1 + 1, m_1 + m_2]$, etc. Suppose $A = ||A_{ij}|| \in \text{Com}(X)$ is a block matrix with corresponding blocks matching the sizes above. Specifically we force A_{ii} to be the same size as Y_i for all i . Computing block matrix products and equating corresponding blocks in the equation $AX = XA$ gives

$$A_{ij} Y_j = Y_i A_{ij}.$$

By Lemma 2.5.2, if $Y_i \not\cong Y_j$, $A_{ij} = 0$. Since equivalent Y_i are grouped together, A is a block diagonal matrix. So $A = A_1 \oplus \cdots \oplus A_k$ for some matrices A_1, \dots, A_k . By

Lemma 2.5.3, if $Y_i \cong Y_j$, $A_{ij} = c_{ij}I$ for some $c_{ij} \in \mathbb{C}$. So

$$A_1 = \begin{bmatrix} c_{11}I_{d_1} & \cdots & c_{1m_1}I_{d_1} \\ \vdots & \ddots & \vdots \\ c_{m_11}I_{d_1} & \cdots & c_{m_1m_1}I_{d_1} \end{bmatrix}$$

is a block diagonal matrix whose blocks are all scalar multiples of the identity. By definition of matrix tensor product, we recognize

$$A_1 = \|c_{ij}\| \otimes I_{d_1}. \quad (2.9)$$

The same argument gives $A_i = M_i \oplus I_{d_i}$ for some $M_i \in \text{Mat}_{m_i}$ which proves the result. □

Theorem 2.6.13. *Let X be a representation of a finite group G with dimension $d > 0$ such that*

$$X = m_1X_1 \oplus \cdots \oplus m_kX_k$$

where the X_i are pairwise inequivalent irreducible representations of G with dimension d_i . Then, $\dim(Z_{\text{Com}(X)}) = k$

Proof. Let $A \in Z_{\text{Com}(X)}$. By Theorem 2.6.12, there exist $M_i \in \text{Mat}_{m_i}$ such that

$$A = \bigoplus_{i=1}^k M_i \otimes I_{d_i} \quad \text{and} \quad B = \bigoplus_{i=1}^k N_i \otimes I_{d_i}. \quad (2.10)$$

Let $N_i \in \text{Mat}_{m_i}$ for all i . Then

$$B = \bigoplus_{i=1}^k N_i \otimes I_{d_i}$$

is in $\text{Com}(X)$. By the definition of center, $AB = BA$. By Theorem 2.2.4,

$$AB = \bigoplus_{i=1}^k (M_i \otimes I_{d_i})(N_i \otimes I_{d_i}).$$

By Theorem 2.2.8,

$$AB = \bigoplus_{i=1}^k (M_i N_i \otimes I_{d_i}).$$

Similarly,

$$BA = \bigoplus_{i=1}^k (N_i M_i \otimes I_{d_i}).$$

Equating blocks in the direct sum, $M_i N_i = N_i M_i$ for all i . Since this is true for any matrices $M_i, N_i \in Z_{\text{Mat}_{m_i}}$, by Proposition 2.6.11, $M_i = c_i I_{m_i}$ for some $c_i \in \mathbb{C}$. Thus

$$A = \bigoplus_{i=1}^k c_i I_{m_i} \otimes I_{d_i} = \begin{bmatrix} c_1 I_{m_1 d_1} & 0 & \dots & 0 \\ 0 & c_2 I_{m_2 d_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & c_k I_{m_k d_k} \end{bmatrix}. \quad (2.11)$$

Since each c_i is a degree of freedom of the center, $\dim(Z_{\text{Com}(X)}) = k$, the number of c_i . □

If we have a representation Y of G not equal to a direct sum of irreducible representations, the previous theorem still holds. By Theorem 2.3.4, $Y = B^{-1}XB$ for some $B \in \text{Mat}_d$ where X satisfies the hypothesis for the above theorem. We observe that $\text{Com}(X) \cong \text{Com}(Y)$ by the map $A \mapsto B^{-1}AB$. So the centers of $\text{Com}(X)$ and $\text{Com}(Y)$ are isomorphic as well and specifically, they have the same dimension.

Theorem 2.6.14. *Let X_1, \dots, X_k be a complete list of pairwise inequivalent representations of a group G . Then k is the number of conjugacy classes of G .*

Proof. Let X be the regular representation of G with character χ . By Theorem 2.5.6,

$$X \cong d_1 X_1 \oplus \cdots \oplus d_k X_k. \quad (2.12)$$

By Theorem 2.6.13, $\dim(Z_{\text{Com}(X)}) = k$. Since the regular representation corresponds to the module $\mathbb{C}[G]$, $\dim(Z_{\text{End}(\mathbb{C}[G])}) = k$. We will define a vector space isomorphism from $G[\mathbb{C}]$ to $\text{End}(\mathbb{C}[G])$. Define ϕ_v to be right multiplication by v . Define $\phi : G[\mathbb{C}] \mapsto \text{End}(\mathbb{C}[G])$ by $\phi(v) = \phi_v$. Computing the kernel of ϕ , if $\phi_v = 0$, then evaluating at e gives $v = 0$. So ϕ is injective. Let $\theta \in \text{End}(\mathbb{C}[G])$. Let $v = \theta(e)$. Then

$$\theta(g) = \theta(ge) = g\theta(e) = gv = \phi_v(g) \quad (2.13)$$

for all $g \in G$. So $\theta = \phi_v$. So ϕ is surjective. Since $\phi(v)\phi(w) = \phi(wv)$, ϕ is an anti-isomorphism of algebras which induces an anti-isomorphism on the centers of these algebras. So $k = \dim(Z_{\mathbb{C}[G]})$.

Examining the elements of this center, let $z = c_1 g_1 + \cdots + c_k g_k \in Z_{\mathbb{C}[G]}$ where g_1, \dots, g_k is a complete list of group elements. By the definition of center, for all $h \in G$, $z = h^{-1}gh$, which gives

$$z = c_1 h^{-1} g_1 h + \cdots + c_k h^{-1} g_k h. \quad (2.14)$$

Since h can be any group element, $h^{-1} g_i h$ takes on all values in the conjugacy class of g_i . So if g_i and g_j share a conjugacy class, $c_i = c_j$. Similarly, if g_i and g_j do not share a conjugacy class, c_i and c_j can vary independently. So we can freely choose one c_i for a representative element g_i of each conjugacy class and then the rest of the c_i are forced. Thus $\dim(Z_{\mathbb{C}[G]})$ equals the number of conjugacy classes of G . \square

Corollary 2.6.15. *Let $C(G)$ be the vector space of class functions for a group G . Then the set of irreducible characters of G is a basis for $C(G)$.*

Proof. The dimension of $C(G)$ is equal to the number of conjugacy classes which by the previous theorem is equal to the number of irreducible characters of G . Since Theorem 2.5.4 gives us that the irreducible characters of G are orthonormal, they must form a basis. \square

2.7 Character Tables

We can list all of the irreducible characters of a group in a table. This will be called the character table of the group.

Example 2.7.1. Once again lets revisit the group S_3 . Since there are three partitions of 3, by Theorem 2.6.14, there will be three irreducible representations of S_3 . As it so happens, we have already found all three irreducible representations of S_3 : the trivial representation, the sign representation, and the standard representation. The character row for the trivial representation is

$$\begin{array}{c|ccc} & K_{111} & K_{21} & K_3 \\ \hline \chi & 1 & 1 & 1 \end{array}.$$

The character row for the sign representation is

$$\begin{array}{c|ccc} & K_{111} & K_{21} & K_3 \\ \hline \chi & 1 & -1 & 1 \end{array}.$$

Recall the standard representation X of S_3 defined by these six mappings:

$$\begin{aligned} X(e) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & X((12)) &= \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \\ X((13)) &= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} & X((23)) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ X((123)) &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} & X((132)) &= \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Computing the value of the character for one representative from each conjugacy class, the character row for the standard representation is

$$\begin{array}{c|ccc} & K_{111} & K_{21} & K_3 \\ \hline \chi & 2 & 0 & -1 \end{array}.$$

By Theorem 2.5.4, the computation

$$\begin{aligned} \langle \chi^{(3)}, \chi^{(3)} \rangle &= \frac{1}{|S_3|} \sum_{\sigma \in S_3} \chi^{(3)}(\sigma) \chi^{(3)}(\sigma) \\ &= \frac{1}{6} (1(2 \cdot 2) + 3(0 \cdot 0) + 2(-1 \cdot -1)) \\ &= 1 \end{aligned}$$

finally confirms that this representation is in fact irreducible. Labeling these three irreducible characters $\chi^{(1)}$, $\chi^{(2)}$ and $\chi^{(3)}$ respectively, we form a table with the irreducible character rows. A table like this listing all of the values of the irreducible characters on the conjugacy classes of a group G is called the *character table* of G .

Table 2.1: Character Table of S_3

	K_{111}	K_{21}	K_3
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	2	0	-1

Since the number of irreducible characters is equal to the number of conjugacy classes for any finite group, character tables are always square.

We can verify that the character relations hold for inner products of irreducible characters of S_3 . We expect

$$\langle \chi^{(1)}, \chi^{(1)} \rangle = \langle \chi^{(2)}, \chi^{(2)} \rangle = \langle \chi^{(3)}, \chi^{(3)} \rangle = 1$$

and

$$\langle \chi^{(1)}, \chi^{(2)} \rangle = \langle \chi^{(1)}, \chi^{(3)} \rangle = \langle \chi^{(2)}, \chi^{(3)} \rangle = 0.$$

It is a good exercise to compute all of these and see that the above is true. We include the computation that

$$\begin{aligned} \langle \chi^{(2)}, \chi^{(3)} \rangle &= \frac{1}{|S_3|} \sum_{\sigma \in S_3} \chi^{(2)}(\sigma) \chi^{(3)}(\sigma) \\ &= \frac{1}{6} ((1 \cdot 2) + 3(-1 \cdot 0) + 2(1 \cdot -1)) \\ &= 0. \end{aligned}$$

We successfully constructed the full character table of S_3 because we had already stumbled across the three irreducible representations of S_3 . However, at this point we could not construct the character table for S_4 . In Chapter 3, we will find the

irreducible representations of S_n for any positive integer n and in Chapter 4, we will learn how to find the character rows for these representations, allowing us to construct the character table of S_n for any positive integer n .

Chapter 3

IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

In this chapter, we utilize the combined power of module theory and combinatorial objects to construct a certain module of S_n associated with each partition of n . We will prove that these modules correspond to irreducible representations and construct an explicit basis for them, revealing their dimensions. This will unveil the holy grail of the symmetric group: a complete list of irreducible representations.

3.1 Tableau and Tabloids

Definition 3.1.1. Given $n \in \mathbb{N}$, a *Young tableau* of shape $\lambda \vdash n$ (also called a λ -*tableau*) is a filling of the Young diagram for λ with the integers 1 to n .

Definition 3.1.2. Two tableaux t_1 and t_2 are *row equivalent*, denoted $t_1 \sim t_2$ if they are of the same shape and their corresponding rows are rearrangements of one another.

Example 3.1.3. If $n = 9$ and $\lambda = 5, 3, 1$,

8					
9	3	1			
6	4	2	7	5	

8					
3	1	9			
2	7	4	6	5	

are examples of Young tableaux. The two tableaux above are row equivalent because taken as sets their rows are the same.

Proposition 3.1.4. *Row equivalence defines an equivalence relation on tableaux.*

Definition 3.1.5. The equivalence class of a λ -tableau t ,

$$\{t\} = \{s : s \text{ is row equivalent to } t\},$$

is a *Young tabloid* of shape λ (also called a λ -*tabloid*).

We use braces around a tableau to denote the corresponding tabloid, not to be confused with set notation. We draw λ -tabloids with thick horizontal lines between the rows to indicate that ordering within rows does not matter. The tabloid containing each of the two tableaux in the previous example is

$$\{t\} = \begin{array}{|c|c|c|c|c|} \hline 8 & & & & \\ \hline 9 & 3 & 1 & & \\ \hline 6 & 4 & 2 & 7 & 5 \\ \hline \end{array}.$$

There is a natural action of S_n on tableaux given by applying the permutation to each entry. For example if $\sigma = (274)(39)$, then

$$\sigma \cdot \begin{array}{|c|c|c|c|c|} \hline 8 & & & & \\ \hline 9 & 3 & 1 & & \\ \hline 6 & 4 & 2 & 7 & 5 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 8 & & & & \\ \hline 3 & 9 & 1 & & \\ \hline 6 & 2 & 7 & 4 & 5 \\ \hline \end{array}$$

In this case, σ yields an equivalent tableau, because it stabilizes the rows. This action extends to an action on tabloids by defining $\sigma \cdot \{t\} = \{\sigma \cdot t\}$ for any tabloid $\{t\}$.

Definition 3.1.6. Let $\lambda = \lambda_1, \dots, \lambda_k$ be a partition of n . The *Young Subgroup* S_λ of S_n is

$$S_\lambda = S_{\{1, \dots, \lambda_1\}} \times S_{\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}} \times \dots \times S_{\{n-\lambda_k+1, \dots, n\}}$$

where S_A is the set of permutations of A for any set A .

Note that $S_\lambda \cong S_{\lambda_1} \times \dots \times S_{\lambda_k}$ for any $\lambda \vdash n$.

Definition 3.1.7. If a tableaux t has rows R_1, \dots, R_k and columns C_1, \dots, C_l , the *row stabilizer* of t is

$$R_t = S_{R_1} \times \dots \times S_{R_k}$$

and the *column stabilizer* of t is

$$C_t = S_{C_1} \times \cdots \times S_{C_l}.$$

Note that $\{t\} = R_t t$ for any tableau t .

Example 3.1.8. Consider the tableau

$$t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline & & 7 \\ \hline \end{array}.$$

The column stabilizer is

$$C_t = \{e, (14), (25), (36), (14)(25), (14)(36), (25)(36), (14)(25)(36)\}.$$

Let us find the column stabilizer after applying a permutation. Let $\sigma = (256)$. Since

$$\sigma \cdot t = \begin{array}{|c|c|c|} \hline 1 & 5 & 3 \\ \hline 4 & 6 & 2 \\ \hline & & 7 \\ \hline \end{array},$$

the new column stabilizer is

$$C_{\sigma \cdot t} = \{e, (14), (56), (23), (14)(56), (14)(23), (56)(23), (14)(23)(560)\}.$$

Note that $C_{\sigma \cdot t}$ could also be obtained by substituting $\sigma(i)$ in for i in the cycles in C_t .

Specifically,

$$C_{\sigma \cdot t} = \sigma C_t \sigma^{-1} \tag{3.1}$$

The same is true for the row stabilizer.

Proposition 3.1.9. *Let t be a λ -tableau and $\sigma \in S_n$. Then $C_{\sigma \cdot t} = \sigma C_t \sigma^{-1}$ and $R_{\sigma \cdot t} = \sigma R_t \sigma^{-1}$.*

3.2 Specht Modules

In this section we the modules that will correspond to the irreducible representations of S_n .

Definition 3.2.1. Let t be a Young tableau. The corresponding *polytabloid* e_t is

$$e_t = \sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{t\}),$$

a formal signed sum.

Example 3.2.2. If

$$t = \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 & 4 \\ \hline \end{array},$$

then

$$\begin{aligned} e_t &= \text{sign}(e) \left(e \cdot \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 & 4 \\ \hline \end{array} \right) + \text{sign}((35)) \left((35) \cdot \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 & 4 \\ \hline \end{array} \right) \\ &\quad + \text{sign}((12)) \left((12) \cdot \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 & 4 \\ \hline \end{array} \right) + \text{sign}((12)(35)) \left((12)(35) \cdot \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 & 4 \\ \hline \end{array} \right) \\ &= \begin{array}{|c|c|} \hline 3 & 1 \\ \hline 5 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 5 & 1 \\ \hline 3 & 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 5 & 1 & 4 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 5 & 2 \\ \hline 3 & 1 & 4 \\ \hline \end{array} \end{aligned}$$

is the associated polytabloid.

The action of S_n on tableau extends naturally to an action on polytabloids, by applying σ to each tabloid in the formal sum.

Theorem 3.2.3. Let t be a tableau. For all $\sigma \in S_n$, $\sigma \cdot e_t = e_{\sigma \cdot t}$.

Proof. By Proposition 3.1.9,

$$\begin{aligned}
e_{\sigma \cdot t} &= \sum_{\tau \in \sigma \cdot C_t} \text{sign}(\tau)(\tau \cdot \{\sigma \cdot t\}) \\
&= \sum_{\tau \in C_t} \text{sign}(\tau)(\sigma \tau \sigma^{-1} \cdot \{\sigma \cdot t\}) \\
&= \sigma \cdot \sum_{\tau \in C_t} \text{sign}(\tau)(\tau \cdot \{t\}) \\
&= \sigma \cdot e_t.
\end{aligned}$$

□

Definition 3.2.4. Let $\lambda \vdash n$ and let $\{t_1\}, \dots, \{t_k\}$ be a list of all tabloids of shape λ . The *Specht Module* S^λ is

$$S^\lambda = \mathbb{C}\{e_{t_1}, \dots, e_{t_k}\},$$

the module generated by all polytabloids of tableaux with shape λ .

We are interested in the irreducible representations of S_n . Let us examine some of these Specht modules and see which representations they correspond to.

Example 3.2.5. Let $\lambda = n$. The Young diagram of λ is a single row with length n . The only tabloid of shape λ is

$$\{t\} = \boxed{1 \ 2 \ \dots \ n},$$

because it contains all tableau of shape λ . Since t has a trivial column stabilizer,

$$S^\lambda = \mathbb{C}\left\{\boxed{1 \ 2 \ \dots \ n}\right\} \cong \mathbb{C}\{1\} \tag{3.2}$$

corresponds to the trivial representation. In general, S^λ corresponds to the representation arising from the action of S_n on the basis elements of S^λ . In the above case, for any $\sigma \in S_n$, $\sigma \cdot e_t = e_t$. Since the action is trivial, the corresponding representation is the trivial representation.

Example 3.2.6. Let us apply this same technique to another Specht module. Let $\lambda = 1, \dots, 1$ be a partition of n . Then every tabloid consists of only one tableau because the row stabilizer is trivial. So there are $n!$ tabloids of this shape. Since the column stabilizer of any of them is the whole group S_n ,

$$\begin{aligned}
\sigma \cdot e_t &= \sigma \cdot \sum_{\tau \in C_t} \text{sign}(\tau)(\tau \cdot \{t\}) \\
&= \sum_{\tau \in S_n} \text{sign}(\tau)(\sigma\tau \cdot \{t\}) \\
&= \sum_{\tau \in S_n} \text{sign}(\sigma^{-1}\sigma\tau)(\sigma\tau \cdot \{t\}) \\
&= \sum_{\tau \in S_n} \text{sign}(\sigma^{-1})\text{sign}(\sigma\tau)(\sigma\tau \cdot \{t\}) \\
&= \text{sign}(\sigma^{-1}) \sum_{\sigma\tau \in S_n} \text{sign}(\sigma\tau)(\sigma\tau \cdot \{t\}) \\
&= \text{sign}(\sigma^{-1})e_t \\
&= \text{sign}(\sigma)e_t
\end{aligned}$$

for any $\sigma \in S_n$ and λ -tableau t . So S^λ corresponds to the representation arising from this action: the sign representation.

3.3 Irreducibility of Specht Modules

We have proven that there is one irreducible representations of S_n for every partition of S_n and we have constructed the Specht Module for each of these partitions. In this

section, we will show that the Specht modules correspond to a complete list of the irreducible representations of S_n .

Lemma 3.3.1. *Let s and t be two λ -tableau. Then*

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{s\})$$

is either equal to 0 or equal to $\pm e_t$.

Proof. Case 1: Suppose there exist two entries a and b in the same column of t , but in the same row of s . Since a and b share a column in t , $C_t = \{\sigma(ab) : \sigma \in C_t\}$. Since a and b share a row in s , $(ab) \cdot \{s\} = \{s\}$. Since the sign of (ab) is -1 ,

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{s\}) = \sum_{\sigma \in C_t} \text{sign}(\sigma(ab))(\sigma(ab) \cdot \{s\}) = - \sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{s\}),$$

which implies

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{s\}) = 0. \tag{3.3}$$

Case 2: Now suppose any two entries in the same column of t are in different rows of s . Then there exists another λ -tableau, row-equivalent to s , whose columns are the same as the columns of t . So there exists $\tau \in C_t$ such that $\{s\} = \tau \cdot \{t\}$. In a similar

argument to Example 3.2.6, observe that

$$\begin{aligned}
\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{s\}) &= \sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot (\tau \cdot \{t\})) \\
&= \sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma\tau \cdot \{t\}) \\
&= \sum_{\sigma \in C_t} \text{sign}(\sigma\tau)\text{sign}(\tau^{-1})(\sigma\tau \cdot \{t\}) \\
&= \text{sign}(\tau^{-1}) \sum_{\sigma \in C_t} \text{sign}(\sigma\tau)(\sigma\tau \cdot \{t\}) \\
&= \text{sign}(\tau^{-1}) \sum_{\sigma\tau \in C_t} \text{sign}(\sigma\tau)(\sigma\tau \cdot \{t\}) \\
&= \text{sign}(\tau^{-1})e_t \\
&= \pm e_t.
\end{aligned}$$

□

Lemma 3.3.2. *Let $\lambda \vdash n$. Let V be a nontrivial subspace of S^λ with nonzero $v \in V$.*

For any λ -tableau t ,

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot v) = c \cdot e_t$$

for some $c \in \mathbb{C}$.

Proof. Since $v \in S^\lambda$, v can be written as a linear combination of its basis elements,

$$v = \sum_i c_i \{t_i\}, \tag{3.4}$$

where $c_i \in \mathbb{C}$ and the $\{t_i\}$ are λ -tabloids. By the previous lemma, for all i ,

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot \{t_i\}) = \pm e_t$$

or equal to zero. Suppose, without loss of generality, that the previous sum is nonzero for c_1, \dots, c_j and is zero for all other c_i . It follows that

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot v) = \pm c_1 e_t \pm c_2 e_t \pm \dots \pm c_j e_t = c e_t \quad (3.5)$$

for some $c \in \mathbb{C}$.

□

Theorem 3.3.3. *The Specht Modules correspond to irreducible representations of S_n .*

Proof. Let $\lambda \vdash n$ and t be a λ -tableau. Let V be a nontrivial subspace of S^λ that is invariant under the action with nonzero $v \in V$. By the previous lemma,

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot v) = c \cdot e_t$$

for some $c \in \mathbb{C}$. Since V is invariant under the action, $\sigma \cdot v \in V$ for all $\sigma \in C_t$ which implies

$$\sum_{\sigma \in C_t} \text{sign}(\sigma)(\sigma \cdot v) \in V.$$

Pairing these two together, $c \cdot e_t \in V$. Since $v \neq 0$, $c \neq 0$. So, $e_t \in V$. Since $e_t \in V$ for all t , $V = S^\lambda$. Thus S^λ contains no non-trivial proper subspaces that are invariant under the action. So by Theorem 2.6.3, S^λ corresponds to an irreducible representation of S_n . □

Definition 3.3.4. We define X^λ with character χ^λ to be the irreducible representation of S_n corresponding to S^λ in the standard basis.

As desired, the X^λ are a complete list of irreducible representations of S_n and the χ^λ are a complete list of irreducible characters of S_n .

3.4 Dimensions of the Irreducible Representations

In this section, we prove that the polytabloids associated with standard tableau form a basis for the Specht modules. This will reveal that the dimension of a X^λ is equal to the number of standard tableaux of the correct shape.

Definition 3.4.1. We say that a λ -tableau t is *standard* if its rows, read left to right, and columns, read bottom to top, form increasing sequences.

For example,

$$t = \begin{array}{|c|c|} \hline 7 & 9 \\ \hline 4 & 6 & 8 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array}$$

is a standard tableau.

Definition 3.4.2. Let $\{t\}$ and $\{r\}$ be λ -tabloids. We write $\{t\} < \{r\}$ if there exists an i such that

1. i occurs in a lower row of $\{t\}$ than of $\{r\}$.
2. for all $j > i$, j is in the same row of $\{t\}$ and $\{r\}$.

For example,

$$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 5 & 1 & 7 \\ \hline 8 & 3 & 6 \\ \hline \end{array} < \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 7 & 3 & 2 \\ \hline 6 & 8 & 4 \\ \hline \end{array}.$$

In other words, suppose $\{s\}$ and $\{t\}$ are both λ -tabloids where $\lambda \vdash n$. Count down from n until we reach the first number that occurs in different rows in $\{s\}$ and $\{t\}$. The tabloid in which this number occurs in a lower row is the smaller tableau. This defines an ordering on λ -tabloids and extends to an ordering on λ -tableaux.

Lemma 3.4.3. *Let t be a standard λ -tableau. If $\sigma \in C_t$, $\{\sigma \cdot t\} \leq \{t\}$*

Proof. Assume $\sigma \in C_t$. If $\sigma = e$, $\{\sigma \cdot t\} = \{t\}$. Suppose $\sigma \neq e$. Let i be the largest number that is not fixed by σ . Any $j > i$ is fixed by σ and is thus in the same row

in both $\{\sigma \cdot t\}$ and $\{t\}$. Since $\sigma(i)$ is not a fixed point, $\sigma(i) < i$. Since $\sigma \in C_t$ and the columns of t are strictly increasing, $\sigma(i)$ occurs below i in a column of t . So, $\{\sigma \cdot t\} < \{t\}$. \square

Theorem 3.4.4. *The set $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ is linearly independent.*

Proof. Let $t_1 < \dots < t_k$ be a complete ordered list of standard λ -tableaux. Assume $c_1 e_{t_1} + \dots + c_k e_{t_k} = 0$ for some $c_i \in \mathbb{C}$. We will prove that $c_i = 0$ for all i by induction. Suppose for some $i \in [k]$, $c_j = 0$ for all $j > i$. Then $c_1 e_{t_1} + \dots + c_i e_{t_i} = 0$. So

$$c_i e_{t_i} = -(c_1 e_{t_1} + \dots + c_{i-1} e_{t_{i-1}}). \quad (3.6)$$

Consider the tabloid t_i . It shows up once in the expansion of e_{t_i} and shows up no times in the expansion of any of e_{t_l} for any $l \in [i]$ by the previous lemma because all the terms in the expansion of e_{t_l} have the form $\pm(\sigma \cdot t_l)$. Equating coefficients of t_i on both sides of Equation 3.6, $c_i = 0$. By induction, $c_1 = \dots = c_k = 0$. \square

To show that the standard tableaux are a spanning set, we will introduce the so-called straightening algorithm by which we take the polytabloid for any tableau t and express it as a linear combination of polytabloids of other tableaux that are closer to being standard. Repeatedly applying this algorithm, we will prove that the polytabloid for any tableaux can be expressed as a linear combination of the polytabloids for the standard tableaux.

Definition 3.4.5. Let t be a tableau. Let A be a subset of column i of t and let B be a subset of column $i + 1$ of t . The *Garnir element* associated with t is

$$g_{A,B} = \sum_{\sigma \in P} \text{sign}(\sigma) \sigma$$

where P is the set of permutations σ of $A \cup B$ such that the elements of $A \cup B$ are strictly increasing in the columns of $\sigma \cdot t$.

Since all permutations in this set P force increasing columns, we have

$$S_{A \cup B} = \{\sigma_1 \sigma_2 \sigma_3 : \sigma_1 \in P, \sigma_2 \in S_A, \sigma_3 \in S_B\} \quad (3.7)$$

with no repeats, which will allow us to factor a sum over $S_{A \cup B}$.

We use Garnir elements in the straightening algorithm described as follows. Let t be a tableau. Let t' be the tableau whose columns are permutations of the columns of t such that t' has increasing columns. For example if

$$t = \begin{array}{|c|c|} \hline 2 & 6 \\ \hline 7 & 3 & 4 \\ \hline 1 & 8 & 5 \\ \hline \end{array},$$

then

$$t' = \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 2 & 6 & 5 \\ \hline 1 & 3 & 4 \\ \hline \end{array}.$$

Scan through t' from left to right, then bottom to top until reaching a cell x such that the cell immediately right of x is smaller than it. Call this smaller cell y . If the entire tableau is scanned through with no occurrences of this, then t' is already standard and we are done. Let A be the set of cells above x , in the same column as x , including x . Let B be the set of cells below y , in the same column as y , including y . In our example, x is the cell containing 6 and y is the cell containing 5. So $A = \{6, 8\}$ and $B = \{4, 5\}$. So,

$$g_{A,B} = e - (56) + (456) + (586) - (4586) + (46)(58).$$

Lemma 3.4.6. *Let t be a tableau. Let A and B be the sets in the straightening algorithm for t . Then $g_{A,B} \cdot e_t = 0$.*

Proof. For any $\tau \in C_t$, there exist $a \in A$ and $b \in B$ such that a and b share a row in τt . Since $(ab) \in S_{A \cup B}$,

$$\begin{aligned} \sum_{\sigma \in S_{A \cup B}} \text{sign}(\sigma)(\sigma \cdot e_t) &= \sum_{\sigma \in S_{A \cup B}} \text{sign}(\sigma(ab))(\sigma(ab) \cdot e_t) \\ &= - \sum_{\sigma \in S_{A \cup B}} \text{sign}(\sigma)(\sigma \cdot ((ab) \cdot e_t)) \\ &= - \sum_{\sigma \in S_{A \cup B}} \text{sign}(\sigma)(\sigma \cdot e_t) \end{aligned}$$

because $(ab) \cdot e_t = e_t$. So the whole sum is zero. We can factor the sum as

$$\begin{aligned} 0 &= \sum_{\sigma \in S_{A \cup B}} \text{sign}(\sigma)(\sigma \cdot e_t) \\ &= \sum_{\sigma_1 \in P} \sum_{\sigma_2 \in S_A} \sum_{\sigma_3 \in S_B} \text{sign}(\sigma_1 \sigma_2 \sigma_3)(\sigma_1 \sigma_2 \sigma_3 \cdot e_t) \\ &= \sum_{\sigma_1 \in P} \text{sign}(\sigma_1) \sigma_1 \cdot \sum_{\sigma_2 \in S_A} \sum_{\sigma_3 \in S_B} \text{sign}(\sigma_2 \sigma_3)(\sigma_2 \sigma_3 \cdot e_t) \\ &= g_{A,B} \sum_{\sigma \in S_A \times S_B} \text{sign}(\sigma)(\sigma \cdot e_t). \end{aligned}$$

But for any $\sigma \in C_t$, $\sigma \cdot e_t = \text{sign}(\sigma)e_t$ which implies $\text{sign}(\sigma)(\sigma \cdot e_t) = e_t$. Since $S_A \times S_B \subseteq C_t$,

$$0 = g_{A,B} \sum_{\sigma \in S_A \times S_B} e_t = |S_A \times S_B| g_{A,B} \cdot e_t$$

by substituting. Dividing by $|S_A \times S_B|$,

$$g_{A,B} \cdot e_t = 0.$$

□

Theorem 3.4.7. *The set $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ spans S^λ .*

Proof. Let t be a λ -tableau. Suppose the Garnir element associated with t is $g_{A,B} = e \pm \sigma_1 \pm \cdots \pm \sigma_k$. By the previous lemma, $0 = g_{A,B} \cdot e_t = e_t \pm \sigma_1 \cdot e_t \pm \cdots \pm \sigma_k \cdot e_t$. So we can express e_t as

$$e_t = \sum_{i=1}^k \pm \sigma_i \cdot e_t, \quad (3.8)$$

a linear combination of other polytabloids. This completes the straightening algorithm. By defining a partial ordering on tableaux, it can be formalized that the polytabloids in this linear combination are closer to being standard. See page 73 of [7] for the full details. We can repeatedly apply the straightening algorithm until we have expressed e_t as a linear combination of polytabloids associated with standard tableaux, completing the proof by induction. \square

Theorem 3.4.8. *The set $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ is basis for S^λ and $\dim(S^\lambda) = \dim(X^\lambda) = f^\lambda$ where f^λ denotes the number of standard tableaux of shape λ .*

Proof. Since the set is linearly independent and spans, it forms a basis. The dimension is the number of basis elements which is equal to the number of standard λ -tableaux. \square

Corollary 3.4.9. *We have*

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Proof. This corollary is a dimension counting argument that is proven by combining Theorem 2.5.6, Theorem 3.4.8, and Theorem 2.2.6. \square

Interestingly enough, this corollary could be interpreted as evidencing a bijection between permutations as counted by the left hand side and pairs of standard tableaux

as counted by the right. In Section 5.3, we include an additional bijective proof of a similar dimension counting corollary that is a generalization of this one.

Chapter 4

SYMMETRIC FUNCTIONS AND THE MURNAGHAN-NAKAYAMA RULE

In this chapter, we develop the theory of symmetric functions and introduce the Frobenius Map revealing the duality of class functions and symmetric functions. The realm of symmetric functions, while fascinating, is not the main focus of this thesis. Most of this chapter is a summary of Dr. Anthony Mendes's writing, particularly in [4], and we omit some proofs for brevity.

The key relevant result is the Murnaghan-Nakayama Rule which will give a complete combinatorial interpretation of the irreducible characters of S_n and allow us to construct the character table of S_n for any positive integer n .

4.1 Standard Bases for Symmetric Functions

Definition 4.1.1. A polynomial f in the indeterminates x_1, \dots, x_N is *symmetric* if for all $\sigma \in S_N$, $f(x_1, \dots, x_N) = f(x_{\sigma(1)} \dots x_{\sigma(N)})$.

We write $\Lambda_n(x_1, \dots, x_N)$ to denote the set of symmetric polynomials with each monomial having total degree n . In practice, we always work with $N \gg n$. For example, $\Lambda_2(x_1, x_2, x_3)$ contains the element $2x_1x_2 + 2x_1x_3 + 2x_2x_3 - x_1^2 - x_2^2 - x_3^2$.

Definition 4.1.2. A function P in the indeterminates x_1, x_2, \dots is *symmetric* if for any N , setting $x_{N+1} = x_{N+2} = \dots = 0$, makes P a symmetric polynomial in the indeterminates x_1, \dots, x_N .

We write $\Lambda_n = \Lambda_n(x_1, x_2, \dots)$ to denote the set of symmetric functions with each monomial having total degree n .

Proposition 4.1.3. *The set Λ_n is a vector space.*

Since $\Lambda_n(x_1, \dots, x_N)$ is what results by setting $x_{N+1} = x_{N+2} = \dots = 0$, $\Lambda_n(x_1, \dots, x_N)$ inherits properties from Λ_n , like also being a vector space.

Definition 4.1.4. For a partition λ , we define the monomial symmetric function m_λ to be the sum over all monomials with exponents that are rearrangements of λ .

Theorem 4.1.5. *The set $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n .*

Proof. No two different elements of this set share a monomial. Equating coefficients of monomials in $\sum_{\lambda \vdash n} c_\lambda m_\lambda$, where $c_\lambda \in \mathbb{C}$ forces $c_\lambda = 0$ for all λ .

Consider $f \in \Lambda_n$. Let c_λ be the coefficient of $x_1^{\lambda_1} \dots x_k^{\lambda_k}$ for any $\lambda = \lambda_1, \dots, \lambda_k \vdash n$. Symmetry forces the coefficient of $x_1^{\sigma(\lambda_1)} \dots x_k^{\sigma(\lambda_k)}$ to also be c_λ for any $\sigma \in S_n$. Since all terms have this form, $f = \sum_{\lambda \vdash n} c_\lambda m_\lambda$. □

We will define four other basis for Λ_n using weighted tableau, all also indexed by partitions. We loosen the definition of tableau to just be any filling of the Young diagram for λ with positive integers.

Definition 4.1.6. If a tableau t has values t_1, \dots, t_n in its n cells, then the weight of t is $\text{wt}(t) = x^{t_1} \dots x^{t_n}$.

Definition 4.1.7. A λ -tableau t is *column-strict* if every row of t forms a weakly increasing sequence from left to right and every column of t forms a strictly increasing sequence from bottom to top.

The object

$$t = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & & & \\ \hline 2 & 3 & 3 & 4 & \\ \hline 1 & 1 & 2 & 2 & 5 \\ \hline \end{array}$$

is an example of a column strict tableaux of shape $\lambda = 5\ 4\ 2$.

Definition 4.1.8. For a partition λ , we define the *power sum symmetric function* p_λ to be $\sum \text{wt}(t)$ where the sum is over all λ -tableaux t with constant rows.

Definition 4.1.9. For a partition λ , we define the *homogeneous symmetric function* h_λ to be $\sum \text{wt}(t)$ where the sum is over all λ -tableaux t with weakly increasing rows.

Definition 4.1.10. For a partition λ , we define the *elementary symmetric function* e_λ to be $\sum \text{wt}(t)$ where the sum is over all λ -tableaux t with strictly increasing rows.

Definition 4.1.11. For a partition λ , we define the *Schur symmetric function* s_λ to be $\sum \text{wt}(t)$ where the sum is over all column strict λ -tableaux t .

We verify that these four are all in fact symmetric functions.

Theorem 4.1.12. For any $\lambda \vdash n$, we have $p_\lambda, h_\lambda, e_\lambda, s_\lambda \in \Lambda_n$.

Proof. Since every permutation can be written as a product of adjacent transpositions, it suffices to show that $p_\lambda, h_\lambda, e_\lambda, s_\lambda$ are unchanged when x_i and x_j are switched where $j = i + 1$. So on each of row constant, non-decreasing, row increasing, and column strict tableaux we need an involution that switches the number of i 's and j 's in the tableau.

For row constant tableaux, replacing all i 's with j and all j 's with i does the trick.

For non-decreasing tableaux and row-increasing tableaux, all occurrences of i in a row, will be immediately followed by all occurrences of j in the same row. For each

row, replace this string of entries with i 's followed by j 's, switching the number of i 's and j 's.

For column strict tableau, the occurrences of i and j in a column strict tableau must look like:

i	i	i	j	j	j	j					
			i	i	i	i	j	j	j		
							i	i	i	j	j

In the section of each row that has no i 's or j 's above or below, switch the number of i 's and j 's. In our example,

i	j	j	j	j	j	j					
			i	i	i	i	j	j	j		
							i	i	i	i	j

would be the i 's and j 's in the new tableaux. This new tableau is also column strict because $j = i + 1$. □

Theorem 4.1.13. *The sets $\{p_\lambda : \lambda \vdash n\}$, $\{h_\lambda : \lambda \vdash n\}$, $\{e_\lambda : \lambda \vdash n\}$, and $\{s_\lambda : \lambda \vdash n\}$ are each a basis for Λ_n .*

Proofs that each of these sets is actually a basis can be found in chapter 2 of [6].

4.2 Rim-Hook Tableaux

In this section we introduce the object that will ultimately appear in our combinatorial interpretation of the irreducible characters of S_n .

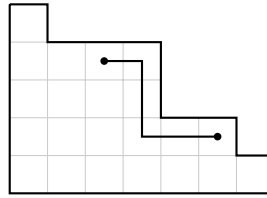
Definition 4.2.1. Let λ be a partition of n . A *rim-hook* of length k is a sequence of adjacent cells x_1, \dots, x_k in the partition diagram of λ that satisfies three conditions. For any $i = 1, \dots, k - 1$,

1. x_1 has no empty cell directly above it.

2. if there is a cell in the Young diagram directly to the right of x_i , it is x_{i+1} .
3. if there is no cell directly to the right of x_i , x_{i+1} is directly below.

Definition 4.2.2. The *sign* of a rim-hook h is $(-1)^m$ where m is the number of cells in h such that the cell directly below them is also in h .

The rim-hook depicted as

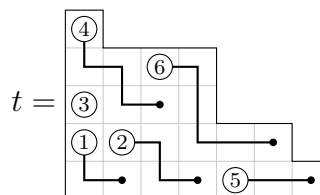


has sign $(-1)^2 = 1$.

Definition 4.2.3. A *composition* of n is an ordered list of n numbers who sum to n .

Definition 4.2.4. Let $\lambda \vdash n$ and $\mu = \mu_1, \dots, \mu_k$ be a composition of n . A *rim-hook tableau* of shape λ and content μ is a filling of the Young diagram for λ with rim-hooks labeled $1, \dots, k$ with lengths μ_1, \dots, μ_k respectively. We require that for any i , the cells covered by rim-hooks $1, \dots, i$ form a valid partition shape.

When constructing a rim-hook tableau, we think of placing the rim-hooks in reverse order. We first place a rim-hook of length μ_k labeled k along the upper right boundary. Then treating the remaining empty cells as a new smaller Young diagram, place a rim-hook of length μ_{k-1} labeled $k - 1$ into this new shape, etc. We write the labels in the head of each rim-hook—the upper left cell. If $n = 22$, $\lambda = 7 \ 6 \ 4 \ 4 \ 1$, and $\mu = 3 \ 4 \ 1 \ 5 \ 3 \ 6$, then



is an example of a rim-hook tableau of shape λ and content μ .

Definition 4.2.5. The *sign of a rim-hook tableau* is the product of the signs of each rim-hook it contains.

In our example, the rim-hooks labeled 1, 2, 3, 4, 5, 6 have signs $-1, -1, 1, 1, 1, 1$ respectively. Multiplying these together, we see that $\text{sign}(t) = 1$. The sign of a rim-hook tableau can also be thought of as -1 to the total number of vertical segments in rim-hooks. For example, t has a total of 6 vertical segments and thus has sign $\text{sign}(t) = (-1)^6 = 1$.

We write $\text{RH}_\lambda(\mu)$ to denote the set of all rim hook tableaux of shape λ and content μ . For example if $\lambda = 3\ 3\ 1$ and $\mu = 1\ 2\ 3\ 1$, $\text{RH}_\lambda(\mu)$ would be the set

$$\text{RH}_\lambda(\mu) = \left\{ \begin{array}{|c|c|c|} \hline \textcircled{4} & & \\ \hline \textcircled{3} & \longrightarrow & \\ \hline \textcircled{1} & \textcircled{2} & \longrightarrow \\ \hline \end{array} , \begin{array}{|c|c|c|} \hline \textcircled{3} & & \\ \hline \textcircled{1} & \longrightarrow & \textcircled{4} \\ \hline \textcircled{1} & \textcircled{2} & \longrightarrow \\ \hline \end{array} , \begin{array}{|c|c|c|} \hline \textcircled{2} & & \\ \hline \textcircled{1} & \longrightarrow & \textcircled{3} \textcircled{4} \\ \hline \textcircled{1} & & \longrightarrow \\ \hline \end{array} \right\}. \quad (4.1)$$

Theorem 4.2.6. Let λ be a partition of n and μ be a composition of n . If ν is a rearrangement of μ , then

$$\sum_{t \in \text{RH}_\lambda(\mu)} \text{sign}(t) = \sum_{t \in \text{RH}_\lambda(\nu)} \text{sign}(t).$$

For a proof of this theorem, see pages 136 through 140 of [5].

Definition 4.2.7. For $\lambda, \mu \vdash n$, we define

$$\chi_\mu^\lambda = \sum_{t \in \text{RH}_\lambda(\nu)} \text{sign}(t)$$

where ν is some ordering of μ into a composition.

We know this is well defined by the previous theorem.

4.3 Relationships Between Bases

Since we have five different bases for the space of symmetric functions, we should be able to convert from one to another. We can prove identities relating the different bases of Λ_n . We will prove a relationship between the power sum and Schur symmetric functions.

In the proof of following theorem it will be convenient to arrange cycles of a permutation in such a way that the parentheses convey no meaningful information. In this way we would be able to represent a permutation sorted by its cycles with a single string of numbers.

We can write a permutation in cycle notation with the largest element first in each cycle and arrange the cycles in order of increasing largest element. Consider the permutation $\sigma = (163)(2)(45)$ from section 1.1. We could reorder and rearrange the cycles in this way to rewrite the permutation as

$$\sigma = (2)(54)(631),$$

which we call the the *implicit cycle notation* for the permutation σ . If we were given just the string of numbers

$$2 \ 5 \ 4 \ 6 \ 3 \ 1,$$

and were told that it corresponded to the implicit cycle notation for σ , we could follow an algorithm to recover where the parenthesis must have been. Scanning from left to right, remember the largest number. Each time we see a new largest number, it will be the start of a new cycle.

The implicit cycle notation is not exactly how we will arrange our cycles in the upcoming proof, but it will be helpful to be familiar with it.

Theorem 4.3.1. *For any $\lambda \vdash n$,*

$$n!s_\lambda = \sum_{\sigma \in S_n} p_{\mu(\sigma)} \chi_{\mu(\sigma)}^\lambda$$

where $\mu(\sigma)$ denotes the cycle type of $\sigma \in S_n$.

Proof. Our strategy will be to first construct a signed and weighted object that gives a combinatorial interpretation of the right hand side. We will then define a sign reversing, weight preserving involution on these objects and prove that the fixed points are in a weight preserving bijective correspondence with pairs of permutations and column strict tableaux, as counted by the left hand side. Once we do this, the result will be proven.

Choose a permutation $\sigma \in S_n$. As an example consider $n = 22$, $\lambda = 7\ 6\ 4\ 4\ 1$ and

$$\sigma = (1\ 4\ 19\ 20\ 7\ 18)(2\ 8\ 15)(3\ 21\ 14\ 22\ 12)(5\ 6\ 13\ 11)(9\ 16\ 17)(10).$$

Reorder each cycle of σ to begin with its largest element. In our example,

$$\sigma = (20\ 7\ 18\ 1\ 4\ 19)(15\ 2\ 8)(22\ 12\ 3\ 21\ 14)(13\ 11\ 5\ 6)(17\ 9\ 16)(10).$$

Since $p_{\mu(\sigma)}$ is the weighted sum over all row constant tableaux of shape $\mu(\sigma)$, we account for this term by assigning the constant value of the row as a label to each of the cycles of σ . For cycles of the same length, we establish the convention that the cycle containing the smaller element will correspond to the lower row in the tableau. We arrange the cycles first in order of increasing label and then by increasing largest element, similar to the arrangement in implicit cycle notation. In our example, if our

row constant tableau was

$$r = \begin{array}{|c|c|c|c|c|c|} \hline 1 & & & & & \\ \hline 3 & 3 & 3 & & & \\ \hline 2 & 2 & 2 & & & \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 5 & 5 & 5 & 5 & 5 & \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array},$$

then the pair of σ and r would correspond to

$$\underbrace{(10)}_1 \underbrace{(15 \ 2 \ 8)}_2 \underbrace{(20 \ 7 \ 18 \ 1 \ 4 \ 19)}_2 \underbrace{(13 \ 11 \ 5 \ 6)}_3 \underbrace{(17 \ 9 \ 16)}_3 \underbrace{(22 \ 12 \ 3 \ 21 \ 14)}_5.$$

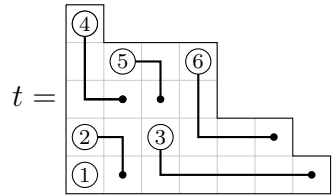
We can entirely recover our row constant tableaux by filling each row with the label on its cycle.

Choose the ordering on $\mu(\sigma)$ by taking the lengths of each cycle of σ as arranged above. In our example,

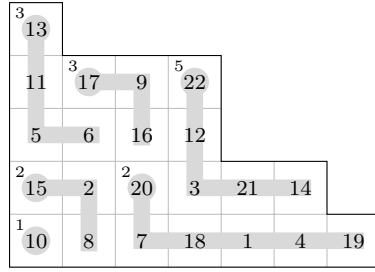
$$\mu(\sigma) = 1 \ 3 \ 6 \ 4 \ 3 \ 5.$$

With this choice, we will be able to insert the cycles into the rim-hooks in a canonical way. Lastly, we combine σ and r with some $t \in \text{RH}_\lambda(\mu(\sigma))$. We write the i^{th} cycle of σ inside the cells of the i^{th} rim-hook of t . We do this in such a way that reading from upper left to bottom right within a rim-hook is the arrangement of the cycle as above. We write the corresponding label in the upper left of the cell containing the head of the rim-hook.

Suppose in the same example,



Then the fully combined object after inserting the labeled permutation is



Notice that we can omit the numbers in the head of the rim-hooks because the order is implicit from the permutation inside the rim-hooks. The *sign* of these objects is defined to be the sign of the underlying rim-hook tableaux and the *weight* is defined to be the weight of the tableau if the rim-hooks and labels are ignored. Let \mathcal{T} be the set of all objects that can be constructed in this way. Then the left hand side is given by $\sum_{t \in \mathcal{T}} \text{sign}(t) \text{wt}(t)$.

We will now define a sign reversing, weight preserving involution ϕ on \mathcal{T} as follows. Consider an object $t \in \mathcal{T}$. Scan from left to right in each row, starting at the bottom row and working to the top row. Call the current cell x and the cell immediately above (if it exists) y . Stop if we reach one of these two cases:

- Case I: The cells x and y are in the same rim-hook.
- Case II: The labels of the rim-hooks containing x and y match and y is the last cell in a rim-hook.

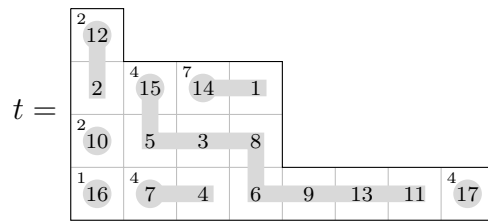
If we scan through every cell and never end up in Case I or Case II, we define $\phi(t) = t$.

Case I: In Case I, split the rim-hook containing x and y by removing the vertical connection between x and y . Then consider the set of cells that appear in the same row as x , including x , whose label matches the label of x , whose value is less than the value in the head of the rim-hook containing x . The label of a cell refers to the label

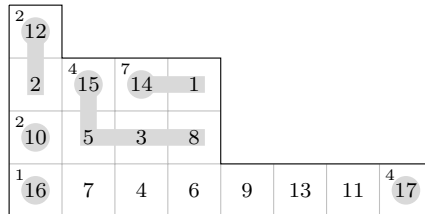
of the rim-hook in which they were contained. Call the cells in this set x_1, \dots, x_k . Remove any rim-hooks on this set.

Now we will add back in rim-hooks in the following way. Run through values of i ranging from 1 to k applying the following rules. If $i = 1$, form a new rim-hook in cell x_i . If $i > 1$, find the value in the first cell of the rim-hook containing x_{i-1} . If the value in x_i is smaller, add x_i to this rim-hook. Otherwise form a new rim-hook in cell x_i . All new rim-hooks should be labeled the same as the rim-hook containing x was originally. In other words, cells keep their labels.

For example, consider applying ϕ to



Scanning from left to right and bottom to top, we arrive first at a Case I. The cell containing 6 shares a rim-hook with the cell directly above it. We remove the vertical segment connecting them. We identify the set in the row with the same labels whose value is less than 15 to be the cells containing 7, 4, 6, 9, 13, and 11. Removing the rim-hooks yields



Now we add rim-hooks back in. The 7 starts a new rim-hook. Since 4 and 6 are both less than 7, they are attached to this same rim-hook. Continuing to move to the right through this row, 9 is the first number we hit that is larger than 7. So we start a new rim-hook. Since 13 is bigger than 9, it again starts a new rim-hook. Since 11 is less

Originally, in choosing the order in which to insert our rim-hooks, we sorted first by increasing label. This ensures that the tableau r will have weakly increasing values in the rows and columns. Since t is a fixed point, we have no occurrences of a cell with a cell above it with the same label that is the end of a rim-hook. Suppose cell x in t has a cell y above and they share label l , but y is *not* the end of a rim-hook. Look down the row at the cell at the end of the rim-hook containing y . The cell below this must also have label l and is thus an occurrence of Case II in t , a contradiction. So r must have strictly increasing columns. The entries in r are exactly the same as the letters of the word c they came from. Thus r is a column strict tableau of the correct weight.

We can verify that this is a bijective correspondence by defining the inverse map from pairs (σ, r) to fixed points of ϕ . Insert the one-line notation expression for σ into the cells of r from left to right and top to bottom. Now there will be both a value and a label in each cell. Within each section of each row that has constant label, add rim-hooks in the unique way that has the largest element at the start of each rim-hook and has rim-hooks ordered by increasing largest element, just like we did with cycles of permutations in implicit cycle notation. Now every rim-hook will be labeled redundantly with the same label occurring in each cell. Instead, just label each rim-hook once in the upper left corner with that constant value. We can verify with the same example that applying this reverse algorithm to the pair (σ, r) recovers t . Since this reverse algorithm is a unique construction, we have a bijection.

Equating the cardinalities of these two sets, the number of fixed points of a certain weight is equal to the number of pairs of permutations and column strict tableau of the same weight. Summing over all possible weights, the result is proven. \square

In a similar manner, interpreting identities combinatorially, other relationships between symmetric functions can be proven.

4.4 The Frobenius Map

Definition 4.4.1. Let $C(S_n)$ be the vector space of class functions for the symmetric group and let K_λ be the conjugacy class of S_n associated with λ . Let 1_λ be the class function whose value is one on permutations with cycle type λ and is zero otherwise. The *Frobenius Map* $F : \Lambda_n \mapsto C(S_n)$ is the linear transformation defined by

$$F(p_\lambda/z_\lambda) = 1_\lambda$$

where $z_\lambda = n!/|K_\lambda|$.

Since $\{1_\lambda : \lambda \vdash n\}$ is a basis for $C(S_n)$ and $\{p_\lambda/z_\lambda : \lambda \vdash n\}$ is a basis for Λ_n , the Frobenius map is invertible.

As we saw in Theorem 2.6.15, the set of irreducible characters of S_n , which we now know to be $\{\chi^\lambda : \lambda \vdash n\}$, is also a basis for $C(S_n)$. So there exists a basis for Λ whose image under the Frobenius Map is this set. Theorem 4.4.2 reveals that this is the basis $\{s_\lambda : \lambda \vdash n\}$.

Theorem 4.4.2. *We have*

$$F(s_\lambda) = \chi^\lambda.$$

The proof of Theorem 4.4.2 relies on symmetric function identities that can be proven combinatorially, in a similar manner to Theorem 4.3.1. For the full proof, see page 58 of [4].

4.5 Character Tables of the Symmetric Group

Theorem 4.5.1 (Murnaghan-Nakayama). *Let μ be a composition of n . If σ is in the conjugacy class associated with the partition corresponding to μ , then*

$$\chi^\lambda(\sigma) = \chi_{\mu(\sigma)}^\lambda$$

where $\mu(\sigma)$ is the cycle type of σ .

Proof. Theorem 4.3.1 can be restated as

$$n!s_\lambda = \sum_{\mu \vdash n} |K_\mu| p_\mu \chi_\mu^\lambda \tag{4.5}$$

where K_μ is the conjugacy class associated with μ . Dividing by $n!$,

$$s_\lambda = \sum_{\mu \vdash n} \chi_\mu^\lambda \frac{p_\mu}{z_\mu}$$

because $|K_\mu|/n! = z_\mu$. Applying the Frobenius map,

$$\chi^\lambda = \sum_{\mu \vdash n} \chi_\mu^\lambda 1_\mu$$

since $F(s_\lambda) = \chi^\lambda$ and $F(p_\mu/z_\mu) = 1_\mu$. Evaluating both sides at a particular $\sigma \in S_n$ proves the result because $1_\mu(\sigma)$ zeros out all terms in the sum except the term where μ is the cycle type of σ . □

This theorem allows us to find all irreducible characters of S_n by counting signed sums of rim-hook tableaux. In the character table for S_n , where columns correspond to conjugacy classes and rows correspond to irreducible representations, the row λ ,

column μ entry is given by the signed sum of all rim-hook tableaux of shape λ and content μ (once we fix an ordering of μ).

Example 4.5.2. Let us use this rule to construct the character table for S_4 . The five partitions of 4 are 4, 31, 22, 211, 1111. We observed that the irreducible characters $\chi^{(4)}$ and $\chi^{(1111)}$ are associated with the trivial and sign representations respectively, which we already know the characters for. The reader can verify that the characters we would get by counting rim-hook tableaux agrees with these.

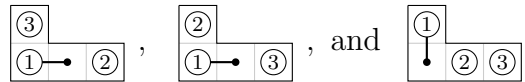
Consider $\lambda = 31$. Let us compute the irreducible character χ^λ by counting rim-hook tableaux of shape λ and content μ for each $\mu \vdash n$. We will assume μ as a composition is ordered in descending order.

For $\mu = 1111$, our tableaux



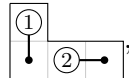
all have sign 1. So $\chi_\mu^\lambda = 1 + 1 + 1 = 3$.

For $\mu = 211$, our tableaux



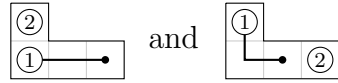
have sign 1, 1, and -1 respectively. So $\chi_\mu^\lambda = 1 + 1 - 1 = 1$.

For $\mu = 22$, we have one tableau,



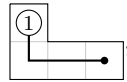
with sign -1. So $\chi_\mu^\lambda = -1$.

For $\mu = 31$, our tableaux



have sign 1 and -1 respectively. So $\chi_\mu^\lambda = 1 - 1 = 0$.

For $\mu = 4$, we have one tableau,



with sign -1. So $\chi_\mu^\lambda = -1$.

Inserting the character row $3, 1, -1, 0, -1$ into the table along with our known characters for the trivial and sign representation, we have:

	K_{1111}	K_{211}	K_{22}	K_{31}	K_4
$\chi^{(4)}$	1	1	1	1	1
$\chi^{(31)}$	3	1	-1	0	-1
$\chi^{(22)}$					
$\chi^{(211)}$					
$\chi^{(1111)}$	1	-1	1	1	-1

We can use the same counting strategy with $\lambda = 22$ and $\lambda = 211$, to fill out the remaining two rows, giving us the completed character table of S_4 .

Table 4.1: Character Table of S_4

	K_{1111}	K_{211}	K_{22}	K_{31}	K_4
$\chi^{(4)}$	1	1	1	1	1
$\chi^{(31)}$	3	1	-1	0	-1
$\chi^{(22)}$	2	0	2	-1	0
$\chi^{(211)}$	3	-1	-1	0	1
$\chi^{(1111)}$	1	-1	1	1	-1

And of course, we can now compute the character table of S_n of any n . The tables of S_5 and S_6 are included below.

Table 4.2: Character Table of S_5

	K_{11111}	K_{2111}	K_{221}	K_{311}	K_{32}	K_{41}	K_5
$\chi^{(5)}$	1	1	1	1	1	1	1
$\chi^{(41)}$	4	2	0	1	-1	0	-1
$\chi^{(32)}$	5	1	1	-1	1	-1	0
$\chi^{(311)}$	6	0	-2	0	0	0	1
$\chi^{(221)}$	5	-1	1	-1	-1	1	0
$\chi^{(2111)}$	4	-2	0	1	1	0	-1
$\chi^{(11111)}$	1	-1	1	1	-1	-1	1

Table 4.3: Character Table of S_6

	K_{111111}	K_{21111}	K_{2211}	K_{222}	K_{3111}	K_{321}	K_{33}	K_{411}	K_{42}	K_{51}	K_6
$\chi^{(6)}$	1	1	1	1	1	1	1	1	1	1	1
$\chi^{(51)}$	5	3	1	-1	2	0	-1	1	-1	0	-1
$\chi^{(42)}$	9	3	1	3	0	0	0	-1	1	-1	0
$\chi^{(411)}$	10	2	-2	-2	1	-1	1	0	0	0	1
$\chi^{(33)}$	5	1	1	-3	-1	1	2	-1	-1	0	0
$\chi^{(321)}$	16	0	0	0	-2	0	-2	0	0	1	0
$\chi^{(3111)}$	10	-2	-2	2	1	1	1	0	0	0	-1
$\chi^{(222)}$	5	-1	1	3	-1	-1	2	1	-1	0	0
$\chi^{(2211)}$	9	-3	1	-3	0	0	0	1	1	-1	0
$\chi^{(21111)}$	5	-3	1	1	2	0	-1	-1	-1	0	1
$\chi^{(111111)}$	1	-1	1	-1	1	-1	1	-1	1	1	-1

Chapter 5

REPRESENTATIONS FROM AN ACTION ON WORDS

We have finally laid sufficient groundwork to enable our exploration of new representations. In this chapter, we turn our attention to specific representations of S_n arising from its action on words.

We first define the action and construct the representations. Next we provide a complete combinatorial interpretation of the irreducible decompositions. Then, we explore generalizations stemming from restricting the action to orbits. Finally, we examine a dimension counting corollary and introduce an algorithm that can alternatively be used to directly prove this and other similar corollaries.

5.1 A Combinatorial Interpretation

Let \mathcal{A} be a finite multi-set with a complete ordering on its elements. Let C be the set of words of length n whose letters come from the alphabet \mathcal{A} . For example, if $n = 3$ and $\mathcal{A} = \{1, 1, 2, 2, 4\}$, then

$$C = \{112, 114, 121, 122, 124, 141, 142, 211, 212, 214, 221, 224, 241, 242, 411, 412, 421, 422\}.$$

We define the action of S_n on C by

$$\sigma \cdot c = c_{\sigma(1)} \cdots c_{\sigma(n)} \tag{5.1}$$

for $\sigma \in S_n$ where $c = c_1 \dots c_n \in C$, permuting the letters in the word. Throughout this section, let X be the representation of S_n with character χ arising from this action.

We write $\text{CS}_\lambda(\mathcal{A})$ to denote the set of all column strict tableaux of shape λ whose entries come from the multiset \mathcal{A} . For example if $\mathcal{A} = \{1, 1, 2, 3\}$ and $\lambda = 2, 1$, $\text{CS}_\lambda(\mathcal{A})$ would be the set

$$\text{CS}_\lambda(\mathcal{A}) = \left\{ \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \right\}. \quad (5.2)$$

Notice that

$$\begin{array}{|c|c|} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array} \notin \text{CS}_\lambda(\mathcal{A})$$

because even though this is a valid column strict tableau, there is only a single 2 available in the multiset \mathcal{A} .

Theorem 5.1.1. *Let $\lambda \vdash n$. If $|\mathcal{A}| = n$, the number of copies of X^λ in the irreducible decomposition of X is equal to $|\text{CS}_\lambda(\mathcal{A})|$.*

Proof. The number of copies of X^λ in X is given by the inner product

$$\langle \chi, \chi^\lambda \rangle = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \chi(\sigma) \chi^\lambda(\sigma) = \frac{1}{n!} \sum_{\sigma \in S_n} |C^\sigma| \chi^\lambda(\sigma) \quad (5.3)$$

where C^σ denotes the fixed point set of σ . By Theorem 4.5.1, $\chi^\lambda(\sigma)$ is the signed sum of rim-hook tableaux of shape λ and content cycle type of σ .

We want to show that this inner product is equal to $|\text{CS}_\lambda(\mathcal{A})|$. Multiplying by $n!$, this is equivalent to proving that

$$\sum_{\sigma \in S_n} |C^\sigma| \chi_{\mu(\sigma)}^\lambda = n! |\text{CS}_\lambda(\mathcal{A})|. \quad (5.4)$$

By Theorem 4.3.1,

$$\sum_{\sigma \in S_n} p_{\mu(\sigma)} \chi_{\mu(\sigma)}^\lambda = n! s_\lambda. \quad (5.5)$$

Let a_1, \dots, a_n be the elements of \mathcal{A} . The coefficient of $x_{a_1} \cdots x_{a_n}$ in s_λ is equal to $|\text{CS}_\lambda(\mathcal{A})|$. So if we can show that the coefficient of $x_{a_1} \cdots x_{a_n}$ in $p_{\mu(\sigma)}$ is equal to $|C^\sigma|$, equating coefficients of $x_{a_1} \cdots x_{a_n}$ and substituting in Equation 5.4 will prove the result.

The coefficient of $x_{a_1} \cdots x_{a_n}$ in $p_{\mu(\sigma)}$ is the number of ways to arrange \mathcal{A} into a tableaux of shape $\mu(\sigma)$ with constant rows. For a given such object, associate each row of these tableaux with a cycle of σ of the correct length, breaking ties by associating the lower row with the cycle containing the smaller element. Create a word by setting the i^{th} letter in the word be the constant value on the row associated with the cycle containing i . For example if $n = 9$, $\mathcal{A} = \{1, 1, 2, 2, 2, 2, 4, 4\}$ and $\sigma = (15)(2783)(46)$, the object

2				
1	1			
4	4			
2	2	2	2	

would correspond to the word 422141222. This is a bijection to the set of ways to arrange \mathcal{A} into a word that is fixed under the action of σ . So the coefficient of $x_{a_1} \cdots x_{a_n}$ in $p_{\mu(\sigma)}$ is equal to $|C^\sigma|$. □

5.2 Generalizations from Restricting the Action to an Orbit

Theorem 5.1.1 can be generalized in multiple ways. To see how, we will develop the theory of representations arising from actions a little further with an additional theorem.

Theorem 5.2.1. *Let X be a representation arising from the action of a group G on a set C . If C has orbits O_1, \dots, O_k , then*

$$X = X_1 \oplus \cdots \oplus X_k$$

where X_i is the representation of G arising from the action of G restricted to the orbit O_i .

Proof. Suppose the O_1, \dots, O_k is a complete list of orbits of C . In the construction of X , we fix an ordering of C . Choose this ordering such that elements of O_i occur before elements of O_j for all $i < j$. Recall that for any $\sigma \in S_n$, the column corresponding to some $c \in C$ contains a single one occurring in row $\sigma(c)$ and zeros everywhere else. Since $\sigma(c)$ shares an orbit with c for any $c \in C$, the only ones in the columns associated with a given orbit occur in the same rows. So the matrix $X(\sigma)$ takes a block diagonal form, with each block corresponding to the action of S_n on an orbit. \square

Now we can state a stronger version of Theorem 5.1.1 without the requirement that $|\mathcal{A}| = n$. Let X , \mathcal{A} , and C be as described in the previous section.

Theorem 5.2.2. *For $\lambda \vdash n$, the number of copies of X^λ in the irreducible decomposition of X is equal to $|CS_\lambda(\mathcal{A})|$.*

Proof. Let A_1, \dots, A_k be a complete list of the subsets of \mathcal{A} of size n . Let C_i be the set of all words from A_i . Observe that C is the disjoint union of the C_i and that the C_i are exactly the orbits of the action. If X_i is the representation arising from the restriction of the action to C_i , then by Theorem 5.1.1, the number of copies of X^λ in X_i is $|CS_\lambda(A_i)|$. So by Theorem 5.2.1 the number of copies of X^λ in X is $\sum_{i=1}^k |CS_\lambda(A_i)|$ which equals $|CS_\lambda(\mathcal{A})|$. \square

Corollary 5.2.3. *We have*

$$|C| = \sum_{\lambda \vdash n} f^\lambda |CS_\lambda(\mathcal{A})|$$

where f^λ is the number of standard tableaux of shape λ .

Proof. We know the $\dim(X) = |C|$, but by Theorem 2.2.6, $\dim(X)$ is also equal to the sum of the dimensions of the X^λ times their multiplicities. By Theorem 3.4.8, the dimension of each X^λ is f^λ and by Theorem 5.2.2, the number of copies of each X^λ is $CS_\lambda(\mathcal{A})$. \square

Example 5.2.4. As an example to demonstrate Theorem 5.2.2, suppose our alphabet is $\mathcal{A} = \{1, 1, 2, 3, 4, 4\}$ and $n = 5$. Let C be the set of all words of length 5 coming from \mathcal{A} and let X be the representation arising from the action of S_5 on C with character χ . We know we can represent X as

$$X \cong \bigoplus_{\lambda \vdash 5} m_\lambda X^\lambda. \tag{5.6}$$

Theorem 5.2.2 claims that $m_\lambda = CS_\lambda(\mathcal{A})$. Let us verify this for a specific $\lambda \vdash 5$.

Consider the partition $\lambda = 32 \vdash 5$. On one hand, we know we can find m_{32} by computing inner products. From the character table of S_5 ,

	K_{11111}	K_{2111}	K_{221}	K_{311}	K_{32}	K_{41}	K_5
$\chi^{(32)}$	5	1	1	-1	1	-1	0

is the character row for $\chi^{(32)}$. To find the character row, of χ , we count the fixed points C for one representative permutation from each conjugacy class. All words are fixed by the permutation e . So the the number of fixed points of e is $|C|$, which in this case equals 180. The permutation (12) fixes all words that have the first two entries

matching. So the number of fixed points of (12) is 24. The permutation (12)(34) has only 4 fixed points:

$$11442, 11443, 44112, 44113,$$

because we require the first two entries to match and the third and fourth entries to match. Any permutation with a cycle of length greater than 2 has no fixed points in this example because we have at most 2 of any character in \mathcal{A} . Interpreting these numbers as the value of χ on conjugacy classes,

	K_{11111}	K_{2111}	K_{221}	K_{311}	K_{32}	K_{41}	K_5
χ	180	24	4	0	0	0	0

is the character row for χ .

We are now in a position to compute

$$\begin{aligned}
 m_{32} &= \langle \chi, \chi^{(32)} \rangle \\
 &= \frac{1}{5!} \sum_K |K| \chi_K \chi_K^{(32)} \\
 &= \frac{1}{120} \left[1(180 \cdot 5) + 10(24 \cdot 1) + 15(4 \cdot 1) \right. \\
 &\quad \left. + 20(0 \cdot (-1)) + 20(0 \cdot 1) + 30(0 \cdot (-1)) + 24(0 \cdot 0) \right] \\
 &= 10.
 \end{aligned}$$

On the other hand, the column strict tableau that have entries from \mathcal{A} are:

3 4 1 1 2	4 4 1 1 2	2 4 1 1 3	4 4 1 1 3	2 3 1 1 4
2 4 1 1 4	3 4 1 1 4	4 4 1 2 3	3 4 1 2 4	2 4 1 3 4

So by Theorem 5.2.2, $m_{32} = |\text{CS}_\lambda(\mathcal{A})| = 10$.

We can also generalize Theorem 5.1.1 in other ways by placing conditions on the words. The original formulation of the representation explored in this chapter was not from an action on words, but an action on compositions. Let us see how we can use this theorem to explain the irreducible decomposition of the representation arising from the action of S_n on compositions of n .

Corollary 5.2.5. *Let X be the representation arising from the natural action of S_n on compositions of k into exactly n parts. The number of copies of X^λ in X is equal to the number of column strict λ -tableau filled with positive integers that sum to k .*

Proof. Permuting compositions changes the order in which the entries occur, but does not change the entries themselves. In other words, the underlying partition remains constant. Also for any two compositions with the same underlying partition, there is a permutation from one to the other. So the orbits of this action correspond to partitions of k into exactly n parts.

By Theorem 5.2.1, $X = \bigoplus X_\mu$ over all partitions $\mu \vdash k$ into n parts where X_μ is the representation of G arising from the action of G restricted to the orbit corresponding to μ . This action restricted to the orbit corresponding to $\mu = \mu_1, \dots, \mu_n$ is identical to the action on words of length n from the alphabet $\mathcal{A} = \{\mu_1, \dots, \mu_n\}$. By Theorem 5.1.1, the number of copies of X^λ in X_μ is the number of column strict λ -tableau with entries μ_1, \dots, μ_n . The number of copies of X^λ in X is the sum of the number of copies contained in X_μ over all partitions $\mu \vdash k$ into n parts, which is the total number of column strict λ -tableaux whose entries sum to k . \square

Example 5.2.6. As an example to demonstrate Corollary 5.2.5, suppose C is the set of compositions of $k = 12$ into exactly $n = 6$ parts. Letting X be the representation arising from this action,

$$X \cong \bigoplus_{\lambda \vdash 6} m_\lambda X^\lambda \tag{5.7}$$

for some m_λ .

Consider the partition $321 \vdash 6$. On one hand, we know we can find m_{321} by computing inner products. From the character table of S_6 ,

	K_{111111}	K_{211111}	K_{2211}	K_{222}	K_{3111}	K_{321}	K_{33}	K_{411}	K_{42}	K_{51}	K_6
$\chi^{(321)}$	16	0	0	0	-2	0	-2	0	0	1	0

is the character row for $\chi^{(321)}$. To find the character row, of χ , we count the fixed points C for one representative permutation from each conjugacy class. Since the character row for $\chi^{(321)}$ has a lot of zeros, our inner product will have a lot of terms that do not matter. We will just compute the values of the χ that are relevant. All compositions are fixed by the permutation e . So the the number of fixed points of e is $|C|$, which in this case equals 462. The permutation (123) fixes all compositions that have the first two entries matching. So the number of fixed points of (123) is 39. The permutation $(123)(456)$ has only 3 fixed points:

$$111333, 333111, 222222,$$

because we require the first three and last three entries to match. The permutation (12345) has only 2 fixed points:

$$111117, 222222,$$

because we require the first five entries to match. The partial character row of χ ,

	K_{111111}	K_{211111}	K_{2211}	K_{222}	K_{3111}	K_{321}	K_{33}	K_{411}	K_{42}	K_{51}	K_6
χ	462				39		3			2	

allows us to compute

$$\begin{aligned}
 m_{321} &= \langle \chi, \chi^{(321)} \rangle \\
 &= \frac{1}{6!} \sum_K |K| \chi_K \chi_K^{(321)} \\
 &= \frac{1}{720} \left[1(462 \cdot 16) + 40(39 \cdot (-2)) + 40(3 \cdot (-2)) + 144(2 \cdot 1) \right] \\
 &= 6.
 \end{aligned}$$

On the other hand, the column strict tableau of shape 321 with entries that sum to 12 are:

$$\begin{array}{|c|c|c|} \hline 5 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 1 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 4 & \\ \hline 1 & 1 & 1 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 2 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 3 \\ \hline \end{array}$$

So by Corollary 5.2.5, $m_{321} = |\text{CS}_\lambda(\mathcal{A})| = 6$.

Another interesting consequence of Theorem 5.2.1 regards the trivial representation.

Theorem 5.2.7. *Let X be a representation arising from the action of a group G on a set C with orbits O_1, \dots, O_k . If*

$$X = X_1 \oplus \dots \oplus X_k$$

where X_i is the representation of G arising from the action of G restricted to the orbit O_i , then each X_i contains exactly one copy of the trivial representation.

Proof. Let χ and χ_i denote the character of X and X_i respectively. The number of copies of the trivial representation in X_i is given by the inner product

$$\langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi_i(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} |C^g|. \quad (5.8)$$

By Theorem 1.4.9, this equals the size of the orbit space of the action of G on O_i , which is one because for all $o_1, o_2 \in O_i$, there exists $g \in G$ such that $go_1 = o_2$. \square

5.3 The Robinson-Schensted-Knuth Correspondence

Corollary 5.2.3 can be viewed as a purely combinatorial identity. In other words, there must exist a bijection between C and the set of pairs of column strict tableau whose entries come from \mathcal{A} with standard tableau of the same shape. The Robinson-Schensted-Knuth Algorithm (RSK) describes this bijection.

We will define what it means to insert a number into a column strict tableau t to yield a new tableau, t^* . The resulting tableau t^* will be the same shape as the original tableau t , except it will have one more cell somewhere. Label the rows of t from bottom to top with $1, 2, 3, \dots$. The insertion algorithm functions as follows.

- **Step 1:** Set b equal to the number we wish to insert and set $m = 1$.
- **Step 2:** If b is greater than or equal to every cell in row m of t , go to step 7.
- **Step 3:** Let y be the smallest value larger than b in row m .
- **Step 4:** Change the value in the leftmost cell containing a y in row m to a b .
- **Step 5:** Set $b = y$.
- **Step 6:** Increment m . Go to step 2.

- **Step 7:** Add a cell containing b to the end of row m .

Call the resulting tableau t^* . If x is the number we inserted, we write $x \rightarrow t = t^*$.

Example 5.3.1. Suppose we want to compute $3 \rightarrow t$ where

$$t = \begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 3 & 5 & 5 & \\ \hline 1 & 2 & 4 & 4 \\ \hline \end{array}.$$

Starting at step 1, we set $b = 3$ and $m = 1$ and since the condition in step 2 fails, we proceed to step 3. Since the smallest value in the row 1 2 4 4, larger than 3, is $y = 4$, we replace the leftmost cell containing a 4 with a 3, in step 4. The bottom row then becomes 1 2 3 4. In steps 5 and 6, we set $b = 4$, and $m = 2$. We say that the 4 has been *bumped* from the bottom row.

Now trying to insert a 4 into the row 3 5 5, the condition in step 2 again fails. So the middle row changes to 3 4 5, because the 4 we are inserting bumps the 5 up a row. We set $b = 5$, $m = 3$, and return to step 2. We are now attempting to insert a 5 into the top row:

$$3 \rightarrow \begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 3 & 5 & 5 & \\ \hline 1 & 2 & 4 & 4 \\ \hline \end{array} \quad 4 \rightarrow \begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 3 & 5 & 5 & \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \quad 5 \rightarrow \begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 3 & 4 & 5 & \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

Since the top row is currently just a single 5, the condition in step 2 is satisfied. We conclude our insertion process at step 7, by adding a cell containing a 5 onto the end of the top row, resulting in the tableau

$$3 \rightarrow t = \begin{array}{|c|c|c|c|} \hline 5 & 5 & & \\ \hline 3 & 4 & 5 & \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}.$$

The RSK algorithm consists of recursively inserting to form a tableau from a word. The tableau we form is called the *insertion tableau* and we also keep track of the order in which the cells were created, forming another tableau of the same shape called *the recording tableau*.

Let $c = c_1 \dots c_n \in C$ be a word. We will construct a sequence of pairs of a standard tableau along with a column-strict tableau whose entries come from c .

- **Step 1:** Let (t_0, r_0) be a pair of empty tableau. Let $i = 1$.
- **Step 2:** Let $t_i = c_i \rightarrow t_{i-1}$
- **Step 3:** Let r_i be a new tableau, the same as r_{i-1} with an extra cell containing i added in the unique way such that shape of t_i and shape of r_i are the same.
- **Step 4:** If $i < |c|$, increment i and go to step 2.
- **Step 5:** The current pair (t_i, r_i) is the output of the algorithm.

We write $\text{RSK}(c)$ to denote the output of the algorithm on a word c . The algorithm as forms a sequence of pairs of tableaux of gradually increasing size culminating in our output.

Example 5.3.2. Consider applying the RSK algorithm to the word $c = 3\ 1\ 2\ 1\ 1$. In step 1, setting our index $i = 1$ indicates that we are currently inserting the first letter of the word—in this case 3. We let t_1 be the result of inserting 3 into an empty tableau depicted below on the left and we let r_1 keep track of which cell was most recently created depicted below on the right:

$$t_1 = \boxed{3} \qquad r_1 = \boxed{1}$$

In step 4, since we have yet to reach the end of the word, $i = 2$ and we return to step 2. Now we find t_2 , depicted below on the left by inserting the second letter of the word, 1, into t_1 . To find r_2 , we add a single cell to r_1 containing a 2 in the unique way that makes the shape of r_2 match the shape of t_2 :

$$t_2 = \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \end{array} \quad r_2 = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}$$

Again in step 4, we have still have i less than the length of our word. So we iterate i and repeat. The next few steps are show below with t_i on the left and r_i on the right for $i = 3, 4, 5$ —inserting 2, then 1, then 1:

$$t_3 = \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array} \quad r_3 = \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}$$

$$t_4 = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 1 \\ \hline \end{array} \quad r_4 = \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$$

$$t_5 = \begin{array}{|c|c|c|} \hline 2 & 3 & \\ \hline 1 & 1 & 1 \\ \hline \end{array} \quad r_5 = \begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline \end{array}$$

At this point of $i = 5$, the condition in step 4 fails, we have reached the end of the word, and the algorithm is complete. Our result is the final pair of tableau:

$$\text{RSK}(c) = \left(\begin{array}{|c|c|c|} \hline 2 & 3 & \\ \hline 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & 4 & \\ \hline 1 & 3 & 5 \\ \hline \end{array} \right) \quad (5.9)$$

We do not include the step by step explanation of the reverse algorithm, but the beauty of RSK is that it is completely reversible. We could start with a pair of tableau and peel off which letter must have been most recently inserted as well as

what the previous pair of tableaux in the sequence must have been. Doing this repeatedly, we could completely reconstruct the word we started with.

So this is a bijective correspondence between the set of all words C from an alphabet \mathcal{A} and the set of all pairs of a standard tableau and a column strict tableau of the same shape. Thus we arrive at an alternative proof that

$$|C| = \sum_{\lambda \vdash n} f^\lambda \text{CS}_\lambda(\mathcal{A}) \tag{5.10}$$

where f^λ is the number of standard tableaux of shape λ . Letting $\mathcal{A} = [n]$, we also have an alternative proof of Corollary 3.4.9.

Chapter 6

REPRESENTATIONS FROM AN ACTION ON MATRICES

In the previous chapter, we were able to provide a clean-cut combinatorial interpretation of our representations. The natural question that arises is how we might generalize these results. In this chapter, we examine an extension of the action from before to an action of $S_m \times S_n$ on matrices.

6.1 Character Tables of $S_m \times S_n$

In order to compute inner products with irreducible representations of $S_m \times S_n$, we will need to construct its character table. Fortunately, with a little bit of additional tensor product theory, we can bootstrap up from knowing the character table for S_n .

Theorem 6.1.1. *If X and Y are representations of groups G and H with characters χ and ψ respectively, then the character of the representation $X \otimes Y$ evaluated at $(g, h) \in G \times H$ is $\chi(g)\psi(h)$.*

Proof. The character of $X \otimes Y$ evaluated at $(g, h) \in G \times H$ is

$$\begin{aligned}\mathrm{tr}((X \otimes Y)(g, h)) &= \mathrm{tr}(X(g) \otimes Y(h)) \\ &= \sum_{ij} X_{ii}(g)Y_{jj}(h) \\ &= \left(\sum_i X_{ii}(g) \right) \left(\sum_j Y_{jj}(h) \right) \\ &= \chi(g)\psi(h)\end{aligned}$$

where $X_{ii}(g)$ denotes the row i , column i entry of $X(g)$. □

Theorem 6.1.2. *Let X and Y be representations of groups G and H respectively. If $\{X_1, \dots, X_k\}$ and $\{Y_1, \dots, Y_l\}$ are complete lists of pairwise inequivalent irreducible representations for G and H respectively, then $\{X_i \otimes Y_j : 1 \leq i \leq k, 1 \leq j \leq l\}$ is a complete list of pairwise inequivalent irreducible representations of $G \times H$.*

Proof. Assume $\{X_1, \dots, X_k\}$ and $\{Y_1, \dots, Y_l\}$ are complete lists of pairwise inequivalent irreducible representations for G and H respectively. Denoting the character of $X_i \otimes Y_j$ as $\chi_i \otimes \psi_j$, we have

$$\begin{aligned} \langle \chi_i \otimes \psi_j, \chi_i \otimes \psi_j \rangle &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} (\chi_i \otimes \psi_j)(g, h) (\chi_i \otimes \psi_j)(g^{-1}, h^{-1}) \\ &= \left(\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_i(g^{-1}) \right) \left(\frac{1}{|H|} \sum_{h \in H} \psi_j(h) \psi_j(h^{-1}) \right) \\ &= \langle \chi, \chi \rangle \langle \psi, \psi \rangle \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

by part 3 of Corollary 2.5.5. Again by part 3 of Corollary 2.5.5, $X_i \otimes Y_j$ is irreducible.

Similarly, denoting the character of $X_p \otimes Y_q$ as $\chi_p \otimes \psi_q$, we can show that

$$\langle \chi_i \otimes \psi_j, \chi_p \otimes \psi_q \rangle = \langle \chi_i, \chi_p \rangle \langle \psi_j, \psi_q \rangle = \delta_{i,p} \delta_{j,q}, \quad (6.1)$$

proving pairwise inequivalence of the $X_i \otimes Y_j$ by Theorem 2.5.4.

Lastly, we know the list is complete by Theorem 2.6.14 because the number of conjugacy classes of $G \times H$ is the number of conjugacy classes of G times the number of conjugacy classes of H . □

With the character table of groups G and H , the previous two theorems allow us to construct the character table of $G \times H$.

Proposition 6.1.3. *Let A and B be the character tables of groups G and H respectively taken as square matrices. Then the matrix tensor product $A \otimes B$ is the character table of $G \times H$.*

By Theorem, 6.1, a complete list of irreducible representations of $S_m \times S_n$ is $\{X^\lambda \otimes X^\mu : \lambda \vdash m, \mu \vdash n\}$ and by Theorem 6.1.2, the character of $X^\lambda \otimes X^\mu$ evaluated at (σ_1, σ_2) is $\chi^\lambda(\sigma_1)\chi^\mu(\sigma_2)$. Since we can construct the character table of S_n for any n , we can now construct the character table of $S_m \times S_n$ for any positive integers m and n .

6.2 An Open Problem

For the remainder of this chapter, fix $m, n \in \mathbb{N}$ and let $G = S_m \times S_n$.

Let \mathcal{A} be a multiset and just as before, and let C be the set of all $m \times n$ matrices whose entries come from the alphabet \mathcal{A} without replacement. We define the action of G on C by

$$(\sigma, \tau) \cdot c = \left\| \|c_{\sigma(i), \tau(j)}\| \right\|_{ij} \tag{6.2}$$

where $\sigma \in S_m$, $\tau \in S_n$ and $c = \left\| \|c_{i,j}\| \right\| \in C$. We permute the rows by σ and then permute the columns by τ . For example

$$((13)(24), (123)) \cdot \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & 0 \\ 2 & 5 & 1 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 0 & 4 \end{bmatrix}.$$

Let X be the representation of G arising from this action.

Ideally, we would be able to find a combinatorial interpretation of the irreducible decomposition of X , as we were in the previous chapter. However, one challenge that presents itself when attempting to break down this representation is the sheer number of matrices in C .

For example, suppose $m = n = 4$ and our alphabet consists of unlimited copies of each of 1, 2, and 3. Then the dimension of our representation is

$$\dim(X) = |C| = 3^{16} = 43046721. \quad (6.3)$$

Theorem 5.2.1 allows to analyze representations like this one, by examining the representation arising from restricting the action to a specific orbit. So we do not need to consider the action on all matrices.

Definition 6.2.1. Let X_A , be the representation arising from the action of G on $G \cdot A$, where $G \cdot A$ denotes the orbit of A .

In this way, instead of choosing an alphabet, we just choose a matrix A to construct X_A . For reference, if again $m = n = 4$,

$$\dim(X_A) = |G \cdot A| \leq |G| = (4!)^2 = 576 \quad (6.4)$$

for any matrix A .

For any matrix B in $G \cdot A$, $X_A \cong X_B$. So we establish a canonical representative element of each orbit.

Definition 6.2.2. We can obtain a word from a matrix by reading off the entries of A from left to right, then top to bottom. Let A be a matrix. If the word obtained from A in this way is lexicographically smaller than the word obtained from $g \cdot A$ for any $g \in G$, we say that A is *minimally ordered*.

For example, the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 2 & 3 & 1 & 1 & 3 \\ 2 & 4 & 1 & 3 & 1 \end{bmatrix}$$

is minimally ordered. Proposition 6.2.3 gives an equivalent definition for minimally ordered.

Proposition 6.2.3. *Let A be a matrix. Let R be a subset of a row of A with constant values in the corresponding subsets of any rows above. Let C be a subset of a column of A with constant values in the corresponding subsets of any columns to the left. The matrix A is minimally ordered if and only if for all such R and C , R weakly increases to the right and C weakly increases down.*

In the previous example, the third and fourth column had constant values in the first and second row, so the value in the third row and third column had to be less than or equal to the value in the third row and fourth column.

We can now precisely breakdown the representation X from the previous section, by Theorem 5.2.1, into

$$X = \bigoplus_A X_A \tag{6.5}$$

where the sum is over all $m \times n$ matrices A that are minimally ordered. If we can understand this much smaller representation X_A , we can translate into an interpretation of the original representation. Furthermore, we can understand other related representations that have additional conditions.

In the previous chapter, we were able to extend to an interpretation of the representations arising from the action on compositions (requiring a fixed sum). One representation of interest would be the representation of G arising from its action on the set of m by n matrices whose rows sum to n and whose columns sum to m . If

we can find the irreducible breakdown of X_A , finding all minimally ordered matrices with the correct row and column sums would give us a complete understanding of this representation as well. Understanding X_A would be the key to breaking down lots of different representations of G with this structure.

The argument that X_A is a representation of critical importance has been made. The open problem is this: For a matrix A , what is a general characterization of the irreducible decomposition of the representation X_A ?

6.3 Current Progress

There are some things we can say about X_A . For example, Theorem 5.2.7 reveals that X_A contains one copy of the trivial representation and as observed in Equation 6.4, $\dim X_A \leq |G|$.

Theorem 6.3.1. *If $\dim(X_A) = |G|$, then X_A is equivalent to the regular representation of G .*

Proof. Since $|G \cdot A| = |G|$, for any matrix $B \in G \cdot A$, $g \cdot B = B$ if and only if $g = e$. So if χ_A denotes the character of X_A , $\chi_A(e) = |G|$ and $\chi_A(g) = 0$ for all $g \neq e$. Since this is the same as the character of the regular representation of G , by part 4 of Corollary 2.5.5, we are done. \square

Since the complete list of irreducible representations of G is the set of all $X^\lambda \otimes X^\mu$ where $\lambda \vdash m$ and $\mu \vdash n$. By Theorem 2.3.4,

$$X_A \cong \bigoplus_{\lambda, \mu} M_{\lambda, \mu} (X^\lambda \otimes X^\mu) \tag{6.6}$$

for some positive integers $M_{\lambda, \mu}$.

Theorem 6.3.2. *If the rows of A , taken as multisets without order, are distinct, then*

$$M_{\lambda, \mu} = f^\lambda M_{(m), \mu}.$$

Proof. Assume the rows of A , taken as multisets without order, are distinct. So no rearrangement of rows can fix any $B \in G \cdot A$. So if $B \in G \cdot A$ is fixed by $(\sigma, \tau) \in G$, then $\sigma = e$. Thus $\chi_A(\sigma, \tau) = 0$ for all $\sigma \neq e$.

For any elements of the form $(e, \tau) \in G$, we have

$$(\chi^\lambda \otimes \chi^\mu)(e, \tau) = \chi^\lambda(e)\chi^\mu(\tau) = f^\lambda \chi^\mu(\tau) \tag{6.7}$$

by Theorem 3.4.8, since evaluating a character at the identity reveals the dimension of the representation. We also have

$$(\chi^{(m)} \otimes \chi^\mu)(e, \tau) = \chi^{(m)}(e)\chi^\mu(\tau) = \chi^\mu(\tau) \tag{6.8}$$

since $\dim(\chi^{(m)}) = 1$.

Since χ_A is zero when $(\sigma, \tau) \in G$ has $\sigma \neq e$, we can ignore these terms and factor out f^λ from the inner product. Observe that

$$\begin{aligned}
M_{\lambda, \mu} &= \langle \chi^\lambda \otimes \chi^\mu, \chi_A \rangle \\
&= \frac{1}{|G|} \sum_{(\sigma, \tau) \in G} (\chi^\lambda \otimes \chi^\mu)(\sigma, \tau) \chi_A(\sigma, \tau) \\
&= \frac{1}{|G|} \sum_{(e, \tau) \in G} (\chi^\lambda \otimes \chi^\mu)(e, \tau) \chi_A(e, \tau) \\
&= \frac{1}{|G|} \sum_{(e, \tau) \in G} f^\lambda \chi^\mu(\tau) \chi_A(e, \tau) \\
&= \frac{f^\lambda}{|G|} \sum_{(e, \tau) \in G} (\chi^{(m)} \otimes \chi^\mu)(e, \tau) \chi_A(e, \tau) \\
&= \frac{f^\lambda}{|G|} \sum_{(\sigma, \tau) \in G} (\chi^{(m)} \otimes \chi^\mu)(\sigma, \tau) \chi_A(\sigma, \tau) \\
&= f^\lambda \langle \chi^{(m)} \otimes \chi^\mu, \chi_A \rangle \\
&= f^\lambda M_{(m), \mu}.
\end{aligned}$$

□

Example 6.3.3. Let us find the irreducible decomposition of X_A for a specific matrix to verify the previous theorem in an example. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Note that A satisfies the hypothesis for Theorem 6.3.2.

To find the character χ_A evaluated at $(\sigma, \tau) \in G$, we count the number of matrices in $G \cdot A$ fixed by σ, τ . We have $\chi_A(e) = |G \cdot A| = |G|/|G_A| = (4!)(4!)/2 = 288$, $\chi_A(e, (12)) = 48$, and χ_A is zero on all other conjugacy classes.

Now we compute the inner product of this character against each row of the character table of $S_4 \times S_4$ to find the $M_{\lambda, \mu}$ from Equation 6.6. As an example, we will compute $M_{(211), (31)}$. Since many of the terms in the character row for χ_A are zero, we will just compute the relevant terms of the character row for $\chi^{(211)} \otimes \chi^{(31)}$. Since there are 3 standard tableaux of shape 2 1 1 and three standard tableaux of shape 3 1, $f^{(211)} = f^{(31)} = 3$. We have

$$(\chi^{(211)} \otimes \chi^{(31)})(e) = \chi^{(211)}(e)\chi^{(31)}(e) = f^{(211)}f^{(31)} = 3 \cdot 3 = 9 \quad (6.9)$$

by Theorem 3.4.8 and we have

$$(\chi^{(211)} \otimes \chi^{(31)})(e, (12)) = \chi^{(211)}(e)\chi^{(31)}((12)) = f^{(211)} = 3 \quad (6.10)$$

by Theorem 3.4.8 and Theorem 4.5.1.

Since the conjugacy classes containing e and $(e, (12))$ have 1 and 6 elements respectively,

$$\begin{aligned} M_{(211), (31)} &= \langle \chi^{(211)} \otimes \chi^{(31)}, \chi_A \rangle \\ &= \frac{1}{4! \cdot 4!} \sum_{(\sigma, \tau) \in G} (\chi^{(211)} \otimes \chi^{(31)})(\sigma, \tau) \chi_A(\sigma, \tau) \\ &= \frac{1}{576} [1(9 \cdot 288) + 6(3 \cdot 48)] \\ &= 6. \end{aligned}$$

We can find the other $M_{\lambda, \mu}$ with similar inner product computations:

Table 6.1: Example Irreducible Decomposition of X_A

	4	31	22	211	1111
4	1	2	1	1	0
31	3	6	3	3	0
22	2	4	2	2	0
211	3	6	3	3	0
1111	1	2	1	1	0

The row λ , column μ entry of Table 6.1 is $M_{\lambda,\mu}$. As anticipated by Theorem 6.3.2, each row is a multiple of the first.

Unfortunately these results fall short of the mark when it comes to answering the open question: How can we interpret the $M_{\lambda,\mu}$? We conclude this chapter with a conversation of an approach that could be taken moving forward.

Equating the dimensions of both sides of Equation 6.6

$$|G \cdot A| = \sum_{\lambda,\mu} f^\lambda f^\mu M_{\lambda,\mu}. \quad (6.11)$$

because $\dim(X^\lambda \otimes X^\mu) = f^\lambda f^\mu$. In Section 5.3, we gave a bijective proof of our dimension counting corollary from the decomposition of our representation from the action on words. An interpretation of the $M_{\lambda,\mu}$ would turn Equation 6.11 into a dimension counting corollary that should also have a bijective proof.

An approach to attempting to understand the $M_{\lambda,\mu}$ would be the following. Find a bijection from $G \cdot A$ that extracts a standard tableau with m entries and a standard tableau with n entries from a matrix. An algorithm of this form will have an additional

term—call it $N_{\lambda,\mu}$. By construction the $N_{\lambda,\mu}$ will satisfy

$$|G \cdot A| = \sum_{\lambda,\mu} f^\lambda f^\mu N_{\lambda,\mu}.$$

There are multiple bijections that could make this work, but there will be a correct bijection that has $M_{\lambda,\mu} = N_{\lambda,\mu}$.

Since the goal is to extract standard tableaux from a matrix, a starting place for determining this mystery bijection could be RSK. We can define a total ordering on all possible columns of A , lexicographically for example. We then run through the RSK algorithm inserting each column of the matrix into an entry of the tableau we are constructing. Partitioning the set $G \cdot A$ by which matrices yield the same shape under column RSK seems to split the sum nicely.

This could be the first step in our desired bijection, accounting for the f^μ term. To account for the f^λ term, applying RSK in a similar way on the rows of A as well seems like a good candidate. Unfortunately, it can be verified through an example that the $N_{\lambda,\mu}$ that arise from this bijection are not equal to the $M_{\lambda,\mu}$. So there must be some other second step in this mystery bijection. Sadly, whatever it is will not be resolved here.

While some progress has been made, the open problem still stands: For a matrix A , what is a general characterization of the irreducible decomposition of the representation X_A ?

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