The effect of quantum fields on black-hole interiors

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Referat (abstract):

Charged or rotating black holes possess an inner horizon beyond which determinism is lost. However, the strong cosmic censorship conjecture claims that even small perturbations will turn the horizon into a singularity beyond which the spacetime is inextendible, preventing the loss of determinism. Motivated by this conjecture, this dissertation studies free scalar quantum fields on various black-hole spacetimes to test whether quantum effects can lead to the formation of a singularity at the inner horizon in cases where classical perturbations cannot. The starting point is the investigation of the behaviour of real-scalar-field observables near the inner horizon of Reissner-Nordström-de Sitter spacetimes. Using semi-analytical methods, we find that quantum effects can indeed uphold the censorship conjecture. Subsequently, we consider charged scalar fields on the same spacetime and observe that a first-principle calculation is essential to accurately describe the quantum effects at the inner horizon. As a first step towards an extension of these results to rotating black holes, we rigorously construct the Unruh state for the real scalar field on slowly rotating Kerr-de Sitter spacetimes. We show that it is a well-defined Hadamard state and can therefore be used to compute expectation values of the stress-energy tensor and other non-linear observables.

Geladene oder rotierende schwarze Löcher besitzen einen inneren Horizont; jenseits dieses Horizonts geht die Vorhersagbarkeit verloren. Dagegen fordert das "strong cosmic censorship conjecture", dass sogar kleinste Störungen den inneren Horizont in eine Singularität verwandeln und so den Verlust der Vorhersagbarkeit verhindern. Vor diesem Hintergrund untersucht die vorliegende Dissertation freie, skalare Quantenfelder auf verschiedenen Raumzeiten mit einem schwarzen Loch. Das Ziel ist es zu überprüfen, ob Quanteneffekte das Entstehen der Singularität in den Fällen herbeiführen können, in denen klassische Störungen dies nicht können. Als Startpunkt dient das Verhalten verschiedener Observablen reeller Skalarfelder am inneren Horizont von Reissner-Nordström-de Sitter-Raumzeiten. Wir bestätigen unter der Verwendung semi-analytischer Methoden, dass Quanteneffekte die "censorship"- Vermutung aufrechterhalten können. Im Anschluss betrachten wir geladene Skalarfelder auf derselben Raumzeit und zeigen, dass quantenfeldtheoretische Berechnungen notwendig sind, um die Quanteneffekte am inneren Horizont genau zu beschreiben. Ein erster Schritt, um diese Ergebnisse auf rotierende schwarze Löcher zu erweitern, ist eine mathematisch exakte Konstruktion des Unruh-Zustands für ein reelles Skalarfeld auf einer langsam rotierenden Kerr-de Sitter-Raumzeit. Wir zeigen, dass dieser Zustand ein wohldefinierter Hadamard-Zustand ist, der für die Berechnung von Erwartungswerten des Energie-Impulstensors und anderer nicht-linearer Observablen verwendet werden kann.

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Contents

1 Introduction

Our universe is filled with fascinating physical phenomena at all scales. At large scales, there are for instance the evolution of our universe under the influence of its matter content or the intriguing phenomena surrounding black holes. All of this is best described by general relativity, Einstein's theory of gravity. At small scales, one enters the realm of quantum mechanics and quantum field theory. These theories have been extremely successful in describing the particles making up all matter and their interactions.

However, we know that the realms of quantum theories and general relativity can also interact with each other, and that physics at the smallest scales can have visible impact on the physics at large scales. This is seen for example in the evaporation of black holes [1] which is only possible due to quantum effects. Fully understanding these effects requires to unify general relativity and quantum field theory into a theory of both gravity and quantum fields – quantum gravity.

As of now, no complete theory of quantum gravity exists, even though there is a broad spectrum of different approaches, see for example [2] for a recent review. The construction of such a theory is possibly one of the biggest open questions in theoretical physics today. With experimental data being mostly in excellent agreement with either general relativity or quantum field theory, and only very little data to constrain potential quantum gravity models, especially at high energies, it is very hard to tell which candidate theory is the most promising.

Even in the absence of a theory of quantum gravity, it is possible to explore the interactions between gravity and quantum theory using the well-established theories of general relativity and quantum field theory. While this does not cover all possible scenarios, there are regimes in which the interactions between gravity and quantum matter should be well-described by these models. In this ansatz, called semi-classical gravity, the matter fields are treated by quantum field theory while gravity is treated classically. The two are then connected by the semi-classical Einstein equations

$$G_{\nu\varrho} + \Lambda g_{\nu\varrho} = 8\pi \left(\langle T_{\nu\varrho} \rangle_{\Psi} + T_{\nu\varrho} \right) . \tag{1.0.1}$$

Here, $G_{\nu\varrho} = R_{\nu\varrho} - g_{\nu\varrho} R/2$ is the Einstein tensor, Λ is the cosmological constant, $T_{\nu\varrho}$ is the stress-energy tensor of the classical matter, and $\langle T_{\nu\varrho} \rangle_{\Psi}$ is the expectation value of the stress-energy tensor of the quantum matter in the state Ψ. Semi-classical gravity is expected to be valid as long as the curvature remains small compared to the Planck scale and the fluctuations of the stress-energy tensor remain small compared to its expectation value[1].

[1]Even in this regime, the semi-classical Einstein equation has conceptual and mathematical problems, but it can still give useful insights.

Making sense of the semi-classical Einstein equations (1.0.1) requires an understanding of quantum field theory in curved spacetime. One needs to be able to compute expectation values of operators requiring renormalization in a local and covariant way and to understand the notion of having the same physics in different spacetimes [3, 4]. In the effort to extend the concepts of quantum field theory to curved spacetimes in a consistent way for this purpose, there has also been considerable progress in understanding quantum field theory itself, introducing important concepts such as the principle of local covariance which lead to a formulation of quantum field theory in the language of categories [5] or the local and covariant renormalization of Wick squares and time-ordered products [6, 7].

Even with these still developing techniques, it remains an extremely difficult task to solve (1.0.1). So far, it has only been accomplished in highly symmetric situations for free quantum field theories [8–13]. Instead of fully solving (1.0.1), a common approach is to compute $\langle T_{\nu\varrho}\rangle_\Psi$ on a fixed background spacetime which solves (1.0.1) for $\langle T_{\nu\varrho}\rangle_\Psi = 0$, neglecting the backreaction of the quantum field onto the spacetime. Even this reduced problem is very challenging, since the un-renormalized quantities and the counterterms required for the renormalization are often given in different forms which are incompatible for numerical evaluation. However, there are methods available, such as pragmatic mode-sum renormalization [14, 15] or state subtraction [16, 17] that allow the computation of expectation values such as $\langle T_{\nu\varrho}\rangle_\Psi$.

While this is not a self-consistent solution to (1.0.1), these results can give hints as to how the geometry of the spacetime may be influenced by the quantum effects and vice versa. Important physical results obtained along these lines include the evaporation of black holes by Hawking radiation [1], the Unruh effect [18], or particle creation in an expanding universe [19, 20]. They illustrate that the method described above can be utilized to investigate the effect of quantum fields on physically interesting spacetimes.

In this thesis, we will focus on quantum effects in black-hole spacetimes. Black holes are astrophysical objects of increasing observational importance. Quite recently, the shadows of the central black holes in our own galaxy [21] and in M87 [22] have been observed for the first time. Moreover, gravitational-wave detectors have recorded a large number of mergers of a black hole with a second black hole or a neutron star over the last couple of years [23]. Observations like these and their future improvements require an increasingly detailed theoretical understanding of black-hole spacetimes, including their interaction with (quantum) matter.

Apart from their observational importance, black-hole spacetimes also pose a variety of conceptual questions. While the observational aspects of black holes concern mainly the black-hole exterior, many of the theoretical questions also involve the interior of the black hole and its structure. One example is the occurrence of a singularity in black holes. It was originally assumed that they are mere artifacts of the high symmetry of the stationary black-hole solutions. However, Penrose [24] showed in a seminal work that the spacetime singularities inside black holes actually appear generically in gravitational collapse under some positivity assumptions on the stress-energy tensor appearing in the Einstein equations. At the singularity, our current physical theories break down, indicating an incompleteness of our present theories of gravity and (quantum) matter. A complete quantum gravity theory should resolve this issue of incompleteness in some way.

Instead of the singularity problem, we will focus on another conceptual question in this work, namely the loss of determinism in the interior of charged or rotating black holes. In contrast to the non-charged, non-rotating black-hole solution derived by Schwarzschild [25], these black holes have a rich structure in their interior. They (or their analytic continuations) contain a second horizon, called the inner horizon, and beyond it a region with a time-like singularity. This does not only lead to the possibility of bypassing the singularity and exiting the black hole into a different exterior universe in the analytically extended spacetime, but also to a breakdown of determinism: the journey through the region beyond the inner horizon is no longer determined by initial data which suffices to describe the complete spacetime up to this horizon. The inner horizon, marking the future boundary of the maximal Cauchy development of this initial data, is therefore also called "Cauchy horizon".

As a solution to this issue, Penrose [26] argued that any small perturbation of the exact black-hole initial data will lead to the formation of a singularity at the inner horizon, making it impossible to extend the spacetime beyond it. This renders the loss of determinism inconsequential.

Since this idea plays a central role as a motivation for this thesis, let us take a more detailed look at it. The heuristic argument presented by Penrose works as follows: Imagine a spacetime with a charged or rotating black hole and imagine there are two observers in this spacetime. Observer A, whose worldline is represented by the blue line in Fig. 1.1, falls into the black hole. Observer B, whose worldline is represented by the red line in Fig. 1.1, remains outside the black hole. B sends signals to A at a constant frequency for the rest of the infinite amount of proper time they have left travelling in the exterior region. In contrast, A only has a finite amount of proper time before reaching the inner horizon of the black hole. Since they must receive all messages sent by B before that point, they will receive the messages at increasing frequency as they approach the inner horizon. In fact, the message frequency will become infinitely blue-shifted.

This blue-shift effect also applies to small perturbations of the initial data for the black-hole spacetime. It will lead to an infinite increase in curvature at the inner horizon, and thus the formation of a singularity. Due to this mechanism, the loss of determinism beyond the inner horizon is an unstable feature of these spacetimes. If the initial data is perturbed slightly, the perturbations will accumulate at the inner horizon and render the metric inextendible across it. This idea is called the strong cosmic censorship conjecture (sCC).

From the heuristic argument one can already guess that sCC will be a more delicate issue in black-hole spacetimes with a positive cosmological constant Λ than in asymptotically flat ones. The reason is that in spacetimes with $\Lambda > 0$, there is an additional effect on the frequency of the perturbation – the cosmological expansion. This expansion leads to a red-shift effect which counteracts the blue-shift. sCC then requires that the blue-shift always overcomes the red-shift.

The exact formulation of the conjecture is a subtle issue. First, one must specify in what sense the metric is required to become inextendible at the Cauchy horizon. A very strong version of sCC would consider inextendibility as a continuous function, corresponding to the formation of a strong singularity that will inevitably destroy any observer

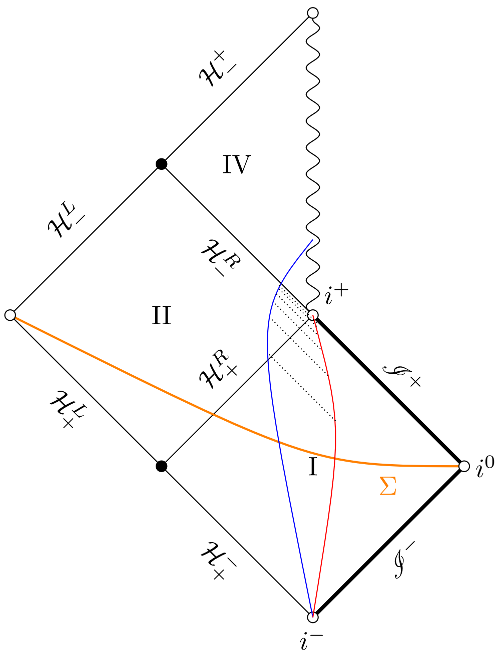


Figure 1.1: Penrose diagram of a partial analytic extension of a charged or rotating black-hole spacetime. The thick lines represent conformal light-like infinity, the white dots are conformal points at infinity. The thin lines represent the outer horizon $\mathcal{H}_+$ and the inner horizon $\mathcal{H}_-$ of the black hole. The orange line is a Cauchy surface for the black-hole exterior (I) and interior up to the Cauchy horizon (II). The blue and red lines represent the world lines of two observers A and B. The dotted lines indicate the signals send from red (B) to blue (A).

approaching it [27]. However, it has been shown that this version of sCC fails for spherically symmetric perturbations of charged black holes [28, 29], and for rotating black holes [30]. Instead, we consider the formulation of sCC due to Christodoulou [31]. This version requires that the metric should fail to have locally square-integrable derivatives at the Cauchy horizon or, in other words, be inextendible as a H^1_{loc}-function. The motivation for this version of sCC is that it renders the metric inextendible as a (weak) solution to the Einstein equations, which is a natural requirement from the viewpoint of the analysis of differential equations. For the more physical criterion of the fate of an observer approaching the horizon, the Christodoulou formulation implies that the horizon turns into a weak singularity [32] at which tidal forces diverge but the tidal deformation an observer suffers while crossing the horizon may remain finite[2]. Hence, from this physical perspective a stronger singularity would be desirable. Nonetheless, we will stick to the requirement of H^1_{loc}-inextendibility.

Second, one needs to decide what kind of perturbations of the initial data are considered. The influence of the smoothness of the initial data can be seen from the early studies [33–35] which obtained different results due to different properties of the initial data [36].

[2] I would like to thank A. Ori for a clarification of this point.

In most of the literature, as well as in this thesis, the initial data is taken to be smooth. However, non-smooth initial data has also been studied [36, 37].

Besides the subtleties in the exact definition of the conjecture, it is worth noting that the sCC demands the breakdown of extendibility of the metric at the Cauchy horizon for *generic* perturbations of the initial data. In particular, finding a specific perturbation that allows for an extension of the metric is not necessarily a contradiction to the sCC. This poses an additional difficulty in proving the conjecture.

Analysing the validity of sCC requires mathematical control of solutions to the Einstein equations. Since the Einstein equations are non-linear, studying their behaviour in full generality is very difficult. For this reason, the studies on sCC often either employ symmetries to simplify computations (e.g. [28]) or consider linearised models as a first step towards controlling the full non-linear equations. In this work, we will focus on the latter.

The simplest linearised model for studying sCC, which will play a central role in this thesis, consists of a scalar field satisfying the scalar wave equation on a charged or rotating black-hole spacetime. The scalar field can be viewed either as a toy model for metric perturbations [38] or as a simple matter model. We focus here on the case of a positive cosmological constant. It was shown [39] that in this case the regularity of solutions to the scalar wave equation at the Cauchy horizon of these spacetimes depends on the quasi-normal modes of the black hole. These modes are purely ingoing at the event horizon and purely outgoing at the cosmological horizon. They can be viewed as generalized resonances of the black hole. More specifically, it was proven [39] that at the Cauchy horizon the forward solutions to the scalar wave equation with a smooth source term lie in $H^{1/2+\beta-\epsilon}$ for any $\epsilon > 0$ with

$$\beta = \frac{\alpha}{\kappa_-} \, .$$

Here, α is the spectral gap of the quasi-normal modes, i.e. the decay rate of the slowest-decaying mode. κ_- is the surface gravity of the Cauchy horizon. Hence, sCC in the Christodoulou formulation is linearly satisfied as long as $\beta \leq 1/2$.

This result allows to test sCC in the linear approximation by (numerically) studying the frequencies of the quasi-normal modes. It was found that sCC can be violated in charged black holes in asymptotically de Sitter spacetimes if the charge of the black hole is sufficiently large [40]. It has also been confirmed in numerical studies of the non-linear Einstein-Maxwell-scalar field equations [41, 42] that these results hold beyond the linear regime.

This leads to an interesting question: since matter is most accurately described by quantum theory, can the inclusion of quantum effects via (1.0.1) lead to a restoration of the sCC conjecture? This question has been addressed in [16]. The authors consider a free scalar quantum field on a fixed spacetime describing a charged black hole in the presence of a positive cosmological constant. They show that for sufficiently large black-hole charge Q, so that $\beta > 1/2$, the renormalized expectation value of the stress-energy tensor, which enters the right-hand side of (1.0.1), may diverge stronger than the stress-energy tensor of the classical field. Moreover, if it does, the leading term of the divergence is independent of the state of the scalar field as long as the state is Hadamard up to the horizon, i.e. a physically reasonable state. State dependence only enters in a sub-leading term which diverges at most like the classical stress-energy tensor. Hence, as long as the state-independent leading divergence term is non-vanishing in general, sCC can be restored. More details will be given in a following chapter.

These results should of course be taken with a grain of salt. When the expectation value of the stress-energy tensor becomes large, calculating the expectation value of the stress-energy tensor on the unperturbed background spacetime will cease to be a good approximation. Moreover, if the large expectation value of the stress-energy tensor leads to large curvature compared to the Planck scale, or if the fluctuations of the stress-energy tensor are of the same order of magnitude as its expectation value, the semi-classical Einstein equations (1.0.1) themselves should no longer be valid. Nonetheless, the results indicate that quantum fields can have a large influence on the structure of the spacetime near the inner horizon and therefore deserve further investigation.

This thesis contributes to this effort. We focus on different aspects of quantum fields related to the validity of the sCC conjecture near the inner horizon of both charged and spinning black-hole spacetimes with a positive cosmological constant.

This introduction is followed in Chapter 2 by a brief introduction to the algebraic approach to quantum field theory and to some aspects of microlocal analysis, focussing on free scalar fields. We will also introduce the Reissner-Nordström-de Sitter (RNdS) and Kerr-de Sitter (KdS) spacetimes describing charged or rotating black holes in the presence of a positive cosmological constant.

Chapter 3 focusses on the real scalar field on a RNdS spacetime. We will present numerical methods for the computation of the energy-flux expectation value of this field at the inner horizon of the black hole and show numerical results obtained with this approach. We will also explain how these methods can be extended to charged fields.

Considering that charged matter is required to form a charged black hole, we will study charged scalar fields in RNdS spacetimes in Chapter 4. We will describe how the results of [16] on the Hadamard property of the Unruh state can be extended from the real scalar to the charged scalar case. A formula for the charge current will be derived and evaluated numerically. The state-independence of the leading divergence of both the energy flux and the charge current at the Cauchy horizon will be shown following [16]. Our numerical results at the inner horizon will give new insight into the (non-)validity of the simple particle picture in black-hole interiors.

Finally, since one expects astrophysical black holes to be rotating rather than carrying significant charge, one would like to extend these results to KdS spacetimes. As a first step in this direction, we will construct the Unruh state on slowly rotating KdS spacetimes in Chapter 5. This state is thought to be a good description of the late-time behaviour in gravitational collapse. We will also show that the extension of the Unruh state to KdS spacetimes is a Hadamard state.

The results are summarized in Chapter 6.

2 An introduction to quantum fields and black holes

In this chapter, we will introduce some background material needed throughout this work. After clarifying some notations and definitions in Section 2.1, we will give a brief introduction to the algebraic approach to quantum field theory in Section 2.2. This introduction will contain brief descriptions of the CCR-algebra, the idea of quasi-free Hadamard states and Hadamard point-split renormalization. In Section 2.3, we will present some basic notions and central theorems of microlocal analysis, and explain how this technique can be used to show the Hadamard property. In Section 2.4, we will introduce the Reissner-Nordström-de Sitter and Kerr-de Sitter spacetimes and discuss some of their properties. We will conclude in Section 2.5 with a brief discussion of states for free scalar field theory on black-hole spacetimes including the Unruh state.

2.1 Notations and conventions

To start, let us summarize some notations and conventions used throughout this thesis.
We work in *natural units* $\hbar = c = k_B = G_N = 1$.
We denote $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_+^* = [0,\infty)$, and analogously $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_-^* = (-\infty, 0]$.
Round brackets around indices will indicate *symmetrisation*:

$$A_{(\mu_1,\dots,\mu_n)} = \frac{1}{n!}\sum_{\pi\in S_n} A_{\mu_{\pi(1)},\dots,\mu_{\pi(n)}}\,,$$

with S_n the permutation group. Indices written between horizontal lines are excluded from the symmetrisation, e.g. $A_{(\mu|\nu|\varrho)} = 1/2(A_{\mu\nu\varrho} + A_{\varrho\nu\mu})$.

We will use the *multi-index notation*: if $x = (x_1, \dots, x_n)$ is an n-tuple, then a multi-index for x is given by $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, and

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i} \quad \text{and} \quad |\alpha| = \sum_{i=1}^n \alpha_i\,.$$

Let $\mathcal{M}$ be a smooth manifold. We denote by $C^\infty(\mathcal{M})$ the *space of smooth, complex-valued functions* on $\mathcal{M}$, and by $C_0^\infty(\mathcal{M})$ the *space of compactly supported, smooth, complex-valued functions*, also referred to as test functions. The *spaces of real-valued functions* are $C^\infty(\mathcal{M};\mathbb{R})$ and $C_0^\infty(\mathcal{M};\mathbb{R})$, respectively. We denote by $\mathcal{D}'(\mathcal{M})$ the *space of distributions*, i.e. the space of continuous linear functionals $u : C_0^\infty(\mathcal{M}) \to \mathbb{C}$. We denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the *spaces of Schwartz functions* and *tempered distributions* on $\mathbb{R}^n$, respectively. The *space of compactly supported distributions* $u : C^\infty(\mathcal{M}) \to \mathbb{C}$ will be denoted by $\mathcal{E}'(\mathcal{M})$.

We will define the *Fourier transform*

$$\begin{aligned}\mathcal{F} : \mathcal{S}(\mathbb{R}^n) &\to \mathcal{S}(\mathbb{R}^n)\\ f &\mapsto \hat{f} = (2\pi)^{-n/2} \int e^{ik\cdot x} f(x)\, \mathrm{d}^n x\,,\end{aligned}$$

where $\cdot$ is the usual product in $\mathbb{R}^n$, and we use the same sign convention for the Fourier transform as [43]. It can be extended to a map $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by duality, i.e. $\hat{u}(f) = u(\hat{f})\, \forall\, u \in \mathcal{S}'(\mathbb{R}^n)$, $f \in \mathcal{S}(\mathbb{R}^n)$ and to a map $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by the Plancherel theorem, see for example [44].

We define a *spacetime* $(\mathcal{M}, g)$ to be a smooth, 4-dimensional, Hausdorff, second-countable, connected manifold with a smooth Lorentzian metric g of mostly-plus signature $(-,+,+,+)$. We also demand that $(\mathcal{M}, g)$ be orientable and time-orientable and that both orientations are fixed. The *volume form* induced by the metric is called $\mathrm{d}vol_g$. We denote the *tangent space* of $\mathcal{M}$ by $T\mathcal{M}$, and the fiber over $x \in \mathcal{M}$ by $T_x\mathcal{M}$. Similarly, the *cotangent space* is denoted $T^*\mathcal{M}$.

The space of smooth sections of the tangent space, or in other words the *space of smooth vector fields* on $\mathcal{M}$, is denoted as $\Gamma(\mathcal{M})$, the *space of smooth covector fields* by $\Gamma^*(\mathcal{M})$. The *zero section* $\{(x,0) \in T^*\mathcal{M} : x \in \mathcal{M}\}$ is denoted by o.

Given a smooth map $\psi : \mathcal{M} \to \tilde{\mathcal{M}}$, we denote by $\psi^* : C^\infty(\tilde{\mathcal{M}}) \to C^\infty(\mathcal{M})$, $\psi^* f(p) = f(\psi(p))$ the *pull-back* of f with respect to ψ. The *push-forward* of vectors $\psi_* : T_p(\mathcal{M}) \to T_{\psi(p)}(\tilde{\mathcal{M}})$ is given by $(\psi_* v)(f)(\psi(p)) = v(\psi^* f)(p)$ for all $v \in T_p(\mathcal{M})$, $f \in C^\infty(\tilde{\mathcal{M}})$. Analogously, the *pull-back* of covectors $\psi^* : T^*_{\psi(p)}(\tilde{\mathcal{M}}) \to T^*_p(\mathcal{M})$ is defined as $\psi^*(w)(v) = w(\psi_* v)$ for all $w \in T^*_{\psi(p)}(\tilde{\mathcal{M}})$ and $v \in T_p(\mathcal{M})$. If ψ is invertible, one can define the push-forward of functions or covectors/ the pull-back of vectors as the corresponding pull-back/ push-forward with respect to the inverse map $\psi^{-1} : \tilde{\mathcal{M}} \to \mathcal{M}$.

Unless stated otherwise, $\nabla^{(g)}$ will denote the *Levi-Civita connection* of $(\mathcal{M}, g)$, and we will drop the superscript g if the metric is clear from the context. We also denote $\Box_g = g^{\nu\varrho}\nabla^{(g)}_\nu \nabla^{(g)}_\varrho$ the *d'Alembert operator* of the spacetime $(\mathcal{M}, g)$. Here, $g^{\nu\varrho}$ denotes the inverse metric.

Given a spacetime $(\mathcal{M}, g)$, we will denote the *future/past lightcone* at $x \in \mathcal{M}$ consisting of all future-directed causal vectors in $T_x\mathcal{M}$ by $V_x^\pm$. We will define a covector $k \in T^*_x\mathcal{M}$ to be *future-pointing*, or future-directed, if $k(v) > 0$ for all time-like future-pointing vectors $v \in V_x^+$. We will write $k \rhd 0$ for a future-pointing covector. The *future/past null cone* in $T^*\mathcal{M}$ is denoted by

$$\mathcal{N}^\pm = \{(x,k) \in T^*\mathcal{M}\backslash o : g^{-1}(x)(k,k) = 0 \text{ and } \pm k \rhd 0\}\,, \tag{2.1.1}$$

and we also set $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$.

A spacetime $(\mathcal{M}, g)$ is *globally hyperbolic* if it contains a hypersurface which is intersected exactly once by each inextendible time-like curve in $\mathcal{M}$. Such a surface is called a

Cauchy surface for $(\mathcal{M}, g)$.

We denote by $J^{\pm}(\mathcal{O})$ the *causal future/past* of the set $\mathcal{O} \subset \mathcal{M}$ in $\mathcal{M}$. This is the set of points that can be reached from $\mathcal{O}$ by a future-/past-directed causal curve. $J(\mathcal{O})$ is the union $J^{+}(\mathcal{O}) \cup J^{-}(\mathcal{O})$. Similarly, we denote by $I^{\pm}(\mathcal{O})$ the *time-like future/past* of $\mathcal{O}$. This is the open set of points that can be reached from $\mathcal{O}$ by a future-/past-directed time-like curve. Note that this excludes curves for which the tangent vector vanishes at some point, while these curves may be included for the definition of the causal future/past. This follows the notation of [45].

Two sets $\mathcal{O}_1 \subset \mathcal{M}$ and $\mathcal{O}_2 \subset \mathcal{M}$ will be called *causally disjoint* or *space-like separated* if there is no causal curve connecting any point $x \in \mathcal{O}_1$ to any point $y \in \mathcal{O}_2$.

A subset $\mathcal{O} \subset \mathcal{M}$ is called *causally convex* if any causal curve with both endpoints in $\mathcal{O}$ is entirely contained in $\mathcal{O}$.

An open neighbourhood $\mathcal{O} \subset \mathcal{M}$ of some $x \in \mathcal{M}$ is called a *geodesically convex neighbourhood*, if any two points in $\mathcal{O}$ can be connected by a unique geodesic contained completely in $\mathcal{O}$.

2.2 A brief introduction to AQFT

In this section, we summarize some of the basic concepts of the algebraic approach to quantum field theory, collecting the most important aspects for this thesis. See also [46–49] and references therein for more complete reviews.

For comparison, let us briefly recap the usual approach to quantum field theories, which is taught in most introductory courses and can be found in well-known textbooks [50–52]. In this approach, one starts with a Hilbert space $\mathcal{H}$, which often has the form of a Fock space. The (complex rays of) normalized elements of $\mathcal{H}$ are the pure states of the theory; mixed states are represented by density matrices, i.e. positive operators ρ on $\mathcal{H}$ with the property $\mathrm{tr}(\rho) = 1$. The observables of the quantum field theory are the Hermitian operators on this particular Hilbert space. In order to take the expectation value of an operator A in some state Ψ, one simply takes the Hilbert-space inner product $\langle \Psi | A\Psi \rangle_{\mathcal{H}}$ or the trace of ρA.

On Minkowski space, the Hilbert space is usually chosen to be the Fock space whose ground state is the Minkowski vacuum. Since the Minkowski vacuum is the unique Poincaré-invariant ground state, this singles out a preferred choice of Hilbert space. However, even in this case there are physically interesting states which do not lie in this or a unitarily equivalent Hilbert space. One example for that are thermal states.

On general curved spacetimes, the situation becomes even more complicated, since in general, one does not have a unique, preferred ground state for the quantum theory. Therefore, there is generally no preferred choice of a Hilbert-space representation of the theory out of the unitarily inequivalent possibilities.

Another idea that often takes a central role in the discussion of quantum field theory is the concept of a particle. However, it is less often discussed that the notion of particle is ambiguous: If one has a Fock space for the quantum theory, then one can define a particle-number operator on the Fock space, but different choices of Fock space will also lead to different particle-number operators.

An example of this can already be seen in Minkowski space $(\mathbb{R}^{3+1}, -\mathrm{d}t^2 + \sum_{i_1}^{3} \mathrm{d}x_i^2)$: Let us restrict to the right Rindler wedge, the region of Minkowski space defined as $\{x_1 - |t| > 0\}$. On this wedge, the boosts in the x_1-direction are an alternative notion of time translation, which is proportional to the proper time of uniformly accelerated observers. The preferred choice of Fock space for such an observer, which one could call the boost Fock space, would be the one containing the ground state for the observer's time evolution. On the right Rindler wedge and with respect to the notion of time translation given by the boosts in x_1-direction, the Minkowski vacuum state is a thermal state and contains infinitely many particles according to the particle definition of the boost Fock space. This is called the Unruh effect [18].

The problem of selecting a representation for the quantum theory from the outset is circumvented by the algebraic approach. Instead of starting with a Hilbert space, one starts with the abstract $*$-algebra of observables of the theory. A $*$-algebra (over $\mathbb{C}$) is an algebra, i.e. a vector space over $\mathbb{C}$ with a bilinear inner product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, together with a $*$-involution, i.e. an anti-linear map $* : \mathcal{A} \to \mathcal{A}$ that satisfies $(A \cdot B)^* = B^* \cdot A^*$ and $(A^*)^* = A$ for all $A, B \in \mathcal{A}$. The algebras we deal with will be unital, i.e. contain the identity element $\mathbf{1}$ for the algebra inner product.

The structure of the theory is already present at the level of the abstract algebra when the locality property of quantum field theory is taken into account. This leads to the Haag-Kastler axioms for algebraic quantum field theory [53], which were initially formulated on Minkowski space and later generalized to curved spacetimes.

Let us assume a globally hyperbolic spacetime $(\mathcal{M}, g)$ and a unital $*$-algebra $\mathcal{A}(\mathcal{M})$. The Haag-Kastler axioms demand that for any open, causally convex subset $\mathcal{O} \subset \mathcal{M}$ with compact closure, there exists an algebra $\mathcal{A}(\mathcal{O})$ containing the observables localized in $\mathcal{O}$ as its Hermitian elements. The $\mathcal{A}(\mathcal{O})$ then collectively generate the algebra $\mathcal{A}(\mathcal{M})$, which is therefore called the algebra of quasi-local observables [49, 53]. In this way, one can also identify all $\mathcal{A}(\mathcal{O})$ with subalgebras of $\mathcal{A}(\mathcal{M})$. The $\mathcal{A}(\mathcal{O})$ form a net of algebras, which is also demanded to satisfy

- **Isotony** If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.
- **Causality** If $\mathcal{O}_1 \subset \mathcal{M}$ and $\mathcal{O}_2 \subset \mathcal{M}$ are causally disjoint, then $[A, B] = 0$ for all $A \in \mathcal{A}(\mathcal{O}_1)$, $B \in \mathcal{A}(\mathcal{O}_2)$.

One may require additional axioms such as the time-slice axiom, which demands that for any $\mathcal{O} \subset \mathcal{M}$ containing a Cauchy surface of $(\mathcal{M}, g)$, $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{M})$, and which guarantees the existence of dynamics.

The embedding of the submanifolds $\mathcal{O}$ into $\mathcal{M}$ is just one example of mappings between manifolds that maintain orientation, time orientation, and causality. Generalizing this idea leads to the definition of a local and covariant quantum field theory in the language of category theory as a functor between the category of globally hyperbolic spacetimes[1], with certain isometric embeddings as morphisms, and the category of unital $*$-algebras with unit-preserving injective $*$-homomorphisms as morphisms, see [5] for

[1]The conditions on a "spacetime" can be somewhat relaxed to allow for disconnected manifolds.

early work and [48] for a more recent review. The categories of spacetimes and algebras can be modified to capture additional structures of the theory, see for example [48, 54].

The states in the algebraic framework are positive, normalized, linear maps from the algebra to $\mathbb{C}$. In other words, a linear map $\omega : \mathcal{A} \to \mathbb{C}$ is a state if it satisfies $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$ and $\omega(\mathbf{1}) = 1$. If the algebra has a topological structure as well, one may additionally require the state to be continuous. A state can be interpreted as a map from operators to expectation values.

Once a state has been chosen, one can get back to the usual Hilbert space formulation by the GNS reconstruction, see e.g. [49]:

Theorem 2.2.1. *For any state ω on the $*$-algebra $\mathcal{A}$, there is a (up to unitary equivalence) unique Hilbert space H_ω, a dense subset D_ω, a representation π_ω of $\mathcal{A}$ by closable operators on D_ω and a vector $\Omega_\omega \in H_\omega$ such that $D_\omega = \pi_\omega(\mathcal{A})\Omega_\omega$ and $\forall A \in \mathcal{A}$*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle_{H_\omega} .$$

In this thesis, we will mostly focus on a free, real scalar field and its CCR-algebra. To construct this algebra, let us begin with the classical theory of the real scalar field. Let $(\mathcal{M}, g)$ be a spacetime, which we assume to be globally hyperbolic. Then the massive Klein-Gordon equation

$$\mathcal{K}\phi = 0 \tag{2.2.1}$$

has a well-defined initial-value problem [45] on $(\mathcal{M}, g)$. Here, we have defined the massive Klein-Gordon operator

$$\mathcal{K} = \Box_g - \mu^2 , \tag{2.2.2}$$

where $\mu \geq 0$ is some constant. Moreover, there are unique retarded and advanced Green's operators [55] $E^\pm : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that satisfy

$$E^\pm \circ \mathcal{K}|_{C_0^\infty(\mathcal{M})} = \mathrm{id}_{C_0^\infty(\mathcal{M})} , \quad \mathcal{K} \circ E^\pm = \mathrm{id}_{C_0^\infty(\mathcal{M})} , \tag{2.2.3a}$$
$$\mathrm{supp}(E^\pm(f)) \subset J^\pm(\mathrm{supp}(f)) \, \forall f \in C_0^\infty(\mathcal{M}) . \tag{2.2.3b}$$

In fact, $E^\pm$ can be extended to maps $E^\pm : C^\infty_{spc/sfc}(\mathcal{M}) \to C^\infty_{spc/sfc}(\mathcal{M})$ and $E^\pm : C^\infty_{pc/fc}(\mathcal{M}) \to C^\infty_{pc/fc}(\mathcal{M})$, where

$$C^\infty_{spc/sfc} \equiv \left\{ f \in C^\infty(\mathcal{M}) : \mathrm{supp}(f) \subset J^\pm(K) \text{ for some compact } K \subset \mathcal{M} \right\}$$
$$C^\infty_{pc/fc} \equiv \left\{ f \in C^\infty(\mathcal{M}) : \exists \text{ smooth, s-like Cauchy surf. } \Sigma \subset \mathcal{M} : \mathrm{supp}(f) \subset J^\pm(\Sigma) \right\}$$

are the spaces of (strictly) past-/future-compact smooth functions on $\mathcal{M}$ [56].

We then define the commutator function $E = E^+ - E^- : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ as the difference between the retarded and advanced Green's operators. By construction, it satisfies

$$\mathcal{K} \circ E = E \circ \mathcal{K}|_{C_0^\infty(\mathcal{M})} = 0\,, \quad \mathrm{supp}(E(f)) \subset J(\mathrm{supp}(f))\,. \tag{2.2.4}$$

We will denote the space of solutions to the Klein-Gordon equation with compact Cauchy data by

$$S(\mathcal{M}) \equiv \{\phi \in C^\infty(\mathcal{M}) : \mathcal{K}\phi = 0,\, \mathrm{supp}(\phi) \cap \Sigma \text{ comp. } \forall \text{ s-like smooth Cauchy surf. } \Sigma\}\ .$$

On this space, there is a non-degenerate symplectic form σ given by

$$\sigma(\phi,\psi) = \int\limits_{\Sigma} (\phi\nabla_a\psi - \psi\nabla_a\phi)\, n_\Sigma^a\, \mathrm{d}vol_\gamma\,, \tag{2.2.5}$$

where Σ is any smooth space-like Cauchy surface, n_Σ^a its future-pointing normal vector and γ the induced metric on Σ. The symplectic form is independent of the choice of Cauchy surface [57][2]. By the support property of $\phi, \psi \in S(\mathcal{M})$, the conservation of the current $J : S(\mathcal{M}) \times S(\mathcal{M}) \to \Gamma^*(\mathcal{M})$,

$$J_\nu[\phi,\psi] = \phi\nabla_\nu\psi - \psi\nabla_\nu\phi\,, \tag{2.2.6}$$

and an application of Gauss' theorem, see e.g. [58], one can actually choose any Cauchy surface for the computation of Σ. This includes Cauchy surfaces which are partially null or only piecewise smooth. See also the discussion in [43, Sec. 2.3].

Defining $E(f,h) \equiv \int\limits_{\mathcal{M}} fE(h)\, \mathrm{d}vol_g$, one can then show that

$$E : C_0^\infty(\mathcal{M})/(\mathcal{K}C_0^\infty(\mathcal{M})) \to S(\mathcal{M})\,, \quad f \mapsto E(f)$$

is a symplectomorphism between the symplectic spaces $(C_0^\infty(\mathcal{M})/(\mathcal{K}C_0^\infty(\mathcal{M})), E(\cdot,\cdot))$ and $(S(\mathcal{M}), \sigma(\cdot,\cdot))$.

The inverse map can be constructed as follows: Let $\Sigma_\pm$ be two Cauchy surfaces of $\mathcal{M}$ satisfying $\Sigma_+ \subset I^+(\Sigma_-)$. Let $\chi_\pm \in C^\infty(\mathcal{M})$ be a partition of unity on $\mathcal{M}$ satisfying $\chi_\pm = 1$ on $J^\pm(\Sigma_\pm)$. Then, for $\phi \in S(\mathcal{M})$, set

$$f_\phi = \mathcal{K}(\chi_+\phi) = [\mathcal{K},\chi_+]\phi = (\Box_g\chi_+)\phi + 2g^{\nu\varrho}\nabla_\nu\chi_+\nabla_\varrho\phi\,.$$

This is a smooth function with compact support in $J^+(\Sigma_-) \cap J^-(\Sigma_+) \cap \mathrm{supp}(\phi)$. Moreover, we have $f_\phi = -\mathcal{K}(\chi_-\phi)$. To show that this is a well-defined map independent of the choice of $\Sigma_\pm$ and $\chi_\pm$, let f'_ϕ be another function obtained with the same map but another partition of unity $\chi'_\pm$ corresponding to a different pair of Cauchy surfaces $\Sigma'_\pm$. Then

$$f_\phi - f'_\phi = \mathcal{K}(\chi_+\phi) - \mathcal{K}(\chi'_+\phi) = \mathcal{K}((\chi_+ - \chi'_+)\phi)\,.$$

is contained in $\mathcal{K}C_0^\infty(\mathcal{M})$, since the support of $(\chi_+ - \chi'_+)\phi$ is compact. It remains to

[2]Note that this reference uses the opposite metric signature, leading to a different sign in σ, see also [45].

show that this map is indeed the inverse of E. Taking into account the extension of $E^\pm$ to future-/past-compact functions, we find

$$E(f_\phi) = E^+(\mathcal{K}(\chi_+\phi)) + E^-(\mathcal{K}(\chi_-\phi)) = (\chi_+ + \chi_-)\phi = \phi\,.$$

Therefore, the map $\phi \mapsto f_\phi$ is a well-defined symplectomorphism and the inverse of E.

After this discussion of the classical theory, let us now outline the quantization procedure. There are different ways to build the algebra of the free, real scalar field out of the classical solution theory described above. In this work, we will consider the CCR-algebra $\mathcal{A}(\mathcal{M})$ constructed in the same way as in [47, 48]:

Definition 2.2.1. The algebra of observables $\mathcal{A}(\mathcal{M})$ for the free scalar field on the spacetime $(\mathcal{M}, g)$ is the free $*$-algebra generated by the unit element $\mathbf{1}$ and the elements $\Phi(f)$, $f \in C_0^\infty(\mathcal{M})$, subject to the relations

- **Linearity** $\Phi(\alpha f + g) = \alpha\Phi(f) + \Phi(g) \quad \forall f, g \in C_0^\infty(\mathcal{M}),\, \alpha \in \mathbb{C}$
- **Klein-Gordon equation** $\Phi(\mathcal{K}f) = 0 \quad \forall f \in C_0^\infty(\mathcal{M})$
- **Hermiticity** $(\Phi(f))^* = \Phi(\bar{f}) \quad \forall f \in C_0^\infty(\mathcal{M})$
- **Commutator property** $[\Phi(f), \Phi(g)] = iE(f,g)\mathbf{1} \quad \forall f, g \in C_0^\infty(\mathcal{M})\,.$

This means that $\mathcal{A}(\mathcal{M})$ is obtained by taking the quotient of the free $*$-algebra generated by $\Phi(f)$ and $\mathbf{1}$ with respect to the ideal defined by the relations above. The elements in the algebra are thus (equivalence classes of) finite sums of finite products of $\Phi(f_i)$. One can interpret the elements $\Phi(f)$ as smeared field operators.

Another equivalent option to construct the algebra is by considering the tensor algebra over $C_0^\infty(\mathcal{M})$ and then taking the quotient by the relations in Definition 2.2.1 which is called the Borchers-Uhlmann algebra [59, 60], see [48].

The net of algebras on the spacetime $\mathcal{M}$ can then be constructed by assigning to each causally convex region $\mathcal{O} \subset \mathcal{M}$ the free $*$-subalgebra $\mathcal{A}(\mathcal{O})$ of $\mathcal{A}(\mathcal{M})$ generated by $\mathbf{1}$ and $\Phi(f)$, $f \in C_0^\infty(\mathcal{O})$.

A short calculation reveals that the resulting net of algebras satisfies the isotony and causality conditions: If $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{M}$ are open sets, then $C_0^\infty(\mathcal{O}_1) \subset C_0^\infty(\mathcal{O}_2)$, and thus $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$. Similarly, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then

$$[\Phi(f_1), \Phi(f_2)] = iE(f_1, f_2) = 0$$

by the support properties of the commutator function for all $f_1 \in \mathcal{O}_1$, $f_2 \in \mathcal{O}_2$. By linearity, this extends to all $A_1 \in \mathcal{A}(\mathcal{O}_1)$ and $A_2 \in \mathcal{A}(\mathcal{O}_2)$.

Moreover, the theory constructed in this way is local and covariant: If $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ are two spacetimes and $\psi : \mathcal{M} \to \tilde{\mathcal{M}}$ is an isometric, causality-preserving embedding, then ψ induces an injective, unit-preserving $*$-homomorphism

$$\alpha_\psi : \mathcal{A}(\mathcal{M}) \to \mathcal{A}(\tilde{\mathcal{M}}) \qquad \Phi_{\mathcal{M}}(f) \mapsto \alpha_\psi(\Phi_{\mathcal{M}}(f)) = \Phi_{\tilde{\mathcal{M}}}(\psi_*(f))\,.$$

Here, $\psi_*(f)$ is the push-forward, which is given by $f(\psi^{-1}(x))$ for $x \in \psi(\mathcal{M})$ and 0 on $\tilde{\mathcal{M}}\backslash\psi(\mathcal{M})$. We have also denoted the smeared field operators on $\mathcal{M}$ by $\Phi_{\mathcal{M}}(f)$ and the ones on $\tilde{\mathcal{M}}$ by $\Phi_{\tilde{\mathcal{M}}}(f)$ to make the spacetime they belong to explicit. The linearity and Hermiticity of the algebra generators $\Phi_{\mathcal{M}}(f)$ are obviously conserved by this map. That α_ψ is compatible with the Klein-Gordon equation and commutator property follows from the fact that for any isometric, causality-preserving embedding $\psi : (\mathcal{M}, g) \to (\tilde{\mathcal{M}}, \tilde{g})$

$$\psi^* \circ \mathcal{K}_{(\tilde{\mathcal{M}},\tilde{g})} = \mathcal{K}_{(\mathcal{M},g)} \circ \psi^* \,, \tag{2.2.7}$$

and therefore

$$\psi^* \circ E^{\pm}_{(\tilde{\mathcal{M}},\tilde{g})} = E^{\pm}_{(\mathcal{M},g)} \circ \psi^* \,, \tag{2.2.8}$$

see [57].

The states on the CCR-algebra are all linear maps $\omega : \mathcal{A}(\mathcal{M}) \to \mathbb{C}$ such that $\omega(\mathbf{1}) = 1$ and $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}(\mathcal{M})$. By its linearity and the structure of the algebra, any such state is determined by its n-point functions

$$W_n^\omega : C_0^\infty(\mathcal{M})^{\otimes n} \to \mathbb{C} \,, \quad (f_1 \otimes \cdots \otimes f_n) \mapsto \omega\left(\Phi(f_1) \ldots \Phi(f_n)\right) \,.$$

The states that we will deal with in this thesis all belong to a particular class of states, called quasi-free or Gaussian states. These states satisfy

$$\omega\left(e^{i\Phi(f)}\right) = e^{-\frac{1}{2}W_2^\omega(f\otimes f)} \,,$$

in the sense that the set of identities obtained by functional differentiation of this equation with respect to f are all satisfied, see e.g. [6, 47]. This entails that all n-point functions for odd n vanish, while all n-point functions for even n are determined from the two-point function using Wick's formula. Thus, these states are determined by their two-point function alone. We will call the two-point function of a quasi-free state w^ω or simply w when no confusion arises. As discussed in [47] and [16], this means in turn

Corollary 2.2.2. *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime and $\mathcal{A}(\mathcal{M})$ the CCR-algebra for a free, massive or massless scalar quantum field on $\mathcal{M}$. Then a bi-distribution $w : C_0^\infty(\mathcal{M}) \otimes C_0^\infty(\mathcal{M}) \to \mathbb{C}$ defines a quasi-free state on $\mathcal{A}(\mathcal{M})$ if it satisfies*

- **Weak bi-solution** $w(\mathcal{K}(f) \otimes g) = w(f \otimes \mathcal{K}(g)) = 0 \quad \forall f, g \in C_0^\infty(\mathcal{M})$
- **Positivity** $w(\bar{f} \otimes f) \geq 0 \quad \forall f \in C_0^\infty(\mathcal{M})$
- **Commutator property** $w(f \otimes g) - w(g \otimes f) = iE(f, g) \quad \forall f, g \in C_0^\infty(\mathcal{M})$.

The Hilbert space obtained by GNS-reconstruction using a quasi-free state is a symmetric Fock space over a Hilbert space $\mathfrak{h}$, which is commoly called the one-particle Hilbert space. The vector Ω_ω is the Fock-space vacuum vector and the smeared field operator $\Phi(f)$ is represented on the Fock space by a sum of creation and annihilation operators, from which one can also construct a particle-number operator on the Fock space [61]. Therefore, one can make sense of the notion of a particle in this context.

Another class of states can be defined if the spacetime $(\mathcal{M}, g)$ has a complete Killing vector field ξ. A complete Killing vector field is a vector field $\xi \in \Gamma(\mathcal{M})$ satisfying the Killing equation

$$\nabla_\nu \xi_\varrho + \nabla_\varrho \xi_\nu = 0$$

which induces the flow $\psi_t : I \times (\mathcal{M}, g) \to (\mathcal{M}, g)$ and for which $I = \mathbb{R}$. Then, the flow ψ_t induces a 1-parameter family of automorphisms

$$\alpha_t : \mathbb{R} \times \mathcal{A}(\mathcal{M}) \to \mathcal{A}(\mathcal{M}) \,, \quad \Phi(f) \mapsto \Phi(\psi^*_{-t}(f)) = \Phi(\psi_{t*} f)$$

on the algebra. These automorphisms satisfy $\alpha_t \circ \alpha_s = \alpha_{t+s}$ and $\alpha_0 = \mathrm{id}$ [47]. We say that a state ω is invariant under the automorphism if for any $A \in \mathcal{A}$, $\omega(\alpha_t A) = \omega(A)$. If the spacetime is stationary, i.e. possesses a complete time-like Killing vector field, one can define a generalization of thermal states and ground states.

Definition 2.2.2 ([62], App. A)**.** Given a complete time-like Killing vector field ξ on $(\mathcal{M}, g)$, denote the induced 1-parameter family of automorphisms on $\mathcal{A}(\mathcal{M})$ by α_t. Let ω be a state on $\mathcal{A}(\mathcal{M})$. Then ω is a

- **KMS state** at inverse temperature $\beta > 0$ if for all $A, B \in \mathcal{A}(\mathcal{M})$, the function $\mathbb{R} \ni t \mapsto \omega(A\alpha_t(B)) \in \mathbb{C}$ is bounded and for any $f \in C_0^\infty(\mathbb{R}; \mathbb{R})$,

$$\int_{-\infty}^{\infty} \hat{f}(t)\omega(A\alpha_t(B))\,\mathrm{d}t = \int_{-\infty}^{\infty} \hat{f}(t + i\beta)\omega(\alpha_t(B)A)\,\mathrm{d}t \,.$$

- **ground state** if $\forall A, B \in \mathcal{A}(\mathcal{M})$, the function $\mathbb{R} \ni t \mapsto \omega(A\alpha_t(B)) \in \mathbb{C}$ is bounded and for all $f \in C_0^\infty(\mathbb{R}_+; \mathbb{R})$,

$$\int_{-\infty}^{\infty} \hat{f}(t)\omega(A\alpha_t(B))\,\mathrm{d}t = 0 \,.$$

Another important class of states are the so-called Hadamard states. To understand the significance of this class of states, let us consider the stress-energy tensor of the scalar field. Classically, it can be written as

$$T_{\nu\varrho}(x) = \partial_\nu \Phi(x) \partial_\varrho \Phi(x) - \frac{1}{2} g_{\nu\varrho} \left(\partial_\sigma \Phi(x) \partial^\sigma \Phi(x) + \mu^2 \Phi(x)^2 \right)$$

if the scalar field is minimally coupled to the Ricci scalar. Note that there is an ambiguity in the classical stress-energy tensor in the case where the Ricci scalar is a constant, since in this case a non-zero mass and a non-minimal coupling have the same effect on the equations of motion as long as backreaction is neglected. Here, and in the rest of the thesis, we consider the stress-energy tensor of a minimally coupled scalar field.

The stress-energy tensor is local and quadratic in the field, and hence it is not in the CCR-algebra of the free scalar field defined above and will require renormalization. In flat spacetime, one can renormalize by subtracting the ground state expectation value or, in other words, by normal ordering with respect to the ground state. However, in a general spacetime we do not have a unique ground state. Instead, one requires that the renormalization procedure satisfies a number of properties. Particularly, for the renormalized Wick squares $:\Phi^2:(f)$ of the quantum field one demands [6]

- **Locality and covariance** Let $(\mathcal{M}, g)$ and $(\tilde{\mathcal{M}}, \tilde{g})$ spacetimes and $\psi : \mathcal{M} \to \tilde{\mathcal{M}}$ an isometric, causality-preserving embedding inducing the $*$-homomorphism α_ψ. Then

$$\alpha_\psi(:\Phi^2_{\mathcal{M}}:(f)) =:\Phi^2_{\tilde{\mathcal{M}}}:(\psi_* f) \quad \forall f \in C_0^\infty(\mathcal{M})\,.$$

- **Expansion** In a distributional sense, $[:\Phi^2:(x), \Phi(y)] = 2iE(x,y)\Phi(x)$.

- **Hermiticity** $:\Phi^2:(f)^* =:\Phi^2:(\bar{f})$ for all $f \in C_0^\infty(\mathcal{M})$.

- **Smoothness** Under certain conditions on the state ω, $\omega(:\Phi^2:(x))$ is a smooth function on $\mathcal{M}$.

- **Continuity and analyticity** $:\Phi^2:(f)$ changes continuously (analytically) under smooth (analytic) changes of the metric, or the mass or coupling of the scalar field.

- **Almost homogeneous scaling** Under a rescaling of the metric, and the mass and coupling of the scalar field, $:\Phi^2:(f)$ scales homogeneously up to logarithmic corrections.

It has been shown [6] that this can be achieved by normal ordering with respect to the Hadamard parametrix:

For x, y in a geodesically convex neighbourhood $\mathcal{U} \subset \mathcal{M}$, the Hadamard parametrix $H(x,y)$ for the scalar field is given by

$$H(x,y) = \lim_{\epsilon\to 0^+} \frac{1}{8\pi^2}\left[\frac{U(x,y)}{\sigma_\epsilon} + \sum_{n=0}^{\infty} V_n(x,y)\sigma(x,y)^n \log\left(\frac{\sigma_\epsilon}{l^2}\right)\right]. \tag{2.2.9}$$

Here, $\sigma(x,y)$ is half the signed, squared geodesic distance, which is also called Synge's world function [63], and $\sigma_\epsilon(x,y) = \sigma(x,y) + 2i\epsilon(T(x) - T(y)) + \epsilon^2$ is an $i\epsilon$-description, with T a local future-directed time coordinate. U and V_n are called the Hadamard coefficients. They are smooth functions and can be determined from the local curvature and the equations of motion using the transport equations. l is a free length scale. Note that the series has to be understood as an asymptotic series in smoothness in the sense that

$$H_m(x,y) = \lim_{\epsilon\to 0^+} \frac{1}{8\pi^2}\left[\frac{U(x,y)}{\sigma_\epsilon} + \sum_{n=0}^{m} V_n(x,y)\sigma(x,y)^n \log\left(\frac{\sigma_\epsilon}{l^2}\right)\right].$$

is a weak bi-solution to the Klein-Gordon equation up to terms in $C^{m-1}(\mathcal{U}\times\mathcal{U})$.

The normal-ordered Wick square is then formally given by

$$:\Phi^2:(f) = \int_{\mathcal{M}} \lim_{y\to x}\left[\Phi(x)\Phi(y) - H(x,y)\mathbf{1}\right] f(x)\,\mathrm{d}vol_g(x)\,.$$

In [6], it is shown further that the algebra $\mathcal{A}(\mathcal{M})$ can be extended to an algebra containing these normal-ordered Wick squares. Furthermore, the only quasi-free states that can be extended from $\mathcal{A}(\mathcal{M})$ to this enlarged algebra and that satisfy the "smoothness" condition for the renormalized Wick squares are the quasi-free Hadamard states.

Definition 2.2.3. Let ω be a quasi-free state on the algebra $\mathcal{A}(\mathcal{M})$. Then ω is Hadamard if for x,y contained in a geodesically convex neighbourhood $\mathcal{U}$, the two-point function $w(x,y)$ can be written as

$$w(x,y) = H(x,y) + w_0(x,y)\,,$$

where $w_0(x,y)$ is smooth. This should be understood in the sense of an asymptotic series: For $x,y\in\mathcal{U}$, $(w(x,y) - H_m(x,y)) \in C^{m-1}(\mathcal{U}\times\mathcal{U})$.

In the same way, one can normal order Wick-squares with derivatives, $\partial^\alpha\Phi(x)\partial^\beta\Phi(x)$, where α and β are multi-indices, and show that they are contained in the enlarged algebra.

Given a Hadamard state ω with two-point function w, the expectation value of the renormalized operator $\partial^\alpha\Phi(x)\partial^\beta\Phi(x)$ can be obtained by the so-called Hadamard point-split renormalization procedure:

$$\omega(:\partial^\alpha\Phi(x)\partial^\beta\Phi(x):) = \lim_{y\to x}\partial_x^\alpha\partial_y^\beta\left[w(x,y) - H_{|\alpha|+|\beta|+1}(x,y)\right].$$

Hadamard point-split renormalization is not the unique local and covariant renormalization scheme. There are always finite renormalization ambiguities which can be constructed from the local metric and the field parameters [6].

Apart from their importance for the definition and evaluation of renormalized Wick powers, Hadamard states guarantee that the fluctuations of all Wick polynomials, i.e. its variance and higher moments, are finite [64], while the fluctuations typically diverge if the state is not Hadamard, even if the expectation value is finite, see e.g. [65].

One may worry that there are globally hyperbolic spacetimes $(\mathcal{M},g)$ for which no Hadamard states on $\mathcal{A}(\mathcal{M})$ exist. That this is not the case has been shown by various methods [66, 67]. In addition, KMS states and ground states, as well as any convex combination of them, called passive states, are Hadamard [62].

2.3 An introduction to microlocal analysis

In order to prove that a certain state is Hadamard, it is often more useful to consider an alternative characterisation of the Hadamard property formulated in the framework of microlocal analysis. In this section, we will introduce some aspects of this framework, following mostly [68].

First, let us define the wavefront set $\mathrm{WF}(u)$ of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$.

Definition 2.3.1. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. Let $(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$, $\chi \in C_0^\infty(\mathbb{R}^n)$ a test function satisfying $\chi(x) \neq 0$ and $V_k \subset \mathbb{R}^n \backslash \{0\}$ an open conic neighbourhood of k, i.e. a set, so that $\lambda l \in V_k$ for all $\lambda > 0$ if $l \in V_k$. Assume χ and V_k can be chosen in such a way that for any $N \in \mathbb{N}$ there is a $C_N > 0$ with [68, Sec. 8.1]

$$|\widehat{\chi u}|(l) \leq C_N (1+|l|)^{-N} \quad \forall l \in V_k\,, \tag{2.3.1}$$

i.e. the function $\widehat{\chi u}$ is rapidly decreasing in $l \in V_k$. Then (x,k) is called a direction of rapid decrease for u. The wavefront set of u is the set of all $(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ which are not of rapid decrease for u.

Another characterization of the wavefront set is given by [69, Prop.2.1]:

Proposition 2.3.1 ([69], Prop. 2.1)**.** *Let $(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$, $u \in \mathcal{D}'(\mathbb{R}^n)$. Then $(x,k) \notin \mathrm{WF}(u)$ iff there exist an open neighbourhood $V \subset (\mathbb{R}^n \backslash \{0\})$ of k, and some test functions $h \in C_0^\infty(\mathbb{R}^n)$ with $h(0) = 1$, and $g \in C_0^\infty(\mathbb{R}^n)$ with $\hat{g}(0) = 1$, such that $\forall p \geq 1$, $\forall N \in \mathbb{N}$, $\exists C_N > 0$, $\lambda_N > 0$ satisfying*

$$\sup_{k' \in V} \left| \int e^{i\lambda^{-1} k' \cdot y} h(y) u\left(g(\lambda^{-p}(\cdot - x - y))\right) d^n y \right| < C_N \lambda^N \quad \forall 0 < \lambda < \lambda_N\,. \tag{2.3.2}$$

The wavefront set has a number of important properties, for example [70, Prop. 6.27]

Proposition 2.3.2. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then*

1. *For any $f \in C^\infty(\mathbb{R}^n)$, we can define $f \cdot u$ as $(f \cdot u)(\phi) = u(f \cdot \phi)$. Then $\mathrm{WF}(f \cdot u) \subset \mathrm{WF}(u)$.*
2. *Let $\alpha \in \mathbb{N}^n$. Then $\mathrm{WF}(\partial_x^\alpha u) \subset \mathrm{WF}(u)$.*
3. *Let $v \in \mathcal{D}'(\mathbb{R}^n)$. Then $\mathrm{WF}(u \pm v) \subset \mathrm{WF}(u) \cup \mathrm{WF}(v)$ and equality is given if $\mathrm{WF}(u) \cap \mathrm{WF}(v) = \emptyset$.*

Let us give a brief proof of the third point, which is not covered by [70, Prop. 6.27]. Let us assume that (x,k) is not in $\mathrm{WF}(u) \cup \mathrm{WF}(v)$. Then one can find test functions ϕ_u and ϕ_v which are non-vanishing at x, and open conic neighbourhoods V_u and V_v of k so that for all $N \in \mathbb{N}$

$$\left|\widehat{\phi_u \cdot u}\right|(l)\,, \; \left|\widehat{\phi_v \cdot v}\right|(l) \leq C_N (1+|l|)^{-N} \quad \forall l \in V_u \cap V_v$$

for some constant $C_N > 0$. Since multiplication by a smooth function cannot increase the wavefront set due to the second point in Proposition 2.3.2, we can find a conic neighbourhood $V \subset V_u \cap V_v$ of k so that for any $N \in \mathbb{N}$

$$\left|\widehat{\phi_u \cdot \phi_v \cdot (u \pm v)}\right|(l) \leq C'_N(1+|l|)^{-N} \quad \forall l \in V$$

for some constants $C'_N > 0$. Hence, (x,k) is not in $\mathrm{WF}(u \pm v)$. Next, we consider the case $\mathrm{WF}(u) \cap \mathrm{WF}(v) = \emptyset$. Assume that (x,k) is in $\mathrm{WF}(u) \cup \mathrm{WF}(v)$, but not in $\mathrm{WF}(u \pm v)$. W.l.o.g. we assume $(x,k) \in \mathrm{WF}(u)$, and therefore $(x,k) \notin \mathrm{WF}(v)$. By using similar arguments as above, we can find a test function ϕ with $\phi(x) \neq 0$, and a conic neighbourhood V of k so that for all $N \in \mathbb{N}$

$$\left|\widehat{\phi \cdot v}\right|(l)\,,\ \left|\widehat{\phi \cdot (u \pm v)}\right|(l) \leq C_N(1+|l|)^{-N} \quad \forall l \in V.$$

Then

$$\left|\widehat{\phi \cdot u}\right|(l) = \left|\widehat{\phi \cdot (u \pm v \mp v)}\right|(l) \leq \left|\widehat{\phi \cdot (u \pm v)}\right|(l) + \left|\widehat{\phi \cdot v}\right|(l) \leq 2C_N(1-|l|)^{-N}$$

for all $l \in V$ and for all $N \in \mathbb{N}$, leading to a contradiction to $(x,k) \in \mathrm{WF}(u)$.

The wavefront set can also be defined for distributions on manifolds. For the rest of this section, let $\mathcal{M}$ be a smooth manifold of dimension n.

Definition 2.3.2. Let $u \in \mathcal{D}'(\mathcal{M})$ be a distribution on $\mathcal{M}$. Then its wavefront set is defined as the subset $\mathrm{WF}(u)$ of $T^*\mathcal{M}\backslash o$ whose restriction (in the base variable) to any coordinate patch $\mathcal{M}_\psi \subset \mathcal{M}$ with the coordinate map $\psi : \mathcal{M}_\psi \to \mathcal{U}_\psi \subset \mathbb{R}^n$ is given by [68, Thm. 8.2.4]

$$\begin{aligned} \mathrm{WF}(u)|_{\mathcal{M}_\psi} &= \psi^* \,\mathrm{WF}(u \circ \psi^{-1}) \\ &= \left\{(x, \psi^* k) : (\psi(x), k) \in \mathrm{WF}(u \circ \psi^{-1})\right\}. \end{aligned} \tag{2.3.3}$$

Later on, we will also consider continuous linear maps $K : C_0^\infty(\mathcal{M}) \to \mathcal{D}'(\tilde{\mathcal{M}})$ from test functions on some manifold $\mathcal{M}$ to distributions on another manifold $\tilde{\mathcal{M}}$. By the Schwartz kernel theorem, every map $K : C_0^\infty(\mathcal{M}) \to \mathcal{D}'(\tilde{\mathcal{M}})$ is in one-to-one correspondence with a distribution on $\tilde{\mathcal{M}} \times \mathcal{M}$, which is called the kernel of K and also denoted by K by an abuse of notation. We will say that K is properly supported if the projection map

$$\pi_{\mathcal{M}} : \mathrm{supp}(K) \to \mathcal{M}\,, \quad (y,x) \mapsto x$$

is proper, i.e. the pre-images of compact sets are compact. Let K be the kernel of such an operator $K : C_0^\infty(\mathcal{M}) \to \mathcal{D}'(\tilde{\mathcal{M}})$. Then we denote

$$\mathrm{WF}'(K) = \{(x,k;y,l) \in T^*(\tilde{\mathcal{M}} \times \mathcal{M})\backslash o : (x,k;y,-l) \in \mathrm{WF}(K)\} \tag{2.3.4a}$$

$$\mathrm{WF}_{\mathcal{M}}(K) = \{(y,l) \in T^*(\mathcal{M}) : (x,0;y,l) \in \mathrm{WF}(K) \text{ for some } x \in \tilde{\mathcal{M}}\} \tag{2.3.4b}$$
$$_{\tilde{\mathcal{M}}}\mathrm{WF}(K) = \{(x,k) \in T^*(\tilde{\mathcal{M}}) : (x,k;y,0) \in \mathrm{WF}(K) \text{ for some } y \in \mathcal{M}\}\,. \tag{2.3.4c}$$

Under certain conditions, two of the maps introduced above can be composed with each other:

Theorem 2.3.3 ([68], Thm. 8.2.14)**.** *Let $\mathcal{M}$, $\tilde{\mathcal{M}}$, and $\mathcal{M}'$ be smooth manifolds, and let $K_1 : C_0^\infty(\mathcal{M}) \to \mathcal{D}'(\tilde{\mathcal{M}})$ and $K_2 : C_0^\infty(\tilde{\mathcal{M}}) \to \mathcal{D}'(\mathcal{M}')$ be continuous linear maps. Assume in addition that the map K_1 is properly supported. Then the composition of maps $K = K_2 \circ K_1 : C_0^\infty(\mathcal{M}) \to \mathcal{D}'(\mathcal{M}')$ defines a continuous map from $C_0^\infty(\mathcal{M})$ to $\mathcal{D}'(\mathcal{M}')$ if*

$$\mathrm{WF}'_{\tilde{\mathcal{M}}}(K_2) \cap {}_{\tilde{\mathcal{M}}}\mathrm{WF}(K_1) = \emptyset\,. \tag{2.3.5}$$

The kernel K of this map satisfies

$$\begin{aligned} \mathrm{WF}'(K) \subset \mathrm{WF}'(K_2) \circ \mathrm{WF}'(K_1) &\cup ({}_{\mathcal{M}'}\mathrm{WF}(K_2) \times \mathcal{M} \times \{0\}) \\ &\cup (\mathcal{M}' \times \{0\} \times \mathrm{WF}'_{\mathcal{M}}(K_1))\,. \end{aligned} \tag{2.3.6}$$

One particular class of operators $K : C_0^\infty(\mathcal{M}) \to C_0^\infty(\mathcal{M})$ which will appear in the subsequent discussion is the class of differential operators. The differential operators of order zero correspond to pointwise multiplication by a smooth function,

$$C_0^\infty(\mathcal{M}) \ni f \mapsto h \cdot f \in C_0^\infty(\mathcal{M})$$

for some $h \in C^\infty(\mathcal{M})$. Thus, we identify $\mathrm{Diff}^0(\mathcal{M}) = C^\infty(\mathcal{M})$.

Next, we use the interpretation of elements in $T\mathcal{M}$ as directional derivatives and set $\mathrm{Diff}^1(\mathcal{M})$ to be the set of all operators $K : C_0^\infty(\mathcal{M}) \to C_0^\infty(\mathcal{M})$ which are of the form $K(f) = A(f) + h \cdot f$ for some $A \in \Gamma(\mathcal{M})$, and $h \in C^\infty(\mathcal{M})$.

For any $m > 1$ we can then define the space of differential operators on $\mathcal{M}$ of order m to consist of all operators $K : C_0^\infty(\mathcal{M}) \to C_0^\infty(\mathcal{M})$ of the form

$$K(f) = \sum_{i=1}^{N} A_{i_1}(\ldots A_{i_{N_i}}(f)\ldots) \text{ for some } A_{i_j} \in \mathrm{Diff}^1(\mathcal{M})\,, \quad N \in \mathbb{N}\,, \quad N_i \leq m\,.$$

All these operators can also be extended to operators from $C^\infty(\mathcal{M})$ to $C^\infty(\mathcal{M})$ [70].

In coordinates, $K \in \mathrm{Diff}^m(\mathcal{M})$ acting on $f \in C_0^\infty(\mathcal{M})$ may be written as

$$K(f)(x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha(f)(x)\,, \tag{2.3.7}$$

where $\alpha \in \mathbb{N}^n$, $a_\alpha \in C^\infty(\mathcal{M})$, and ∂_{x^ν} are the usual coordinate derivatives forming the coordinate basis for $T_x\mathcal{M}$ for all $x \in \mathcal{M}$ covered by that particular coordinate chart. The dual basis for $T_x^*\mathcal{M}$, with elements denoted $\mathrm{d}x^\nu$, satisfies $\mathrm{d}x^\nu(\partial_{x^\varrho}) = \delta_{\varrho\nu}$, and allows to express any $k \in T_x^*\mathcal{M}$ as $k = k_\nu\, \mathrm{d}x^\nu$.

The principal symbol of the differential operator K in (2.3.7) in these coordinates is then given by

$$\sigma_m(K)(x,k) = \sum_{|\alpha|=m} a_\alpha(x)k^\alpha \in C^\infty(T^*\mathcal{M})\,. \tag{2.3.8}$$

The principal symbol of a differential operator is an important tool in the analysis of differential equations, since it already captures a large part of the behaviour of the operator, see for example [70] for an introduction to the symbol calculus.

Given the principal symbol $a = \sigma_m(A)$ of $A \in \mathrm{Diff}^m(\mathcal{M})$, the Hamiltonian vector field $H_a \in \Gamma(T^*\mathcal{M})$ corresponding to a is, in a local trivialization of $T^*\mathcal{M}$, given by

$$H_a(x,k) = \partial_{k_\nu} a(x,k)\partial_{x^\nu} - \partial_{x^\mu} a(x,k)\partial_{k_\mu}\,. \tag{2.3.9}$$

It satisfies $H_a(a) = 0$. Hence, if $a(x,k) = 0$, it remains zero along the flow induced on $T^*\mathcal{M}$ by H_a through (x,k).

With the help of this, one can define the bicharacteristic $B[A](x,k)$ of $A \in \mathrm{Diff}^m(\mathcal{M})$ through (x,k) as the integral curve of H_a in $T^*\mathcal{M}$ lying in $\{a = 0\}$ and intersecting (x,k) [70].

In this thesis, we are mostly interested in a particular differential operator, namely the Klein-Gordon operator (2.2.2). It is easy to see that its principle symbol is given by

$$\sigma_2(\mathcal{K}) = g^{\nu\varrho}k_\nu k_\varrho\,. \tag{2.3.10}$$

This is a real and homogeneous function of k and vanishes only when k is a null covector. Thus, the bicharacteristics of $\mathcal{K}$ are

$$B(x,k) = B[\mathcal{K}](x,k) = \{(x',k') \in T^*\mathcal{M} : (x,k) \sim (x',k')\} \tag{2.3.11}$$

for any $x \in \mathcal{M}$ and any null covector $k \in T^*_x\mathcal{M}$. The relation $(x,k) \sim (y,l)$ means that x and y can be connected by a null geodesic to which k is cotangent at x and l agrees with k parallel transported along the geodesic to y. We will also denote $B(x,0) = \{(x,0)\}$ for the zero covector. The projection of the bicharacteristics to the manifold,

$$B_\mathcal{M}(x,k) = \pi(B(x,k)) \subset \mathcal{M}\,, \tag{2.3.12}$$

with $\pi : T^*\mathcal{M} \to \mathcal{M}$, are the null geodesics on $\mathcal{M}$.

As we have discussed in the previous section, the two-point functions for states of the free scalar field will be distributional bi-solutions to the Klein-Gordon equation. Due to the form of the principal symbol of $\mathcal{K}$, one can use the real-principal-type Propagation of Singularities Theorem to make some statements on the wavefront sets of such bi-solutions. Here, we will work with the version given in [71, Lemma 6.5.5][3]:

[3]We state the lemma only for differential operators, but it applies to a more general class of operators, called Fourier Integral Operators [71]

Theorem 2.3.4 ([71], Lemma 6.5.5)**.** *Let $K \in \mathrm{Diff}^m(\mathcal{M})$ on some spacetime $(\mathcal{M}, g)$ with a real homogeneous principal symbol $\sigma_m(K) \in C^\infty(T^*\mathcal{M})$. Assume $u \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ satisfies $K \circ u \in C^\infty(\mathcal{M} \times \mathcal{M})$, where u is considered as a map from $C_0^\infty(\mathcal{M})$ to $\mathcal{D}'(\mathcal{M})$. If $(x, k; y, l) \in \mathrm{WF}'(u)$ and $k \neq 0$, then $\sigma_m(K)(x, k) = 0$ and $B[K](x, k) \times \{(y, l)\}$ is contained in $\mathrm{WF}'(u)$.*

Finally, let us return to quasi-free Hadamard states on the CCR-algebra. Therefore, let $(\mathcal{M}, g)$ now be a spacetime, and $\mathcal{A}(\mathcal{M})$ the CCR-algebra of a free scalar field on $\mathcal{M}$. In the previous section, we defined quasi-free Hadamard states by the singular behaviour of their two-point functions. However, it has been shown [72] that this definition is equivalent to a condition on the wavefront set of the two-point function, called the microlocal spectrum condition:

Theorem 2.3.5 ([72], Thm. 5.1)**.** *Let ω be a quasi-free state on $\mathcal{A}(\mathcal{M})$ with two-point function $w \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$. Then ω is a Hadamard state iff w satisfies the microlocal spectrum condition,*

$$\mathrm{WF}'(w) = \mathcal{C}^+ \,, \tag{2.3.13a}$$
$$\mathcal{C}^\pm = \{(x, k; y, l) \in T^*(\mathcal{M} \times \mathcal{M}) : (x, k) \sim (y, l) \text{ and } \pm k \rhd 0\} \,. \tag{2.3.13b}$$

2.4 An introduction to black-hole spacetimes

In this thesis, we will consider quantum fields on spacetimes describing charged or rotating black holes in the presence of a positive cosmological constant. In this section, we will introduce these spacetimes, define some relevant coordinate systems and summarize their most significant features.

2.4.1 The Reissner-Nordström-de Sitter spacetime

Let us start with RNdS spacetimes which describe eternal, charged, non-rotating black holes in the presence of a cosmological constant Λ. Astrophysical black holes are not expected to carry significant electric charge. Nonetheless, RNdS spacetimes are interesting because they share many features with rotating black-hole spacetimes while being easier to handle mathematically. They can thus serve as toy models for the more realistic rotating black-hole spacetimes. Moreover, they are interesting in their own right due to the classical violation of sCC in highly charged RNdS black holes.

A discussion of these spacetimes, their analytical extensions, and the Kruskal-type coordinates introduced below can be found in [73].

The RNdS spacetimes are vacuum solutions to the coupled Einstein-Maxwell system, i.e. they solve

$$G_{\nu\varrho} + \Lambda g_{\nu\varrho} = 8\pi \left(\mathcal{E}_{\nu\varrho} + T_{\nu\varrho}\right) \,, \tag{2.4.1a}$$
$$\nabla_\nu F^{\nu\varrho} = -4\pi J^\varrho \,, \tag{2.4.1b}$$

with $T_{\nu\varrho} = J^\nu = 0$. Here, J^ν is the charge current of the matter and the stress-energy tensor of the Maxwell field is given by

$$\mathcal{E}_{\nu\varrho} = \frac{1}{4\pi}\left(F_{\nu\alpha}F_{\varrho}{}^{\alpha} - \frac{1}{4}g_{\nu\varrho}F_{\alpha\beta}F^{\alpha\beta}\right) .$$

In Boyer-Lindquist coordinates, the metric takes the form

$$g = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \,, \tag{2.4.2}$$

where $\mathrm{d}\Omega^2$ is the usual metric on the two-dimensional unit sphere, and the function $f(r)$ is given by

$$f(r) = \frac{\Delta_r}{r^2} = -\frac{\Lambda}{3}r^2 + 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \,. \tag{2.4.3}$$

The constant M can be identified with the mass of the black hole, while Q represents the charge of the black hole. This becomes even more apparent when one considers the vector potential, which represents the electromagnetic part of this solution to the Einstein-Maxwell system, and which is of the form

$$A = -\frac{Q}{r}\,\mathrm{d}t \,. \tag{2.4.4}$$

The RNdS spacetimes thus form a three-parameter family of spacetimes, characterized by Q, Λ, and M. Unless specified otherwise, we will rescale to $M = 1$, so that we are left with (Λ, Q) as the free spacetime parameters.

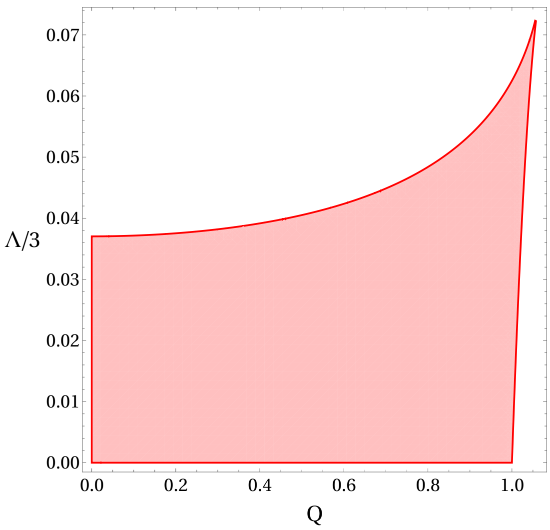


Figure 2.1: The parameter space for RNdS spacetimes. The red region is the subextremal parameter region, in which $f(r)$ has three real, distinct positive roots.

The Boyer-Lindquist coordinates are only well-defined away from $r = 0$ and the roots of $f(r)$[4]. We are interested in the parameter range of (Λ, Q) in which $f(r)$ has three real, positive and distinct roots, $r_- < r_+ < r_c$. This parameter space is illustrated in Fig. 2.1, see also [74] for a detailed discussion. The locations of the roots of $f(r)$ correspond to the locations of the inner (r_-) and outer (r_+) horizon of the black hole and the cosmological horizon (r_c). The Boyer-Lindquist coordinates cover the exterior region up to the cosmological horizon, $\mathrm{I} = \mathbb{R}_t \times (r_+, r_c) \times \mathbb{S}^2$, the interior region up to the inner horizon, $\mathrm{II} = \mathbb{R}_t \times (r_-, r_+) \times \mathbb{S}^2$, the region beyond the cosmological horizon, $\mathrm{III} = \mathbb{R}_t \times (r_c, \infty) \times \mathbb{S}^2$, and the interior beyond the inner horizon, $\mathrm{IV} = \mathbb{R}_t \times (0, r_-) \times \mathbb{S}^2$.

The Penrose diagram for this spacetime is shown in Fig. 2.2.

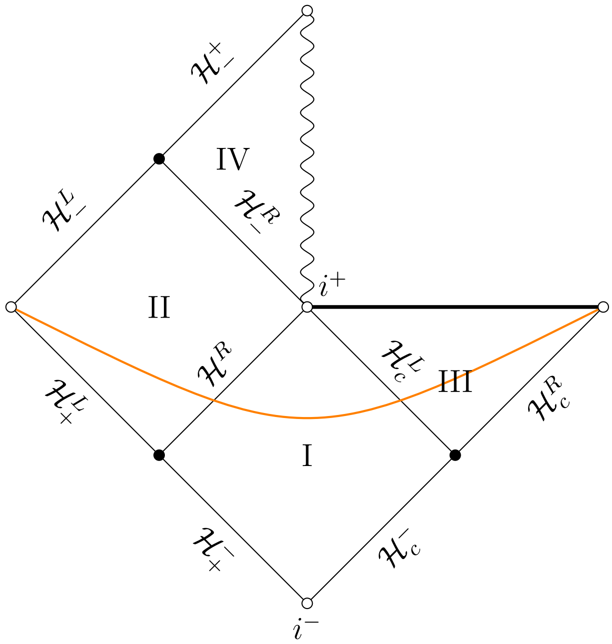


Figure 2.2: Penrose diagram for the Reissner-Nordström-de Sitter (RNdS) spacetime. The wiggled line represent the curvature singularity, the thick line correspond to conformal infinity. All other lines represent the different horizons. Filled dots stand for bifurcation surfaces, while empty dots indicate conformal points at infinity. The orange line indicates a possible Cauchy surface for the region $\mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$.

Let us also note that the metric (2.4.2) is spherically symmetric and independent of t. Hence, the blocks I and IV, in which $f(r)$ is positive and ∂_t is a time-like Killing vector field, are static. Moreover, the horizons are bifurcate Killing horizons generated by ∂_t.

In order to get rid of the coordinate singularities at the horizons and to extend the metric through the horizon, we construct Kruskal-type coordinates. As a first step, we introduce the tortoise radial coordinate r_*, which is defined by

$$\mathrm{d}r_* = f(r)^{-1}\,\mathrm{d}r\,.$$

[4]Strictly speaking, they do also not cover the axis of the two-spheres $\{t, r = \text{const.}\}$, where $\sin\theta = 0$.

One can choose the integration constant such that in each of the regions I, II, III and IV,

$$r_*(r) = -\frac{1}{2\kappa_-}\log|r-r_-| + \frac{1}{2\kappa_+}\log|r-r_+| \\ -\frac{1}{2\kappa_c}\log|r-r_c| + \frac{1}{2\kappa_o}\log|r-r_o| \,, \tag{2.4.5}$$

where $r_o = -(r_- + r_+ + r_c)$ is the fourth root of $f(r)$, and

$$\kappa_i = \frac{1}{2}\left|\partial_r f(r)\right|_{r=r_i} \tag{2.4.6}$$

is the surface gravity at the corresponding horizon.

As a next step, we introduce a set of double null coordinates in each region,

$$v = t + r_* \,, \qquad\qquad u = t - r_* \,.$$

In region I, they are both increasing towards the future. In these coordinates, the metric takes the form

$$g = -2f(r)\,\mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\Omega^2 \,.$$

Note that the metric still degenerates when $f(r) = 0$, and $u \to \pm\infty$ at outgoing horizons, while $v \to \pm\infty$ at ingoing horizons.

Let us now focus on the outer horizon of the black hole. In block I, we define

$$U_+ = -e^{-\kappa_+ u} \,, \qquad\qquad V_+ = e^{\kappa_+ v} \,.$$

By construction, U_+ approaches zero as $u \to \infty$ towards the event horizon $\mathcal{H}_+^R$ of the black hole. Similarly, $V_+ \to 0$ towards $\mathcal{H}_+^-$, see Fig. 2.2.

Combining the definition of U_+ and V_+ with (2.4.5), one finds that $f(r)/(U_+V_+)$ is a smooth, positive, non-vanishing function of r on (r_-, r_c). As a result, the metric in the Kruskal coordinates is smooth and non-degenerate as $U_+ \to 0$ or $V_+ \to 0$ and can be extended from $\mathbb{R}_{-,U_+} \times \mathbb{R}_{+,V_+} \times \mathbb{S}^2$ to $\mathcal{M}_+ = \mathbb{R}_{U_+} \times \mathbb{R}_{V_+} \times \mathbb{S}^2$. We will call this a Kruskal block. One can then identify II with $\mathbb{R}_{+,U_+} \times \mathbb{R}_{+,V_+} \times \mathbb{S}^2$, and connect (u, v) and (U_+, V_+) in II via

$$U_+ = e^{-\kappa_+ u} \,, \qquad\qquad V_+ = e^{\kappa_+ v} \,.$$

The Kruskal block $\mathcal{M}_+$ also contains a second copy of both I and II, but with reversed time orientation. We will denote these time-reversed regions I$'$ and II$'$.

Similarly, we can define on I

$$U_c = e^{\kappa_c u} \,, \qquad\qquad V_c = -e^{-\kappa_c v} \,.$$

This set of Kruskal-type coordinates leads to a smooth, non-degenerate metric at the cosmological horizon $\{r = r_c\}$, allowing us to extend through the this horizon. In III, these

Kruskal-type coordinates are then related to u and v by

$$U_c = e^{\kappa_c u}\,, \qquad\qquad V_c = e^{-\kappa_c v}\,.$$

The associated Kruskal block $\mathcal{M}_c = (\mathbb{R}_{U_c} \times \mathbb{R}_{V_c}) \cap \{U_c V_c < 1\} \times \mathbb{S}^2$ additionally contains the regions I' and III'.

Finally, we can also construct a set of Kruskal coordinates which allows extension of the metric through the inner horizon. They are related to u and v by

$$\begin{aligned} U_- &= -e^{\kappa_- u}\,, & V_- &= -e^{-\kappa_- v} & &\text{in II}\,,\\ U_- &= -e^{\kappa_- u}\,, & V_- &= e^{-\kappa_- v} & &\text{in IV}\,. \end{aligned}$$

With the Kruskal-type coordinates, one can patch together an atlas covering the whole physical RNdS spacetime $\mathcal{M}$ up to the Cauchy horizon, consisting of the block I joined via $\mathcal{H}_+^R$ to II and via $\mathcal{H}_c^L$ to III. In fact, this atlas will also cover the boundaries of the RNdS spacetime when embedded into its analytical extension $\mathcal{M}_+ \cup \mathcal{M}_c$, with region I in the two blocks identified. This analytic extension of the RNdS spacetime up to the Cauchy horizon is globally hyperbolic [73]. Moreover, by considering $\mathcal{M}$ as a submanifold of the analytical extension and taking into account the causal structure of RNdS, one can see that the physical spacetime $\mathcal{M} = \mathrm{I} \cup \mathcal{H}_+^R \cup \mathrm{II} \cup \mathcal{H}_c^L \cup \mathrm{III}$ must be globally hyperbolic as well.

2.4.2 The Kerr-de Sitter spacetime

Next, we discuss Kerr-de Sitter (KdS) spacetimes, which describe eternal, rotating, electrically neutral black holes in the presence of a positive cosmological constant Λ. These spacetimes provide a good model for isolated astrophysical black holes. They are solutions to the vacuum Einstein equations and depend on three parameters: the cosmological constant Λ, the mass M of the black hole, and the angular momentum parameter a of the black hole.

In Boyer-Lindquist coordinates (t, r, θ, φ), the metric is given by [5]

$$\begin{aligned} g = \frac{\Delta_\theta a^2 \sin^2\theta - \Delta_r}{\rho^2\chi^2}\,\mathrm{d}t^2 + \left[\Delta_\theta (r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta\right] \frac{\sin^2\theta}{\rho^2\chi^2}\,\mathrm{d}\varphi^2 \\ + \frac{\rho^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\,\mathrm{d}\theta^2 + 2\frac{a\sin^2\theta}{\rho^2\chi^2}[\Delta_r - \Delta_\theta(r^2+a^2)]\,\mathrm{d}t\,\mathrm{d}\varphi\,, \end{aligned} \tag{2.4.7}$$

where

$$\Delta_r = (1-\lambda r^2)(r^2+a^2) - 2Mr\,, \qquad \Delta_\theta = 1 + a^2\lambda\cos^2\theta\,, \tag{2.4.8a}$$
$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \chi = 1 + a^2\lambda\,, \tag{2.4.8b}$$

[5]This coordinate system does not cover the axis where $\sin\theta = 0$. However, it can be shown that this metric can be extended to the axis [75].

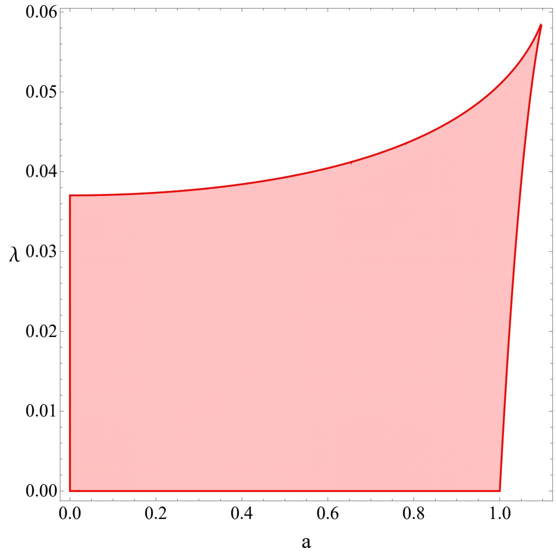


Figure 2.3: The parameter region in which Δ_r has three distinct real, positive roots in the (a, λ)-plane for $M = 1$.

and $\lambda = \Lambda/3$. From here on, we will rescale to $M = 1$. We will also restrict to the parameter region in which Δ_r has three distinct real, positive roots $r_- < r_+ < r_c$. The admissible parameter range in the (a, λ)-plane is depicted in Fig. 2.3. In this case, the Boyer-Lindquist blocks

$$\begin{aligned} \mathrm{I} =& \mathbb{R}_t \times (r_+, r_c) \times (\mathbb{S}^2_{\theta,\varphi}) \,, \\ \mathrm{II} =& \mathbb{R}_t \times (r_-, r_+) \times (\mathbb{S}^2_{\theta,\varphi}) \,, \text{ and} \\ \mathrm{III} =& \mathbb{R}_t \times (r_c, \infty) \times (\mathbb{S}^2_{\theta,\varphi}) \end{aligned}$$

are all non-empty. The blocks I and III make up the exterior of the black hole, with III being the region beyond the cosmological horizon. The block II is the interior of the black hole up to its inner horizon.

The metric is independent of both t and φ, and the surfaces of constant t are space-like in block I. Moreover, for any fixed point x_0 in I, the Killing vector field $\partial_t + c(x_0)\partial_\varphi$ with the constant $c(x_0) = a/(r(x_0)^2 + a^2)$ is time-like at x_0. However, none of these Killing vector fields will be time-like in all of I. Hence, Kerr-de Sitter is axisymmetric, and block I is manifestly time-invariant and locally stationary, but not globally stationary [76].

The horizons $\{r = r_i\}$, $i \in \{-, +, c\}$, are bifurcate Killing horizons generated by

$$\partial_{t_i} = \partial_t + \frac{a}{r_i^2 + a^2}\partial_\varphi \,. \tag{2.4.9}$$

Using the so-called $*KdS$- and $KdS*$-coordinates, one can continue the metric through the ingoing or outgoing pieces of the horizons respectively [75]. In particular, we define the tortoise coordinate r_* by

$$\mathrm{d}r_* = \frac{\chi(r^2+a^2)}{\Delta_r}\,\mathrm{d}r\,.$$

Thus, after fixing the integration constant, r_* is given by (2.4.5), where the surface gravities κ_i are now

$$\kappa_i = \frac{|\partial_r \Delta_r|_{r=r_i}}{2\chi(r_i^2+a^2)}\,. \tag{2.4.10}$$

In addition, we set

$$A(r) = \int \mathrm{d}r \frac{\chi a}{\Delta_r}\,.$$

Then, the $KdS*$-coordinates are given by

$$v = t + r_*(r)\,, \quad r^* = r\,, \quad \theta^* = \theta\,, \quad \varphi^* = \varphi + A(r)\,, \tag{2.4.11}$$

while the $*KdS$-coordinates are given by

$$u = t - r_*(r)\,, \quad {}^*r = r\,, \quad {}^*\theta = \theta\,, \quad {}^*\varphi = \varphi - A(r)\,. \tag{2.4.12}$$

In the $KdS*$-coordinates, the metric takes the form

$$\begin{aligned} g =& g_{tt}\,\mathrm{d}v^2 + \frac{2}{\chi}\,\mathrm{d}v\,\mathrm{d}r + \frac{\rho^2}{\Delta_\theta}\,\mathrm{d}\theta^2 + g_{\varphi\varphi}\,\mathrm{d}\varphi^{*2} \\ &+ 2g_{t\varphi}\,\mathrm{d}v\,\mathrm{d}\varphi^* - \frac{2a\sin^2\theta}{\chi}\,\mathrm{d}r\,\mathrm{d}\varphi^*\,, \end{aligned} \tag{2.4.13}$$

where the components $g_{\nu\varrho}$ are those in the Boyer-Lindquist coordinates, see (2.4.7). One obtains the metric in $*KdS$-coordinates by replacing $\mathrm{d}v \to -\mathrm{d}u$ and $\mathrm{d}\varphi^* \to -\mathrm{d}^*\varphi$.

We will refer to block I joint to block II via $\mathcal{H}_+^R \subset \{r = r_+\}$ in the $KdS*$-coordinates and to III via $\mathcal{H}_c^L \subset \{r = r_c\}$ in the $*KdS$-coordinates as the Kerr-de Sitter spacetime $\mathcal{M}$.

Next, let us construct Kruskal-type coordinates for this spacetime. For each horizon $i \in \{-,+,c\}$, we start by defining an adapted azimuthal coordinate

$$\varphi_i = \varphi - \frac{a}{r_i^2+a^2}t\,.$$

The metric in the $(u, v, \theta, \varphi_i)$-coordinates with u and v as in the $*KdS$- and $KdS*$-

coordinates, takes the form

$$
\begin{aligned}
g =& \frac{1}{4\chi^2\rho^2(r_i^2+a^2)^2}\left[\Delta_\theta \sin^2\theta a^2\left(r_i^2-r^2\right)^2-\Delta_r\rho_i^4\right](\,\mathrm{d}u+\,\mathrm{d}v)^2 \\
&+\frac{\Delta_r\rho^2}{4\chi^2(r^2+a^2)^2}(\,\mathrm{d}u-\,\mathrm{d}v)^2+\frac{\rho^2}{\Delta_\theta}\,\mathrm{d}\theta^2+g_{\varphi\varphi}\,\mathrm{d}\varphi_i^2 \\
&+\frac{a\sin^2\theta}{\chi^2\rho^2(r_i^2+a^2)}\left[\Delta_r\rho_i^2-(r^2+a^2)\Delta_\theta\left(r_i^2-r^2\right)\right]\,\mathrm{d}\varphi_i(\,\mathrm{d}u+\,\mathrm{d}v)\,.
\end{aligned}
\tag{2.4.14}
$$

Here, we defined $\rho_i^2=\rho^2(r_i,\theta)$, and $g_{\varphi\varphi}$ is the corresponding component of the metric in Boyer-Lindquist coordinates. Note that in contrast to the RNdS spacetime, (u,v,θ,φ_i) is not a double-null coordinate system.

The Kruskal-type coordinates are defined from u and v in region I in the same way as for the RNdS spacetime with κ_i replaced by the κ_i for the KdS spacetime (2.4.10): on I, they are related by

$$
U_+=-e^{-\kappa_+u}\,,\qquad V_+=e^{\kappa_+v}\,,\qquad U_c=e^{\kappa_cu}\,,\qquad V_c=-e^{-\kappa_cv}\,.
$$

The metric in these coordinates is given by [75, Eq. (66)]

$$
\begin{aligned}
g =& f_1^i\,(V_i\,\mathrm{d}U_i-U_i\,\mathrm{d}V_i)^2+f_2^i\left(V_i^2\,\mathrm{d}U_i^2+U_i^2\,\mathrm{d}V_i^2\right)+f_3^i\,\mathrm{d}U_i\,\mathrm{d}V_i \\
&+f_4^i\,\mathrm{d}\varphi_i\,(V_i\,\mathrm{d}U_i-U_i\,\mathrm{d}V_i)+\frac{\rho^2}{\Delta_\theta}\,\mathrm{d}\theta^2+g_{\varphi\varphi}\,\mathrm{d}\varphi_i^2\,,
\end{aligned}
\tag{2.4.15a}
$$

$$
f_1^i=\frac{a^2\sin^2\theta\Delta_\theta G_i^2}{4\kappa_i^2\chi^2\rho^2(r_i^2+a^2)^2}\,,\tag{2.4.15b}
$$

$$
f_2^i=\frac{a^2\sin^2\theta(r_i+r)G_i^2\Delta_r}{4\kappa_i^2\chi^2\rho^2(r_i^2+a^2)(r^2+a^2)(r-r_i)}\left(\frac{\rho_i^2}{r_i^2+a^2}+\frac{\rho^2}{r^2+a^2}\right)\,,\tag{2.4.15c}
$$

$$
f_3^i=\frac{G_i\Delta_r}{2\kappa_i^2\chi^2\rho^2(r-r_i)}\left(\frac{\rho_i^4}{(r_i^2+a^2)^2}+\frac{\rho^4}{(r^2+a^2)^2}\right)\,,\tag{2.4.15d}
$$

$$
f_4^i=s_i\frac{a\sin^2\theta G_i}{\kappa_i\chi^2\rho^2(r_i^2+a^2)}\left[(r^2+a^2)(r+r_i)\Delta_\theta+\frac{\rho_i^2\Delta_r}{r-r_i}\right]\,,\tag{2.4.15e}
$$

with $G_i=(r-r_i)/(U_iV_i)$ an analytic, non-vanishing function as long as r is not equal to any other horizon radius r_j, $j\neq i$ [75, Lemma 14]. Here, $s_i=1$ for $i\in\{-,c\}$, and $s_+=-1$.

Note that $(U_i,V_i,\theta,\varphi_i)$ is not a double-null coordinate system, but ∂_{U_i} is null on the hypersurface $\{V_i=0\}$ and vice versa.

In the coordinate system $(U_i,V_i,\theta,\varphi_i)$, the spacetime can be extended through the horizon at $r=r_i$ to the Kruskal blocks [75] $\mathcal{M}_-=\mathbb{R}_{U_-}\times\mathbb{R}_{V_-}\times\mathbb{S}^2_{(\theta,\varphi_-)}\backslash\{r=0,\,\theta=\frac{\pi}{2}\}$, $\mathcal{M}_+=\mathbb{R}_{U_+}\times\mathbb{R}_{V_+}\times\mathbb{S}^2_{(\theta,\varphi_+)}$, and $\mathcal{M}_c=\{U_cV_c<1\}\times\mathbb{S}^2_{(\theta,\varphi_c)}$.

The manifold $\tilde{\mathcal{M}}=\mathcal{M}_+\cup\mathcal{M}_c$, where the blocks I in $\mathcal{M}_+$ and $\mathcal{M}_c$ are identified with each other, will be referred to as the extended Kerr-de Sitter spacetime. $\mathcal{M}$ can be embedded into $\tilde{\mathcal{M}}$ by realizing that $\mathcal{M}\cap\mathcal{M}_+=\{V_+>0\}$ and $\mathcal{M}\cap\mathcal{M}_c=\{U_c>0\}$.

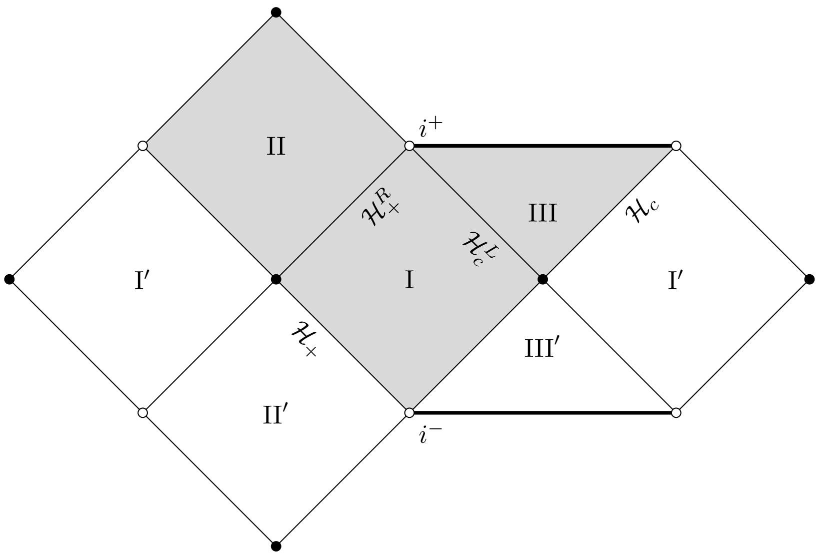


Figure 2.4: Penrose diagram of the $(\theta, \varphi) =$ const.-surface of the extended spacetime $\tilde{\mathcal{M}}$. The gray area corresponds to $\mathcal{M}$, the union of the blocks I, II and III. The prime indicates a reversal of the time orientation. The horizons $\mathcal{H}_+^R$ and $\mathcal{H}_c^L$ are part of $\mathcal{M}$, while the long horizons $\mathcal{H}_+$ and $\mathcal{H}_c$ are the boundary of $\mathcal{M}$ in $\tilde{\mathcal{M}}$.

The boundary of the embedding of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ consists of the horizons $\mathcal{H}_+ = \{V_+ = 0\}$ and $\mathcal{H}_c = \{U_c = 0\}$. $\mathcal{H}_+$ can be split into $\mathcal{H}_+^L = \mathcal{H}_+ \cap \{U_+ > 0\}$, $\mathcal{H}_+^- = \mathcal{H}_+ \cap \{U_+ < 0\}$ and the bifurcation surface $\mathcal{B}_+ = \{V_+ = U_+ = 0\}$, while $\mathcal{H}_c$ splits into $\mathcal{H}_c^R = \mathcal{H}_c \cap \{V_c > 0\}$, $\mathcal{H}_c^- = \mathcal{H}_c \cap \{V_c < 0\}$ and $\mathcal{B}_c = \{U_c = V_c = 0\}$. These surfaces will play an important role later on.

The Penrose-Carter diagram of the (extended) Kerr-de Sitter spacetime is shown in Fig. 2.4. A primed region has a reversed time orientation in comparison to the corresponding region in $\mathcal{M}$.

Before we move on, let us also briefly discuss null geodesics on the KdS spacetime.

There are three constants of motion: the energy

$$E = -g(\gamma', \partial_t)\,, \tag{2.4.16}$$

the angular momentum in the direction of the rotation axis

$$L = g(\gamma', \partial_\varphi)\,, \tag{2.4.17}$$

and the Carter constant K [77]. Here, γ' is the tangent vector of the geodesic γ.

Using the vector fields

$$V = (r^2 + a^2)\partial_t + a\partial_\varphi\,, \quad W = \partial_\varphi + a\sin^2\theta\partial_t\,,$$

the principal null directions of the Kerr-de Sitter spacetime are given by $\pm\partial_r + (\chi/\Delta_r)V$

[75, Prop. 1]. In addition, for any null geodesic γ with tangent vector γ', we set

$$\mathbb{P}(r) = -g(\gamma', V) = (r^2 + a^2)E - La \tag{2.4.18a}$$
$$\mathbb{D}(\theta) = -g(\gamma', W) = l - Ea\sin^2\theta\,. \tag{2.4.18b}$$

Since V is a future-pointing time-like vector field in region I, we conclude that $\mathbb{P}$ must be positive in region I.

With the help of the constants of motion, the geodesic equation can be separated and written as [75, 78, 79]

$$\rho^4\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \chi^2\mathbb{P}^2(r) - K\Delta_r \equiv R(r) \tag{2.4.19a}$$
$$\rho^4\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 = K\Delta_\theta - \chi^2\mathbb{D}^2(\theta) \equiv \Theta(\theta) \tag{2.4.19b}$$
$$\rho^2\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\chi^2(r^2+a^2)\mathbb{P}(r)}{\Delta_r} + \frac{\chi^2 a\mathbb{D}(\theta)}{\Delta_\theta} \tag{2.4.19c}$$
$$\rho^2\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{\chi^2 a\mathbb{P}(r)}{\Delta_r} + \frac{\chi^2\mathbb{D}(\theta)}{\sin^2\theta\Delta_\theta} \tag{2.4.19d}$$

for light-like geodesics. From the structure of these equations, one can infer that solutions to the geodesic equation can only exist when both $R(r) \geq 0$ and $\Theta(\theta) \geq 0$. This immediately entails $K \geq 0$. It is also simple to see that the turning points of the geodesics in r are given by the roots of $R(r)$,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0 \quad \Leftrightarrow \quad R(r) = 0\,,$$

while geodesics with $r = \text{const.}$ can exist at double roots of $R(r)$,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0 \text{ and } \frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = 0 \quad \Leftrightarrow \quad R(r) = 0 \text{ and } \partial_r R(r) = 0\,.$$

Moreover, the geodesics contained in the horizon can be extended through the bifurcation sphere and, in the corresponding Kruskal-type coordinates, take the form $(\tau, 0, \theta_0, \varphi_{i,0})$ for ingoing or $(0, \tau, \theta_0, \varphi_{i,0})$ for outgoing null geodesics [75, Sec. 4.4.2]. This generalizes [80, Lemma 3.4.10] from Kerr to Kerr-de Sitter.

2.5 Free scalar fields in black-hole spacetimes

In this section, we will return to the theory of the free real scalar quantum field and focus in particular on the case where the spacetime is a black-hole spacetime.

As discussed earlier, if we want to compute physical observables of the quantum field, we need to pick a state. If we also want to consider observables which are local and nonlinear in the quantum field, we have to choose a Hadamard state. Furthermore, we would like the state to adequately describe the physical situation under consideration.

For the real scalar field on a black-hole spacetime, there are a number of states which are physically well-motivated.

One common way to define these states [47] is to consider the local quantum field $\Phi(x)$. To do so, let us assume that we are working on a fixed globally hyperbolic spacetime $(\mathcal{M}, g)$. $\Phi(x)$ can be considered as an operator-valued distribution

$$\Phi : C_0^\infty(\mathcal{M}) \to \mathcal{A}(\mathcal{M}) \, , \quad \Phi(f) = \int_{\mathcal{M}} \Phi(x) f(x) \, \mathrm{d}vol_g(x) \, .$$

Assume we have a (generalized) basis $(\phi_i(x), \overline{\phi}_i(x))$ of the solution space $S(\mathcal{M})$ which satisfies

$$\begin{aligned} \sigma(\phi_i, \overline{\phi}_j) &= i\delta_{ij} \, , \\ \sigma(\phi_i, \phi_j) &= \sigma(\overline{\phi}_i, \overline{\phi}_j) = 0 \, , \end{aligned}$$

where δ_{ij} is a δ-distribution in the case of continuous indices and the Kronecker-Delta for discrete indices. Then, one may write

$$\Phi(x) = \sum_i \left[\phi_i(x) a_i + \overline{\phi}_i(x) a_i^* \right] \, ,$$

where the a_i are operators, and where the sum should be replaced by an integral for continuous indices. Next, define the state by $\pi_\omega(a_i)\Omega_\omega = 0$ for all i. This state corresponds to the vacuum state of the Fock space on which the a_i and a_i^* act as the usual annihilation- and creation operators, or the quasi-free state with two-point function

$$w(x, y) = \sum_i \phi_i(x) \overline{\phi}_i(y) \, . \tag{2.5.1}$$

Thus, in order to choose a state in this way, one needs to choose a generalized basis for $S(\mathcal{M})$ [47].

For concreteness, let us consider a Schwarzschild spacetime which can be obtained from the RNdS spacetime by setting $\Lambda = Q = 0$. More specifically, its metric takes the form

$$g = -(1 - 2M/r) \, \mathrm{d}t^2 + (1 - 2M/r)^{-1} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \, .$$

In this spacetime, there is only one bifurcate Killing horizon, the event horizon of the black hole at $r = 2M$, and the surface gravity of the horizon is $\kappa = (4M)^{-1}$. As on RNdS, one can also define $u = t - r_*$ and $v = t + r_*$ with tortoise coordinate

$$r_* = r - 2M \log |r - 2M| \, ,$$

and obtain the Kruskal coordinates U and V from u and v. The Kruskal coordinates allow to extend the metric through the event horizon of the black hole.

One possibility to uniquely identify an element $\phi \in S(\mathcal{M})$ is by giving initial data $\phi_0 = \phi|_\Sigma$ and $\phi_1 = n^a\partial_a\phi|_\Sigma$ on some Cauchy surface Σ. However, it has been shown in [43, Thm. 2.1] that one can also describe elements $\phi \in S(\mathcal{M})$ on this spacetime by their asymptotic data on the past event horizon $\mathcal{H} = \{V = 0\}$ and at past null infinity $\mathcal{I}^- = \{U = -\infty\}$.

On the Schwarzschild spacetime, there are three well-studied states that can be defined in this way.

The first one is the Boulware vacuum state [81]. It is of the form (2.5.1), with $i = (\omega, \ell, m, \lambda)$ running over $\mathbb{R}_+^* \times \mathbb{N} \times [-\ell, \ell] \times \{\text{in}, \text{up}\}$. The modes are of the form

$$\phi^\lambda_{\omega\ell m} = \mathcal{N}^\lambda_{\omega\ell m} Y_{\ell m}(\theta, \varphi) h^\lambda_{\omega\ell}(t, r)\,.$$

$Y_{\ell m}$ are the spherical harmonics, $\mathcal{N}^\lambda_{\omega\ell m}$ is a normalization constant, and $h^\lambda_{\omega\ell}(t, r)$ are determined by $h^{up}_{\omega\ell m}(t, r) \sim r_+^{-1}e^{-i\omega u}$ at $\mathcal{H}^- = \{V = 0, U < 0\}$ and ~ 0 at $\mathcal{I}^-$, while $rh^{in}_{\omega\ell m}(t, r) \sim e^{-i\omega v}$ at $\mathcal{I}^-$ and ~ 0 at $\mathcal{H}^-$.

The Boulware state is invariant under the automorphism induced by ∂_t, under time reversal $t \to -t$, and under automorphisms generated by the Killing fields of the $SO(3)$-symmetrie of the spacetime. It is a ground state with respect to ∂_t [82], and therefore a Hadamard state, in the exterior of the black hole [62]. Physically, it can be considered as the state that contains no particles incoming from $\mathcal{I}^-$ or outgoing to $\mathcal{I}^+ = \{V = \infty\}$ from the viewpoint of a static observer far away from the black hole [83]. A static observer is one that follows the orbits of the Killing field ∂_t. For $\Lambda = 0$ and in the limit $r \to \infty$, this agrees with a freely falling observer, i.e. an observer following a time-like geodesic.

One important shortcoming of this state is the fact that it is not Hadamard across the black-hole event horizon. Hence, if one is interested in the interior of the black hole, one has to choose a different state.

An alternative state for the free scalar field on the Schwarzschild spacetime is the Hartle-Hawking state [84, 85]. It can be constructed in the same way as the Boulware state, with the difference that now $h^{\text{in}}_{\omega\ell m} \sim r_+^{-1}e^{-i\omega U_+}$ on the whole past horizon $\mathcal{H}$ and ~ 0 on $\mathcal{I}^-$, while $rh^{\text{up}}_{\omega\ell m} \sim e^{-i\omega V_+}$ on $\mathcal{I}^-$ and ~ 0 on $\mathcal{H}$.

Similar to the Boulware state, the Hartle-Hawking state is invariant under the automorphisms generated by ∂_t and the $SO(3)$ Killing fields, as well as under time reversal. But instead of being a ground state, the Hartle-Hawking state restricted to the exterior of the black hole is a KMS-state at inverse temperature $\beta = 2\pi\kappa^{-1}$, the inverse of the surface gravity of the black hole times 2π [82]. In contrast to the Boulware state, it extends as a Hadamard state not only through the event horizon $\mathcal{H}^R = \{U = 0, V > 0\}$ of the black hole, but to the full Kruskal extension of the Schwarzschild spacetime. In fact, it is the only quasi-free state which has this property and is invariant under the automorphisms generated by ∂_t [61]. The Hartle-Hawking state has been rigorously constructed, and has been shown to be a Hadamard state not only on Schwarzschild spacetimes but on any spacetime with static [86] or even stationary [87] bifurcate Killing horizon.

Physically, the Hartle-Hawking state describes a black body, namely the black hole, at the Hawking temperature immersed in a thermal bath of the same temperature. Thus, this state does not seem to be the optimal description for the scalar field outside the black hole – we would expect that, from the perspective of a static observer far from the black hole, a physical state would not contain any particles incoming from $\mathcal{I}^-$, similar to the Boulware state.

This can be seen as a motivation for the third state, the Unruh state [18]. It is defined in the same way as the other two, with the asymptotic behaviour of the functions $h^\lambda_{\omega\ell m}$ chosen as $rh^{\mathrm{in}}_{\omega\ell m} \sim e^{-i\omega v}$ at $\mathcal{I}^-$ and ~ 0 at $\mathcal{H}$, while $h^{\mathrm{up}}_{\omega\ell m} \sim r_+^{-1}e^{-i\omega U_+}$ at $\mathcal{H}$ and ~ 0 at $\mathcal{I}^-$.

The Unruh state, in contrast to the other two, is no longer an equilibrium state, and is no longer invariant under the time-reversal symmetry. Nonetheless, it is still invariant under the automorphisms generated by ∂_t and the $SO(3)$ Killing fields. In addition, it extends as a Hadamard state through the event horizon $\mathcal{H}^R$ [43]. This is sufficient, since one is usually not interested in what happens at the past event horizon $\mathcal{H}$. The reason is that in the more realistic case of a gravitational collapse, the past horizon lies inside the collapsing body and is absent from the spacetime. Therefore, a breakdown of the Hadamard property at the past event horizon does not cause any problems.

The Unruh state can be interpreted physically as containing no incoming particles from $\mathcal{I}^-$ as viewed from a static observer far from the black hole, while the black hole radiates at inverse Hawking temperature $\beta = 2\pi\kappa^{-1}$. In fact, at $\mathcal{I}^+$ one finds an outgoing thermal flux of energy at the Hawking temperature which is in agreement with the expected Hawking radiation [1, 83, 88, 89]. For this reason, the Unruh state is considered a good description of the late-time behaviour of the scalar quantum field in the case of spherically symmetric gravitational collapse.

Analogues of the Unruh state have been applied for the computation of observables of the scalar field on the Reissner-Nordström (RNdS with $\Lambda = 0$) [90–92] and Kerr [17, 93–95] spacetime. Furthermore, its rigorous construction and the proof of its Hadamard property [43] have been extended to Schwarzschild-de Sitter [96] and Reissner-Nordström-de Sitter [16]. In this thesis, we will demonstrate that it can also be extended to Kerr-de Sitter spacetimes under certain conditions on the angular momentum a of the black hole and the cosmological constant Λ.

Finally, we discuss how the formulation of the Unruh two-point function used in the rigorous results [16, 43, 96] is, at least formally, related to the mode-sum expression. Let us take the case of RNdS [16] as an example. Note that the following computation is purely formal and serves a purely illustrative purpose. Therefore, we will not be careful when interchanging different integrals with each other or with infinite sums, or when taking limits.

On the one hand, in [16, Eq. (66)], the two-point function for the Unruh state is given as

$$
\begin{aligned}
w(f,h) = &-\frac{r_+^2}{\pi} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{E(f)|_{\mathcal{H}_+}(U_+,\Omega)E(h)|_{\mathcal{H}_+}(U'_+,\Omega)}{(U_+ - U'_+ - i0^+)^2}\,\mathrm{d}^2\Omega\,\mathrm{d}U_+\,\mathrm{d}U'_+ \\
&-\frac{r_c^2}{\pi} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{E(f)|_{\mathcal{H}_c}(V_c,\Omega)E(h)|_{\mathcal{H}_c}(V'_c,\Omega)}{(V_c - V'_c - i0^+)^2}\,\mathrm{d}^2\Omega\,\mathrm{d}V_c\,\mathrm{d}V'_c\,,
\end{aligned}
$$

with $\Omega = (\theta, \varphi)$ and $\mathrm{d}^2\Omega$ the usual volume element of the unit sphere $\mathbb{S}^2$.

On the other hand, using the mode-sum description introduced above, one can write the two-point function of the Unruh state on RNdS as in (2.5.1), with the modes given by a set of up-modes $\sim e^{-i\omega U_+}$ at $\mathcal{H}_+$, and a set of in-modes $\sim e^{-i\omega V_c}$ at $\mathcal{H}_c$ (and both vanishing at the other horizon respectively), resulting in the two-point function

$$
w(f,h) = \int\limits_{\mathcal{M}} \mathrm{d}vol_g(x) \int\limits_{\mathcal{M}} \mathrm{d}vol_g(y) \sum_{\lambda,\ell,m} \int\limits_0^\infty \mathrm{d}\omega \psi^\lambda_{\omega\ell m}(x) f(x) \overline{\psi^\lambda_{\omega\ell m}}(y) h(y)\,.
$$

Combining [57, Lemma A.1] with a change of integration and summation order, one can bring this into the form

$$
w(f,h) = \sum_{\lambda,\ell,m} \int\limits_0^\infty \mathrm{d}\omega \sigma(\psi^\lambda_{\omega\ell m}, E(f)) \sigma(\overline{\psi^\lambda_{\omega\ell m}}, E(h))\,.
$$

Since the symplectic form σ is independent of the Cauchy surface it is evaluated on, we evaluate it on $\mathcal{H}_+ \cup \mathcal{H}_c$ which strictly speaking is only the limit of a sequence of Cauchy surfaces. But this limit can be taken thanks to the decay of the solutions in $S(\mathcal{M})$ towards i^- proven in [39]. After a partial integration and plugging in the asymptotic behaviour of the modes we obtain

$$
\begin{aligned}
w(f,h) = &\frac{r_+^2}{\pi} \int\limits_0^\infty \mathrm{d}\omega \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \sum_{\ell,m} \left(\int\limits_{\mathbb{S}^2} \overline{Y_{\ell m}}(\Omega) E(f)|_{\mathcal{H}_+}(U_+,\Omega)\,\mathrm{d}^2\Omega \right) Y_{\ell m}(\Omega') \\
&\times E(h)|_{\mathcal{H}_+}(U'_+,\Omega') \omega e^{-i\omega(U_+ - U'_+)}\,\mathrm{d}U_+\,\mathrm{d}U'_+\,\mathrm{d}^2\Omega' \\
&+ \frac{r_c^2}{\pi} \int\limits_0^\infty \mathrm{d}\omega \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \sum_{\ell,m} \left(\int\limits_{\mathbb{S}^2} \overline{Y_{\ell m}}(\Omega) E(f)|_{\mathcal{H}_c}(V_c,\Omega)\,\mathrm{d}^2\Omega \right) Y_{\ell m}(\Omega') \\
&\times E(h)|_{\mathcal{H}_c}(V'_+,\Omega') \omega e^{-i\omega(V_c - V'_c)}\,\mathrm{d}V_c\,\mathrm{d}V'_c\,\mathrm{d}^2\Omega'
\end{aligned}
$$

One can now use the completeness relation of the spherical harmonics to identify the integral in the brackets with the coefficients of the Laplace series for $E(f)|_{\mathcal{H}_+}(U_+,\cdot)$ or $E(f)|_{\mathcal{H}_c}(V_c,\cdot)$, implying that after taking the sum over ℓ and m one is left with the corresponding function evaluated at Ω'. It then only remains to apply the fact that the distribution $\Theta(\omega)\omega$, where $\Theta(\omega)$ is the Heaviside distribution, is the Fourier transform of $-(x - i0^+)^{-2}$. In this way, one reaches the form given in [16, Eq. (66)].

This illustrates the connection between the different formulations of the Unruh state.

3 Computing the energy flux of the real scalar field

The main objective of this chapter is to check whether quantum effects can restore the validity of the sCC on RNdS spacetimes in cases when it is classically violated [40]. This requires the numerical computation of the leading divergence of the energy flux at the inner horizon of the RNdS black hole. We begin the chapter by recalling the connection between the energy flux and the sCC conjecture in Section 3.1, giving also a formula for the flux in terms of scattering coefficients. Since these coefficients need to be computed numerically, we continue by describing a semi-analytical method for the computation of these coefficients in Section 3.2, and we extend these methods to the charged scalar field on RNdS in Section 3.3. The numerical results obtained in this way for the real scalar field are presented in Section 3.4.

3.1 Strong cosmic censorship on RNdS

Before we start with the computation, let us give a motivation by recalling the results of [16]. In this paper, the authors considered a real, scalar field on a RNdS spacetime. They were particularly interested in the parameter region in which a study of the corresponding classical field shows a violation of sCC [40]. The main goal of their work was to examine whether quantum effects could change this result. To do so, they studied the behaviour of the expectation value of the energy flux $T_{V_-V_-}$ of the scalar field near the Cauchy horizon $\mathcal{H}_-^R$ in some state Ψ. The only requirement on Ψ was that it is Hadamard in the regions I, II and III.

The significance of $\Psi(:T_{V_-V_-}:)$, which we will denote as $\left\langle T_{V_-V_-}\right\rangle_\Psi$ in the following, can be understood by considering the backreaction of the quantum field on the spacetime via the semi-classical Einstein equations (1.0.1). If we assume that the leading correction to the metric will be spherically symmetric, we can make an ansatz for the corrected metric of the form

$$g = -e^{\sigma}\, \mathrm{d}u\, \mathrm{d}v + r^2\, \mathrm{d}\Omega^2\,, \tag{3.1.1}$$

where σ and r are unknown functions of u and v. Plugging this into the vv-component of the semi-classical Einstein equations (1.0.1), one obtains

$$\partial_v^2 r - \partial_v r \partial_v \sigma = -4\pi r \left\langle T_{vv}\right\rangle_\Psi\,. \tag{3.1.2}$$

By the tensor-transformation law, $\langle T_{vv} \rangle_\Psi$ is related to $\left\langle T_{V_-V_-} \right\rangle_\Psi$ by

$$\left\langle T_{V_-V_-} \right\rangle_\Psi = (\kappa_- V_-)^{-2} \langle T_{vv} \rangle_\Psi \ .$$

As a next step, we decompose σ and r into a background part r_0, σ_0 and a perturbation r_1, σ_1. The background parts will be chosen such that they correspond to the RNdS spacetime on which $\langle T_{vv} \rangle_\Psi$ was computed. In other words, we set

$$r_0 = r_{RNdS} \ , \qquad e^{\sigma_0} = \frac{f(r)}{2} \ .$$

We then assume that the backreaction is sufficiently weak so that we can expand (3.1.2) to first order in the perturbation, considering $\langle T_{vv} \rangle_\Psi$ as a first-order perturbation as well. Since we are interested in the results near the inner horizon, we assume that we may evaluate the background functions at the inner horizon. In this limit $\partial_v r_0 \to 0$, $r_0 \to r_-$ and $\partial_v \sigma_0 \to -\kappa_-$ as defined in (2.4.6). As a result, we obtain [92]

$$\partial_v^2 r + \kappa_- \partial_v r = -4\pi r_- \langle T_{vv} \rangle_\Psi \ . \tag{3.1.3}$$

This equation has the solution

$$\partial_v r = -\frac{4\pi r_-}{\kappa_-} \langle T_{vv} \rangle_\Psi \tag{3.1.4}$$

plus an exponentially decaying term. Thus, when the scalar field is considered as a matter model, the vv-component of the current decides whether nearby geodesics approaching the Cauchy horizon are accelerated towards or away from each other. See also [92] for a similar discussion with vanishing cosmological constant.

The results of [16] show that in the case where sCC is classically violated, the expectation value of $T_{V_-V_-}$ has a state-independent quadratic leading divergence,

$$\left\langle T_{V_-V_-} \right\rangle_\Psi \sim C V_-^{-2} \ .$$

The state dependence only enters through sub-leading terms, which behave not worse than the classical results [16, Prop. 5.1].

The prefactor C of the leading divergence is given by the expectation value in a reference state, which is chosen to be the Unruh state. To simplify computations, the expectation value is computed using state subtraction with respect to a comparison state,

$$C = \frac{1}{\kappa_-^2} \left(\langle T_{vv}(r_-) \rangle_{\mathrm{U}} - \langle T_{vv}(r_-) \rangle_{\mathrm{C}} \right) \equiv \frac{1}{\kappa_-^2} \langle T_{vv} \rangle_{\mathrm{U-C}} \ . \tag{3.1.5}$$

Here, the subscripts U and C signify that the expectation values are computed in the Unruh state and in a comparison state. The latter is constructed such that it is a Hadamard state in a neighbourhood of the Cauchy horizon. The evaluation at r_- indicates the Cauchy-horizon limit of the expectation values. Changing the reference- or the comparison state will only modify the sub-leading contributions to $\langle T_{V_-V_-}\rangle_\Psi$.

The constant C can be calculated in terms of a mode-sum formula. This formula contains the scattering coefficients of the Boulware-type solutions to the Klein-Gordon equation on the RNdS spacetime. More concretely, it is given by the formula

$$\langle T_{vv}\rangle_{\mathrm{U-C}} = \sum_\ell \frac{2\ell+1}{16\pi^2 r_-^2}\int_0^\infty \mathrm{d}\omega\,\omega n_\ell(\omega)\,, \tag{3.1.6a}$$

$$\begin{aligned} n_\ell(\omega) &= \coth\left(\pi\tfrac{\omega}{\kappa_c}\right)\left|\mathcal{T}^{\mathrm{I}}_{\omega\ell}\right|^2\left|\mathcal{T}^{\mathrm{II}}_{\omega\ell}\right|^2 + \coth\left(\pi\tfrac{\omega}{\kappa_+}\right)\left(\left|\mathcal{R}^{\mathrm{I}}_{\omega\ell}\right|^2\left|\mathcal{T}^{\mathrm{II}}_{\omega\ell}\right|^2 + \left|\mathcal{R}^{\mathrm{II}}_{\omega\ell}\right|^2\right) \\ &+ 2\,\mathrm{csch}\left(\pi\tfrac{\omega}{\kappa_+}\right)\mathrm{Re}\left[\overline{\mathcal{R}}^{\mathrm{I}}_{\omega\ell}\mathcal{T}^{\mathrm{II}}_{\omega\ell}\mathcal{R}^{\mathrm{II}}_{\omega\ell}\right] - \coth\left(\pi\tfrac{\omega}{\kappa_-}\right)\,. \end{aligned} \tag{3.1.6b}$$

The scattering coefficients $\mathcal{R}^N_{\omega\ell}$ and $\mathcal{T}^N_{\omega\ell}$ with $N \in \{\mathrm{I},\mathrm{II}\}$ will be introduced below in (3.2.4) and (3.2.5).

Unfortunately, these coefficients are not known analytically, but must be computed numerically. In this chapter, we will therefore describe a formalism that allows the computation of these scattering coefficients, and subsequently a computation of C, for general masses of the scalar field. We will also emphasize how this procedure can accommodate for a non-vanishing scalar-field charge.

The question we want to answer with the numerical computation is whether C is generically, i.e. for a broad range of both spacetime and scalar-field parameters, non-zero. This would indicate that quantum effects can restore sCC if it is classically violated.

3.2 The Klein-Gordon equation on RNdS

In this section, we will recapitulate how the massive wave equation on RNdS can be reduced to an ordinary differential equation (ODE) for the radial function. We will present two different ways to write the radial ODE. One of them will allow us to reformulate the radial ODE in a form that reduces to the Heun equation in the case of conformal coupling. Most of the content of this section has been published in [97]. For this paper, I derived the analytical reformulation of the radial ODE for general masses, implemented its numerical solution as described below, and produced the numerical results with the help of Dr. Zahn and Prof. Hollands.

Let us start with the massive Klein-Gordon equation (2.2.1) on RNdS. In order to solve this equation, one can write the equation in the usual Boyer-Lindquist coordinates and make the ansatz

$$\phi_{\omega\ell m} = \mathcal{N}_{\omega\ell m}e^{-i\omega t}Y_{\ell m}(\theta,\varphi)R_{\omega\ell m}(r)\,, \tag{3.2.1}$$

where $Y_{\ell m}(\theta,\varphi)$ are the spherical harmonics and $\mathcal{N}_{\omega\ell m}$ is a normalization constant. Plugging this ansatz into (2.2.1), one is left with an ODE for $R_{\omega\ell m}(r)$,

$$\left[\partial_r \Delta_r \partial_r + \frac{r^4\omega^2}{\Delta_r} - \mu^2 r^2 - \ell(\ell+1)\right] R_{\omega\ell m}(r) = 0\,. \tag{3.2.2}$$

We will refer to (3.2.2) as the radial equation. There are two different, useful ways to rewrite this ODE.

The first one is particularly useful to identify the Boulware-type modes by considering the asymptotic behaviour of the solutions as $r \to r_j$, $j \in \{+, c\}$. Let us define

$$R_{\omega\ell m} = r^{-1} F_{\omega\ell}\,.$$

We can drop the m-subscript, since the radial function does not depend on it. In addition, let us change from the radial coordinate r to the tortoise coordinate r_* defined in (2.4.5). Then (3.2.2) can be written as

$$\left[\partial_{r_*}^2 - V_\ell(r) + \omega^2\right] F_{\omega\ell}(r_*) = 0\,, \tag{3.2.3a}$$

$$V_\ell(r) = f(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{\partial_r f(r)}{r} + \mu^2\right)\,, \tag{3.2.3b}$$

thus taking the form of a Schrödinger-type equation. One can show that $V_\ell \to 0$ exponentially fast in r_* as $r_* \to \pm\infty$: Let us consider $V_\ell(r)$ in the interval $[r_- + \epsilon, r_c - \epsilon]$ around r_+ for some $\epsilon > 0$. The other cases can be handled analogously. Then we notice that the term in the brackets in (3.2.3b) is a polynomial in r divided by r^3. Hence, this term is bounded on $[r_- + \epsilon, r_c + \epsilon]$ by some constant. Similarly, the prefactor $f(r)$ can be bounded by some constant times $|r - r_+|$. Taking into account the definition of r_* in (2.4.5), one finds

$$|r - r_+| \leq e^{2\kappa_+ r_*} e^{\sum\limits_{j\neq +} \frac{s_j \kappa_+}{\kappa_j} \log\epsilon}$$

with $s_j = 1$ for $j \in \{-, c\}$, and $s_o = -1$. This shows the exponential decay of $V_\ell(r)$ towards r_+ as a function of r_*.

Hence, the radial equation takes the form of a one-dimensional scattering problem with a localized potential. One expects to find solutions with an asymptotic behaviour of the form

$$F^{\mathrm{I}}_{\omega\ell}(r_*) \to \begin{cases} e^{i\omega r_*} + \mathcal{R}^{\mathrm{I}}_{\omega\ell} e^{-i\omega r_*} & r_* \to -\infty \\ \mathcal{T}^{\mathrm{I}}_{\omega\ell} e^{i\omega r_*} & r_* \to \infty \end{cases} \tag{3.2.4}$$

in the exterior region of the black hole, $r \in (r_+, r_c)$ and

$$F^{\mathrm{II}}_{\omega\ell}(r_*) \to \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \\ \mathcal{T}^{\mathrm{II}}_{\omega\ell} e^{-i\omega r_*} + \mathcal{R}^{\mathrm{II}}_{\omega\ell} e^{i\omega r_*} & r_* \to \infty \end{cases} \tag{3.2.5}$$

in the interior region $r \in (r_-, r_+)$. These solutions are exactly the ones we will be looking for. Let us also note that, since the radial differential equation does not contain a first-order derivative term, the Wronskian

$$W[F(r_*), G(r_*)] = F(r_*)\partial_{r_*}G(r_*) - G(r_*)\partial_{r_*}F(r_*)$$

is independent of r_* for any two solutions $F(r_*)$, $G(r_*)$ of the radial equation. Computing the Wronskian between the modes in (3.2.4) or (3.2.5) and their complex conjugates in the two asymptotic limits then results in the identities

$$\left|\mathcal{R}^{\mathrm{I}}_{\omega\ell}\right|^2 + \left|\mathcal{T}^{\mathrm{I}}_{\omega\ell}\right|^2 = 1\,, \tag{3.2.6a}$$

$$\left|\mathcal{T}^{\mathrm{II}}_{\omega\ell}\right|^2 - \left|\mathcal{R}^{\mathrm{II}}_{\omega\ell}\right|^2 = 1\,. \tag{3.2.6b}$$

In particular, the coefficients $\mathcal{R}^{\mathrm{I}}_{\omega\ell}$ and $\mathcal{T}^{\mathrm{I}}_{\omega\ell}$ behave like expected for reflection- and transmission coefficients. The behaviour of $\mathcal{R}^{\mathrm{II}}_{\omega\ell}$ and $\mathcal{T}^{\mathrm{II}}_{\omega\ell}$ is different, since r_* is a time-like coordinate in the black hole interior.

The second way of rewriting the radial equation will be the key to the numerical computation of the modes defined in (3.2.4) and (3.2.5). To rewrite the radial equation in the second way, we follow [98] and introduce the dimensionless variable

$$x = x_\infty \frac{r - r_+}{r - r_o} = \frac{r_- - r_o}{r_- - r_+}\frac{r - r_+}{r - r_o}\,, \tag{3.2.7}$$

where $x_\infty = \lim\limits_{r\to\infty} x$. Writing $x_c = x(r_c)$, we note that this definition entails

$$1 - x = \frac{r_+ - r_o}{r_+ - r_-}\frac{r - r_-}{r - r_o}\,, \qquad \frac{x - x_c}{1 - x_c} = \frac{r_- - r_o}{r_- - r_c}\frac{r - r_c}{r - r_o}\,. \tag{3.2.8}$$

Let us also introduce the coefficients a_i, $i \in \{o, -, +, c\}$, given by

$$a_i = \operatorname{sign}\left(\partial_r \Delta_r|_{r_i}\right)\frac{i\omega}{2\kappa_i}. \tag{3.2.9}$$

These coefficients satisfy the relation $\sum_i a_i = 0$, see [98]. We can make an ansatz for the radial function $R_{\omega\ell}$ of the form

$$R_{\omega\ell}(x) = |x|^{a_+}\,|1 - x|^{a_-}\left|\frac{x - x_c}{1 - x_c}\right|^{a_c}\frac{x - x_\infty}{1 - x_\infty}h(x)\,. \tag{3.2.10}$$

Using the above relations and the definition of r_* in (2.4.5), we note that we can identify

$$|x|^{a_+}|1 - x|^{a_-}\left|\frac{x - x_c}{1 - x_c}\right|^{a_c} = e^{i\omega D}e^{i\omega r_*}\,, \tag{3.2.11}$$

where the constant D is given by

$$D = \frac{1}{2\kappa_+} \ln|x_\infty| - \frac{1}{2\kappa_-} \ln\left(\frac{r_+ - r_o}{r_+ - r_-}\right) - \frac{1}{2\kappa_c} \ln\left(\frac{r_- - r_o}{r_c - r_-}\right) . \tag{3.2.12}$$

Moreover,

$$\frac{x - x_\infty}{1 - x_\infty} = \frac{r_- - r_o}{r - r_o}$$

is a strictly positive, smooth, bounded function of r on the interval $[r_-, r_c]$ we are interested in.

Next, let us turn to the unknown function $h(x)$. Applying the results obtained so far, one can follow [98] and derive the differential equation for $h(x)$. It reads

$$\begin{aligned} &\partial_x^2 h(x) + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x - x_c}\right] \partial_x h(x) && (3.2.13)\\ &+ \left[\frac{\sigma_+\sigma_- x - q}{x(x-1)(x-x_c)} + \frac{\Delta_1 x - \Delta_2}{x(x-1)(x-x_c)(x-x_\infty)^2}\right] h(x) = 0 . \end{aligned}$$

The constants in the first order differential term are given by

$$\gamma = 1 + 2a_+ , \qquad \delta = 1 + 2a_- , \qquad \epsilon = 1 + 2a_c . \tag{3.2.14}$$

Together with the coefficients $\sigma_+ = 1 - 2a_o$ and $\sigma_- = 1$, they satisfy the relation

$$\gamma + \delta + \epsilon = \sigma_+ + \sigma_- + 1 , \tag{3.2.15}$$

as would be the case for the corresponding coefficients in a Heun differential equation [99]. The parameter q can be written as (compare [98], note however the modification due to the more general mass μ^2)

$$q = x_\infty + [(1 + x_c)a_+ + x_c a_- + a_c] + \frac{3\left[\ell(\ell+1) + \mu^2 r_o^2\right]}{\Lambda(r_+ - r_-)(r_c - r_o)} . \tag{3.2.16}$$

The last two parameters

$$\Delta_1 = \left(2 - \frac{3\mu^2}{\Lambda}\right) \frac{2r_o(r_o - r_+)}{(r_+ - r_-)(r_c - r_o)} , \tag{3.2.17a}$$

$$\Delta_2 = \left(2 - \frac{3\mu^2}{\Lambda}\right) \frac{x_\infty^2\left(r_o^2 - r_+^2\right)}{(r_+ - r_-)(r_c - r_o)} , \tag{3.2.17b}$$

are the only ones that do not fit into the frame of the Heun differential equation. However, they both vanish when $\mu^2 = 2\Lambda/3$, so that the differential equation for $h(x)$ reduces to a Heun equation. As we have mentioned in Chapter 2, the Ricci scalar of this spacetime is a constant, and hence a non-zero mass and a non-miminal coupling have the same effect on the equation of motion. Throughout this section, we will assume minimal coupling. But since the equations of motion with mass $\mu^2 = 2\Lambda/3$ are the same as those for a massless, conformally coupled scalar field, we will refer to this case as "conformal coupling" to stress the distinction of this parameter choice.

In order to solve for $h(x)$ in the case of general mass, we make a power-series ansatz for $h(x)$,

$$h(x) = \sum_{n=-\infty}^{\infty} h_n x^n \,.$$

Since we have shown that our ansatz already contains the oscillatory behaviour we are looking for, we will look for solutions $h(x)$ that are regular at $x = 0$. Further, we take $h(0) = 1$. This amounts to setting $h_n = 0$ for $n < 0$ and $h_0 = 1$.

Plugging the ansatz into the differential equation for $h(x)$ then yields a 5-term recurrence relation for the coefficients h_n,

$$\begin{aligned} &x_\infty^2 a(n+2)h_{n+2} - [x_\infty^2 b(n+1) + 2x_\infty a(n+1) + \Delta_2]h_{n+1} \\ &+ [x_\infty^2 c(n) + 2x_\infty b(n) + a(n) + \Delta_1]h_n - [2x_\infty c(n-1) + b(n-1)]h_{n-1} \\ &+ c(n-2)h_{n-2} = 0\,, \end{aligned} \tag{3.2.18}$$

with

$$a(n) = x_c n(n-1+\gamma) \tag{3.2.19a}$$

$$b(n) = n\left[(x_c+1)(n-1+\gamma) + x_c\delta + \epsilon\right] + q \tag{3.2.19b}$$

$$c(n) = (n+\sigma_+)(n+\sigma_-)\,. \tag{3.2.19c}$$

By reorganizing (3.2.18) as

$$\begin{aligned} &x_\infty^2\left[a(n+2)h_{n+2} - b(n+1)h_{n+1} + c(n)h_n\right] \\ &- 2x_\infty\left[a(n+1)h_{n+1} - b(n)h_n + c(n-1)h_{n-1}\right] \\ &+ a(n)h_n - b(n-1)h_{n-1} + c(n-2)h_{n-2} = -\Delta_2 h_{n+1} - \Delta_1 h_n\,, \end{aligned} \tag{3.2.20}$$

one can immediately see that this reduces to the known three-term recurrence relation of the Heun equation [99],

$$a(n+1)h_{n+1} - b(n)h_n + c(n-1)h_{n-1} = 0\,,$$

when $\Delta_1 = \Delta_2 = 0$, i.e. $\mu^2 = 2\Lambda/3$. Hence, we have reduced the radial equation to a 5-term recurrence relation, which can be evaluated numerically to arbitrary order.

However, before we get to the numerical implementation, let us estimate the radius of convergence of the power series in our ansatz. To this end, let us note that for $n \geq 0$, $x_\infty^2 a(n+2) \neq 0$. Hence, we can divide (3.2.18) by $x_\infty^2 a(n+2)$, and shift the label from

n to $n-2$. This brings the equation into the form

$$h_{n+4} + \sum_{i=0}^{3} \alpha_i h_{n+i} = 0\,,$$

where α_i are the coefficients of h_{n+i} divided by $x_\infty^2 a(n+4)$. In the limit of $n \to \infty$, the coefficients α_i approach the finite limits

$$\beta_0 \equiv \lim_{n\to\infty} \alpha_0 = \frac{1}{x_\infty^2 x_c^2} \tag{3.2.21a}$$

$$\beta_1 \equiv \lim_{n\to\infty} \alpha_1 = -\frac{2x_\infty + x_c + 1}{x_\infty^2 x_c} \tag{3.2.21b}$$

$$\beta_2 \equiv \lim_{n\to\infty} \alpha_2 = \frac{x_\infty^2 + 2x_\infty(x_c+1) + x_c}{x_\infty^2 x_C} \tag{3.2.21c}$$

$$\beta_3 \equiv \lim_{n\to\infty} \alpha_3 = -\frac{x_\infty^2(x_c+1) + 2x_\infty x_c}{x_\infty^2 x_c}\,. \tag{3.2.21d}$$

The roots of the characteristic polynomial $\lambda^4 + \sum_{i=0}^{3} \beta_i \lambda^i$ are given by $\lambda_1 = 1$, $\lambda_2 = 1/x_c$ and $\lambda_3 = 1/x_\infty$, and λ_3 is a double root. Since $|x_\infty| > 1$ by construction, λ_3 cannot be the root of maximal absolute value. By the definition of [100], this recurrence relation is therefore "maxmod-generic". Hence, by [100, Lemma 3], combined with [101, Thm. 1], the limit $\lim_{n\to\infty} \frac{h_{n+1}}{h_n}$ exists and is given by one of the λ_i. In particular, it is bounded by $\max(1, |x_c|^{-1})$. Thus, our ansatz for $h(x)$ can, for sufficiently large n, be estimated by a geometric series, and we find that the radius of convergence for $h(x)$ is generally given by $\min(1, |x_c|)$.

As a result, the solution obtained in this way is only valid in a neigbourhood of the event horizon. In order to find solutions in neighbourhoods of the other two horizons, we can apply two of the coordinate-change transformations mapping Heun equations to Heun equations [99], $x \to y = 1-x$ and $x \to z = (x_c - x)/(x_c - 1)$. Under these coordinate changes, the form of the differential equation for h remains invariant, but the parameters change according to

$$\gamma_y = \delta_x\,, \qquad \delta_y = \gamma_x\,, \tag{3.2.22a}$$

$$\epsilon_y = \epsilon_x\,, \qquad \sigma_{+,y} = \sigma_{+,x}\,, \tag{3.2.22b}$$

$$\sigma_{-,y} = \sigma_{-,x}\,, \qquad q_y = \sigma_{+,x}\sigma_{-,x} - q_x\,, \tag{3.2.22c}$$

$$\Delta_{1,y} = \Delta_{1,x}\,, \qquad \Delta_{2,y} = \Delta_{1,x} - \Delta_{2,x}\,, \tag{3.2.22d}$$

$$y_c = 1 - x_c\,, \qquad y_\infty = 1 - x_\infty \tag{3.2.22e}$$

and

$$\gamma_z = \epsilon_x\,, \qquad \delta_z = \delta_x\,, \tag{3.2.23a}$$

$$\epsilon_z = \gamma_x\,, \qquad \sigma_{+,z} = \sigma_{+,x}\,, \tag{3.2.23b}$$

$$\sigma_{-,z} = \sigma_{-,x}\,, \qquad q_z = \frac{x_c\sigma_{+,x}\sigma_{-,x} - q_x}{x_c - 1}\,, \tag{3.2.23c}$$

$$\Delta_{1,z} = \frac{\Delta_{1,x}}{(x_c-1)^2}\,, \qquad \Delta_{2,z} = \frac{x_c\Delta_{1,x} - \Delta_{2,x}}{(x_c-1)^3}\,, \tag{3.2.23d}$$

$$z_c = \frac{x_c}{x_c - 1}\,, \qquad z_\infty = \frac{x_c - x_\infty}{x_c - 1}\,. \tag{3.2.23e}$$

Note that in contrast to x_c and y_c, which are the values of x and y at $r = r_c$, see (3.2.7) and (3.2.8), z_c is the value of z at $r = r_+$.

Repeating the analysis of the resulting recurrence relation and its large-n limit, we find that the roots of the characteristic polynomials are again given by 1, $(y_c)^{-1}$, and y_∞^{-1}, and 1, z_c^{-1}, and z_∞^{-1}. y_∞^{-1} and z_∞^{-1} are the double roots. One can check that these cannot be the roots with the largest absolute value. Hence, by the same arguments as before, the radius of convergence of the solution near the inner horizon in terms of the coordinate y is given by $\min(1, |y_c|)$, while the radius of convergence of the solution near the cosmological horizon in terms of z is given by $\min(1, |z_c|)$.

In this way, we can numerically obtain solutions $h_i(x)$ of the corresponding Heun equation in a neighbourhood of each of the horizons r_i. We can then plug the solution $h_i(x)$ into the ansatz (3.2.10), and normalize by adding a factor $N_i = e^{-i\omega D} r_i^{-1} \frac{1-x_\infty}{x_i - x_\infty}$, with $x_i = x(r_i)$. Thus, we obtain solutions to the radial equation which are defined in the neighbourhood of one of the horizons r_i and behave as $\sim r_i^{-1} e^{i\omega r_*}$ for $r \to r_i$. We will call these solutions $R^i_{\omega\ell}(r)$.

Once we have these solutions, we can obtain the scattering coefficients in (3.2.4) and (3.2.5) as follows: We express the functions $r^{-1}F^N_{\omega\ell}$, $N \in \{\mathrm{I}, \mathrm{II}\}$, in terms of the $R^i_{\omega\ell}$ and the scattering coefficients. In this way, one obtains two expressions for each $r^{-1}F^N_{\omega\ell}$ from the two asymptotic limits of these functions. Comparing the two expressions as well as their first derivatives in a region where they are both well-defined then yields equations for the scattering coefficients. If one solves the equations for the scattering coefficients, one finds that they are given by

$$\mathcal{R}^{\mathrm{I}} = \frac{W[R^+, R^c]}{W[R^c, \overline{R}^+]}\,, \qquad \mathcal{T}^{\mathrm{I}} = \frac{W[R^+, \overline{R}^+]}{W[R^c, \overline{R}^+]}\,, \tag{3.2.24a}$$

$$\mathcal{R}^{\mathrm{II}} = \frac{W[\overline{R}^-, \overline{R}^+]}{W[R^-, R^-]}\,, \qquad \mathcal{T}^{\mathrm{II}} = \frac{W[\overline{R}^+, R^-]}{W[\overline{R}^-, R^-]}\,. \tag{3.2.24b}$$

Here, we have dropped the $\omega\ell$-subscripts for brevity.

Thus, this formalism allows us to numerically compute the scattering coefficients even in the case of a general scalar-field mass when other methods based on the Heun equation such as the MST-method [102], which was applied to asymptotically-de Sitter spacetimes in [98, 103], are not applicable. For the Heun-equation case, recent results indicate that it is even possible to obtain analytical expressions for the scattering coefficients by using methods based on 2-dimensional conformal quantum field theory [104, 105]. Nonetheless, for our purposes the applicability to a general scalar-field mass is a decisive advantage of the numerical formalism presented here.

3.3 Extension to the charged scalar field on RNdS

In this section, we will demonstrate how the method described in the previous section can be extended to charged scalar fields on RNdS.

The charged scalar field satisfies the equation

$$\left[D_\nu D^\nu - \mu^2\right]\phi = 0\,, \tag{3.3.1}$$

where $D_\nu = \nabla_\nu - iqA_\nu$ is the (gauge-)covariant derivative, and q is the charge of the scalar field. Note that the Klein-Gordon operator is no longer real. Hence, if ϕ solves (3.3.1), then $\bar{\phi}$ solves the complex conjugate equation, which is not the same.

Making an ansatz of the form (3.2.1), and changing to r_* and $F_{\omega\ell}(r_*)$, the radial equation takes the form

$$\left[\partial_{r_*}^2 - V_\ell(r) + \left(\omega - \frac{qQ}{r}\right)^2\right] F_{\omega\ell}(r_*) = 0\,, \tag{3.3.2}$$

with $V_\ell(r)$ of the same form as in (3.2.3b). For the charged scalar field, we have a gauge freedom of the form

$$A_\nu(x) \to A_\nu(x) + \partial_\nu\chi(x)\,, \qquad\qquad \phi(x) \to e^{iq\chi(x)}\phi(x)\,, \tag{3.3.3}$$

with χ any smooth function. One can see that transformations of the form $\chi = ct$, with c some constant, leave the radial equation invariant, while they change the t-dependence of ϕ to $e^{-i(\omega-c)t}$, and change A to $(-Q/r+c)\,\mathrm{d}t$. For the rest of this section, we will choose $c = Q/r_+$, so that $A_t = Q\left(r_+^{-1} - r^{-1}\right)$ vanishes at the event horizon. We will also define a new frequency $\tilde{\omega} = \omega - c$, and drop the tilde from here on out. The radial equation then reads

$$\left[\partial_{r_*}^2 - V_\ell(r) + \left(\omega - \frac{qQ}{r} + \frac{qQ}{r_+}\right)^2\right] F_{\omega\ell}(r_*) = 0\,, \tag{3.3.4}$$

and the solutions we are looking for now take the form

$$F^{\mathrm{I}}_{\omega\ell}(r_*) \to \begin{cases} e^{i\omega r_*} + \mathcal{R}^{\mathrm{I}}_{\omega\ell} e^{-i\omega r_*} & r_* \to -\infty \\ \mathcal{T}^{\mathrm{I}}_{\omega\ell} e^{i(\omega+\omega_{\mathrm{I}})r_*} & r_* \to \infty \end{cases} \tag{3.3.5}$$

in the exterior region of the black hole, $r \in (r_+, r_c)$, or

$$F^{\mathrm{II}}_{\omega\ell}(r_*) \to \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \\ \mathcal{T}^{\mathrm{II}}_{\omega\ell} e^{-i(\omega-\omega_{\mathrm{II}})r_*} + \mathcal{R}^{\mathrm{II}}_{\omega\ell} e^{i(\omega-\omega_{\mathrm{II}})r_*} & r_* \to \infty \end{cases} \tag{3.3.6}$$

in the interior region $r \in (r_-, r_+)$. Here, we have defined $\omega_{\mathrm{I}} = qQ(r_+^{-1} - r_c^{-1})$ and $\omega_{\mathrm{II}} = qQ(r_-^{-1} - r_+^{-1})$. Using the Wronskian, one can show that the coefficients satisfy

$$|\mathcal{R}^{\mathrm{I}}_{\omega\ell}|^2 + \frac{\omega + \omega_{\mathrm{I}}}{\omega}|\mathcal{T}^{\mathrm{I}}_{\omega\ell}|^2 = 1 \tag{3.3.7a}$$

$$\left|\mathcal{T}^{\mathrm{II}}_{\omega\ell}\right|^2 - \left|\mathcal{R}^{\mathrm{II}}_{\omega\ell}\right|^2 = \frac{\omega}{\omega - \omega_{\mathrm{II}}} \,. \tag{3.3.7b}$$

Next, we change to the radial coordinate x, see (3.2.7), and make the ansatz (3.2.10), but with a_i given by

$$a_i = \mathrm{sign}\left(\partial_r \Delta_r|_{r_i}\right) i \frac{\omega + qA_t(r_i)}{2\kappa_i} \,. \tag{3.3.8}$$

These coefficients still satisfy $\sum_i a_i = 0$ [98]. Then, up to functions which are smooth in a neighbourhood of the corresponding horizon, the ansatz divided by $h(x)$ behaves as $e^{i\omega r_*}$ as $r \to r_+$, as $e^{i(\omega+\omega_{\mathrm{I}})r_*}$ as $r \to r_c$ and as $e^{i(\omega-\omega_{\mathrm{II}})r_*}$ as $r \to r_-$. This can be seen by a computation as in (3.2.11). Note that due to the modification of the a_i, the D in (3.2.11) is a function of r which is smooth at the horizon under consideration for the charged scalar. Nonetheless, we can again look for solutions with regular $h(x)$ at the corresponding horizon.

The equation for $h(x)$ is of the same form as for the real scalar field, with the a_i adapted accordingly. Thus, it can be solved in the same way as before adopting a power-series ansatz, which yields a recurrence relations for the coefficients of the power series of the same form and with the same large-n limit as in (3.2.18). Consequently, the analysis of the radius of convergence for the power-series solutions carries over to the charged scalar field. Once the local solutions obtained this way are properly normalized, we obtain the scattering coefficients analogously to the real scalar field.

3.4 The energy flux at the Cauchy horizon

In this section, we present numerical results for the difference of the expectation value of the energy flux of the real scalar quantum field between the Unruh- and comparison state evaluated at the Cauchy horizon of a RNdS spacetime, $\langle T_{vv}\rangle_{\mathrm{U-C}}$. As discussed in Section 3.1, sCC on the RNdS spacetime will be restored by quantum effects if this flux is generically non-vanishing. Moreover, the flux will influence the fate of an observer approaching the Cauchy horizon. To obtain the results we apply the formalism introduced in Section 3.2.

Before we get to the results themselves, let us outline the numerical implementation. We use Mathematica 12.1. We start by picking a data set $(\Lambda, M, Q, \mu^2, \omega, \ell)$ with $M = 1$, consisting of a choice of spacetime parameters (Λ, M, Q), a mass μ^2 for the scalar field, and a set of parameters (ω, ℓ) in $\mathbb{R}_+ \times \mathbb{N}$[1]. As an alternative set of spacetime parameters

[1]We avoid setting ω to exactly zero due to a divergence of the scattering coefficients $\mathcal{R}^{\mathrm{II}}$ and $\mathcal{T}^{\mathrm{II}}$ at $\omega = 0$, see the discussion in [16] and [106].

one can also use the three horizon radii, leading to the data set $(r_-, r_+, r_c, \mu^2, \omega, \ell)$. For the chosen data set we solve (3.2.18) with initial conditions $h_0 = 1$ and $h_n = 0$ for $n < 0$ up to some large value of n for each of the three horizons. In the special case of a scalar field with mass $\mu^2 = 2\Lambda/3$, which satisfies the same equations of motions as the massless, conformally coupled scalar field, we can instead use the functions "HeunG" which have been implemented in Mathematica 12. This is more efficient than solving the recurrence relation in terms of RAM allocation and computation time. The application of the function "HeunG" to the wave equation on black-hole spacetimes has been introduced in [107]. From this approximation of $h_i(x)$, we can then obtain an approximation for the normalized solutions $R^i_{\omega\ell}$ as described above.

We then evaluate the normalized solutions in their overlap region. For the cases of small Λ and large Q considered here, one has $|x_c| > 1/2$. Therefore, we choose the evaluation point $x = 1/2$ in the overlap of $R^+_{\omega\ell}$ and $R^-_{\omega\ell}$, and $x = -1/2$ in the overlap of $R^c_{\omega\ell}$ and $R^+_{\omega\ell}$. For cases where $|x_c| \gg 1$, it is more convenient to choose $x = -0.8$ as the evaluation point in the overlap of $R^c_{\omega\ell}$ and $R^+_{\omega\ell}$.

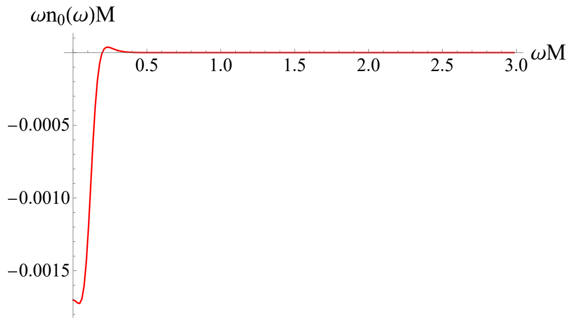
In order to test our approximation and estimate the error due to the cut-off of the power series in $h_i(x)$, we calculate the relative contribution of the last term in the expansion of h_i to R^i at the evaluation point. We find that about 5000 terms are sufficient to keep the error below $\mathcal{O}(10^{-15})$. The numerical precision for the evaluation of the recurrence relation is chosen such that numerical errors remain of order $\mathcal{O}(10^{-40})$. When the "HeunG"-functions are utilized, the numerical precision for their evaluation is chosen of the same order or higher.

In the next step, we plug $R^i_{\omega\ell}$ into (3.2.24) to compute the scattering coefficients. Before we use the scattering coefficients to compute the energy flux of the scalar field at the Cauchy horizon according to (3.1.6a), we can perform a number of consistency tests. First of all, we make sure that the scattering coefficients satisfy (3.2.6) up to errors of order 10^{-15}. Second, for some example parameters, we compared our results to results of numerical integration of the radial equation. The results are in agreement to the 0.02%-level for $\omega r_+ \sim 1$, and their agreement is even better for smaller ω. Subsequently, we check that also the second derivatives of the different expressions for $r^{-1}F^N_{\omega\ell}$ agree at the evaluation point. We find the error to be of the order $\mathcal{O}(10^{-15})$, the same order as our estimate for the cut-off error.

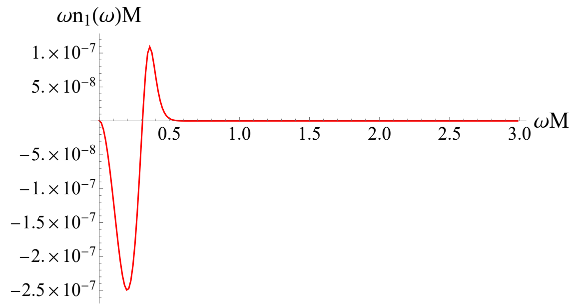
Finally, we use the results for the scattering coefficients to compute the integrand in (3.1.6a). We repeat this process for different values of ω and ℓ. The numerical results show that the integrand $\omega n_\ell(\omega)$ decreases rapidly with ℓ. The decrease from ℓ to $\ell + 1$ can be as large as four orders of magnitude for small values of Λ and μ^2, as for example indicated in Fig. 3.1a and Fig. 3.1b. However, for larger values of Λ and μ, the decrease from ℓ to $\ell + 1$ can become even less than one order of magnitude, see Fig. 3.1c and Fig. 3.1d for an example. For the results presented here, we used all ℓ-modes with

$$\max |M\omega n_\ell(\omega)| > 10^{-15}\,.$$

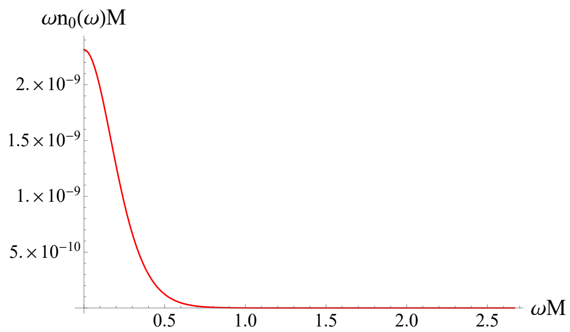
The integral over ω is estimated by the middle Riemann-sum, while the discretization error is estimated by the difference to the upper Riemann-sum. We scan over ω up to



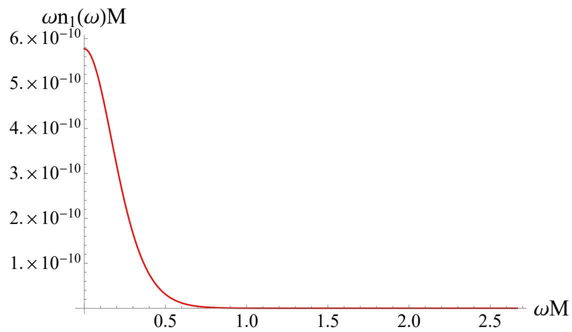
(a) $\omega n_0(\omega)$, $\mu^2 = 0$, $\Lambda M^2 = 0.02$ and $Q/M = 0.9917$



(b) $\omega n_1(\omega)$, $\mu^2 = 0$, $\Lambda M^2 = 0.02$ and $Q/M = 0.9917$



(c) $\omega n_0(\omega)$, $\mu^2 = 1000\Lambda/3$, $\Lambda M^2 = 0.14$ and $Q/M = 0.9945$



(d) $\omega n_1(\omega)$, $\mu^2 = 1000\Lambda/3$, $\Lambda M^2 = 0.14$ and $Q/M = 0.9945$

Figure 3.1: The integrands $\omega n_0(\omega)$ and $\omega n_1(\omega)$ for a massless scalar with $\Lambda M^2 = 0.02$ and $Q/M = 0.9917$ (Fig. 3.1a and Fig. 3.1b), as well as for a scalar of mass $1000\Lambda/3$ with $\Lambda M^2 = 0.14$ and $Q = 0.9945$ (Fig. 3.1c and Fig. 3.1d).

some maximal value ω_{max} which is determined by the condition $\omega n_\ell(\omega) < 10^{-15}$ for $\omega > \omega_{\text{max}}$. For large Q/M, $\omega_{\text{max}} = 3M^{-1}$ is sufficient, while for some comparably small values of Q/M we choose $\omega_{\text{max}} = 4.5M^{-1}$. In this way, the dominant contribution to the error estimate comes from the discretization of the integral, which gives us a rather conservative estimate of the numerical error of the whole computation.

Another way to observe the decay in ℓ is to compute the approximate integral over ω but not the sum over ℓ. In other words, we compute the contribution to $\langle T_{vv}\rangle_{\text{U}-\text{C}}$ by modes of a fixed angular momentum,

$$T_{vv}^{(\ell)} = \frac{2\ell+1}{16\pi^2 r_-^2} \int_0^\infty \omega n_\ell(\omega)\, \mathrm{d}\omega\,. \tag{3.4.1}$$

Here, $n_\ell(\omega)$ is as defined in (3.1.6b).

Fig. 3.2 shows $T_{vv}^{(\ell)}$ for the conformally coupled scalar field at $\Lambda M^2 = 0.02$ as a function of Q/M for $\ell = 0, 1, 2$. One can see clearly that in most of the range depicted here, the $(\ell = 0)$-term dominates, and all other terms only give small corrections. The exception is the region around Q_0 where $T_{vv}^{(0)}$ vanishes. Here, the higher ℓ- modes become

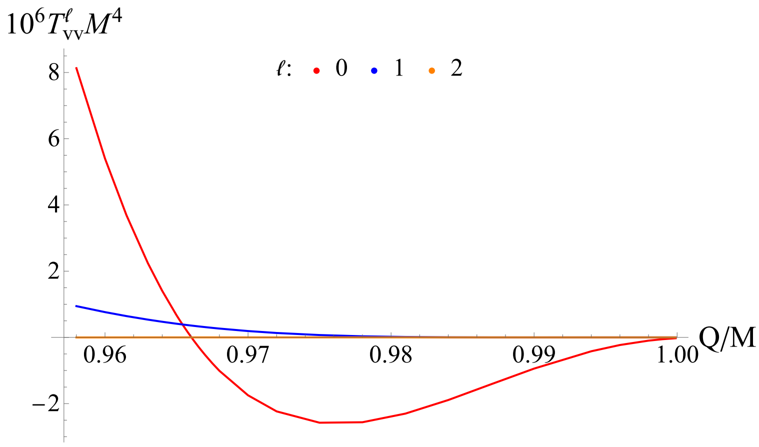


Figure 3.2: The contribution of modes with different angular momentum ℓ to $\langle T_{vv} \rangle_{\mathrm{U-C}}$ for $\ell = 0$ (red), $\ell = 1$ (blue) and $\ell = 2$ (orange) as a function of Q/M. The mass of the scalar field is given by $\mu^2 = 2\Lambda/3$, the cosmological constant is $\Lambda M^2 = 0.02$.

relevant, with the dominant contribution coming from the $(\ell = 1)$-mode. Therefore, the behaviour is always dominated by the low ℓ-modes, justifying our rather low cutoff in ℓ.

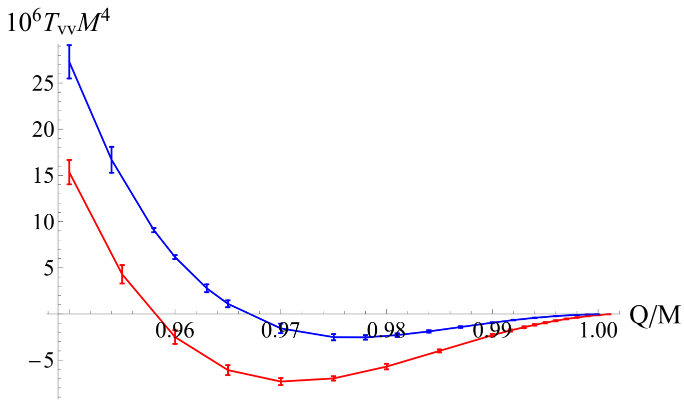
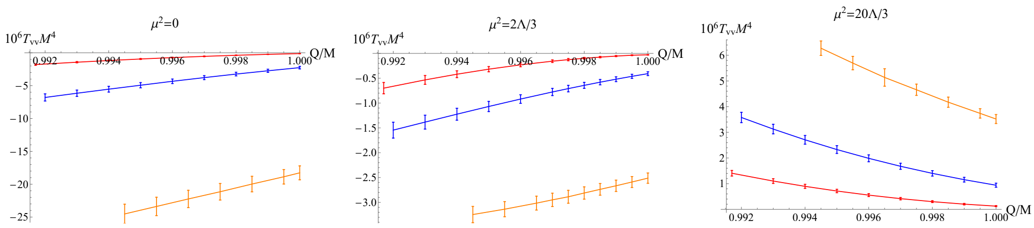


Figure 3.3: The energy flux $\langle T_{vv} \rangle_{\mathrm{U-C}}$ as a function of the black-hole charge Q/M at the Cauchy horizon for a massless (red) and conformally coupled (blue) scalar field, with cosmological constant $\Lambda M^2 = 0.02$.

In Fig. 3.3, the flux $\langle T_{vv} \rangle_{\mathrm{U-C}}$ at the Cauchy horizon is shown for the massless and conformally coupled scalar field. The cosmological constant is set to $\Lambda M^2 = 0.02$, the lowest value considered in [40], and Q/M ranges from 0.95 to 1.001. We see that indeed, $\langle T_{vv} \rangle_{\mathrm{U-C}}$ is non-zero in general. Another interesting feature is that $\langle T_{vv} \rangle_{\mathrm{U-C}}$ passes through zero and becomes negative for Q/M sufficiently large. Consequently, while the energy flux diverges generically as V_-^{-2}, it depends on the spacetime parameters whether it diverges to $+\infty$ or $-\infty$. In turn, nearby geodesics could be accelerated towards or away from each other when approaching the horizon, and hence finite-size observers could be destroyed by squeezing or stretching correspondingly, see Section 3.1, and [16, 92] for similar discussions.

Indeed, the results are very similar to the results obtained for the massles scalar field on the Reissner-Nordström spacetime in [92]. Comparing to T^U_{vv} in [92, Fig. 1], we see that their result has similar features, including the change of sign as well as the decrease towards extremality. In contrast to the computation presented here, the results in [92] were obtained by using the t-splitting variant of the pragmatic mode-sum renormalization scheme [14]. While state subtraction is arguably simpler on the Cauchy horizon, the advantage of the pragmatic mode-sum method is that it is possible to compute expectation values off the horizon without significant complications, see e.g. [108]. The energy flux of the massless scalar near the inner horizon of Reissner-Nordström has likewise been analysed analytically in the near-extremal regime [109], revealing that it behaves like $\langle T_{vv}\rangle_{\mathrm{U}} \sim -(1-(Q/M)^2)^2$ in the extremal limit. While such a result is not known for the real scalar field on RNdS yet, it seems possible that it can be obtained given the behaviour of our numerical results in the near-extremal limit.

We note also that the results in Fig. 3.3, in contrast to those in [92], show $\langle T_{vv}\rangle_{\mathrm{U-C}}$ for two different masses μ of the scalar field, indicating that $\langle T_{vv}\rangle_{\mathrm{U-C}}$ increases with μ. Consequently, the value of Q/M at which $\langle T_{vv}\rangle_{\mathrm{U-C}}$ becomes negative increases. This leads to the question whether there is a mass μ_c of the scalar field, possibly dependent on ΛM^2, above which the sign switch in $\langle T_{vv}\rangle_{\mathrm{U-C}}$ as a function of Q/M is absent. To answer this, a more detailed examination of the dependence of $\langle T_{vv}\rangle_{\mathrm{U-C}}$ on the scalar field mass is necessary.



(a) (b) (c)

Figure 3.4: $\langle T_{vv}\rangle_{\mathrm{U-C}}$ as a function of Q/M in the part of the parameter regime, where sCC is violated classically [40]. The three subplots show the massless (3.4a), conformally coupled (3.4b), and $\mu^2 = 20\Lambda/3$ (3.4c) case. The cosmological constant is fixed to $\Lambda M^2 = 0.02$ (red)/ 0.06 (blue)/ 0.14 (orange). The result is restricted to $Q \leq M$.

In Fig. 3.4, the flux as a function of Q/M is shown in the near-extremal regime for different masses μ of the scalar field and all three values of the cosmological constant Λ considered in [40]. This limit is of particular interest, since this is the parameter region in which the classical violation of sCC occurs [40]. One observes that in the last plot, Fig. 3.4c, $\langle T_{vv}\rangle_{\mathrm{U-C}}$ is positive in the near-extremal regime. This is an indication that there is indeed a mass $\mu_c(\Lambda)$, so that for $\mu > \mu_c$, the quadratic divergence $\langle T_{vv}\rangle_{\mathrm{U-C}}$ of the energy flux remains positive for all Q/M.

To explore this dependence on the scalar-field mass further, we plot the mass depen-

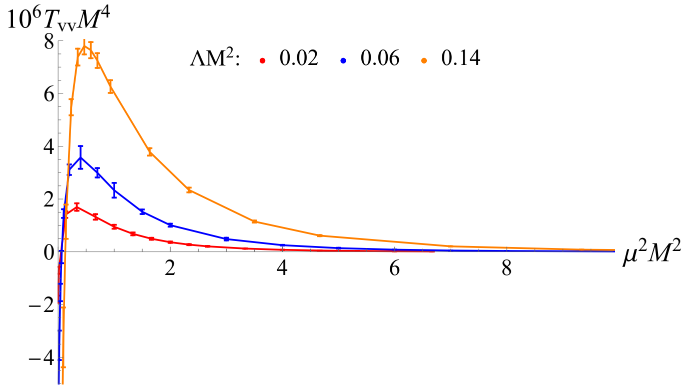


Figure 3.5: $\langle T_{vv} \rangle_{\mathrm{U-C}}$ as a function of the mass squared μ^2 of the scalar field. The spacetime parameters are the ones which correspond approximately to the lowest values of Q/M with $\Lambda M^2 = 0.02$ (red), 0.06 (blue), and 0.14 (orange), such that sCC is classically violated [40].

dence of $\langle T_{vv} \rangle_{\mathrm{U-C}}$ for fixed Q/M and ΛM^2 in Fig. 3.5. The spacetime parameters are fixed to the smallest Q/M at which classical violation of sCC occurs [40] for the corresponding Λ: we set $Q/M = 0.9917$ for $\Lambda M^2 = 0.02$, $Q/M = 0.992$ for $\Lambda M^2 = 0.06$, and $Q/M = 0.9945$ for $\Lambda M^2 = 0.14$. We observe that, again, the flux is generically non-vanishing, but can have either sign depending on the mass of the scalar field, even when the spacetime parameters are fixed.

The mass dependence of $\langle T_{vv} \rangle_{\mathrm{U-C}}$ for fixed spacetime parameters has a number of interesting features: for small values of μ^2, $\langle T_{vv} \rangle_{\mathrm{U-C}}$ increases rapidly with μ^2, leading to the change of sign. At intermediate μ^2, $\langle T_{vv} \rangle_{\mathrm{U-C}}$ reaches a maximum, and at large μ^2 it is asymptotically approaching zero from above.

If one reinstates the gravitational constant G explicitly, one can write the mass μ in units more commonly used in particle physics,

$$\mu^2 = \left(\mu^2 M^2\right) \frac{1.785 \cdot 10^{-38}}{(M[M_\odot])^2} \left(\frac{\mathrm{GeV}}{\mathrm{c}^2}\right)^2 . \tag{3.4.2}$$

It is thus clear that the scalar-field masses tested here are very small compared to, for example, the Higgs mass, at least for black holes of solar mass.

Overall, our results show that, at least for small enough masses of the scalar field, the energy flux of the quantum scalar field diverges faster than its classical counterpart as the Cauchy horizon is approached. This divergence indicates that sCC in the formulation by Christodoulou [31] can be restored by the quantum effects. Yet, the leading divergence of the flux can have either sign, depending on the parameters of the spacetime and the scalar field. Thus, the final fate of an observer approaching the horizon will depend on the spacetime- and scalar-field parameters. Recently, it has even been argued [110] that the strong quantum effects at the inner horizon may altogether alter our understanding of black-hole evaporation.

One may worry that the restoration of sCC breaks down when $\langle T_{vv} \rangle_{\mathrm{U-C}}$ vanishes, which is the case as it changes sign and hence crosses through zero. But since this is an isolated point in Fig. 3.3, achieving $\langle T_{vv}\rangle_{\mathrm{U-C}} = 0$ for given masses of the black hole and of the scalar field and for given cosmological constant will require a fine tuning of the black-hole charge. Thus, this will not be the generic situation, and its existence is not a contradiction to sCC.

However, there is another way of looking at that issue which is interesting in its own regard and which we will discuss hereafter. A publication of this discussion together with J. Zahn is in preparation. The numerical calculations have been performed by me.

Let us consider the correlations of the stress-energy tensor, renormalized by Hadamard point-splitting, near the inner horizon. If we take the correlation between two points x and y outside the Cauchy horizon where the Unruh state is well-defined, a straight-forward computation yields

$$\langle T^{\mathrm{ren}}_{vv}(x)T^{\mathrm{ren}}_{vv}(y)\rangle_{\mathrm{U}} - \langle T^{\mathrm{ren}}_{vv}(x)\rangle_{\mathrm{U}}\,\langle T^{\mathrm{ren}}_{vv}(y)\rangle_{\mathrm{U}} = 2\,\langle \partial_v\Phi(x)\partial_v\Phi(y)\rangle^2_{\mathrm{U}}\,. \tag{3.4.3}$$

The same formula can be obtained for the comparison state as defined in [16].

For the question above, we are particularly interested in the correlation on the horizon, so we will choose $x = (U_-, 0, \theta, \varphi)$ to be some point on the Cauchy horizon, and $y = (U_-, 0, \theta + \delta\theta, \varphi)$ separated from x in the θ-direction. We note that the comparison state in [16] is constructed to be stationary. Hence, the right hand side of (3.4.3) for the comparison state can be computed on either part of the inner horizon, $\mathcal{H}^L_-$ or $\mathcal{H}^R_-$. If we compute it on $\mathcal{H}^L_-$, setting for example $x = (0, V_-, \theta, \varphi)$ and $y = (0, V_-, \theta + \delta\theta, \varphi)$, it formally takes the form [16]

$$\langle \partial_v\Phi(\theta)\partial_v\Phi(\theta+\delta\theta)\rangle_{\mathrm{C}} = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{16\pi^2 r_-^2} P_\ell(\cos\delta\theta) \int_0^\infty \omega \coth\left(\pi\tfrac{\omega}{\kappa_-}\right)\,\mathrm{d}\omega\,, \tag{3.4.4}$$

where P_ℓ are the Legendre polynomials, or, equivalently,

$$\langle \partial_v\Phi(\theta)\partial_v\Phi(\theta+\delta\theta)\rangle_{\mathrm{C}} = -\lim_{\epsilon\to 0^+}\frac{r_-^2}{\pi}\int \frac{E(V,0,\Omega;x)E(V',0,\Omega;x+\delta\theta)}{(V-V'-i\epsilon)^2}\,\mathrm{d}V\,\mathrm{d}V'\,\mathrm{d}^2\Omega\,,$$

compare [16, Eq. (66)] and the discussion in Section 2.5. Taking into account the support properties of E, one can see that this vanishes as long as $\delta\theta \neq 0$. Therefore, we can subtract the right hand side of (3.4.3) in the comparison state from the result in the Unruh state without changing the result. In fact, considering (3.4.4), this is the same as subtracting a "blind spot" [15] from the corresponding expression in the Unruh state, which does not alter the result, but can improve the convergence of the numerical computation significantly.

Using also the stationarity of the Unruh state, we obtain

$$\langle \partial_v\Phi(\theta)\partial_v\Phi(\theta+\delta\theta)\rangle_{\mathrm{U}} = \sum_{\ell=0}^{\infty} P_\ell(\cos\delta\theta) T^{(\ell)}_{vv}\,, \tag{3.4.5}$$

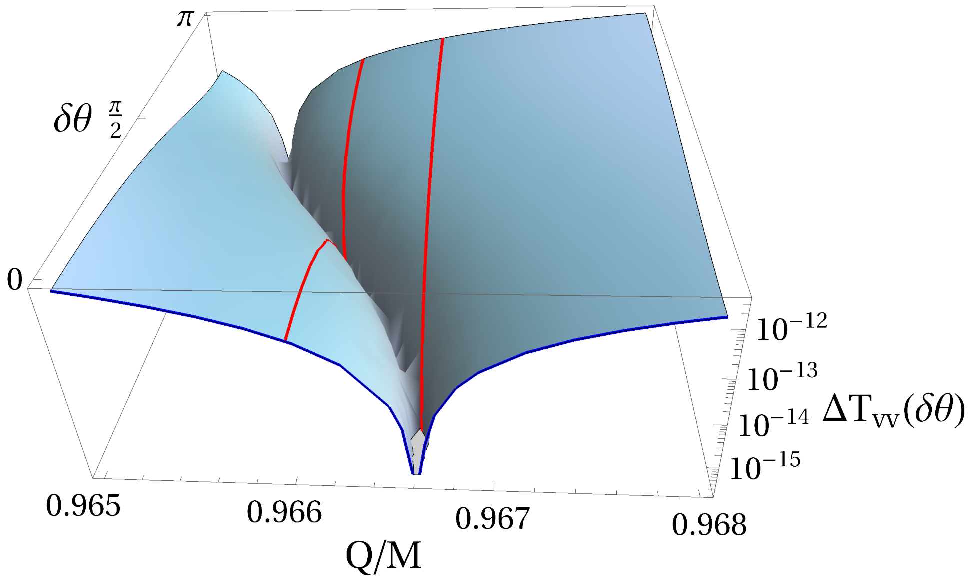


Figure 3.6: The correlation of the energy flux T_{vv} in the Unruh state at the Cauchy horizon at angular separation $\delta\theta$. The two red lines indicate the Q/M-values Q_0 at which $T^{(0)}_{vv} = 0$ (left) and Q_* at which $\langle T_{vv} \rangle_{\mathrm{U-C}} = 0$ (right).

where $T^{(\ell)}_{vv}$ is as defined in (3.4.1). Taking these results together, we can already see that for $\delta\theta \to 0$, the leading divergence of the correlation of the energy flux at $\mathcal{H}^R_-$ will approach $2 \langle T_{vv} \rangle^2_{\mathrm{U-C}}$. Note, however, that the correlation is no longer well-defined in the limit $\delta\theta \to 0$. In this limit, the two stress-energy tensors must, in addition to the θ-direction, be smeared in a time-like direction. This can be seen by computing the wave-front set of the squared Unruh-state two-point function, $w^{\mathrm{U}}(x,y)^2$, by an application of [68, Thm. 8.2.10].

Recalling the analysis of $T^{(\ell)}_{vv}$ in Fig. 3.2, one can guess that for Q/M sufficiently far away from Q_0, where $T^{(0)}_{vv}$ vanishes, the right-hand side of (3.4.5) will only depend very weakly on $\cos\delta\theta$, while it will be approximately linear in $\cos\delta\theta$ in a neighbourhood of Q_0. We would expect that, consequently, the correlation between the energy flux at θ and $\theta + \delta\theta$ at the inner horizon is only weakly dependent on $\delta\theta$ except for a narrow parameter range in Q/M around Q_0. In this range, we expect the dependence to be dominated by the behaviour of $P_1(\cos\delta\theta)^2$, which behaves approximately quadratic in $\cos\delta\theta$.

The numerical result for the correlation in a small neighbourhood of Q_0 are shown in Fig. 3.6. One can clearly see that there is only a very weak dependence of the correlation on $\delta\theta$ except for a very narrow range of Q/M around Q_0 which is indicated by the left red line in Fig. 3.6. Indeed, the behaviour of the correlations along this line seems to be compatible with the expected $\cos^2\delta\theta$-behaviour. Another point of interest is Q_* where $\langle T_{vv} \rangle_{\mathrm{U-C}}$ vanishes and which is indicated by the right red line in Fig. 3.6. At this value of Q/M, the correlation is of the same order of magnitude as for other nearby values of Q/M for sufficiently large $\delta\theta$.

These results have a number of implications. First of all, since the correlation is essentially independent of θ for most of the range in Q/M, it is approximately given by $2\left\langle T_{vv}\right\rangle_{\mathrm{U-C}}^2$. Thus, changing to the Kruskal-type coordinate V_-, the covariance of $T_{V_-V_-}$ diverges as V_-^{-4} as the Cauchy horizon is approached, and the prefactor is of the same order of magnitude as $\left\langle T_{V_-V_-}\right\rangle_{\mathrm{U-C}}^2$. As a result, there will be huge fluctuations in the energy flux correlated over macroscopic distances, and these fluctuations will have a size comparable to that of $\left\langle T_{V_-V_-}\right\rangle_{\mathrm{U-C}}^2$. Since the validity of semi-classical gravity requires that the fluctuations of the stress-energy tensor of the quantum field remain small compared to the expectation value, so that the expectation value is a good approximation, this signals its breakdown near the inner horizon. Of course, the divergence of the expectation value also signals a breakdown of the semi-classical theory, since it will lead to a divergence of the curvature, while the semi-classical theory is only expected to be valid as long as the curvature remains small compared to the Planck scale. A more detailed analysis of the stress-energy off the horizon would be required to resolve the question in which way the semi-classical theory breaks down first as the horizon is approached.

Second, it is very remarkable that near Q_*, the correlations of $T_{V_-V_-}$ are actually larger at larger angular separation. The reason for this behaviour is the dominance of the low-ℓ modes: at Q_* one has approximately $T_{vv}^{(0)} = -T_{vv}^{(1)}$, while higher ℓ-modes do not play a significant role. At large angular separation, $P_1(\cos\delta\theta) \approx -1$, and the cancellation between the $(\ell = 0)$-mode and the $(\ell = 1)$-mode is lifted. Nonetheless, this behaviour is very counter-intuitive to physical expectations.

Third, at Q_*, where $\left\langle T_{vv}\right\rangle_{\mathrm{U-C}}$ vanishes, there are still correlations at large angular separation comparable in size to $\left\langle T_{vv}\right\rangle_{\mathrm{U-C}}^2$ at other nearby values of Q/M. Thus, even though the expectation value of the energy flux might vanish there, the typical values of T_{vv} in an actual realization will be of the same order of magnitude as for other nearby values of Q/M. But when the average over different realizations is taken, the positive and negative realizations of T_{vv} are spread such that they cancel on average.

This last observation relates back to the question whether the zero of $\left\langle T_{vv}\right\rangle_{\mathrm{U-C}}$ at the Cauchy horizon as a function of Q/M poses a problem for sCC. Even when the expectation value vanishes, the stress-energy tensor in a typical realization will not, and sCC is restored by quantum effects.

To summarize, we have shown that in the setup of linear perturbations by a real scalar field on a RNdS spacetime, quantum effects can restore sCC in the case where it is classically violated. In particular, the expectation value of the energy flux $T_{V_-V_-}$ has a quadratic divergence at the Cauchy horizon. This effect is independent of the quantum state, as long as it is a Hadamard state in the RNdS spacetime up to the Cauchy horizon.

Nonetheless, we found that the sign of the prefactor of the quadratic divergence does depend on the parameters Q/M and ΛM^2 of the spacetime and the mass $\mu^2 M^2$ of the scalar field. Via backreaction on the spacetime described by the semi-classical Einstein equations (1.0.1), this leads either to infinite stretching or squeezing of observers approaching the horizon. While sCC is restored in any case, the final fate of an observer falling into the black hole therefore depends on the parameters of the black hole and the scalar field.

Further, we find that at the Cauchy horizon, there are sizeable correlations of the stress-energy tensor over macroscopic distances, which are moreover divergent. This is not only a counter-intuitive behaviour of the correlations, but also signals an additional reason for the breakdown of semi-classical gravity.

4 The charged scalar field in Reissner-Nordström-de Sitter

In the previous chapter we presented a method for the numerical computation of scattering coefficients on the RNdS spacetime. Our motivation for this computation was to study the quantum scalar field near the Cauchy horizon of this spacetime and to see whether its effects could restore sCC.

Since we consider a spacetime with a charged black hole, it seems reasonable to also consider a matter model that can accommodate for the creation of such a black hole. In other words, we would like to consider a matter model of charged particles. The simplest example thereof would be a charged scalar field.

These considerations were also made in [111], as an alternative remedy to the classical sCC violation. However, it was found in [112] that even when charged scalar fields are considered, there is still a parameter region of the black hole close to extremality, albeit much smaller than for the real scalar field, in which sCC is classically violated. This leaves the question whether sCC can again be restored if the quantum effects of the charged scalar field are taken into account.

However, considering a charged scalar field on this spacetime raises another interesting question. Since there is a non-vanishing background electromagnetic field in the RNdS spacetime, a charged scalar field on this spacetime should induce a charge current j_ν. This current influences the energy of the electromagnetic field and thereby contributes to the backreaction of the quantum field onto the spacetime through the semi-classical Einstein-Maxwell system

$$G_{\nu\varrho} + \Lambda g_{\nu\varrho} = 8\pi \left(\langle T_{\nu\varrho} \rangle_\Psi + \mathcal{E}_{\nu\varrho} \right) \tag{4.0.1a}$$

$$\nabla^\nu F_{\nu\varrho} = -4\pi \langle j_\varrho \rangle_\Psi \; . \tag{4.0.1b}$$

Here, Ψ is an appropriate Hadamard state for the charged scalar field. Assuming that the corrections to the spacetime maintain spherical symmetry, we can make the ansatz (3.1.1) for the metric and

$$F = -\frac{Q}{2r^2} e^\sigma \, \mathrm{d}u \wedge \, \mathrm{d}v \tag{4.0.2}$$

for the field-strength tensor F, where we also take Q as an unknown function of u and v. One can see by Gauß's law that this function corresponds to the charge contained in a surface of constant u and v with area $4\pi r^2$.

The vv-component of the semi-classical Einstein equations then leads to the solution

(3.1.4) for $\partial_v r$, while the v-component of (4.0.1b) can be written as

$$\partial_v Q = -4\pi r^2 \langle j_v \rangle_\Psi \, , \tag{4.0.3}$$

using the weak-backreaction assumption as explained in Section 3.1. Thus, the sign of the current decides whether the (local) charge of the black hole decreases ($\langle j_v \rangle_\Psi > 0$) or increases ($\langle j_v \rangle_\Psi < 0$).

One can also consider the electromagnetic field strength

$$\sqrt{-\frac{1}{2} F^{\nu\varrho} F_{\nu\varrho}} = \frac{Q}{r^2} \, .$$

Combining (3.1.4) and (4.0.3), one finds that near the Cauchy horizon, the change of the field strength in the weak-backreaction regime is governed by

$$\partial_v \left(\frac{Q}{r^2} \right) = -4\pi \langle j_v \rangle_\Psi + 8\pi \frac{Q}{r_-^2 \kappa_-} \langle T_{vv} \rangle_\Psi \, . \tag{4.0.4}$$

In the intuitive particle picture, the occurrence of the charge current can be understood as follows: Pairs of particles and antiparticles are spontaneously created from the vacuum at a rate which can for example be estimated by the Schwinger formula [113]. Due to the background electromagnetic field, the particles and antiparticles are then accelerated in opposite directions so that over time they eliminate the charge of the black hole. This has been the starting point for the study of the current in [114, 115].

But the interior of the black hole is not stationary, and hence there is no preferred choice amongst the different possible notions of particle there, see the discussion in Section 2.2. Moreover, near the Cauchy horizon, even the behaviour of classical fields is determined by non-local effects, namely the competition between the cosmological red-shift in region I and the blue-shift in region II. It seems reasonable to expect that these effects will influence the behaviour of the quantum fields as well.

Therefore, a first-principle calculation of the current j_ν in quantum field theory may be necessary to entirely capture the quantum effects of the charged scalar field near the Cauchy horizon of a RNdS spacetime. This calculation will be presented in this chapter. We will start by demonstrating that the results for the Unruh state of the real scalar field in [16] can be extended to the charged scalar field and we introduce a mode-sum representation of the Unruh state. Afterwards, we will derive a formula for the current in the Unruh state using Hadamard point-split renormalization. We will numerically study the current with this formula in regions I, II, and at the event horizon. Finally, we will consider the charged scalar near the inner horizon. We will demonstrate that the leading divergence of both the current and the stress-energy tensor is state-independent and show numerical results for this leading divergence.

4.1 The Unruh state for the charged scalar field

In this section, we sketch how the Unruh state can be defined for the charged scalar and how the proof of its Hadamard property can be obtained from the case of a real scalar field on RNdS [16]. This has been shown in the supplementary material to [116]. This supplementary material was developed and written mostly by me, with helpful discussions with J. Zahn and S. Hollands. Moreover, we introduce the mode-sum notation of the Unruh state, which can be found in [116, 117]. For these two papers, I did both the analytical as well as numerical computations under the supervision of J. Zahn and S. Hollands.

Before we begin by constructing the operator algebra for the charged scalar quantum field, we should mention that even for the classical charged scalar field, there exist instabilities in the black-hole exterior of RNdS spacetimes [112, 118–120]. These instabilities are quasi-normal modes which are non-decreasing or even increasing with time. Quasi-normal modes are solutions to (3.3.1) in region I which are purely ingoing at $r \to r_+$ and purely outgoing at $r \to r_c$ and which describe if and how perturbations of the black hole, in this instance by a charged scalar field, decay away over time. These instabilities only appear for small charge q and mass μ of the scalar field, as well as small cosmological constant Λ. A condition for the absence of the instability is given in [112, Eq. (4.6)]. In this work, the authors use the fact that the instability appears for small values of q, and treat this case analytically using perturbation theory. They find that for $r_+/r_c > \sqrt{2} - 1$, there is no instability near extremality. This condition can be reformulated as a lower bound on ΛM^2 for given Q/M. Since the classical instabilities obstruct the construction of the quantum theory (as well as making it a somewhat academic exercise, since the spacetime will already be significantly altered by the classical field), we will restrict all the considerations in the following to a parameter region in which no classical instabilities arise.

As mentioned in the previous chapter, the equation of motion for the charged scalar field ϕ is given by (3.3.1), while the complex conjugate field $\overline{\phi}$ obeys the complex conjugate equation. We will call the corresponding Klein-Gordon operator in (3.3.1) $\mathcal{K}_q$. We take our spacetime $(\mathcal{M}, g)$ to be a globally hyperbolic region of the RNdS spacetime, namely the union of regions I, II and III together with the horizons $\mathcal{H}_+^R$ and $\mathcal{H}_c^L$ between them, see Fig. 2.2.

We have already discussed that the charged scalar field allows for gauge transformations of the form (3.3.3). One way to deal with the gauge in a neat and systematic way is to consider the charged scalar field as a smooth section of the complex line bundle associated to the principal bundle $P(\mathcal{M}, \mathbb{R})$, with the representation of any $a \in \mathbb{R}$ on $\mathbb{C}$ given by the multiplication operator e^{iqa}, see [121]. In this way, scalar fields differing only by a gauge transformation are identified. However, for explicit calculations, a representative of the equivalence class has to be chosen, which corresponds to fixing a gauge. Therefore, in this work, we instead keep track of the gauge explicitly throughout the constructions and computations.

The gauge transformation (3.3.3) induces a gauge transformation of $\mathcal{K}_q$ of the form

$$\mathcal{K}_q \to e^{iq\chi}\mathcal{K}_q e^{-iq\chi}\,. \tag{4.1.1}$$

In the following, we will use the notation

$$\begin{aligned}
A^{(\chi)} &= -\frac{Q}{r}\,\mathrm{d}t + \mathrm{d}\chi\,,\\
D^{(\chi)}_\nu &= \nabla_\nu - iqA^{(\chi)}_\nu\,,\\
\mathcal{K}^{(\chi)}_q &= g^{\nu\varrho}D^{(\chi)}_\nu D^{(\chi)}_\varrho - \mu^2\,,
\end{aligned}$$

whenever we wish to make the gauge explicit.

Note that the A defined in (2.4.4), which corresponds to $A^{(0)}$ in the above notation, ceases to be regular at the horizons. However, this can be seen as a result of the choice of gauge, since the gauge-invariant field-strength tensor $F_{\mu\nu}$ is regular. At $\{r = r_i\}$, $i \in \{-,+,c\}$, this can be remedied by setting $\chi = \chi_i = tQ/r_i$.

If we want the charged Klein-Gordon operator to be a differential operator on $\mathcal{M}$ as defined in Section 2.3, then we have to demand that $\chi|_{\{r=r_i\}} = \chi_i + h_i$ for all i and for some smooth functions $h_i \in C^\infty(\mathcal{M})$, so that the irregularity of $\mathrm{d}\chi$ cancels the irregularity in A. As a consequence, the difference of two such gauge functions χ and χ' is a smooth function on the whole spacetime $\mathcal{M}$. We will refer to gauges satisfying this condition as "regular" gauges. A regular gauge can for example be constructed as follows: Let $\zeta_i \in C^\infty(\mathbb{R})$, with $i \in \{-,+,c\}$ be supported in $[r_i - \delta, r_i + \delta]$ for some $\delta > 0$. We also assume that $\zeta_i = 1$ in a neighbourhood of r_i. Let the gauge function be $\chi = \sum_i t\zeta_i(r)Q/r_i$. Then χ is smooth in the interior of I, II, and III, and it satisfies the condition for a regular gauge at all horizons.

Let us also note that we have

$$\partial_t\left(\mathcal{K}^{(\chi)}_q e^{iq\chi}\phi(x)\right) = \mathcal{K}^{(\chi)}_q \partial_t\left(e^{iq\chi}\phi\right)(x) - iqe^{iq\chi}\left[\mathcal{K}^{(0)}_q, \partial_t\chi\right]\phi(x)\,.$$

Hence, as long as we choose a gauge such that $\partial_\mu\partial_t\chi = 0$, or in other words a gauge of the form $\chi = ct + h(r,\theta,\varphi)$ for some constant c, the Killing field ∂_t maps solutions of (3.3.1) in the gauge χ to solutions in the same gauge. Therefore, it will sometimes be useful to choose a gauge of this form, and we will refer to a gauge satisfying this condition as a "static" gauge. Note, however, that we cannot make A smooth at all horizon radii $r = r_i$ with a gauge of this form, but only at one of them. In other words, static gauges are not regular on all of $\mathcal{M}$.

Let us assume for the moment that we have fixed a regular gauge χ . Since $(\mathcal{M}, g)$ is globally hyperbolic, and the principal symbol of $\mathcal{K}^{(\chi)}_q$ is of the form $g^{\nu\varrho}(x)k_\nu k_\varrho$ (i.e. $\mathcal{K}^{(\chi)}_q$ is normally hyperbolic), we can again find unique retarded and advanced Green's operators $E^{(\chi)\pm} : C^\infty_{(s)pc/(s)fc}(\mathcal{M}) \to C^\infty_{(s)pc/(s)fc}(\mathcal{M})$ satisfying (2.2.3) with $\mathcal{K}$ replaced by $\mathcal{K}_q^{(\chi)}$ [55, Thm. 3.3.1]. Their integral kernels also satisfy

$$E^{(\chi)\pm}(x,y) = \overline{E^{(\chi)\mp}}(y,x)\,.$$

From them, we can construct the commutator function

$$E^{(\chi)} = E^{(\chi)+} - E^{(\chi)-} : C_0^\infty(\mathcal{M}) \to S_q^{(\chi)}(\mathcal{M})\,.$$

Here, $S_q^{(\chi)}(\mathcal{M})$ is the space of solutions to (3.3.1) in gauge (χ) with compact Cauchy data. The commutator function satisfies (2.2.4) with $\mathcal{K}$ replaced by $\mathcal{K}_q^{(\chi)}$ and

$$E^{(\chi)}(f,h) \equiv (f|E^{(\chi)}h) = -(E^{(\chi)}f|h) = -\overline{(h|E^{(\chi)}f)} = -\overline{E^{(\chi)}(h,f)}\,, \tag{4.1.2}$$

for any $f, h \in C_0^\infty(\mathcal{M})$, where

$$(f|h) = \int\limits_{\mathcal{M}} \overline{f(x)}h(x)\,\mathrm{d}vol_g(x)\,.$$

Thus, the space $(C_0^\infty(\mathcal{M})/\mathcal{K}_q^{(\chi)}C_0^\infty(\mathcal{M}), E^{(\chi)})$ is a charged symplectic space with charged symplectic form $E^{(\chi)}$, following the notation and convention of [122].

The gauge transformation of $\mathcal{K}_q^{(\chi)}$ induces a gauge transformation of (the kernel of) $E^{(\chi)}$ of the form

$$E^{(\chi)}(x,y) \to E^{(\chi')}(x,y) = e^{iq(\chi'-\chi)(x)}E(x,y)e^{-iq(\chi'-\chi)(y)}\,. \tag{4.1.3}$$

Therefore, the gauge map

$$\psi_G^{C_0^\infty}(\chi'-\chi) : C_0^\infty(\mathcal{M})/\mathcal{K}_q^{(\chi)}C_0^\infty(\mathcal{M}) \ni [f] \mapsto e^{iq(\chi'-\chi)}[f] \in C_0^\infty(\mathcal{M})/\mathcal{K}_q^{(\chi')}C_0^\infty(\mathcal{M})$$

leaves the charged symplectic form invariant.

On $S_q^{(\chi)}(\mathcal{M})$, we can define

$$\sigma(\psi,\phi) = \int\limits_{\Sigma} n_\Sigma^a \left(\overline{\psi}D_a^{(\chi)}\phi - \overline{D_a^{(\chi)}\psi}\phi\right)\mathrm{d}vol_\gamma\,, \quad \psi,\,\phi \in S_q^{(\chi)}(\mathcal{M})\,, \tag{4.1.4}$$

where Σ is any space-like Cauchy surface with future-pointing unit normal n_Σ^a and induced volume element $\mathrm{d}vol_\gamma$. This is a charged symplectic form on $S_q^{(\chi)}(\mathcal{M})$ satisfying

$$\sigma(E^{(\chi)}(f), E^{(\chi)}(h)) = E^{(\chi)}(f,h) \quad \forall f,h \in C_0^\infty(\mathcal{M})\,.$$

Similar to the real scalar case, the last property can be utilized to show that the map $E^{(\chi)} : (C_0^\infty(\mathcal{M})/\mathcal{K}_q^{(\chi)}C_0^\infty(\mathcal{M}), E^{(\chi)}) \to (S_q^{(\chi)}(\mathcal{M}), \sigma)$ is an isomorphism of charged symplectic spaces.

In addition, σ is invariant under the gauge-transformation map

$$\psi_G^{S_q}(\chi' - \chi) : S_q^{(\chi)}(\mathcal{M}) \ni \phi \mapsto e^{iq(\chi'-\chi)}\phi \in S_q^{(\chi')}(\mathcal{M}) \,,$$

which can thus be seen to be an isomorphism of charged symplectic spaces. Therefore, we can use any of the spaces discussed above to construct the CCR-algebra.

Let us take our algebra of observables to be the free unital $*$-algebra generated by $\Phi^{(\chi)}(f)$ and $\Phi^{*(\chi)}(f)$, with χ a fixed but arbitrary regular gauge and $f \in C_0^\infty(\mathcal{M})$. We impose the relations

- **(Anti-)linearity** $\Phi^{(\chi)}(\alpha f + h) = \bar{\alpha}\Phi^{(\chi)}(f) + \Phi^{(\chi)}(h)$,
  $\Phi^{*(\chi)}(\alpha f + h) = \alpha\Phi^{*(\chi)}(f) + \Phi^{*(\chi)}(h)$
- **Klein-Gordon equation** $\Phi^{(\chi)}\left(\mathcal{K}_q^{(\chi)} f\right) = \Phi^{*(\chi)}\left(\mathcal{K}_q^{(\chi)} f\right) = 0$
- **Star-involution** $\left(\Phi^{(\chi)}(f)\right)^* = \Phi^{*(\chi)}(f)$, $\left(\Phi^{*(\chi)}(f)\right)^* = \Phi^{(\chi)}(f)$
- **Commutator property** $\left[\Phi^{(\chi)}(f), \Phi^{*(\chi)}(h)\right] = iE^{(\chi)}(f,h)\mathbf{1}$,
  $\left[\Phi^{(\chi)}(f), \Phi^{(\chi)}(h)\right] = \left[\Phi^{*(\chi)}(f), \Phi^{*(\chi)}(h)\right] = 0$

for all $f, h \in C_0^\infty(\mathcal{M})$ and $\alpha \in \mathbb{C}$. Note that this choice makes the field Φ antilinear, while Φ^* is linear, compare [122, 123].

In the construction of the algebra, we have made an arbitrary choice of gauge. Let us call this algebra $\mathcal{A}^{(\chi)}(\mathcal{M})$. Then the gauge isomorphism $\psi_G^{C_0^\infty}(\chi' - \chi)$ induces a $*$-automorphism $\alpha(\chi' - \chi) : \mathcal{A}^{(\chi)}(\mathcal{M}) \to \mathcal{A}^{(\chi)}(\mathcal{M})$ by

$$\alpha(\chi' - \chi)\left(\Phi^{(\chi)}(f)\right) = \Phi^{(\chi)}\left(\psi_G^{C_0^\infty}(\chi' - \chi)^{-1} f\right) = \Phi^{(\chi)}\left(e^{-iq(\chi'-\chi)} f\right) \,, \tag{4.1.5a}$$
$$\alpha(\chi' - \chi)\left(\Phi^{*(\chi)}(f)\right) = \Phi^{*(\chi)}\left(\psi_G^{C_0^\infty}(\chi' - \chi)^{-1} f\right) = \Phi^{*(\chi)}\left(e^{-iq(\chi'-\chi)} f\right) \,. \tag{4.1.5b}$$

Let us define $\Phi^{(\chi')}(f) = \alpha(\chi'-\chi)\left(\Phi^{(\chi)}(f)\right)$ and $\Phi^{*(\chi')}(f) = \alpha(\chi'-\chi)\left(\Phi^{*(\chi)}(f)\right)$. Then $\Phi^{(\chi')}(f)$ and $\Phi^{*(\chi')}(f)$ satisfy the same relations as the generators $\Phi^{(\chi)}(f)$, $\Phi^{*(\chi)}(f)$ but with χ replaced by χ'. Thus, they generate the algebra $\mathcal{A}^{(\chi')}(\mathcal{M})$. Therefore, the algebras for different (regular) gauge choices are isomorphic, and it does not matter which one we use to describe the theory. We will thus drop the χ-superscript unless confusion arises.

The physical observables are the gauge-invariant elements of $\mathcal{A}(\mathcal{M})$, namely those $a \in \mathcal{A}(\mathcal{M})$ that are invariant under the gauge automorphisms $\alpha(h)$, $\alpha(h)a = a$, for all $h \in C^\infty(\mathcal{M})$.

Quasi-free states on this algebra are determined by the two-point functions

$$w_+^\omega(f,h) = \omega\left(\Phi(f)\Phi^*(h)\right) \,, \tag{4.1.6a}$$
$$w_-^\omega(f,h) = \omega\left(\Phi^*(h)\Phi(f)\right) \,. \tag{4.1.6b}$$

They are sesquilinear functionals on $C_0^\infty(\mathcal{M}) \times C_0^\infty(\mathcal{M})$, which are related by

$$w_+^\omega(f,h) - w_-^\omega(f,h) = iE(f,h) \,, \tag{4.1.7}$$

and must satisfy $w^{\omega}_{\pm}(f,f) \geq 0$, and $w^{\omega}_{\pm}(\mathcal{K}_q f, h) = w^{\omega}_{\pm}(f, \mathcal{K}_q h) = 0$, see also [123]. We will often omit the ω-superscript indicating the state unless it is required for clarity.

After this preliminary discussion, let us now define the Unruh state for the charged scalar field on RNdS and sketch how the proof of its Hadamard property can be obtained by generalising the results from [16]. The Uhruh state for the charged scalar field on RNdS will be defined in analogy to the one for the real scalar field presented at the end of Chapter 2 by

$$
\begin{aligned}
&w^{(\chi)}_{+}(f,h) \qquad\qquad (4.1.8)\\
&= - \lim_{\epsilon\to 0^+} \frac{r_+^2}{\pi} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\overline{E^{(+)}(\overline{e^{iq(\chi_+ - \chi)} f})|_{\mathcal{H}_+}(U_+,\Omega)} E^{(+)}(e^{iq(\chi_+ - \chi)} h)|_{\mathcal{H}_+}(U'_+,\Omega)}{(U_+ - U'_+ - i\epsilon)^2} \,\mathrm{d}^2\Omega \,\mathrm{d}U_+ \,\mathrm{d}U'_+\\
&\quad - \lim_{\epsilon\to 0^+} \frac{r_c^2}{\pi} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \frac{\overline{E^{(c)}(\overline{e^{iq(\chi_c - \chi)} f})|_{\mathcal{H}_c}(V_c,\Omega)} E^{(c)}(e^{i(\chi_c - \chi)} h)|_{\mathcal{H}_c}(V'_c,\Omega)}{(V_c - V'_c - i\epsilon)^2} \,\mathrm{d}^2\Omega \,\mathrm{d}V_c \,\mathrm{d}V'_c\,.
\end{aligned}
$$

Here, we have introduced the notation (i), $i \in \{-,+,c\}$, for the gauges with χ_i as defined at the beginning of the section. Note that these gauges are not regular, but we have $\mathcal{K}^{(+)}_q, \mathcal{K}^{(c)}_q \in \mathrm{Diff}^2(\mathrm{I})$, so they are regular on I. Therefore, we first restrict to test functions f, h supported in I. However, at least for these test functions, $w^{(\chi)}_{+}(f,h)$ can also be written in the form

$$
\begin{aligned}
&w^{(\chi)}_{+}(f,h) \qquad\qquad (4.1.9)\\
&= - \lim_{\epsilon\to 0^+} \frac{r_+^2}{\pi} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} g^{(\chi)}_{+}(U_+,\Omega;U'_+,\Omega) \frac{\overline{E^{(\chi)}(\overline{f})|_{\mathcal{H}_+}(U_+,\Omega)} E^{(\chi)}(h)|_{\mathcal{H}_+}(U'_+,\Omega)}{(U_+ - U'_+ - i\epsilon)^2} \,\mathrm{d}^2\Omega \,\mathrm{d}U_+ \,\mathrm{d}U'_+\\
&\quad - \lim_{\epsilon\to 0^+} \frac{r_c^2}{\pi} \int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} g^{(\chi)}_{c}(V_c,\Omega;V'_c,\Omega) \frac{\overline{E^{(\chi)}(\overline{f})|_{\mathcal{H}_c}(V_c,\Omega)} E^{(\chi)}(h)|_{\mathcal{H}_c}(V'_c,\Omega)}{(V_c - V'_c - i\epsilon)^2} \,\mathrm{d}^2\Omega \,\mathrm{d}V_c \,\mathrm{d}V'_c\,,
\end{aligned}
$$

where

$$
g^{(\chi)}_i(x,y) = e^{iq((\chi-\chi_i)(x)-(\chi-\chi_i)(y))}
$$

is a smooth, bounded function on $\mathcal{H}_i \times \mathcal{H}_i$. Since the gauge χ is regular, the commutator functions $E^{(\chi)}$ can be uniquely extended from $C^\infty_0(\mathrm{I})$ to $C^\infty_0(\mathcal{M})$. Therefore, it becomes apparent that $w^{(\chi)}_{+}(f,h)$ can be extended to $C^\infty_0(\mathcal{M}) \times C^\infty_0(\mathcal{M})$.

The two-point function $w^{(\chi)}_{-}$ can be obtained from $w^{(\chi)}_{+}$ using (4.1.7). It has the same form as $w^{(\chi)}_{+}$, but with $-i\epsilon \to i\epsilon$ in both denominators.

The integrals in (4.1.8) can only be well-defined if the elements of $S_q(\mathcal{M})$ restricted to $\mathcal{H}_+$ or $\mathcal{H}_c$ decay sufficiently fast for $U_+, V_c \to -\infty$. This is the case, because the results of [39] continue to hold also for a charged scalar field, as long as the charge is sufficiently small, see also [111, App. A]:

In the following, we assume that there is a spectral gap $\alpha > 0$, i.e. the parameters (Λ, Q, q, μ) are restricted to a parameter region without classical instabilities of the black-hole spacetime under charged scalar perturbations [112, 118–120]. In this case, a forward solution $F^{(\chi)} = E^{+(\chi)} f$ to the differential equation $\mathcal{K}_q^{(\chi)} F^{(\chi)} = f$ with smooth source $f \in C_0^\infty(\Omega^o)$ satisfies [39, Eq. (2.59)],

$$F^{(\chi)} \in H^{1/2+\beta-0}\left(I; \tau^\alpha H_b^N(\mathbb{R}^*_{+\tau} \times \mathbb{S}^2)\right) . \tag{4.1.10}$$

Herein, $\beta = \frac{\alpha}{\kappa_-}$, $I \subset \mathbb{R}$ is an interval containig r_-, and Ω^o is a neighbourhood of i^+ as in [39, Fig. 9]. $N \in \mathbb{N}$ is arbitrary. The time coordinate t_* defined in [39, Sec. 2.1] is related to τ by $t_* = -\log \tau$. It behaves as $t_* \sim t$ away from the horizons, $t_* \sim v$ near $\mathcal{H}_+^R$, $t_* \sim u$ near $\mathcal{H}_-^R$ and $\mathcal{H}_c^L$, and $t_* \to \infty$, i.e. $\tau \to 0$, towards i^+. In addition to that, as long as $r_+ \leq r \leq r_c$, one can estimate [16, Eq. (75)]

$$\left|\partial^N F^{(\chi)}(t, r, \theta, \varphi)\right| \leq C \tau^\alpha \|f\|_{C^m(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III})} , \quad \partial \in \{\tau\partial_\tau, \partial_r, \partial_\theta, \partial_\varphi\} , \tag{4.1.11}$$

for any $N \in \mathbb{N}$ and sufficiently large m depending on N. The constant C will depend on the support of f.

These results can be transferred to results on the backwards solution near i^- with the help of the $t \to -t$-symmetry [16]. They are sufficient to show the well-definedness of the integrals. It is also straightforward to see that $w_+^{(\chi)}$ satisfies positivity, the equations of motion and the commutator property following the same steps as in [16].

In the final step, one can show that the proof for the Hadamard property of the Unruh state, [16, Prop. 4.5], can be adapted to the charged scalar field. First, notice that the wavefront set of the commutator function remains unchanged, while the form of the two-point function only changes by the addition of smooth, gauge-related terms when compared to the real scalar case. Thus, case 1) of the proof of [16, Prop.4.5] also applies to the Unruh state for the charged scalar field.

For case 2) and 3), let us define the maps

$$\begin{aligned} &K_j^{(\chi)} : C_0^\infty(\mathcal{O}) \to L^2(\mathbb{R}_\omega \times \mathbb{S}^2_\Omega) , \\ &K_j^{(\chi)} f(\omega, \Omega) = r_j \sqrt{\frac{\omega e^{\pi \frac{\omega}{\kappa_j}}}{\pi \sinh \frac{\pi\omega}{\kappa_j}}} \int_{\mathbb{R}} E^{(j)}(e^{iq(\chi_j - \chi)} f)|_{\mathcal{H}_j^-}(s_j, \Omega) e^{i\omega s_j} \,\mathrm{d}s_j , \end{aligned} \tag{4.1.12}$$

with $j \in \{+, c\}$, $\mathcal{O} \subset \mathrm{I}$ an open connected set, $s_+ = u$, and $s_c = v$. One can demonstrate by a change of coordinates and Fourier-Laplace transform that

$$C_0^\infty(\mathcal{O}) \ni f \mapsto K^{(\chi)} f \equiv K_+^{(\chi)} f \oplus K_c^{(\chi)} f \in L^2(\mathbb{R}_\omega \times \mathbb{S}^2_\Omega) \oplus L^2(\mathbb{R}_\omega \times \mathbb{S}^2_\Omega)$$

satisfies

$$w_+^{(\chi)}(f, h) = \left\langle K^{(\chi)} f, K^{(\chi)} h \right\rangle_{L^2 \oplus L^2} ,$$

compare [16, Eq. (78)].

Moreover, we note that from our discussion of the Klein-Gordon operator in the (j)-gauges, one can conclude

$$E^{(j)}(r,t,\Omega;r',t'+s,\Omega') = E^{(j)}(r,t-s,\Omega;r',t',\Omega')\,. \tag{4.1.13}$$

Together with the decay results (4.1.11), see also [16, Thm. 4.4], this implies that the kernel of $K^{(\chi)}$ is a Hilbert-space valued distribution on $\mathcal{O}$ whose values are in the space $L^2(\mathbb{R}\times\mathbb{S}^2)\oplus L^2(\mathbb{R}\times\mathbb{S}^2)$. Moreover, the distribution has an analytic extension to the strip $\{t+is : s\in(0,\min\{\frac{\pi}{\kappa_+},\frac{\pi}{\kappa_c}\})\}$ as long as the functions

$$\Delta\chi_j(b) = (\chi_j-\chi)(r,t+b,\theta,\varphi) - (\chi_j-\chi)(r,t,\theta,\varphi)$$

with $(r,t,\theta,\varphi)\in\mathcal{O}$ have an analytic extension to this strip. This shows that $K^{(\chi)}$ has the same properties as K defined in [16]. Since the spacetime is still the same, this means that the rest of the proof of [16, Prop. 4.5] also continues to hold for the charged scalar field, demonstrating that the Unruh state for the charged scalar field on RNdS is a Hadamard state as long as $\alpha > 0$.

Note that (4.1.13) also entails that the two-point function is stationary in region I, i.e. invariant under ∂_t, as long as χ is a static gauge.

For computational purposes, we would also like to give a mode-sum expression of the Unruh state. To this end, we need the definition of the modes ϕ_i that we would like to utilize for the expansion of the local fields $\Phi(x)$ and $\Phi^*(x)$. The mode solutions to the Klein-Gordon equation on RNdS for the charged scalar field, (3.3.1), are obtained as described in Section 3.3. In particular, we make the ansatz

$$\phi^\lambda_{\omega\ell m} = (4\pi)^{-1/2} r^{-1} Y_{\ell m}(\theta,\varphi) h^\lambda_{\omega\ell}(t,r)\,, \tag{4.1.14}$$

with λ running over "in" and "up".

When working with mode-sum expressions, we will restrict ourselves to gauges of the form $\chi_{r_0} = tQ/r_0$, which were already discussed in Section 3.3. These gauges are static and only modify $h^\lambda_{\omega\ell}(t,r)$. Moreover, they allow us to set $A = 0$ at any fixed but arbitrary radius r_0. They are thus especially simple to use for calculations. Since we are only computing gauge-invariant observables and making sure to always use a gauge that is regular in a neighbourhood of the point at which we evaluate the mode solutions, choosing a gauge which is not regular everywhere should not cause any problems. We will denote the χ_{r_0}-gauge by a (r_0)-superscript on $h^\lambda_{\omega\ell}(t,r)$ for a general r_0 or an (i)-superscript for $r_0 = r_i$, with $i\in\{-,+,c\}$.

We can now specify the modes used for the definition of the Unruh state. We will call these solutions Unruh modes. Similar to the discussion in Section 2.5, they are defined by the asymptotic behaviour of $h^\lambda_{\omega\ell}$ on $\mathcal{H}\cup\mathcal{H}_c$, and behave like

$$h^{\mathrm{in}(c)}_{\omega\ell} \sim \begin{cases} |\omega|^{-1/2}\, e^{-i\omega V_c} & \text{on } \mathcal{H}_c\,,\\ 0 & \text{on } \mathcal{H}_+\,, \end{cases} \tag{4.1.15a}$$

$$h^{\mathrm{up}(+)}_{\omega\ell} \sim \begin{cases} 0 & \text{on } \mathcal{H}_c\,, \\ |\omega|^{-1/2}\, e^{-i\omega U_+} & \text{on } \mathcal{H}_+\,. \end{cases} \tag{4.1.15b}$$

The normalization is chosen such that

$$i\sigma\left(\phi^{\lambda}_{\omega\ell m}, \phi^{\lambda'}_{\omega'\ell'm'}\right) = \delta_{\lambda\lambda'}\delta_{\ell\ell'}\delta_{mm'}\delta(\omega-\omega')$$

for all positive frequency modes ($\omega > 0$).

As mentioned above, we assume that we are in a parameter region in which there are no classical instabilities. In other words, we assume that in region I, solutions to (3.3.1) of the form (4.1.14) with asymptotic behaviour

$$h^{(+)}(r_*, t) \sim e^{-i\omega v} \qquad \text{for } r_* \to -\infty, \tag{4.1.16a}$$

$$h^{(c)}(r_*, t) \sim e^{-i(\omega+\omega_{\mathrm{I}})u} \qquad \text{for } r_* \to +\infty, \tag{4.1.16b}$$

exist only for ω with negative imaginary part.

In this case, we can expand the charged scalar field in terms of positive frequency Unruh modes as

$$\Phi^{(r_0)}(x) = \sum_{\lambda,\ell,m} \int_0^\infty \mathrm{d}\omega \left(\phi^{(r_0)\lambda}_{\omega\ell m}(x) a^{\lambda}_{\omega\ell m} + \phi^{(r_0)\lambda}_{-\omega\ell m}(x) b^{\lambda\dagger}_{\omega\ell m}\right)\,, \tag{4.1.17}$$

with λ running over "in" and "up". The coefficients $a^{\lambda}_{\omega\ell m}$ and $b^{\lambda}_{\omega\ell m}$ are then taken to be the usual annihilation operators on a Fock space satisfying the canonical commutation relations

$$\left[a^{\lambda}_{\omega\ell m}, a^{\lambda'\dagger}_{\omega'\ell'm'}\right] = \left[b^{\lambda}_{\omega\ell m}, b^{\lambda'\dagger}_{\omega'\ell'm'}\right] = \delta_{\lambda\lambda'}\delta_{\ell\ell'}\delta_{mm'}\delta(\omega-\omega')\,, \tag{4.1.18}$$

and all other commutators vanish. The Unruh state is defined by

$$a^{\lambda}_{\omega\ell m}|0\rangle_{\mathrm{U}} = b^{\lambda}_{\omega\ell m}|0\rangle_{\mathrm{U}} = 0\,.$$

Hence, it is the vacuum state of the Fock space on which the $a^{\lambda}_{\omega\ell m}$ and $b^{\lambda}_{\omega\ell m}$ act.

Similar to the computation at the end of Chapter 2, one can convince oneself that this state indeed agrees with the Hadamard state defined by the two-point function (4.1.8).

4.2 The renormalized current

In this section, we derive a formula for the charge current of the charged scalar field on RNdS in the Unruh state using point-split renormalization. We will utilize the mode-sum formulation of the Unruh state. The contents of this section were published in [117]. For this paper, I computed the analytic and numerical results displayed below and in the next section under the supervision of J. Zahn.

Formulas for the renormalized charge current have been obtained previously. In [124], a formula for the renormalized charge current in a general curved spacetime was derived. However, the computation seems to contain a sign error, leading to a divergent counterterm when applied to the RNdS spacetime. [125] also presents a formula for the charge current with point-split renormalization and discusses current conservation and the renormalization ambiguities of this formula, but without referring to any particular spacetime. A mode-sum formula for j_r for the massless scalar on Reissner-Nordström spacetimes in the Boulware state using Hadamard point-split renormalization is presented, and evaluated numerically, in [126]. In contrast to this, we use the Unruh state, making our formula valid in the black-hole exterior as well as in the interior. Moreover, our formula will be applicable for any component of the current and for any mass of the scalar field.

The formula for the charge current of a classical charged scalar field is

$$j_\nu(x) = iq\left(\Phi(x)D^*_\nu\Phi^*(x) - \Phi^*(x)D_\nu\Phi(x)\right) , \tag{4.2.1}$$

where $D^*_\nu = \nabla_\nu + iqA_\nu$. Since this is local and quadratic in the field, the corresponding observable for the quantum scalar field requires renormalization. Here, we will apply Hadamard point-split renormalization, which was introduced in Section 2.2 and which is local and (gauge) covariant [6, 7, 54]. For the current, the renormalization ambiguities are proportional to the current J_ν of the background electromagnetic field [54, 125]. As mentioned in Section 2.4, the current corresponding to the background electromagnetic field in RNdS vanishes, and hence Hadamard point-split renormalization gives a unique result for the renormalized current. Other applications of this renormalization scheme in the context of curved spacetime or background electromagnetic fields can be found for example in [16, 47, 92] and [127, 128].

In the following, we derive a formula for the renormalized current in the Unruh state,

$$\begin{aligned}\langle j_\nu\rangle_{\mathrm{U}} = \lim_{x'\to x}\big(&iq\left\langle\{\Phi(x),\partial_\nu\Phi^*(x')\} - \{\Phi^*(x),\partial_\nu\Phi(x')\}\right\rangle_{\mathrm{U}} \\ &+ 2q\partial'_\nu \mathrm{Im}H(x,x')\big) .\end{aligned} \tag{4.2.2}$$

Here, we have introduced a symmetrisation $\{A, B\} = 1/2(AB + BA)$ for computational convenience. In addition, we used the gauge independence of the current to choose a gauge for the evaluation in which $A(x') = 0$. As a result, the gauge-covariant derivatives reduce to partial derivatives.

$H(x, y)$ is the Hadamard parametrix for the charged scalar field, which is of the same form as that for the real scalar field shown in (2.2.9). The Hadamard coefficients for the charged scalar field satisfy the symmetry relations

$$\overline{U(x,x')} = U(x',x), \qquad \overline{V_n(x,x')} = V_n(x',x). \tag{4.2.3}$$

This property allows us to reduce the parametrix for the current in (4.2.2) to a multiple of $\partial'_\nu\,\mathrm{Im}H(x,x')$.

The derivation of the formula will proceed in two steps. In a first step, we will compute a mode-sum formula for the first term on the right-hand side of (4.2.2) for x and x' at non-zero geodesic distance. Since the point splitting can be viewed as a regularization of the current, we will refer to this term as the regularized current $\langle j_\nu(x,x')\rangle_{\mathrm{U}}$. In a second step, we will compute the contribution of the Hadamard parametrix, which one could call the counterterms. We find that the counterterms in the present case are finite, vanish at the horizon and only contribute to the t-component of the current.

Let us start with the mode-sum formula for the regularized current. From (4.1.17) and (4.1.18), we find that it takes the form

$$\begin{aligned}&\langle j_\nu(x,x')\rangle_{\mathrm{U}} \\ &= 2q\sum_{\lambda,\ell,m}\int_0^\infty \mathrm{d}\omega \mathrm{Im}\left(\overline{\phi}^\lambda_{\omega\ell m}(x)\partial_\nu\phi^\lambda_{\omega\ell m}(x')+\overline{\phi}^\lambda_{-\omega\ell m}(x)\partial_\nu\phi^\lambda_{-\omega\ell m}(x')\right)\,.\end{aligned} \tag{4.2.4}$$

In order to make the formula easier to evaluate numerically, we would like to expand the Unruh modes in terms of another set of modes. If we choose $h^\lambda_{\omega\ell}$ of the form

$$\tilde{h}^\lambda_{\omega\ell} = |\omega|^{-1/2}\,e^{-i\omega t}R^\lambda_{\omega\ell}(r)\,, \tag{4.2.5}$$

then the Klein-Gordon equation separates and reduces to an ODE for $R^\lambda_{\omega\ell}(r)$. This ODE can be rewritten as in (3.3.2) and solved numerically as described in Section 3.3. Thus, this ansatz is particularly useful for computational purposes. We define the so-called Boulware modes utilizing this ansatz separately for regions I and II. In region I, they are defined by their asymptotic behaviour on $\mathcal{H}^-_+\cup\mathcal{H}^-_c$,

$$|\omega|^{1/2}\,\tilde{h}^{(c)\mathrm{inI}}_{\omega\ell}\sim e^{-i\omega v}\quad \text{on } \mathcal{H}^-_c\,,\qquad |\omega|^{1/2}\,\tilde{h}^{(+)\mathrm{upI}}_{\omega\ell}\sim e^{-i\omega u}\quad \text{on } \mathcal{H}^-_+\,, \tag{4.2.6}$$

and vanishing boundary condition on the other horizon, correspondingly. In region II, the asymptotic behaviour is given on $\mathcal{H}^L_+\cup\mathcal{H}^R_+$, and the non-vanishing parts are

$$|\omega|^{1/2}\,\tilde{h}^{(+)\mathrm{inII}}_{\omega\ell}\sim e^{-i\omega v}\quad \text{on } \mathcal{H}^R_+\,,\qquad |\omega|^{1/2}\,\tilde{h}^{(+)\mathrm{upII}}_{\omega\ell}\sim e^{-i\omega u}\quad \text{on } \mathcal{H}^L_+\,. \tag{4.2.7}$$

The modes from region I can be extended to region II by comparing their asymptotic behaviour near $\mathcal{H}^R_+$ to the corresponding behaviour of the modes from II.

In terms of the Boulware modes, the Unruh modes in I $\cup$ II can be expanded as

$$|\omega|^{1/2}h^\lambda_{\omega\ell}(r_*,t) = \sum_{\mathrm{N}\in\{\mathrm{I,II}\}}\int_{-\infty}^\infty \frac{\mathrm{d}\omega'}{2\pi}|\omega'|^{1/2}\alpha^{\lambda\mathrm{N}}_{\omega\omega'}\tilde{h}^{\lambda\mathrm{N}}_{\omega'\ell}(r_*,t)\,. \tag{4.2.8}$$

The coefficients $\alpha^{\lambda\mathrm{N}}_{\omega\omega'}$ can be computed by a double Fourier transform of the asymptotic behaviour of the Unruh modes [90]. The resulting integral has a divergence for $\omega'\to 0$, which has to be regularized by introducing a small imaginary part for ω'. As a result, the regularized coefficients take the form

$$\alpha^{\mathrm{inI}}_{\omega\omega'} = \lim_{\epsilon\to 0} \frac{1}{\kappa_c} |\omega|^{i\frac{\omega'}{\kappa_c}} e^{\mathrm{sgn}(\omega)\frac{\pi\omega'}{2\kappa_c}} \Gamma\left(-i\frac{\omega'+i\epsilon}{\kappa_c}\right) \tag{4.2.9a}$$

$$\alpha^{\mathrm{upI}}_{\omega\omega'} = \lim_{\epsilon\to 0} \frac{1}{\kappa_+} |\omega|^{i\frac{\omega'}{\kappa_+}} e^{\mathrm{sgn}(\omega)\frac{\pi\omega'}{2\kappa_+}} \Gamma\left(-i\frac{\omega'+i\epsilon}{\kappa_+}\right) \tag{4.2.9b}$$

$$\alpha^{\mathrm{upII}}_{\omega\omega'} = \overline{\alpha^{\mathrm{upI}}_{\omega\omega'}} \tag{4.2.9c}$$

$$\alpha^{\mathrm{inII}}_{\omega\omega'} = 0\,. \tag{4.2.9d}$$

Before inserting these results into (4.2.4), let us argue that the regularization in $\alpha^{\lambda\mathrm{N}}_{\omega\omega'}$ can be dropped. To this end, we take a closer look at the Boulware modes $\tilde{h}^{\lambda\mathrm{N}}_{\omega\ell}$, focussing on the upI-modes as an example. These modes enter I from $\mathcal{H}^-_+$. They are then either scattered back to $\mathcal{H}^R_+$ or are transmitted to $\mathcal{H}^L_c$. Consequently, their radial component $R^{\mathrm{upI}}_{\omega\ell}(r)$ behaves like

$$R^{\mathrm{upI}}_{\omega\ell}(r) \sim \begin{cases} e^{i\omega r_*} + \mathcal{R}^{\mathrm{I}}_{\omega\ell} e^{-i\omega r_*} & r_* \to -\infty \\ \mathcal{T}^{\mathrm{I}}_{\omega\ell} e^{i(\omega+\omega_{\mathrm{I}})r_*} & r_* \to \infty\,, \end{cases} \tag{4.2.10}$$

with ω_{I} defined below (3.3.6). While the scattering coefficients $\mathcal{R}^{\mathrm{I}}_{\omega\ell}$ and $\mathcal{T}^{\mathrm{I}}_{\omega\ell}$ are not known in closed form, it is possible to obtain information on their asymptotic behaviour for small ω. To this end, one can employ a first-order expansion of the radial equation (3.3.4) in $(r-r_+)$ near the event horizon [91]. Demanding the solution to this approximate equation to vanish as $r_* \to \infty$ and to behave like $R^{\mathrm{upI}}_{\omega\ell}$ as $r_* \to -\infty$, one finds that for small ω

$$\mathcal{R}^{\mathrm{I}}_{\omega\ell} \sim -1 + \mathcal{O}(\omega)\,.$$

One can combine this result with the relation (3.3.7) to find $\mathcal{T}^{\mathrm{I}}_{\omega\ell} \sim \mathcal{O}(\omega)$. Thus, $R^{\mathrm{upI}}_{\omega\ell}$ vanishes pointwise as $\omega \to 0$ near the boundaries of region I. By differentiating (4.2.10) with respect to r_*, one can see that the same is true for $\partial_{r_*} R^{\mathrm{upI}}_{\omega\ell}$. Since $R^{\mathrm{upI}}_{\omega\ell}$ is a solution to (3.3.4), this implies that $R^{\mathrm{upI}}_{\omega\ell} \to 0$ pointwise as $\omega \to 0$.

Similarly, one can demonstrate that the inI-Boulware modes on $\mathrm{I}\cup\mathrm{II}$ and the combination of the upI-Boulware modes reflected into the black-hole region II together with the upII-modes vanish pointwise for $\omega \to 0$. Since these are exactly the combination of modes appearing in the expansion (4.2.8), these results show that the regularization for the coefficients $\alpha^{\lambda\mathrm{N}}_{\omega\omega'}$ can be safely neglected.

We can now insert the expansion (4.2.8) into (4.2.4). We will use a point splitting in the θ-direction [15]. This means we take $x = (t, r, \theta, \varphi)$ and $x' = (t+\delta, r, \theta+\epsilon, \varphi)$. The small offset in the t-direction is used to guarantee the convergence of the integral over ω, and the limit $\delta \to 0$ is taken before the limit of $\epsilon \to 0$.

One can convince oneself that by the spherical symmetry of both the spacetime and the state, the radial components of the current should vanish. This can also be seen by considering the mode-sum expression for the regularized current and realizing that with the derivatives in the θ- or φ-direction, the expression vanishes in the coinciding-point limit either mode-wise or when summed over m respectively. Therefore, we assume that the partial derivative in (4.2.4) acts either on r or t. The formula then reads

$$\begin{aligned}\langle j_\nu(x,x')\rangle_{\mathrm{U}} =& \frac{q}{16\pi^3} \sum_{\ell,m,\lambda,\mathrm{N},\mathrm{N}'} \int_0^\infty \frac{\mathrm{d}\omega}{\omega} \int_{-\infty}^{\infty} \mathrm{d}\omega' \, |\omega'|^{\frac{1}{2}} \int_{-\infty}^{\infty} \mathrm{d}\omega'' \, |\omega''|^{\frac{1}{2}} \\ & \times \left(\overline{\alpha^{\lambda\mathrm{N}}_{\omega\omega'}} \alpha^{\lambda\mathrm{N}'}_{\omega\omega''} + \overline{\alpha^{\lambda\mathrm{N}}_{-\omega\omega'}} \alpha^{\lambda\mathrm{N}'}_{-\omega\omega''} \right) \frac{1}{r} \overline{Y_{\ell m}}(\theta,\varphi) Y_{\ell m}(\theta+\epsilon,\varphi) \\ & \times \mathrm{Im}\left[\overline{\tilde{h}^{\lambda\mathrm{N}}_{\omega'\ell}}(r,t) \partial_\nu \left(\frac{1}{r} \tilde{h}^{\lambda\mathrm{N}'}_{\omega''\ell}(r,t+\delta) \right) \right] . \end{aligned} \tag{4.2.11}$$

Here and in the following, the Boulware modes should be evaluated in the (r_0)-gauge with $r_0 = r(x')$. We have omitted the gauge superscript here and in the following to avoid notational clutter.

The expression in (4.2.11) can be simplified further by realising that the derivative acting on r^{-1} will lead to an expression that vanishes mode-wise when the limit $\delta \to 0$ is taken. Since the spherical harmonics are the only part in (4.2.11) which depends on m, one can use the identity

$$\sum_{m=-\ell}^{\ell} \overline{Y_{\ell m}}(\theta,\phi) Y_{\ell m}(\theta+\epsilon,\phi) = \frac{2\ell+1}{4\pi} P_\ell(\cos\epsilon) \tag{4.2.12}$$

to perform the sum over m. Moreover, one can explicitly perform the integral over ω, which only involves the α-coefficients. Dropping the regularization of the coefficients, one finds

$$\int_0^\infty \frac{d\omega}{\omega} \left(\overline{\alpha^{\lambda\mathrm{N}}_{-\omega\omega'}} \alpha^{\lambda\mathrm{N}'}_{-\omega\omega''} + \overline{\alpha^{\lambda\mathrm{N}}_{\omega\omega'}} \alpha^{\lambda\mathrm{N}'}_{\omega\omega''} \right) = 4\pi^2 C_{\lambda\mathrm{N}\mathrm{N}'}(\omega') \delta(\omega'-\omega'') \,. \tag{4.2.13}$$

The $C_{\lambda\mathrm{N}\mathrm{N}'}(\omega')$ are real coefficients given by

$$C_{\mathrm{in},\mathrm{I},\mathrm{I}}(\omega) = \coth\left(\pi \tfrac{\omega}{\kappa_c}\right) \omega^{-1} \,, \tag{4.2.14a}$$

$$C_{\mathrm{up},\mathrm{I},\mathrm{I}}(\omega) = C_{\mathrm{up},\mathrm{II},\mathrm{II}}(\omega) = \coth\left(\pi \tfrac{\omega}{\kappa_+}\right) \omega^{-1} \,, \tag{4.2.14b}$$

$$C_{\mathrm{up},\mathrm{I},\mathrm{II}}(\omega) = C_{\mathrm{up},\mathrm{II},\mathrm{I}}(\omega) = \left[\omega \sinh\left(\pi \tfrac{\omega}{\kappa_+}\right)\right]^{-1} \,, \tag{4.2.14c}$$

and all other $C_{\lambda\mathrm{N}\mathrm{N}'}(\omega)$ vanish.

Thus, the renormalized current takes the form

$$\langle j_\nu(x,x')\rangle_{\mathrm{U}} = \frac{q(2\ell+1)}{16\pi^2 r^2} \sum_{\ell,\lambda,\mathrm{N}\mathrm{N}'} P_\ell(\cos\epsilon) \int_{-\infty}^{\infty} \mathrm{d}\omega \, |\omega| \, \mathrm{Im}\left[C_{\lambda\mathrm{N}\mathrm{N}'}(\omega) \overline{\tilde{h}^{\lambda\mathrm{N}}_{\omega\ell}}(r,t) \partial_\nu \tilde{h}^{\lambda\mathrm{N}'}_{\omega\ell}(r,t+\delta) \right] . \tag{4.2.15}$$

Let us take a closer look at the integrand. The splitting in t introduces an oscillatory term $\sim e^{-i(\omega-\omega_g)\delta}$, where ω_g is the shift between the gauge in which the mode is defined and the one in which it is evaluated. Thus, for the inI-modes ω_g is equal to $\omega_{r,\text{in}} = qQ\left(r_0^{-1} - r_c^{-1}\right)$, while for the up-modes $\omega_g = \omega_{r,\text{up}} = qQ\left(r_0^{-1} - r_+^{-1}\right)$. We will shift the integrals over ω by ω_g to make sure that all oscillatory terms have the same frequency, so that the limit $\delta \to 0$ can be taken later.

Finally, let us discuss the convergence of the integral over ω. For this, we consider the large-$|\omega|$-limit of the radial equation. In this limit, (3.3.4) reduces to

$$\left[\partial_{r_*}^2 + \omega^2\right] R_{\omega\ell}(r_*) = 0\,.$$

Thus, in this limit, $R_{\omega\ell}(r_*) \sim e^{\pm i\omega r_*}$. As a result, the integrand takes the form

$$\coth\left(\pi\tfrac{\omega}{\kappa_c}\right)(\partial_\nu t + \partial_\nu r_*)\cos(\delta\omega) + \coth\left(\pi\tfrac{\omega}{\kappa_+}\right)(\partial_\nu t - \partial_\nu r_*)\cos(\delta\omega)\,. \tag{4.2.16}$$

Since this is antisymmetric in ω, the contributions to the integral from $\omega \to \infty$ and $\omega \to -\infty$ cancel. This alone is not sufficient to conclude the convergence of the integral. However, results on similar one-dimensional scattering problems and the numerical results presented in the next two sections indicate that the cancellation happens not only at leading order in $|\omega|^{-1}$, but is strong enough to make the integral converge. Assuming that this is indeed true, one can take the limit $\delta \to 0$. We will also split the integral at $\omega = 0$, and change the integration variable from ω to $-\omega$ on $\mathbb{R}_-$ to make full use of this cancellation in the numerical computation.

Next, we turn to the computation of the counterterm. This means, we want to compute (2.2.9) for the charged scalar field on the RNdS spacetime. In particular, we need the imaginary part of $H(x,x')$ for x and x' separated by ϵ in the θ-direction. We can assume ϵ to be sufficiently small so that x and x' are in a geodesically convex neighbourhood. The Hadamard coefficients $U(x,x')$ and $V_n(x,x')$ for the charged scalar field can be determined from the transport equations induced by the Klein-Gordon equation of the charged scalar field. Using the notation $\sigma^\nu = \nabla^\nu\sigma$ for the derivatives of Synge's world function, they can be written as

$$\left[\sigma^\nu D_\nu + \frac{1}{2}\Box\sigma - 2\right] U = 0, \tag{4.2.17a}$$

$$2\left[\sigma^\nu D_\nu + \frac{1}{2}\Box\sigma - 1\right] V_0 = -\left[D_\nu D^\nu - \mu^2\right] U, \tag{4.2.17b}$$

$$2(n+1)\left[\sigma^\nu D_\nu + \frac{1}{2}\Box\sigma + n\right] V_{n+1} = -\left[D_\nu D^\nu - \mu^2\right] V_n\,, \tag{4.2.17c}$$

compare [125]. As in the real scalar case, the correct behaviour of the leading divergence is ensured by the initial condition $U(x,x) = 1$. This way, $U(x,x')$ is uniquely determined to be of the form $U(x,x') = \Delta^{1/2}(x,x')P(x,x')$. Here, $\Delta(x,x')$ is the Van Vleck-Morette determinant [63], and $P(x,x')$ is the parallel transport along the geodesic from x to x' with respect to D_ν. It is determined by $\sigma^\nu D_\nu P(x,x') = 0$ with initial condition $P(x,x) = 1$.

As discussed in Section 2.2, the series expansion of the logarithmic divergences does not converge in general, but for the normalization of the current we only need the imaginary parts of terms up to order $n = 1$.

Since we are ultimately interested in the coinciding-point limit, we compute approximations of the Hadamard coefficients by expanding their imaginary parts in a covariant Taylor series around x of the form

$$F(x, x') = F^{(0)}(x) + F^{(1)}_{\alpha}(x)\sigma^{\alpha}(x, x') + F^{(2)}_{\alpha\beta}(x)\sigma^{\alpha}(x, x')\sigma^{\beta}(x, x') + \dots \tag{4.2.18}$$

by successively evaluating higher covariant derivatives of $U(x, x')$ and $V_n(x, x')$ in the coinciding-point limit. Plugging the expansion into the transport equations (4.2.17), one obtains [125]

$$\text{Im}\left(U^{(1)}_{\alpha}\right) = qA_{\alpha}\,, \tag{4.2.19a}$$

$$\text{Im}\left(U^{(2)}_{\alpha\beta}\right) = -\frac{q}{2}\nabla_{(\alpha}A_{\beta)}\,, \tag{4.2.19b}$$

$$\text{Im}\left(U^{(3)}_{\alpha\beta\gamma}\right) = \frac{q}{6}\text{Re}\left(D_{(\alpha}D_{\beta}A_{\gamma)}\right) + \frac{q}{12}A_{(\alpha}R_{\beta\gamma)}\,, \tag{4.2.19c}$$

$$\text{Im}\left(V^{(1)}_{0\alpha}\right) = \frac{q}{2}\left[\mu^2 - \frac{1}{6}R\right]A_{\alpha} - \frac{q}{12}\nabla^{\nu}F_{\nu\alpha}\,, \tag{4.2.19d}$$

$$\begin{aligned}\text{Im}\left(V^{(2)}_{0\alpha\beta}\right) = &-\frac{q}{4}\left[\mu^2 - \frac{1}{6}R\right]\text{Re}\left(D_{(\alpha}A_{\beta)}\right) \\ &+ \frac{q}{24}A_{(\alpha}\nabla_{\beta)}R - \frac{q}{24}\nabla_{(\alpha}\nabla^{\nu}F_{\beta)\nu}\,,\end{aligned} \tag{4.2.19e}$$

$$\text{Im}\left(V^{(0)}_{1}\right) = 0\,. \tag{4.2.19f}$$

When we take into consideration that we compute the current in a gauge such that $A(x)$ vanishes in the coinciding-point limit where $x = x'$, we can immediately eliminate all terms containing A with no derivatives acting on it. The most important consequence is that this entails $U^{(1)}_{\alpha} = 0$.

To relate the expansion in terms of σ^{ν} to an expansion in terms of the angular separation ϵ, we need to expand σ^{ν} in terms of ϵ. This expansion can be obtained from the one of σ in terms of $\Delta x^{\alpha} = x^{\alpha} - x'^{\alpha}$,

$$\begin{aligned}\sigma(x, x') =&\frac{1}{2}g_{\alpha\beta}\Delta x^{\alpha}\Delta x^{\beta} + A_{\alpha\beta\gamma}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma} + B_{\alpha\beta\gamma\delta}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}\Delta x^{\delta} \\ &+ C_{\alpha\beta\gamma\delta\epsilon}\Delta x^{\alpha}\Delta x^{\beta}\Delta x^{\gamma}\Delta x^{\delta}\Delta x^{\epsilon} + \dots\,,\end{aligned} \tag{4.2.20}$$

with coefficients

$$A_{\alpha\beta\gamma} = -\frac{1}{4}\partial_{(\alpha}g_{\beta\gamma)} \tag{4.2.21a}$$

$$B_{\alpha\beta\gamma\delta} = -\frac{1}{3}\Bigg(\partial_{(\alpha}A_{\beta\gamma\delta)} + g^{\nu\rho}\Bigg(\frac{1}{8}\partial_{\nu}g_{(\alpha\beta}\partial_{|\rho|}g_{\gamma\delta)} + \frac{3}{2}\partial_{\nu}g_{(\alpha\beta}A_{|\rho|\gamma\delta)} \tag{4.2.21b}$$

$$+ \frac{9}{2} A_{\nu(\alpha\beta} A_{|\rho|\gamma\delta)} \Big) \Big)$$

$$C_{\alpha\beta\gamma\delta\epsilon} = -\frac{1}{4} \left(\partial_{(\alpha} B_{\beta\gamma\delta\epsilon)} + g^{\nu\rho} \left(12 A_{\nu(\alpha\beta} B_{|\rho|\gamma\delta\epsilon)} + 3 A_{\nu(\alpha\beta} \partial_{|\rho|} A_{\gamma\delta\epsilon)} \right.\right. \tag{4.2.21c}$$
$$\left.\left. + 2\partial_\nu g_{(\alpha\beta} B_{|\rho|\gamma\delta\epsilon)} + \frac{1}{2} \partial_\nu A_{(\alpha\beta\gamma} \partial_{|\rho|} g_{\delta\epsilon)} \right) \right) ,$$

by taking covariant derivatives [129]. Plugging in $\Delta x^\alpha = -\epsilon \delta^\alpha_\theta$ and the RNdS metric, the result is

$$\sigma = \frac{r^2}{2}\epsilon^2 - \frac{fr^2}{24}\epsilon^4 + \mathcal{O}\left(\epsilon^6\right) \tag{4.2.22a}$$

$$\sigma_\theta = -r^2\epsilon + \frac{fr^2}{6}\epsilon^3 + \mathcal{O}\left(\epsilon^4\right) \tag{4.2.22b}$$

$$\sigma_r = \frac{r}{2}\epsilon^2 + \mathcal{O}\left(\epsilon^4\right) \tag{4.2.22c}$$

$$\sigma_t \approx \sigma_\varphi = 0 + \mathcal{O}\left(\epsilon^4\right) . \tag{4.2.22d}$$

Combining this with the imaginary parts of the Hadamard coefficients, we find the counterterm

$$-2q\partial'_\nu \mathrm{Im}\left[H(x,x')\right] = -\frac{1}{4\pi^2}\frac{q^2 Q f(r)}{6r^3}\delta^t_\nu + \mathcal{O}(\epsilon) . \tag{4.2.23}$$

Notably, the counterterm only gives a finite contribution to the current and only affects the t-component. Moreover, it vanishes at the horizons. Similar finite counterterms have been obtained for the current previously, compare [127, 128].

This is in contrast to the result in [124], where a divergent parametrix was obtained. However, this seems to be due to an error in [124, Eq. (8)], where two terms with the derivatives acting on different variables should be subtracted rather than added. If their result is corrected accordingly, it agrees with the one obtained here.

The fact that the counterterm only gives a finite contribution indicates that the coinciding-point limit can be taken and that the sum over ℓ as well as the integral in the regularized current (4.2.15) converge. This is also supported by the numerical results presented in the next section. Our final formula for the renormalized current in the Unruh state is thus given by

$$\langle j_\nu(x)\rangle_{\mathrm{U}} = \sum_\ell \frac{q(2\ell+1)}{16\pi^2 r^2} \int_0^\infty \mathrm{d}\omega \sum_{\lambda,\mathrm{N},\mathrm{N}'} \Big(|\omega + \omega_{r,\lambda}|\, C_{\lambda\mathrm{N}\mathrm{N}'}(\omega+\omega_{r,\lambda}) \tag{4.2.24}$$
$$\times \ \mathrm{Im}\left[\overline{\tilde{h}^{\lambda\mathrm{N}}_{(\omega+\omega_{r,\lambda})\ell}}(r,t)\partial_\nu \tilde{h}^{\lambda\mathrm{N}'}_{(\omega+\omega_{r,\lambda})\ell}(r,t)\right] + \omega \leftrightarrow -\omega \Big)$$
$$+ \frac{1}{4\pi^2}\frac{q^2 Q f(r)}{6r^3}\delta^t_\nu.$$

Before presenting numerical results, let us briefly examine the conservation of the renormalized current. As discussed above, due to the spherical symmetry of the spacetime and the state $\langle j_\theta \rangle_{\mathrm{U}} = \langle j_\varphi \rangle_{\mathrm{U}} = 0$. Moreover, the expectation values are independent of t. As a result, the current conservation equation $\nabla_\nu j^\nu = 0$ reduces to [126]

$$\partial_{r_*} \left(r^2 \langle j_{r_*} \rangle_{\mathrm{U}} \right) = 0 \,.$$

It can be seen immediately that this is satisfied by the contribution from the counterterm, but it is less obvious that the contribution of the mode sum satisfies this equation as well. Therefore, this will be tested numerically in the next section.

4.3 The current in the Unruh state - numerical results

In this section we present numerical results for the current up to the inner horizon of a RNdS black hole. Most of the results in this section have been published in [117]. The numerical results were obtained by me under the supervision of J. Zahn.

As mentioned in the previous section, one important consistency check for the formula of the charge current is to test current conservation. To test this numerically we compute $r^2 \langle j_{r_*} \rangle_{\mathrm{U}}$ at different radii r in the black-hole exterior region I, as well as in the black-hole interior II for the charged scalar of mass $\mu^2 = 2\Lambda/3$. As in the previous chapter, we will also refer to this choice of mass as "conformal coupling", since the equation of motion is the same as for a massless, conformally coupled scalar field. This choice of μ^2 has the advantage that with an ansatz of the form (3.2.10) discussed in Chapter 3, the computation of the Boulware modes reduces to finding local solutions to particular Heun equations [98]. Hence, their implementation in Mathematica 12 allows for very efficient computation of the current in this case [107]. In addition, one can re-express the radial function $R^{\mathrm{inI}}_{\omega\ell}(r)$ of the in-modes in terms of $R^{\mathrm{upI}}_{\omega\ell}(r)$ and its complex conjugate to further reduce the computational effort.

$r^2 \langle j_{r_*} \rangle_{\mathrm{U}}$ can then be computed numerically by evaluating (4.2.24) with the methods described in Chapter 3. Note that we restrict the computations to small charges q and masses μ of the scalar field due to the limitations of the numerical computations. In particular, for most of the calculations we choose the mass to be $\mu^2 = 2\Lambda/3$. We have already discussed in Section 3.4 that this is much smaller that any typical particle masses for a black hole of at least solar mass. Similarly, if one assumes that q agrees with the elementary charge e, then $qQ \sim 10^{36} M/M_\odot$, which is too large to be handled by our numerical code. As a result of the small values of q and μ^2, we need to choose the cosmological constant Λ sufficiently large in order to avoid the appearance of classical instabilities, see the discussion at the beginning of Section 4.1. We will choose it to be $\Lambda M^2 = 0.14$ for comparability with the results for the real scalar field and [111]. We checked that this lies outside the classical instability regime for all tested values of Q/M according to the condition derived in [112]. As a result, we obtain a cosmological horizon radius r_c of the same order of magnitude as the event horizon r_+. In fact, for this value of Λ, $Q/M > 0.755$ is required to ensure $r_c > r_+$, compare the parameter-region plot in Fig. 2.1. This means that

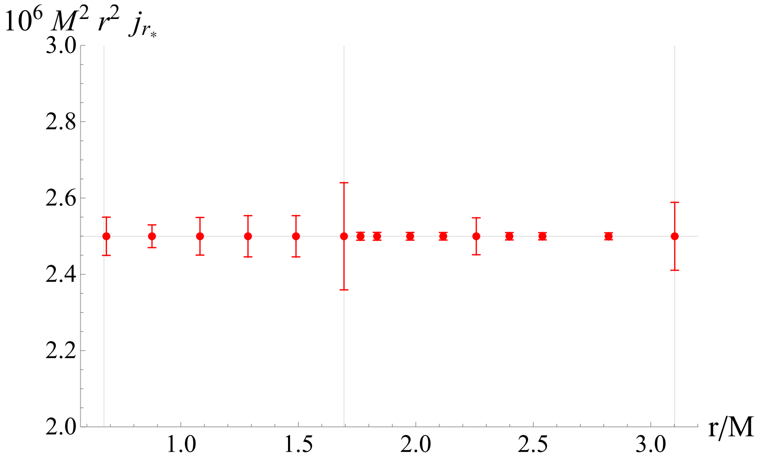


Figure 4.1: $r^2 \langle j_{r_*} \rangle_{\mathrm{U}}$ for the conformally coupled scalar at different radii r. This quantity must be constant for current conservation to be satisfied. The vertical lines mark r_-, r_+ and r_c respectively. We set $Q/M = 0.95$, $\Lambda M^2 = 0.14$, and $qQ = 0.1$.

not only is our cosmological constant unrealistically high, but also the charge of the black hole, which is expected to be very small [130] for astrophysical black holes. However, the large-charge regime of the black hole is interesting due to the occurrence of sCC violation [112] and as a toy model for rapidly rotating black holes.

More details on the numerical computation and in particular the error estimation can be found in the previous chapter.

In Fig. 4.1 we present $r^2 \langle j_{r_*} \rangle_{\mathrm{U}}$ as a function of r in the regions I and II of the black-hole spacetime for $qQ = 0.1$, $\Lambda M^2 = 0.14$, and $Q/M = 0.95$. One can see clearly that it is constant within the error margins of the numerical calculation. This confirms the conservation of the renormalized charge current in the form of (4.2.24).

Next, we wish to study the behaviour of the current at the event horizon. At the horizon, we can use the asymptotic behaviour of $R^{\mathrm{upI}}_{\omega\ell}$ and the reformulation of $R^{\mathrm{inI}}_{\omega\ell}$ in terms of $R^{\mathrm{upI}}_{\omega\ell}$ and its complex conjugate to obtain a formula for the current in terms of the scattering coefficients $\mathcal{R}^{\mathrm{I}}_{\omega\ell}$ and $\mathcal{T}^{\mathrm{I}}_{\omega\ell}$ defined in (3.3.5) and (4.2.10). The formula for the current then reads

$$\langle j_v \rangle_{\mathrm{U}} = -\sum_{\ell} \frac{q(2\ell+1)}{16\pi^2 r^2} \int_0^\infty \mathrm{d}\omega \;\left(F(\omega) + F(-\omega)\right) \tag{4.3.1a}$$

$$F(\omega) = \coth\left(\pi \tfrac{\omega+\omega_{\mathrm{I}}}{\kappa_c}\right)\left(1 - \left|\mathcal{R}^{\mathrm{I}}_{\omega\ell}\right|^2\right) + \coth\left(\pi \tfrac{\omega}{\kappa_+}\right)\left|\mathcal{R}^{\mathrm{I}}_{\omega\ell}\right|^2 . \tag{4.3.1b}$$

The numerical evaluation of this formula is done along the lines of the previous chapter. The error estimate for the numerical computation is implemented as described in detail in Section 3.4. Fig. 4.2 shows the v-component of the current at the event horizon for the conformally coupled scalar field as a function of the scalar-field charge qQ for different values of the black-hole charge Q/M. The cosmological constant has been set to

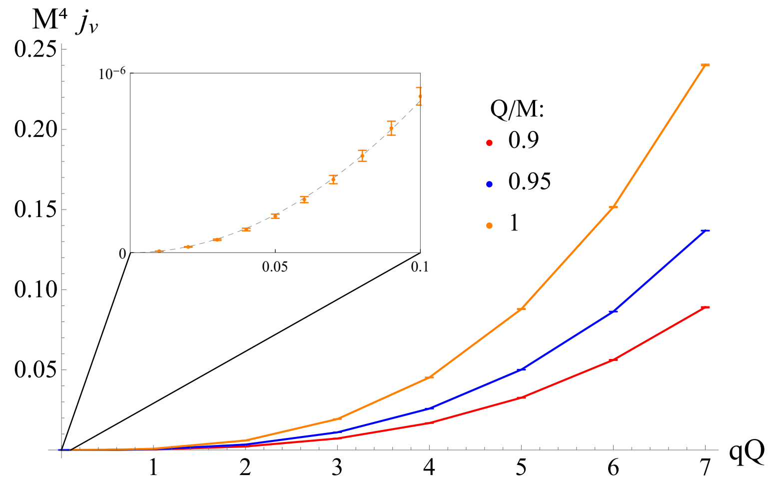


Figure 4.2: The v-component of the current of a conformally coupled scalar field on the event horizon as a function of the scalar field charge in the Unruh vacuum for $\Lambda M^2 = 0.14$. The smaller graph shows the results for $Q/M = 0.95$ for small qQ, the dashed line indicates a q^2-fit.

$\Lambda M^2 = 0.14$, which is outside the classical instability region for the scalar-field charges under consideration. For small charges q, the current $\langle j_v \rangle_{\mathrm{U}}$ behaves approximately as $\sim q^2$. This can be seen from the smaller graph in Fig. 4.2, which shows the current for $Q/M = 0.95$ and $qQ \leq 0.1$. The current has been fitted with a function of the form $a(qQ)^2$, which is represented by the gray dashed line. An intuitive explanation for this behaviour is that the current is caused by a particle of charge q being created near the horizon and subsequently accelerated away from the black hole with an acceleration of order q. Another way to model the current that has been used frequently in the literature on black-hole discharge [131–133] is by an application of Schwinger's pair creation formula [113]. According to this approach, the pair creation rate Γ is non-perturbative in q. In particular, it involves a factor of the form

$$\Gamma \sim \exp\left[-\frac{\pi\mu^2 r_+^2}{qQ}\right] . \tag{4.3.2}$$

For the parameters used in the small window in Fig. 4.2, one finds $\pi\mu^2 r_+^2 \sim 0.84$, and the deviation from $\sim q^2$ should be clearly visible. That this is not the case indicates that the estimate by the Schwinger formula is not applicable for such small masses and charges of the scalar field. This is not surprising, since for the conformal mass considered here the Compton wavelength coincides with the Hubble wavelength, and the flat-space approximation implied in the application of the Schwinger formula is not appropriate in this parameter range.

Looking at the dependence of the current on Q/M, we see that the current increases with the charge of the black hole, at least in this near-extremal regime. This result is in agreement with the findings of [134] that the pair-production rate of an extremal black hole is larger that that of a near-extremal one.

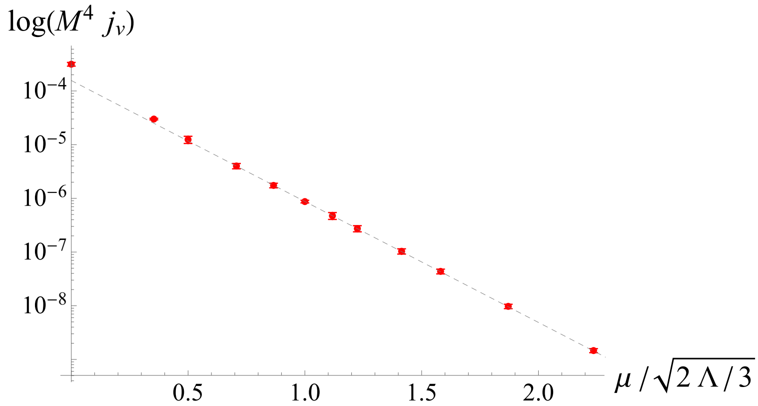


Figure 4.3: The v-component of the current of a charged scalar field of charge $qQ = 0.1$ at the event horizon as a function of the scalar-field mass squared in the Unruh vacuum for $Q/M = 0.95$ and $\Lambda M^2 = 0.14$.

It remains to explore the dependence of the current on the scalar-field mass μ. Fig. 4.3 shows the v-component of the current at the event horizon as a function of μ in units of the conformal mass $\sqrt{2\Lambda/3}$ for $\Lambda M^2 = 0.14$, $Q/M = 0.95$ and $qQ = 0.1$. We find that the current $\langle j_v \rangle_{\mathrm{U}}$ decays exponentially with μ. This can be seen from the corresponding fit function, which is represented in the plot by the gray dashed line. We have excluded the point for $\mu = 0$ from the fit, since it seems that it does not follow this law. In comparison, the result obtained from the Schwinger formula (4.3.2) indicates an exponential decay in the scalar-field mass squared. This agrees with the analysis of the previous case and is a further indication that the Schwinger formula is not applicable for scalar-field masses of the order $\sqrt{2\Lambda/3}$.

Next, let us discuss the other components of the current at the horizon. Since the Unruh state is Hadamard across $\mathcal{H}_+^R$, the expectation value of the renormalized current must be regular across it. Changing to a Kruskal coordinate system which is regular across the horizon, one can see that this requires $\langle j_u \rangle_{\mathrm{U}}$ to vanish at $\mathcal{H}_+^R$. Thus, at the event horizon we have $\langle j_v \rangle_{\mathrm{U}} = \langle j_{r_*} \rangle_{\mathrm{U}} = \langle j_t \rangle_{\mathrm{U}}$ and the results presented here also apply to these components. Therefore, the charge density with respect to the Killing field ∂_t, which is proportional to $\langle j^t \rangle_{\mathrm{U}}$, is negative at the event horizon and its absolute value decreases exponentially as the inverse of the Compton wavelength μ^{-1}.

Moreover, the results shown in Fig. 4.2 and Fig. 4.3 agree with the conserved quantity $r^2 \langle j_{r_*} \rangle_{\mathrm{U}}$ up to a rescaling by r_+^{-2}. Comparing Fig. 4.2 to the results for the corresponding quantity on Reissner-Nordström in [126, Fig. 3], one observes that the results are qualitatively similar. A more detailed comparison is difficult due to the difference in parametrization and choice of parameter range. Note, however, that [126] defines the current with an additional minus sign relative to (4.2.1) or (4.2.2).

Finally, we want to compute the t-component of the current in regions I and II. Together with the r_*-component of the current, this will allow us to compute the u- and v-components. Unfortunately, the t-component of the current, and similarly the u- and

v-components, are very difficult to compute numerically unless one considers them at one of the horizons: First of all, the integrands in (4.2.24) for fixed ℓ have prominent features in a small ω- window that shifts to higher values of ω as ℓ increases in region I and grows broader with increasing ℓ in region II. Moreover, for higher values of ℓ, the result of the ω-integral shrinks due to strong cancellations, even though the amplitude (width) of the features increases in region I (II). This makes the higher ℓ-modes very challenging to compute numerically: they require not only control over the large-ω tail of the integrand for the precise cancellation, but also very high precision throughout the whole calculation.

Second, omitting the higher ℓ-modes is not possible since the decay in ℓ is much slower than the exponential decay observed in the computation of $\langle j_{r_*} \rangle_{\mathrm{U}}$. Therefore, a simple error estimate using the Riemann upper- or lower sum as in Section 3.4 is insufficient. Instead of using this estimate we will discuss all error sources and numerical challenges separately for this case.

We start by considering fixed ℓ and computing the integral over ω in (4.2.24). Since we compute the integrand numerically, we can only integrate up to some cutoff $\omega_{\max}$. Due to the importance of the large-ω tail of the integral described above, one would like $\omega_{\max}$ to be as large as possible. At the same time, at low ω, a small step-size in ω is required to achieve a sufficiently precise estimate of the integrand. Both requirements can be met reasonably well by using two different step sizes in ω for the computation of the integrand: a small step size $(1/2000 - 1/1000)$ for small ω, and a slightly larger step size $(1/100 - 1/20)$ for larger ω. Since a lot of numerical precision is lost in the calculation of the integrand for the higher ℓ-modes, we also increase the numerical precision in the calculation of the integrand for small ω compared to the other calculations.

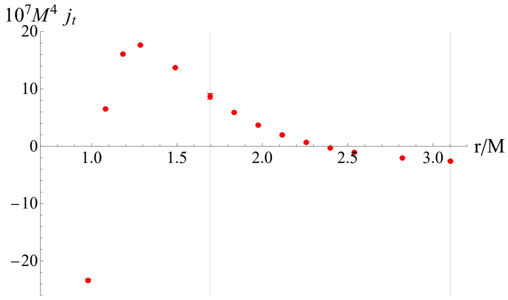
To further increase the precision of the numerical integration, we interpolate the obtained values for the integrand for fixed ℓ using a 10th-order interpolation. One can then use numerical integration to obtain the integral up to $\omega_{\max}$. We check the stability of the result under variation of the interpolation order and the working precision of the numerical integration. With the working precision of the same order as the numerical precision of the interpolating function, we find that the result for the integral is stable up to at least 5 digits.

To compensate for the missing large-ω tail of the numerical integral, we fit the integrand from approximately $\omega_{\max} - 5$ to $\omega_{\max}$ and integrate the fit function from $\omega_{\max}$ to infinity. In all cases, we find that the leading order of the large-ω tail is ω^{-3}, and the best fit by a polynomial in ω^{-1} with two fit parameters is obtained by choosing a fit function of the form

$$f(\omega) = a\omega^{-3} + b\omega^{-5} \,.$$

This power-law decay for the t-component can be understood from the fact that we only subtracted a finite-order approximation of the Hadamard parametrix. Therefore, we expect that the regularized current will not necessarily be a smooth function, but rather a function of some finite regularity, see the discussion in Section 2.2. This will then lead to a power-law decay instead of an exponential decay in the mode sum.

We test the stability of the resulting integral under a change of the fit function by al-



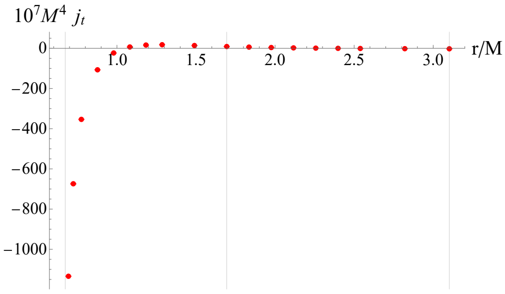


Figure 4.4: $\langle j_t \rangle_{\mathrm{U}}$ for the conformally coupled scalar in region I and II as a function of r for $qQ = 0.1$, $Q/M = 0.95$ and $\Lambda M^2 = 0.14$. The vertical gray lines correspond to the locations of the three horizons r_-, r_+ and r_c respectively. The points on these lines are obtained using the formula for the current in terms of scattering coefficients, including error estimates as described in the previous chapter.

lowing additional higher-order terms of the form $c\omega^{-n}$ in $f(\omega)$. We combine the resulting uncertainty with the error estimates of the fit and the change of the result under a variation of the number of data points used for the fit. We find that the estimate for the tail has at least 3 significant digits.

Due to the strong cancellations in the integral for larger ℓ, this tail becomes increasingly important. For large ℓ, its contribution to the total integral over ω can become even larger than the contribution of the numerical integral up to $\omega_{\max}$. Therefore, it is crucial to include this correction when the convergence in ℓ is considered.

Next, we want to take the sum over the contributions of the individual ℓ-modes of the current. In comparison to $\langle j_{r_*} \rangle_{\mathrm{U}}$, the decay in ℓ is very slow in the present case. In particular, it is not exponential but rather follows a power law. This may again be attributed to the finite regularity discussed above. For a power-law decay, convergence is given as long as the decay is faster than ℓ^{-1}.

In order to test convergence and to correct for the finite cutoff in ℓ, we fit a power law of the form

$$f(\ell) = a\ell^{-b}$$

to the decay of the fixed-ℓ contributions to $\langle j_t \rangle_{\mathrm{U}}$, including the corrections for the large-ω tails, and use the Hurwitz ζ-function to compute an estimate of the sum over $f(\ell)$ from $\ell_{\max} + 1$ to infinity based on this fit. We also estimate the uncertainty of the large-ℓ tail by taking into account variations due to uncertainties in the individual ℓ-modes arising both from the numerical integration and the estimates of the large-ω tails as well as variations of the fit under a change of the fit range and the error estimates for the fit parameters. It turns out that the estimate of the large-ℓ tail has a large uncertainty, but constitutes only up to approximately one percent of the final result for $\langle j_t \rangle_{\mathrm{U}}$.

The results for $\langle j_t \rangle_{\mathrm{U}}$ for $qQ = 0.1$, $Q/M = 0.95$, $\mu^2 = 2\Lambda/3$, and $\Lambda M^2 = 0.14$ are shown in Fig. 4.4, where the evaluation on the horizons uses the formula in terms of scattering coefficients as described above. We have combined all the error sources identified above into our error estimate for $\langle j_t \rangle_{\mathrm{U}}$ displayed in the figure. The largest relative errors are approximately 1.5%. It turns out that the dominant contribution to the error is given by the uncertainty of the estimate of the large-ℓ tail.

Note that $\langle j_t \rangle_{\mathrm{U}}$ starts positive at the event horizon and becomes negative towards the cosmological horizon. This is in agreement with the fact that the Unruh state is Hadamard across $\mathcal{H}_c^L$: The regularity of the Unruh state at $\mathcal{H}_c^L$ requires that $\langle j_v \rangle_{\mathrm{U}} \to 0$ at $\mathcal{H}_c^L$, and thus $\langle j_t \rangle_{\mathrm{U}} \to -\langle j_{r_*} \rangle_{\mathrm{U}}$. By current conservation, we know that $\langle j_{r_*} \rangle_{\mathrm{U}} \to r_c^{-2} r^2 \langle j_{r_*} \rangle_{\mathrm{U}}$ is positive, and therefore $\langle j_t \rangle_{\mathrm{U}}$ must be negative at the cosmological horizon.

The negativity of $\langle j_t \rangle_{\mathrm{U}}$ on the cosmological horizon can also be concluded from a physical argument. ∂_t restricted to region III is space-like and directed inwards, as can be seen by, for example, expressing it in terms of the in- and outgoing radial null vectors. Thus, $-j_t$ is the outward current, which is expected to be positive for a black hole of positive charge.

Throughout most of region II, $\langle j_t \rangle_{\mathrm{U}}$ is positive and only decreases very slowly with r. However, near the inner horizon, there is a drastic change of behaviour: around $r \sim 1.3$, $\langle j_t \rangle_{\mathrm{U}}$ reaches a maximum and starts decreasing rapidly as the inner horizon is approached. Near the inner horizon the absolute value of $\langle j_t \rangle_{\mathrm{U}}$ is significantly larger than in the rest of the spacetime. Comparing to the results for the r_*-component in Fig. 4.1, we expect that the u- and v-component will behave qualitatively similar to the t-component. This indicates that the current at the inner horizon will likely show very different qualitative behaviour from the current at the event horizon and from the behaviour expected from an intuitive particle picture [114]. This will be the subject of the next section.

Overall, we observe that the black hole is always discharged by the scalar field, and our formula for the current satisfies the conservation equation, as already observed in [130]. The deviation from the Schwinger formula can be expected for the range of parameters that we have tested. Thus, the results obtained with our formula agree with expectations. However, we have also seen first hints that a very different behaviour can appear near the inner horizon of the black hole.

4.4 The charged scalar field at the inner horizon

In the following section we will study the scalar field at the inner horizon of the RNdS black hole. In order to do so, we first indicate how the result on the state-independence of the leading divergence of the energy flux at the Cauchy horizon in [16] extends to the charged scalar field, and how a corresponding result can be found for the leading divergence of the current. After that, we introduce a state-subtraction formula for the stress-energy tensor in the Unruh state at the inner horizon and show numerical results for both the current and the stress-energy tensor. This section is based on the paper [116], for which I performed the computations, and its supplementary material [135], which I developed and wrote under the supervision of S. Hollands and J. Zahn.

We start by showing that the result on the state-independence of the leading divergence of the energy flux near the Cauchy horizon, particularly the first part of [16, Prop. 5.1], can be extended to the charged scalar field. In addition, we will argue that such a result can also be found for the charge current.

To this end, let us consider the expectation value of the observable $A(x)$ in the Cauchy-horizon limit in some state Ψ. Here, $A(x)$ is a placeholder for either the charge current or the stress-energy tensor of the charged scalar field. We assume that the state Ψ is Hadamard on $\mathcal{M}$. Performing a null addition, this expectation value can be written as [16]

$$\langle A(x)\rangle_{\Psi} = \langle A(x)\rangle_{\mathrm{U-C}} + \langle A(x)\rangle_{\Psi-\mathrm{U}} + \langle A(x)\rangle_{\mathrm{C}} \ ,$$

where we have used the same notation as in previous sections to denote differences of expectation values between states, and the U- and C- subscripts for the Unruh- and comparison state correspondingly.

As for the real scalar field in Chapter 3, we take the comparison state to be a stationary state which is Hadamard in a neighbourhood of the Cauchy horizon. For its construction, we modify the spacetime beyond the Cauchy horizon so that the central singularity is replaced by the origin of spherical coordinates. We then define the comparison state via mode expansion along the lines of (4.1.17). The modes for the expansion are determined by initial data on $\mathcal{H}^L_- \cup \mathcal{H}^+_-$. Concretely, they satisfy

$$\phi^{C(-)}_{\omega\ell m} \sim \left(4\pi^2 r^2 \left|\omega\right|\right)^{-1/2} Y_{\ell m}(\theta,\varphi) e^{-i\omega V_-} \tag{4.4.1}$$

at $\mathcal{H}^L_- \cup \mathcal{H}^+_-$. The proof of the Hadamard property of this state in a neighbourhood of the Cauchy horizon $\mathcal{H}^R_-$ follows along the same lines as for the real scalar [16], see also the discussion for the Unruh state above. The modification of the spacetime does not affect the expectation values of observables at $\mathcal{H}^L_-$ computed in the comparison state. Therefore, this description of the comparison state and the modification of the spacetime is sufficient for the purposes of this work.

By construction, the expectation values of the current and the stress-energy tensor in the comparison state should be finite at the Cauchy horizon. The difference between the Unruh- and the comparison state expectation values is independent of the state Ψ. To show that this state-independent term is the leading divergence of $\langle A\rangle_{\Psi}$ at the Cauchy horizon, one has to show an upper limit for the potential divergence of $\langle A(x)\rangle_{\Psi-\mathrm{U}}$, which contains the state dependence. This amounts to showing

Proposition 4.4.1. *Let $x \in \mathcal{H}^R_-$ be a point on the Cauchy horizon. Let $\mathcal{U}$ be a small open neighbourhood of x with compact closure in the analytic extension of $\mathcal{M}$ and contained in a coordinate chart of the form $(x^\nu) = (V_-, y^i)$. Here, y^i are coordinates parametrizing $\mathcal{H}^R_-$ in a neighbourhood of x. Let $\langle\cdot\rangle_{\Psi-\mathrm{U}}$ denote the difference of expectation values between an arbitrary but fixed Hadamard state on $\mathcal{M}$ and the Unruh state. Assume that $\beta = \frac{\alpha}{\kappa_-} > \frac{1}{2}$, where α is the spectral gap. Then $\langle T_{\nu\varrho}\rangle_{\Psi-\mathrm{U}}(V_-,\cdot)$ and $\langle j_\nu\rangle_{\Psi-\mathrm{U}}(V_-,\cdot)$ are smooth functions of y^i on $\mathcal{U}_{\mathcal{M}} \equiv \mathcal{U}\cap\mathcal{M}$. In addition, $\langle T_{\nu\varrho}\rangle_{\Psi-\mathrm{U}}(\cdot,y^i)$, considered as a function of $V_- < 0$, is locally in $L^p(\mathbb{R}_-)$, $p = (2-2\beta')^{-1}$, for any β' with $\frac{1}{2} < \beta' < \min(1,\beta)$, while $\langle j_\nu\rangle_{\Psi-\mathrm{U}}(\cdot,y^i)$ is locally in $L^{2p}(\mathbb{R}_-)$. Their norm is uniformly bounded in y^i within $\mathcal{U}_{\mathcal{M}}$.*

To prove this proposition, we first show the ensuing lemma, which is an adaptation of [136, Lemma 3.7] to the charged scalar field and the geometric situation at hand.

Lemma 4.4.2. *Let $(\mathcal{M}, g)$ be the RNdS spacetime, $\mathcal{U}$ as in Proposition 4.4.1, and let $W \in C^\infty(\mathcal{M} \times \mathcal{M})$ satisfy $\mathcal{K}_q(x)W(x,y) = \overline{\mathcal{K}_q}(y)W(x,y) = 0$. Then one can find a $B \in C_0^\infty(\mathcal{M} \times \mathcal{M})$ such that*

$$\begin{aligned}&\int\limits_{\mathcal{M}\times\mathcal{M}} \bar{f}(x)W(x,y)h(y)\, dvol_g(x)\, dvol_g(y) \\ &= \int\limits_{\mathcal{M}\times\mathcal{M}} \overline{E(f)}(x)B(x,y)E(h)(y)\, dvol_g(x)\, dvol_g(y)\end{aligned} \tag{4.4.2}$$

for all $f, h \in C_0^\infty(\mathcal{U}_\mathcal{M})$.

Proof. Let $\sigma_\pm$ be two Cauchy surfaces of $\mathcal{M}$ so that $\sigma_+ \subset I^+(\sigma_-)$ and $\mathcal{U}_\mathcal{M} \subset I^+(\sigma_+)$. Let $f, h \in C_0^\infty(\mathcal{U}_\mathcal{M})$. Then $J(\mathrm{supp}(f) \cup \mathrm{supp}(h)) \cap J^+(\sigma_-) \cap J^-(\sigma_+)$ is contained in the closure of $J(\mathcal{U}_\mathcal{M}) \cap J^+(\sigma_-) \cap J^-(\sigma_+)$ in $\mathcal{M}$, which we will call G and which is a compact subset of $\mathcal{M}$.

We start by noting that for $\mathcal{K}_q$, a generalization of Green's formula takes the form

$$\int\limits_{\mathcal{L}} (\overline{u}\mathcal{K}_q v - v\overline{\mathcal{K}_q u})\, \mathrm{d}vol_g = \int\limits_{\partial\mathcal{L}} (\overline{u}D_\mu v - v\overline{D_\mu u})n^\mu\, \mathrm{d}vol_\gamma\,,$$

where u, v are smooth functions and we assume that the intersection of their supports with $\mathcal{L}$ is compact. $\mathcal{L} \subset \mathcal{M}$ is any closed subset of $\mathcal{M}$ and its boundary $\partial\mathcal{L}$ in $\mathcal{M}$ has outward-pointing unit normal n^μ and induced metric γ.

Next, we set

$$\tilde{f} = \mathcal{K}_q(\chi E(f))$$

for $f \in C_0^\infty(\mathcal{U}_\mathcal{M})$, and equivalently we define $\tilde{h}$. Here, $\chi \in C^\infty(\mathcal{M})$ is equal to one on $J^+(\sigma_+)$ and vanishes on $J^-(\sigma_-)$. As discussed in Section 2.2, $\tilde{f} \in C_0^\infty(\mathcal{M})$ with support contained in G, and $E(\tilde{f}) = E(f)$. This allows the replacement of f and h by $\tilde{f}$ and $\tilde{h}$ on the right-hand side of (4.4.2). By the property of the kernel of E, we can find an $f_0 \in C_0^\infty(\mathcal{M})$ so that $\tilde{f} = f + \mathcal{K}_q(f_0)$. Since $W(x,y)$ satisfies the (complex conjugate) Klein-Gordon equation in the first (second) variable, and all other functions are compactly supported within $\mathcal{M}$, we have by an application of Green's formula

$$\int\limits_{\mathcal{M}\times\mathcal{M}} \bar{\tilde{f}}(x)\tilde{h}(y)W(x,y)\, \mathrm{d}vol_g(x)\, \mathrm{d}vol_g(y) = \int\limits_{\mathcal{M}\times\mathcal{M}} \bar{f}(x)h(y)W(x,y)\, \mathrm{d}vol_g(x)\, \mathrm{d}vol_g(y)\,.$$

Therefore, one can replace f and h with $\tilde{f}$ and $\tilde{h}$ on the left- and right-hand side of (4.4.2). Hence, it is sufficient to show that the lemma holds for $\tilde{f}$ and $\tilde{h}$.

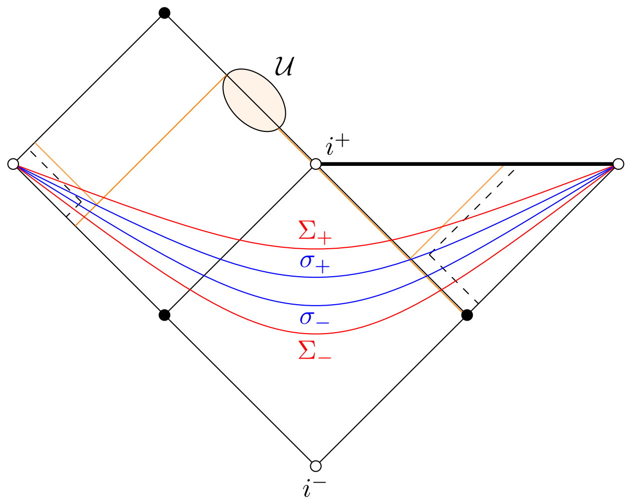


Figure 4.5: Sketch of the setup in the proof of Lemma 4.4.2. The orange region is $\mathcal{U}$, the orange lines emanating from $\mathcal{U}$ mark $J^-(\mathcal{U}_\mathcal{M})$. The blue lines are $\sigma_\pm$, the red ones $\Sigma_\pm$. The region between the blue lines in $J^-(\mathcal{U}_\mathcal{M})$ is (the interior of) G and $J(G)$ is also marked in orange. The region between the red lines is $\mathcal{L}$. The dashes lines mark $J(\hat{G})$.

One can now follow the proof of [136, Lemma 3.7] for $\tilde{f}, \tilde{h} \in C_0^\infty(G)$. We choose Cauchy surfaces $\Sigma_\pm \subset I^\pm(\sigma_\pm)$ to the past of $\mathcal{U}_\mathcal{M}$, and a compact set $\hat{G} \subset \sigma_+$ so that $G \cap \sigma_+$ is contained in the interior of $\hat{G}$. We set $\mathcal{L} = J^-(\Sigma_+) \cap J^+(\Sigma_-)$ and construct an open cover of $\mathcal{M}$ consisting of

$$\mathcal{M}_0 = \mathcal{M} \backslash J(G \cap \sigma_+) \quad \text{and} \quad \mathcal{M}_\pm = I(\hat{G}) \cap I^\pm(\Sigma_\mp)$$

with a subordinate partition of unity (ψ_0, ψ_+, ψ_-). A sketch is shown in Fig. 4.5. We then set $B \in C_0^\infty(\mathcal{M} \times \mathcal{M})$ to be given by

$$B(x, y) = \zeta(x, y) K_q(x) \overline{K}_q(y) \psi_-(x) \psi_-(y) W(x, y)$$

for some function $\zeta \in C_0^\infty(\mathcal{M} \times \mathcal{M})$ which is equal to one on $(J(G) \cap \mathcal{L}) \times (J(G) \cap \mathcal{L})$. Since B is supported in $(J(\hat{G}) \cap \mathcal{L}) \times (J(\hat{G}) \cap \mathcal{L})$, one can ignore the cutoff function ζ for $x, y \in J(G)$. Following the proof of [136, Lemma 3.7], one can show that this $B(x, y)$ satisfies (4.4.2) by using the extension of Green's formula and the support properties of the various functions. □

This result allows us to prove Proposition 4.4.1.

Proof of Proposition 4.4.1. As a first step in the proof, let us note that both the current and the stress-energy tensor are gauge-invariant observables, and hence we may choose any gauge to evaluate them. After fixing an evaluation point x_0, we choose a gauge with $A_\nu(x_0) = 0$. For example, we may set $\chi = tQ/r_0$ with r_0 the fixed value of r at x_0.

In the rest of the proof, we will work in this gauge and drop the gauge-superscripts for convenience.

The stress-energy tensor of the charged scalar field can then be written as

$$T_{\nu\varrho}(x) = \partial_{(\nu}\Phi^*(x)\partial_{\varrho)}\Phi(x) - \frac{1}{2}g_{\nu\varrho}\left(\partial_\gamma\Phi^*(x)g^{\gamma\delta}\partial_\delta\Phi(x) + \mu^2\Phi^*(x)\Phi(x)\right) . \tag{4.4.3}$$

One can then follow the proof of [16, Prop. 5.1] step by step: By [136, App. B], and the symmetry properties of $W(x,y) = w_+^\Psi(x,y) - w_+^{\mathrm{U}}(x,y)$, one can find a sequence $(b_j)_{j\in\mathbb{N}} \subset C_0^\infty(\mathcal{L}\cap J(\hat{G}))$ so that the $B \in C_0^\infty((\mathcal{L}\cap J(\hat{G}))\times(\mathcal{L}\cap J(\hat{G})))$, which is obtained by an application of Lemma 4.4.2 to $W(x,y)$, can be expanded in terms of the b_j as in [16, Eq. (91)][1],

$$B(x,y) = \sum_j c_j b_j(x)\overline{b_j}(y)\,, \quad c_j \in \{1,-1\}\,, \quad \sum_j \|b_j\|^2_{C^m(\mathcal{L}\cap J(\hat{G}))} < \infty\,.$$

This entails that $W(x,y)$ restricted to $\mathcal{U}_\mathcal{M}\times\mathcal{U}_\mathcal{M}$ can be expanded in terms of forward solutions ψ_j to $\mathcal{K}_q\psi_j = b_j$ as noted in [16, Eq. (92)].

As a result, the stress-energy tensor of the charged scalar field can be written in terms of the ψ_j as in [16, Eq. (94)], while the charge current $\langle j_\nu\rangle_{\Psi-\mathrm{U}}$ can be written as

$$\begin{aligned}\langle j_\nu(x)\rangle_{\Psi-\mathrm{U}} &= iq\sum_j c_j\left\{\partial_\nu\overline{\psi_j(x)}\psi_j(x) - \overline{\psi_j(x)}\partial_\nu\psi_j(x)\right\} \\ &= \sum_j c_j q\,\mathrm{Im}\left\{\overline{\psi_j(x)}\partial_\nu\psi_j(x)\right\} .\end{aligned} \tag{4.4.4}$$

The results of [39] discussed in Section 4.1, see also [111, App. A], and [16, Thm. 4.4], imply that $\psi_j(\cdot,y^i) \in H^{1/2+\beta'}(I)$, and $\partial_y^n\psi_j \in H^{1/2+\beta'}(\mathcal{U}_\mathcal{M})$ for any $1/2 < \beta' < \min(1,\beta)$. From [16, Thm. 4.4] one can also glean that [16, Eq. (93)] still holds[2],

$$\|\psi_j\|_{H^{1/2+\beta'}(\mathcal{U}_\mathcal{M})} \leq C\,\|b_j\|_{C^m(\mathcal{L}\cap J(\hat{G}))}$$

for sufficiently large m and some $C > 0$. Together with Sobolev embedding and the estimate [16, Eq. (95)], one has

$$\begin{aligned}\|\psi_j(\cdot,y_i)\|_{L^\infty(\mathbb{R}_-)} &\leq C\|(-\partial_{V_-}^2+1)^{\frac{1+\epsilon}{4}}\psi_-(\cdot,y^i)\|_{L^2(\mathbb{R}_-)} \\ &\leq C'\|(-\partial_y^2+1)^{\frac{N}{4}}\psi_j\|_{H^{1/2+\beta'}(\mathcal{U})} \leq C''\|b_j\|_{C^m(\mathcal{L}\cap J(\hat{G}))}\,,\end{aligned} \tag{4.4.5}$$

as well as

$$\|\partial_\nu\psi_j\|_{L^{2p}(\mathbb{R}_-)} \leq \tilde{C}\,\|b_j\|_{C^m(\mathcal{L}\cap J(\hat{G}))}\,,$$

[1] Note that in contrast to the claim in [16], the support of b_j lies in $\mathcal{L}\cap J(\hat{G})$ rather than in G, see [136].
[2] Note that $\mathcal{O}$ should be replaced by $J(\hat{G})\cap\mathcal{L}$ here as well.

for some $C, C', C'', \tilde{C} > 0$, compare also the estimates in [16, Eq. (96)] and [16, Eq. (97)].

For the stress-energy tensor this means that the proof can be completed in the same way as in [16]. For the charge current, one can find by an application of the Hoelder inequality

$$\begin{aligned} \|\langle j_\nu(\cdot, y^i)\rangle_{\Psi-\mathrm{U}}\|_{L^{2p}(\mathbb{R}_-)} &\leq C \sum_j \|\mathrm{Im}(\overline{\psi_j}\partial_\nu\psi_j)\|_{L^{2p}(\mathbb{R}_-)} \\ &\leq C' \sum_j \|\psi_j\|_{L^\infty(\mathbb{R}_-)}\|\partial_\nu\psi_j\|_{L^{2p}(\mathbb{R}_-)} \\ &\leq C'' \sum_j \|b_j\|^2_{C^m(\mathcal{L}\cap J(\hat{G}))} < \infty\,, \end{aligned} \tag{4.4.6}$$

for some constants C, C', $C'' > 0$ and consequently $\langle j_\nu(\cdot, y^i)\rangle_{\Psi-\mathrm{U}} \in L^{2p}(\mathbb{R}_-)$. In the same way, one can obtain bounds for $\left\|\left\langle \partial_y^n j_\nu(\cdot, y^i)\right\rangle_{\Psi-\mathrm{U}}\right\|_{L^{2p}(\mathbb{R}_-)}$, proving that $\langle j_\nu\rangle_{\Psi-\mathrm{U}}$ is a smooth function of the local coordinates y^i parametrizing the Cauchy horizon, while it is in $L^{2p}(\mathbb{R}_-)$ as a function of V_-. □

Thus, as long as $\langle T_{vv}\rangle_{\mathrm{U-C}}$ and $\langle j_v\rangle_{\mathrm{U-C}}$ in the Cauchy-horizon limit are non-vanishing, they constitute the leading divergences of the stress-energy tensor and the current at the Cauchy horizon in any state that is Hadamard in $\mathcal{M}$. In other words, in this case the leading divergences of the stress-energy tensor and the current at the Cauchy horizon are universal in the sense that they do not depend on the choice of Hadamard state. It remains to confirm that $\langle T_{vv}\rangle_{\mathrm{U-C}}$ and $\langle j_v\rangle_{\mathrm{U-C}}$ are indeed non-vanishing at the Cauchy horizon by numerical computations.

In the following, we compute the leading divergences of the stress-energy tensor and the current in the $\mathcal{H}^L_-$-limit and subsequently infer the result at $\mathcal{H}^R_-$ using the stationarity of the Unruh- and comparison state and assuming that the computation is done in a static gauge.

For the current, we can use the extension of the Boulware modes from I to II and the asymptotic behaviour of the modes in II, see (3.3.6), to obtain the limit of (4.2.24) towards $\mathcal{H}^L_-$. The formula for the current then reads

$$\langle j_v\rangle_{\mathrm{U}} = -\sum_{\ell=0}^{\infty} \frac{q(2\ell+1)}{16\pi^2 r_-^2} \int_0^\infty \mathrm{d}\omega\, [F_\ell(\omega) + F_\ell(-\omega)]\,, \tag{4.4.7a}$$

$$\begin{aligned} F_\ell(\omega) = &\frac{\omega(\omega_- + \omega_{\mathrm{I}})}{(\omega_-)^2} \coth\left(\pi\tfrac{\omega_-+\omega_{\mathrm{I}}}{\kappa_c}\right)\left|\mathcal{T}^{\mathrm{I}}_{\omega_-\ell}\right|^2\left|\mathcal{T}^{\mathrm{II}}_{\omega_-\ell}\right|^2 \\ &+ \frac{\omega\coth\left(\pi\frac{\omega_-}{\kappa_+}\right)}{\omega_-}\left[\left|\mathcal{R}^{\mathrm{II}}_{\omega_-\ell}\right|^2 + \left|\mathcal{R}^{\mathrm{I}}_{\omega_-\ell}\right|^2\left|\mathcal{T}^{\mathrm{II}}_{\omega_-\ell}\right|^2\right] \\ &+ \frac{2\omega\,\mathrm{csch}\left(\pi\frac{\omega_-}{\kappa_+}\right)}{\omega_-}\mathrm{Re}\left(\overline{\mathcal{R}^{\mathrm{I}}_{\omega_-\ell}}\mathcal{T}^{\mathrm{II}}_{\omega_-\ell}\mathcal{R}^{\mathrm{II}}_{\omega_-\ell}\right)\,, \end{aligned} \tag{4.4.7b}$$

where we have defined $\omega_- = \omega + \omega_{\mathrm{II}} = \omega + qQ\left(r_-^{-1} - r_+^{-1}\right)$.

We have already discussed that the integrand $F_\ell(\omega)$ is regular as $\omega_- \to 0$. However, there is an additional potential infrared divergence in the current near the inner horizon. This divergence appears for $\omega \to 0$. In this case, the asymptotic behaviour of the Boulware-modes in region II near r_- is $R_{\omega_{\mathrm{II}}\ell} \sim A + Br_*$, which diverges as the horizon is approached. One can translate this divergence of the solution into a linear divergence of the scattering coefficients $\mathcal{T}^{\mathrm{II}}_{\omega_-\ell}$ and $\mathcal{R}^{\mathrm{II}}_{\omega_-\ell}$ as $\omega \to 0$. However, from the way the integral is written above we see that it can be read as a the principal value of the ω-integral over $\omega F_\ell(\omega)/\omega$, which is finite.

Note that for the current we did not subtract the contribution from the comparison state. The reason is that the comparison state that is used for the computation of the stress-energy tensor does not give a contribution to the v-component of the current at the Cauchy horizon.

For the stress-energy tensor, we can construct a mode-sum formula using the results from Section 4.2. In contrast to the result for the current, we do not expect to find a finite contribution from the Hadamard parametrix in this case. Since the application of the Hadamard point-split renormalization combined with the mode-sum formula is very challenging, it is computationally simpler to perform a state-subtraction of the unrenormalized expectation values for the stress-energy tensor with respect to the comparison state just as for the real scalar field.

Taking into account that $g_{vv} = 0$, the vv-component of the classical stress-energy tensor is given by

$$T_{vv}(x) = \partial_v \Phi^*(x) \partial_v \Phi(x)\,. \tag{4.4.8}$$

Symmetrizing with respect to $\Phi(x)$ and $\Phi^*(x)$, compare (4.2.2), and following the same steps as for the regularized charge current in Section 4.2, we find a mode-sum expression for $\langle T_{vv}\rangle_{\mathrm{U-C}}$ at $\mathcal{H}^L_-$,

$$\langle T_{vv}\rangle_{\mathrm{U-C}} = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{32\pi^2 r_-^2} \int_0^\infty \mathrm{d}\omega\, \omega \left[F_\ell(\omega) - F_\ell(-\omega) - 2\coth\left(\tfrac{\pi\omega}{\kappa_-}\right)\right]\,, \tag{4.4.9}$$

with $F_\ell(\omega)$ as in (4.4.8). The formulas for the current and energy flux can now be evaluated numerically as described in sections 3.3, 3.4 and 4.3.

First, we discuss the results for the stress-energy tensor. In Fig. 4.6, we plot the energy flux for $\mu^2 = 2\Lambda/3$ and $\Lambda M^2 = 0.14$ as a function of Q/M for different values of qQ. The results we obtain are compatible with those found for the real scalar field, [92, 97], see also Section 3.4. In particular, the results in the small plot zooming in on large Q/M and small qQ look very similar to those in [92, 97], and converge to those in [97] for $qQ \to 0$. Another interesting feature is the fact that for qQ sufficiently large, $\langle T_{vv}\rangle_{\mathrm{U-C}}$ remains positive, at least for $\mu^2 = 2\Lambda/3$. This is in contrast to the result for the real scalar, where the energy flux changes sign and becomes negative near extremality in the case of a conformally coupled scalar field. To demonstrate the increase of the energy flux with qQ, we have also plotted $\langle T_{vv}\rangle_{\mathrm{U-C}}$ as a function of qQ for different values of Q/M. This

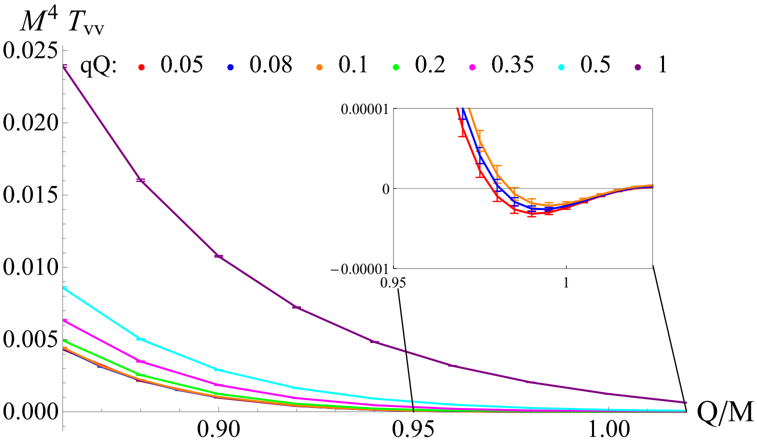


Figure 4.6: $\langle T_{vv}\rangle_{\text{U-C}}$ evaluated on $\mathcal{CH}^R$ as a function of Q/M for different values of qQ and $\mu^2 = 2\Lambda/3$, $\Lambda M^2 = 0.14$.

plot is shown in Fig. 4.7.

Note that by Proposition 4.4.1, $\langle T_{vv}\rangle_{\text{U-C}}$ is the state-independent leading divergence of the energy flux as long as sCC is classically violated for the chosen spacetime parameters [112]. Since it is generically non-zero, these results indicate that just as for the real scalar field, quantum effects can restore sCC in this setup when it is classically violated.

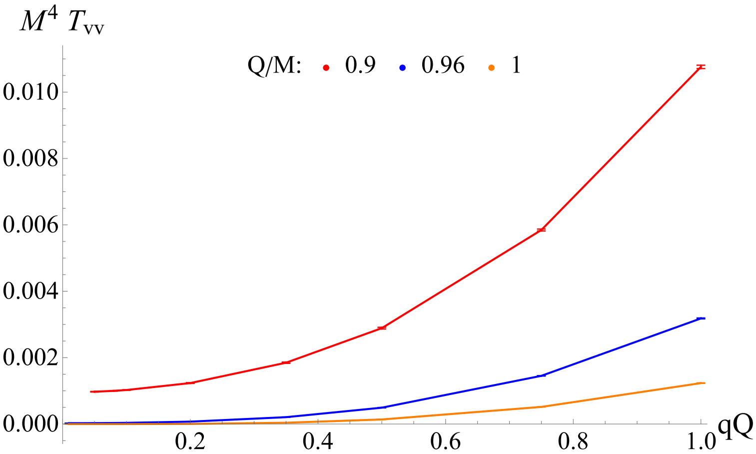


Figure 4.7: The difference of the vv-component of the stress-energy tensor of the charged scalar field between the Unruh-and the comparison state near $\mathcal{CH}^R$ as a function of qQ for different values of Q/M and $\mu^2 = 2\Lambda/3$, $\Lambda M^2 = 0.14$.

Next, let us discuss the results for the charge current. Fig. 4.8 shows the charge current $\langle j_v\rangle_{\text{U}}$ at the Cauchy horizon for $\mu^2 = 2\Lambda/3$ and $\Lambda M^2 = 0.14$ as a function of Q/M for different values of qQ. Comparing this result to the charge current at the event horizon, Fig. 4.2, the most prominent difference is that the current at the Cauchy horizon can change its sign while the sign at the event horizon is fixed. Considering the (weak) backreaction of the current onto the charge as described in (4.0.3), this means that in the

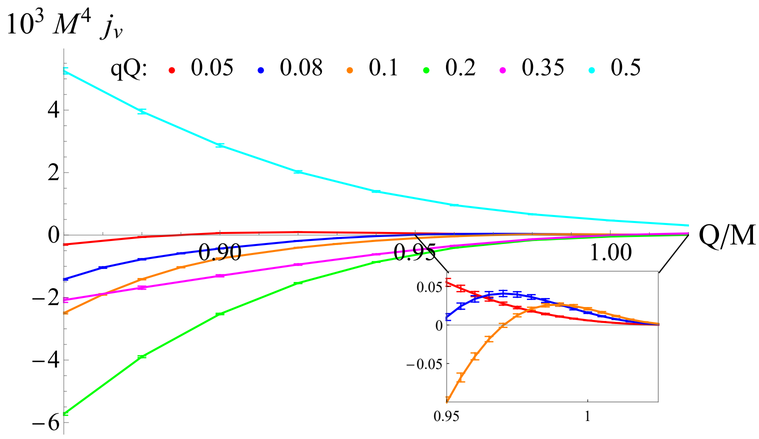


Figure 4.8: $\langle j_v \rangle_{\mathrm{U}}$ evaluated on $\mathcal{CH}^R$ as a function of Q/M for different values of qQ and $\mu^2 = 2\Lambda/3$, $\Lambda M^2 = 0.14$.

parameter range in which $\langle j_v \rangle_{\mathrm{U}} < 0$, the charge of the black hole within the inner horizon increases.

This result is very surprising, since it contrasts both the intuitive particle picture [114, 115] and the results at the event horizon presented in the previous section and discussed in the literature [130]. One possible reason is the scattering of modes entering the black hole through the event horizon in the gravitational potential in the black-hole interior. This scattering can be seen best in the radial ODE (3.3.2).

We also note that the current is always positive when the black hole is close to extremality. This means that even though the inner horizon may be charged by quantum effects, the charge cannot be increased beyond its extremal value and the black hole cannot be turned into a naked singularity in this manner.

Unfortunately, the parameters tested here are not quite realistic. As discussed already in the previous section, in order to achieve a reasonable performance of the numerics, comparability with results in [111] and the results on the real scalar, and to avoid the classical instability regime, we have chosen an unrealistically large cosmological constant Λ. Moreover, the charge and mass of the scalar field have also been chosen very small for solar mass black holes. Nonetheless, our results show that the intuitive particle picture does not fully capture the behaviour of the quantum effects in the black-hole interior, and that the assumption that the black hole is always discharged is false.

One may then ask whether a similar effect can be observed for the electromagnetic field strength Q/r^2. As seen in (4.0.4), the change of the field strength depends on both $\langle j_v \rangle_{\mathrm{U}}$ and $\langle T_{vv} \rangle_{\mathrm{U-C}}$. Hence, the field strength might decrease even if $\langle j_v \rangle_{\mathrm{U}} < 0$ or the other way around, depending on the sign and relative magnitude of the two terms. Its v-derivative as a function of Q/M for different values of qQ is plotted in Fig. 4.9. In fact, we find that even if $\langle j_v \rangle_{\mathrm{U}} > 0$ and the Cauchy horizon is discharged, the field strength does in many cases increase with v, because its v-derivative is dominated by the positive contribution proportional to $\langle T_{vv} \rangle_{\mathrm{U-C}}$. The interpretation of this is that in most of this parameter regime, the area of the fixed (u, v)-surface shrinks faster than the charge contained in it.

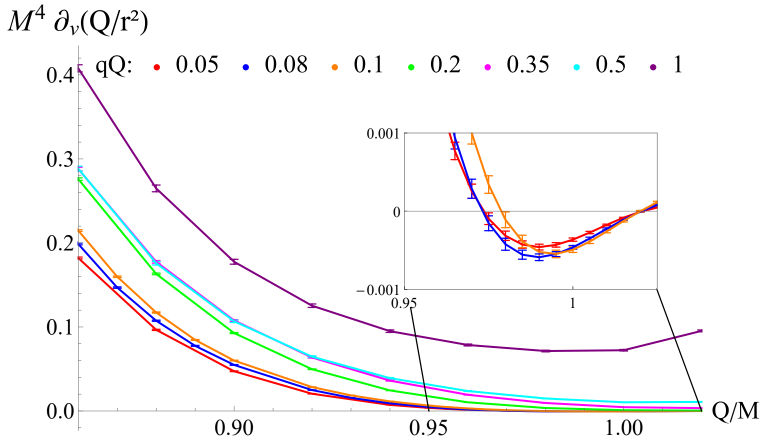


Figure 4.9: The change of the field strength $\partial_v \frac{Q}{r^2}$ near $\mathcal{H}_-^R$ as a function of Q/M for different values of qQ and $\mu^2 = 2\Lambda/3$, $\Lambda M^2 = 0.14$.

To summarize, we observed that the energy flux of the charged scalar quantum field is generically non-vanishing at the inner horizon of a RNdS black hole. Therefore, quantum effects can restore sCC when it is classically violated in this scenario as well. Additionally, we have found a potential increase of the black-hole charge due to quantum effects at the Cauchy horizon. This result could not have been predicted from the particle picture alone. This demonstrates that first-principle calculations of the different observables in the black-hole interior are important if we want to understand how the presence of the quantum field modifies it.

5 The Unruh state on Kerr-de Sitter

In the previous chapter, we have seen that the current induced by a charged scalar field near the Cauchy horizon of a RNdS black hole can have either sign. Via backreaction, this current can thus charge the Cauchy horizon instead of discharging it. However, realistic astrophysical black holes are expected to be rotating rather than charged. It would therefore be interesting to see whether such an effect also occurs for rotating black holes described by the Kerr metric or, in the presence of a positive cosmological constant, Kerr-de Sitter metric. Here, the charge or discharge would correspond to the speed-up or slow-down of the black-hole rotation.

To study this effect, one needs at least one Hadamard state for the quantum theory under consideration on the Kerr or Kerr-de Sitter spacetime. Preferably, one would like this state to be physically well-motivated. As discussed in Section 2.5, one such state, the Unruh state [18], has been rigorously constructed in a number of black-hole spacetimes [16, 43, 96] and is thought to be a good description for the behaviour of the quantum field arising from gravitational collapse at late times.

In this chapter, we will show that the Unruh state can be defined for the free scalar field on the Kerr-de Sitter spacetime as well, and that it is a Hadamard state across both the event horizon and the cosmological horizon. While it is clear how the Unruh state on Kerr-de Sitter should be defined in terms of mode-sums, see for example [137], its rigorous construction and the proof of its Hadamard property have been an open problem until now.

The largest difficulty arises from the fact that the Killing field ∂_t becomes space-like outside the black hole in the so-called ergoregion[1]. Thus, region I is not static, but only locally stationary [76]. Since the staticity of the black-hole exterior is needed for the proof of the Hadamard property of the Unruh state as given in [43, 96] for the Schwarzschild (-de Sitter) spacetime, this proof cannot be adapted directly to the Kerr-de Sitter spacetime.

Instead, we will combine the ideas from [43] with ideas developed in [138], in which the authors demonstrate a rigorous construction of and prove the Hadamard property for the Unruh state for free, massless fermions on the Kerr spacetime.

This ansatz requires us to generalize some geometrical results from the Kerr- to the Kerr-de Sitter spacetime. These generalizations will be shown in the first section. After that, we will define the Unruh two-point function and show that it indeed defines a quasi-free state for the real massive scalar field on Kerr-de Sitter by the criteria described in Section 2.2. Finally, we will prove that the Unruh state is Hadamard. The results in this section have been published in [139].

[1] There is in fact a second "ergoregion" just inside the cosmological horizon.

5.1 Null geodesics in the Kerr-de Sitter spacetime

In this section, we want to show some results on the geometry and the behaviour of geodesics on Kerr-de Sitter. In particular, we want to extend some of the results shown in [138, App. C] from Kerr to Kerr-de Sitter, beginning with the behaviour of null geodesics on this spacetime.

We start from the results on the geodesics collected in Section 2.4.2. One can already see that the geodesic equation (2.4.19) has the same form as the geodesic equation for Kerr [80], apart from factors of χ. With the identities collected in Section 2.4.2, we can show that a light-like geodesic with vanishing Carter constant, $K = 0$, is a principle null geodesic, and hence [80, Cor. 4.2.8] generalizes from Kerr to Kerr-de Sitter.

Using the results from [75] regarding the structure of the extended Kerr-de Sitter spacetime and recalling the definition of the conserved energy E and angular momentum L in (2.4.16) and (2.4.17), one can also generalize [80, Lemma 4.2.9] to Kerr-de Sitter:

Lemma 5.1.1. *A null geodesic γ on a Kerr-de Sitter spacetime*

1. *is principal iff $K = 0$.*
2. *is contained in a time-like polar plane $\{t = t_0, \varphi = \varphi_0\}$, i.e. a polar plane in a region where $\Delta_r \leq 0$, iff $L = E = 0$ but $K \neq 0$.*
3. *is contained in $\{\sin\theta = 0\} \setminus (\{r = r_-\} \cup \{r = r_+\} \cup \{r = r_c\})$ iff $K = L = 0$ but $E \neq 0$.*
4. *is a null geodesic generating one of the horizons iff $K = L = E = 0$.*

In addition, we note that [80, Cor. 4.3.2] holds with small modifications to the equations for $\rho^2\, \mathrm{d}v/\,\mathrm{d}\tau$, $\rho^2\, \mathrm{d}u/\,\mathrm{d}\tau$, $\rho^2\, \mathrm{d}\varphi^*/\,\mathrm{d}\tau$ and $\rho^2\, \mathrm{d}^*\varphi/\,\mathrm{d}\tau$,

$$\rho^2 \frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{a\chi^2\mathbb{D}}{\Delta_\theta} + \frac{\chi^2(r^2+a^2)}{\Delta_r}\left[\mathbb{P} \oplus \left(\pm\sqrt{\frac{R(r)}{\chi^2}}\right)\right], \tag{5.1.1a}$$

$$\rho^2 \frac{\mathrm{d}\varphi^*}{\mathrm{d}\tau} = \frac{\chi^2\mathbb{D}}{\sin^2\theta\Delta_\theta} + \frac{\chi^2 a}{\Delta_r}\left[\mathbb{P} \oplus \left(\pm\sqrt{\frac{R(r)}{\chi^2}}\right)\right]. \tag{5.1.1b}$$

Here, $\oplus$ is a plus in the equations above for v and φ^*, but will be replaced by a minus in the corresponding equations for u and $^*\varphi$, while the $\pm$ is the sign of $\mathrm{d}r/\,\mathrm{d}\tau$. $\mathbb{P}$ and $\mathbb{D}$ are defined in (2.4.18), $R(r)$ is defined in (2.4.19a). In comparison to [80, Cor. 4.3.2], (5.1.1) has additional factors χ and Δ_θ. Still, these factors are both bounded by $1 < \chi, \Delta_\theta < 2$ and do not destroy the separability. Hence, the calculations presented in [80, Sec. 4.3 and 4.4] restricted to null geodesics hold with only a minor modification due to these factors.

Concluding, we find in analogy to [80, Prop. 4.3.9] that any null geodesic on the Kerr-de Sitter spacetime $\mathcal{M}$ (or its extension $\tilde{\mathcal{M}}$, compare Fig. 2.4) that is not completely contained in a horizon or the axis $\{\sin\theta = 0\}$ can be extended to $\tau \in \mathbb{R}$, or ends at a horizon or on the axis in a finite amount of proper time. Analogously to the results in [80, Sec. 4.4], geodesics can only cross horizons transversally at finite proper time. For blocks II and III, this includes the possibility that the geodesic approaches the corresponding bifurcation sphere of the horizon in finite proper time. Whether a geodesic approaches the ingoing or outgoing part of the horizon depends on the signs of $\mathrm{d}r/\mathrm{d}\tau$ and $\mathbb{P}$ as described in [80, Prop. 4.4.6].

The axis is a closed, totally geodesic submanifold of the (extended) Kerr-de Sitter spacetime by [80, Thm. 1.7.12]. The geodesics contained in the axis satisfy $K = L = 0$ and $E \neq 0$, and hence $R(r)$ reduces to $\chi^2 E^2 (r^2 + a^2)^2 > 0$, meaning that there are no turning points in r. Geodesics that approach the axis can be extended through it and cross it transversally, similar to the behaviour of null geodesics near the horizons [79].

To conclude our study of null geodesics, we want to take a more detailed look at geodesics in region I that approach either i^+ or i^- (or both). Considering the structure of $R(r)$, this must be due to a double root of $R(r)$. To study the possible locations of these double roots, we show a version of [138, Lemma C.1+C.2] for the Kerr-de Sitter spacetime.

Lemma 5.1.2. *1. There are $\lambda_0 > 0$ and $a_0 > 0$ so that for all $0 < \lambda < \lambda_0$ and all $0 < a < a_0$, the double roots r_0 of $R(r)$ satisfy*

$$r_+ < 3 - C(\lambda)a + \mathcal{O}(a^2) \leq r_0 \leq 3 + C(\lambda)a + \mathcal{O}(a^2) < r_c \tag{5.1.2}$$

with $C(\lambda) = 2\sqrt{1/3 - 9\lambda}$.

2. Under the same conditions as above, for any double root r_0 of $R(r)$, we have $\rho^2 \frac{dt}{d\tau}|_{r_0} \neq 0$ for all $\theta \in [0, \pi]$.

Proof. Let us start with the first point, and let us begin by noting that roots of $R(r)$ can only occur in regions where $\Delta_r \geq 0$. We can thus focus on $r \in [r_+, r_c]$.

We first consider the case $E = 0$. With this, $\Theta(\theta)$ as defined in (2.4.19b) takes the form

$$\Theta(\theta) = \frac{K}{\sin^2\theta}\left(-a^2\lambda\cos^4\theta - (1 - a^2\lambda)\cos^2\theta + 1 - \frac{L^2\chi^2}{K}\right).$$

Since $a^2\lambda < 1$ in the whole subextremal parameter range, see Fig. 2.3, a solution for the geodesic equation can only exist if $\frac{L^2\chi^2}{K} \leq 1$. Furthermore, for $E = 0$ the conditions for a double root of $R(r)$ reduce to $\Delta_r(r_0) = a^2\frac{L^2\chi^2}{K}$ and $\partial_r\Delta_r(r_0) = 0$. Together with the first condition, we see that this can only have a solution if at the local maximum $r_+ < r_m < r_c$ of Δ_r one has

$$\Delta_r(r_m) \leq a^2 .$$

One finds that there is a $\lambda_1 > 0$, so that this condition cannot be met as long as $\lambda < \lambda_1$. This case can thus be avoided by demanding that $\lambda_0 < \lambda_1 \approx 0.0332$.

Going forward, we can therefore assume $E \neq 0$. This allows us to introduce the rescaled quantities $l = L/E$ and $k = K/(\chi^2 E^2)$. With these, one can write $R(r)$ and

$\Theta(\theta)$ as

$$R(r) = \chi^2 E^2 \left(\beta r^4 + \gamma r^2 + 2kr - a^2 q\right) \tag{5.1.3a}$$

$$\Theta(\theta) = \frac{\chi^2 E^2}{\sin^2\theta} \left(-a^2\beta\cos^4\theta + \gamma\cos^2\theta + q\right) \tag{5.1.3b}$$

with

$$\beta = 1 + \lambda k\,, \qquad \gamma = 2a(a-l) - k(1-a^2\lambda)\,, \qquad q = k - (a-l)^2\,.$$

One can see that if $q < 0$, one has $\Theta(\theta) < 0$ for all θ unless $\gamma > 0$. However, if $\gamma > 0$ and $q < 0$, then $R(r) > 0$ for all $r > 0$, and hence no double root of $R(r)$ can appear. This means that a double root of $R(r)$ requires $q \geq 0$.

In a next step, we can solve $R(r) = \partial_r R(r) = 0$ for l and k in terms of r, a, and λ. We find

$$l = \frac{\Delta_r'(r^2+a^2) - 4r\Delta_r}{a\Delta_r'}|_{r=r_0}\,, \qquad k = \frac{16r^2\Delta_r}{\Delta_r'^2}|_{r=r_0}\,, \tag{5.1.4}$$

where a prime denotes a derivative with respect to r. Plugging this into the definition of q, we find that the highest-order contributions in r cancel,

$$\begin{aligned} q &= \frac{r^2}{a^2\Delta_r'^2}\left(16\Delta_r(a^2-\Delta_r) + r\Delta_r'(8\Delta_r - r\Delta_r')\right)|_{r=r_0} \\ &= \frac{4r^3}{a^2\Delta_r'^2}\left(4a^2 - r(r-3)^2 - a^2\lambda r^2(2(r+3) + a^2\lambda r)\right)|_{r=r_0}\,. \end{aligned} \tag{5.1.5}$$

Therefore, the condition that $q \geq 0$ for any double root of $R(r)$ translates to the condition

$$4a^2 - r_0(r_0-3)^2 - a^2\lambda r_0^2(2(r_0+3) + a^2\lambda r_0) \geq 0\,.$$

Notice that in comparison to [138, Eq. (C.7)], the additional terms proportional to λ all enter with a minus-sign, and thus reduce the range of r in which the double root may be located. The double root must either lie in $r < r_1$ for some $r_1 < r_+$ or in the interval

$$\left[3 - 2\sqrt{\tfrac{1}{3} - 9\lambda}\,a + \mathcal{O}(a^2), 3 + 2\sqrt{\tfrac{1}{3} - 9\lambda}\,a + \mathcal{O}(a^2)\right]\,.$$

Since no roots of $R(r)$ can exist in (r_-, r_+), this concludes the proof of the first point.

For the second point, we consider the right hand side of (2.4.19c) and evaluate it at a double root r_0 of $R(r)$. This means that we can assume $E \neq 0$, and can set l to the expression in (5.1.4). Then, after some simplification, in which we use that $\Delta_r'(r_0) \neq 0$ if $E \neq 0$, we find

$$\rho^2 \frac{\mathrm{d}t}{\mathrm{d}\tau}(r_0,\theta) = \frac{2\chi^2 E}{\Delta_\theta\Delta_r'(r_0)}\left[r_0(\chi r_0 + 3) + a^2\cos^2\theta(\chi r_0 - 1)\right]\,. \tag{5.1.6}$$

The first factor is positive, because $\operatorname{sign}(E) = \operatorname{sign}(\Delta_r'(r_0))$, and one can check that the second factor is also positive for all possible values of r_0 using the condition obtained in the first part of the proof. $\square$

With these results at hand, let us now collect some properties of future-directed (inextendible) null geodesics on each Boyer-Lindquist block separately. We assume that the geodesics are parametrized by $\tau \in (\tau_-, \tau_+)$, and we will sort them by the type of radial motion, indicated by $(r(\tau_-), r(\tau_+))$. Since $\mathbb{P} > 0$ for future-directed null geodesics in I, the horizon crossed by the geodesic and whether u or v remains finite as the horizon is approached depends only on $\mathrm{d}r/\mathrm{d}\tau$, compare the extension of [80, Prop. 4.3.4]. Then, we have the following types of future-directed null geodesics in region I:

- (r_c, r_+) or (r_+, r_c): $|\tau_\pm| < \infty$, $\mathrm{d}r/\mathrm{d}\tau < (>)0$, $\lim\limits_{\tau \to \tau_\pm} u = \pm\infty$ for $\mathrm{d}r/\mathrm{d}\tau > 0$ and $\lim\limits_{\tau \to \tau_\pm} v = \pm\infty$ for $\mathrm{d}r/\mathrm{d}\tau < 0$. The geodesic crosses I from $\mathcal{H}_+^-$ ($\mathcal{H}_c^-$) to $\mathcal{H}_c^L$ ($\mathcal{H}_+^R$).

- (r_c, r_c): $|\tau_\pm| < \infty$, $\mathrm{d}r/\mathrm{d}\tau$ starts negative and becomes positive at a simple zero of $R(r)$, $\lim\limits_{\tau \to \tau_-} u = -\infty$, $\lim\limits_{\tau \to \tau_+} v = \infty$. The geodesic starts at $\mathcal{H}_c^-$, is reflected at a simple root of $R(r)$ and ends at $\mathcal{H}_c^L$.

- (r_+, r_+): $|\tau_\pm| < \infty$, $\mathrm{d}r/\mathrm{d}\tau$ starts positive and becomes negative at a simple zero of $R(r)$, $\lim\limits_{\tau \to \tau_-} v = -\infty$, $\lim\limits_{\tau \to \tau_+} u = \infty$. The geodesic starts at $\mathcal{H}_+^-$, is reflected at a simple root of $R(r)$ and ends at $\mathcal{H}_+^R$.

- (r_+, r_0) or (r_c, r_0): $|\tau_-| < \infty$, $\tau_+ = \infty$, $\mathrm{d}r/\mathrm{d}\tau \to 0$ from above (below), $\lim\limits_{\tau \to \tau_-} v(u) = -\infty$. The geodesic starts at $\mathcal{H}_+^-$ ($\mathcal{H}_c^-$) and asymptotically approaches $r = r_0$ and therefore i^+.

- (r_0, r_+) or (r_0, r_c): $\tau_- = -\infty$, $|\tau_+| < \infty$, $\mathrm{d}r/\mathrm{d}\tau < (>)0$ starts from zero at τ_-, $\lim\limits_{\tau \to \tau_+} u(v) = \infty$. The geodesic asymptotically approaches $r = r_0$, and therefore i^-, to the past and ends at $\mathcal{H}_+^R$ ($\mathcal{H}_c^L$).

- (r_0, r_0): $\tau_\pm = \pm\infty$, $\mathrm{d}r/\mathrm{d}\tau = 0$. This geodesic corresponds to a circular orbit at $r = r_0$. It approaches $i^\pm$ to the future/past.

In region II and III, $R(r)$ is always positive, and the sign of $\mathrm{d}r/\mathrm{d}\tau$ is dictated by the choice of time orientation, see [75] and Section 2.4.2. In particular, all future-pointing geodesics in II are of the form (r_+, r_-) and have $|\tau_\pm| < \infty$, while those in III are of the form (r_c, ∞) and satisfy $|\tau_-| < \infty$, $\tau_+ = \infty$. It only remains to remark that the geodesic will cross into region I at $r = r_i$ if $\mathbb{P}(r_i) > 0$. When we also take into account the extended spacetime $\tilde{\mathcal{M}}$, the other cases are crossing into a copy of I$'$, corresponding to I with reversed time orientation, if $\mathbb{P}(r_i) < 0$ and through the bifurcation sphere into III$'$ or II$'$ if $\mathbb{P}(r_i) = 0$.

With the above results, one can follow the proof of case 2 and 3 in [138, Lemma C.4, i)] to show that

$$\lim_{\tau \to \tau_\pm} t(\tau) = \pm\infty$$

for all future-pointing inextendible null geodesics in I.

This analysis of the null geodesics also enables the proof of the following result:

Proposition 5.1.3. *$\mathcal{M}$ and $\tilde{\mathcal{M}}$ are globally hyperbolic.*

Proof. By direct computation, one can see that the function $x(r) = r_*(r) - r$ is strictly monotonic on (r_+, r_c), and ranges from $-\infty$ for $r \to r_+$ to ∞ for $r \to r_c$. Hence, for any $T \gg 1$, there is a unique solution r_T of $x(r) = -T$ near r_+ and a unique solution r'_T of $x(r) = T$ near r_c. Let us choose T sufficiently large, so that all double roots r_0 of $R(r)$ are contained in (r_T, r'_T), and set

$$u_T = \begin{cases} u + c(r) + T & : \ r'_T \leq r \\ t & : \ r_T < r < r'_T \\ v + T - c(r) & : \ r \leq r_T \end{cases},$$

with

$$\partial_r c(r) = 1 + \phi(r) \frac{1}{r - r_-}, \qquad c(r_+) = r_+ .$$

$\phi \in C^\infty(\mathbb{R}; \mathbb{R}_+^*)$ is equal to 1 on $(-\infty, r_- + \epsilon]$, and $\phi = 0$ on $(1/2(r_+ + r_-), \infty)$, see [138, App. C.6.2]. One can then show that u_T satisfies the conditions in [138, Cor. C.7]. In particular, for $r_T < r < r'_T$, we have

$$\begin{aligned} g^{-1}(\mathrm{d}u_T, \mathrm{d}u_T) = g^{tt} &\leq \frac{\chi^2}{\rho^2} \left(a^2 - \frac{(r^2 + a^2)^2}{\Delta_r} \right) \\ &= -\frac{\chi^2}{\rho^2} \frac{\chi r^4 + \chi r^2 a^2 + 2a^2 r}{\Delta_r} < 0 , \end{aligned} \tag{5.1.7}$$

since $\Delta_r > 0$ in this interval. Outside of this interval, we can use the inverse metric in the $KdS*$- and $*KdS$-coordinates respectively. We combine it with the fact that $\partial_r c(r) = 1$ in $[r_+, \infty)$ and $c'(r) \geq 1$ on $[r_-, r_+)$, where $\Delta_r \leq 0$. From this, we find that on $[r_-, r_T] \cup [r'_T, \infty)$

$$g^{-1}(\mathrm{d}u_T, \mathrm{d}u_T) \leq \frac{1}{\rho^2} \left(-\lambda r^4 - (1 + 3\lambda a^2) r^2 - 2r + \lambda^2 a^6 \right) .$$

The expression in brackets can have at most one root r_0 in $r > 0$ and is negative in $r > r_0$. At r_-, the expression in brackets reduces to $\chi(\chi - 2)a^2 - 2\chi r^2$. Since $\lambda a^2 < 1$ and hence $1 < \chi < 2$ in the whole subextremal parameter region, compare Fig. 2.3, this is strictly negative. Thus, ∇u_T is time-like on $\mathcal{M}$.

In the next step, we check that $\sup_\gamma u_T = \infty$ and $\inf_\gamma u_T = -\infty$ for any inextendible future-directed null geodesic on $\mathcal{M}$. As long as the geodesic does not intersect region III, the proof follows exactly as in the proof of [138, Prop. C.11]. We should remark that a geodesic intersecting none of the regions I, II or III can either be contained in $\mathcal{H}_+^R$, in which case $\inf_\gamma v = -\infty$ and $\sup_\gamma v = \infty$, or in $\mathcal{H}_c^L$, in which case $\inf_\gamma u = -\infty$ and $\sup_\gamma u = \infty$. This follows from the results in [75, Sec. 4.4.2].

It remains to discuss the future-directed null geodesics intersecting III. It follows from the preceding discussion of the null geodesics that for all such geodesics $r(\tau)$ diverges to infinity towards the future, and hence $\sup_\gamma u_T = \infty$. Towards the past, if γ does not intersect I, it must either approach $\mathcal{H}_c^R$ or $\mathcal{B}_c$. From the above discussion of the null geodesics one obtains $\inf_\gamma v = -\infty$ in both cases. When γ crosses through $\mathcal{H}_c^L$ into I, let $\tilde{\gamma} = \gamma \cap \mathrm{I} : (\tau_-, \tau_+) \to \mathrm{I}$. By the foregoing analysis, $\tilde{\gamma}$ can either approach r_+, r_c or r_0 as $\tau \to \tau_-$. In the first case $\lim\limits_{\tau \to \tau_-} v = -\infty$, in the second case $\lim\limits_{\tau \to \tau_-} u = -\infty$, and in the third case $\lim\limits_{\tau \to \tau_-} t = -\infty$. Taking into account the definition of u_T, this shows the desired property. Hence $\mathcal{M}$ is globally hyperbolic by [138, Cor. C.7].

Next, let us also discuss $\tilde{\mathcal{M}}$. The proof of its global hyperbolicity follows along the lines of [138, Prop. C.12]: we set

$$\Sigma = \{U_+ = -V_+\} \sqcup \{U_c = -V_c\} / \sim \,,$$

where $\sim$ is the identification of $\mathrm{I} \subset \mathcal{M}_+$ with $\mathrm{I} \subset \mathcal{M}_c$ in $\tilde{\mathcal{M}}$. One can check that under this identification $\{U_+ = -V_+\} \cap \mathrm{I}$ and $\{U_c = -V_c\} \cap \mathrm{I}$ indeed agree. In fact, Σ can also be characterized by

$$\Sigma = (\{t=0\} \cap (\mathrm{I}' \cap \mathcal{M}_+)) \cup \mathcal{B}_+ \cup (\{t=0\} \cap \mathrm{I}) \cup \mathcal{B}_c \cup (\{t=0\} \cap (\mathrm{I}' \cap \mathcal{M}_c)) \,.$$

Going back to the previous formulation, one can see that $\tilde{\mathcal{M}} \backslash \Sigma$ is disconnected.

On $\Sigma \cap \mathcal{M}_+ \cap \mathrm{I}^{(\prime)}$, we have $\mathrm{d}(U_+ + V_+) = 2\kappa_+ V_+ \,\mathrm{d}t$, while on $\Sigma \cap M_c \cap \mathrm{I}^{(\prime)}$, one finds $\mathrm{d}(U_c + V_c) = 2\kappa_c U_c \,\mathrm{d}t$. Since $g^{tt} < 0$ when $\Delta_r > 0$, see (5.1.7),

$$g^{-1}(\,\mathrm{d}(U_{+/c} + V_{+/c}),\, \mathrm{d}(U_{+/c} + V_{+/c})) < 0 \quad \text{on } (\Sigma \cap \mathcal{M}_{+/c}) \backslash \mathcal{B}_{+/c} \,.$$

On $\mathcal{B}_+$ and $\mathcal{B}_c$ the metric in the Kruskal-type coordinates, see (2.4.15a), simplifies and one obtains

$$g^{-1}(\,\mathrm{d}(U_{+/c} + V_{+/c}),\, \mathrm{d}(U_{+/c} + V_{+/c})) < 0 \quad \text{on } \mathcal{B}_{+/c} \,.$$

As a result, Σ is also space-like. By [138, Thm. C.6(2)], it is achronal.

Next, we want to show that any inextendible null geodesic enters $I^+(\Sigma)$ and $I^-(\Sigma)$. Together with [138, Thm. C.6(1)], this will show that Σ is a Cauchy surface for $\tilde{\mathcal{M}}$. Since $t \to \pm\infty$ on any inextendible null geodesic in I or I$'$, any inextendible null geodesics in $\tilde{\mathcal{M}}$ intersecting I or I$'$ enters $I^\pm(\Sigma)$. It thus remains to consider geodesic that do not intersect I or I$'$.

If the geodesic intersects neither II$^{(\prime)}$ nor III$^{(\prime)}$, it must be contained in one of the horizons, and thus intersects $I^{\pm}(\Sigma)$ by the results of [75, Sec. 4.4.2]. If the geodesic γ intersects II, but neither I nor I′, the geodesic must cross $\{r = r_+\}$ through the bifurcation sphere $\mathcal{B}_+$. Since $R(r_+) = 0$ in this case, $\mathrm{d}r/\,\mathrm{d}\tau$ must change sign. This is only allowed if γ crosses into II′. The same argument holds for geodesics in III. Since $\mathrm{II}\cup\mathrm{III} \subset I^+(\Sigma)$ and $\mathrm{II}' \cup \mathrm{III}' \subset I^-(\Sigma)$, this concludes the proof. □

By the definition of u_T above, $r_T \to r_+$ and $r_T' \to r_c$ for $T \to \infty$. Moreover, for any finite t_0, the surfaces $\{u_T = t_0\}_{T\gg 1\in\mathbb{N}}$ form a family of Cauchy surfaces which approaches

$$\Sigma_{t_0} = \mathcal{H}_+^L \cup \mathcal{B}_+ \cup (\{t = t_0\} \cap \mathrm{I}) \cup \mathcal{B}_c \cup \mathcal{H}_c^R \tag{5.1.8}$$

as $T \to \infty$. This can be seen as follows:

For u_T to remain finite as $T \to \infty$ in $r < r_T$, either $v \to -\infty$ or $c(r) \to \infty$. However, $c(r)$ is bounded from above on $[r_-, r_T]$, and hence the only possibility is $v \to -\infty$. Since $r_T \to r_+$, this part of $\{u_T = t_0\}$ approaches $\mathcal{H}_+^L$. Similarly, for $r > r_T'$, u_T can only remain finite if $u \to -\infty$ and hence this part of $\{u_T = t_0\}$ approaches $\mathcal{H}_c^R$. By going to Kruskal-type coordinates, one can see that these parts will be connected to $\{t = t_0\} \cap \mathrm{I}$ via the bifurcation surfaces. See also the discussion in [138].

After this extensive analysis of the null geodesics, we now conclude this section by showing two more results that will be important later. The first one is a KdS-version of [138, Lemma C.4(2)]:

Lemma 5.1.4. *There exists a $\lambda_0 > 0$ and an $a_0 > 0$, such that for all $0 < \lambda < \lambda_0$ and all $0 < a < a_0$, any inextendible null geodesic on $\mathcal{M}$ that does not approach $\mathcal{H}_+$ or $\mathcal{H}_c$ in the past must intersect the region in which the vector fields ∂_{t_i}, $i \in \{+, c\}$, are both time-like.*

Proof. As discussed above, any null geodesic on $\mathcal{M}$ that does not approach $\mathcal{H}_+$ or $\mathcal{H}_c$ in the past must either approach a double root r_0 of $R(r)$ or have $r(\tau) = r_0$. Moreover, any double root r_0 must lie in the interval identified in Lemma 5.1.2.

We thus focus on the vector fields ∂_{t_i}. They satisfy

$$g(\partial_{t_i}, \partial_{t_i}) = \frac{a^2 \sin^2\theta \Delta_\theta (r_i^2 - r^2)^2 - \Delta_r \rho_i^4}{\chi^2 \rho^2 (r_i^2 + a^2)^2} \,. \tag{5.1.9}$$

The denominator is strictly positive on $\mathcal{M}$, while the numerator can be written as

$$\begin{aligned}&- \left[\lambda(r_i^2 - r^2)^2 + \Delta_r\right] a^4 \cos^4\theta - \left((1 - a^2\lambda)(r_i^2 - r^2)^2 + 2\Delta_r r_i^2\right) a^2 \cos^2\theta \\ &+ a^2 (r_i^2 - r^2)^2 - r_i^4 \Delta_r \,,\end{aligned}$$

which is monotonically decreasing in $\cos^2\theta$ for $\Delta_r \geq 0$.

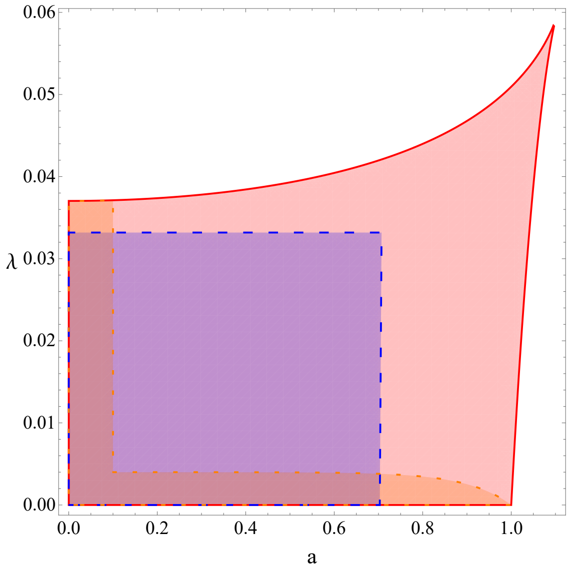


Figure 5.1: The parameter region in the (a, λ)-plane. The region surrounded by the solid line is the subextremal range of the parameters. The region within the dashed line shows the parameter region in which Lemma 5.1.4 is valid according to our numerical results. The region within the dotted lines is a sketch of the parameter region in which mode stability for the wave equation on Kerr-de Sitter has been shown [140, 141], see in particular [141, Fig. 1.1].

Therefore, we obtain the estimate

$$\begin{aligned}\chi^2\rho^2(r_i^2+a^2)^2 g(\partial_{t_i},\partial_{t_i})|_{r_0} &\leq a^2(r_i^2-r_0^2)^2-\Delta_r(r_0)r_i^4 \\ &\leq (1-27\lambda)r_i^4|_{a=0}\left[-3+\frac{8\sqrt{1-27\lambda}}{\sqrt{3}}a\right]+\mathcal{O}(a^2)\,.\end{aligned}$$

In the last line, we have taken into account that $(r-3)$ is of order a for any doble root r_0 of $R(r)$. By a continuity argument as in [138], there must be an $a_0 > 0$ such that for all $0 < a < a_0$ and $0 < \lambda < \lambda_0 < 0.0332$, $g(\partial_{t_i}, \partial_{t_i})|_{r_0} < 0$ for all possible values of r_0. $\square$

We have also checked the validity of Lemma 5.1.4 numerically by computing the quantity $a^2(r_i^2-r_0^2)^2-\Delta_r(r_0)r_i^4$ for $i \in \{+, c\}$ for different fixed values of λ and the potential range of r_0 identified in Lemma 5.1.2. The results indicate that in the whole range of λ, a_0 is of the order ~ 0.7, with a percent-level variation over the range in λ. The parameter region in which the above lemma holds according to this result is shown in Fig. 5.1. It covers a large part of the subextremal parameter range. However, the physically interesting case of small λ and a close to extremality is unfortunately not covered.

The second result we want to show concerns null geodesics approaching $\mathcal{H}_+$. It is a Kerr-de Sitter version of [123, Lemma 5.1]. The same result for geodesics approaching $\mathcal{H}_c$ can be obtained by interchanging $U \leftrightarrow V$ and $+ \leftrightarrow c$.

Lemma 5.1.5. *Denote by $\psi_+ : \mathcal{M}_+ \to \mathbb{R}^2 \times \mathbb{S}^2$ the coordinate map of the +-Kruskal coordinates. If $(U_+, \theta, \varphi_+, \xi, \sigma_\theta, \sigma_\varphi) \in T^*(\mathbb{R} \times \mathbb{S}^2)$, then there is a unique $\eta(\xi, \sigma_\theta, \sigma_\varphi) \in \mathbb{R}$ such that $\psi_+^*(U_+, 0, \theta, \varphi_+, \xi, \eta, \sigma_\theta, \sigma_\varphi)$ is null and does not lie in the conormal space $N^*(\mathcal{H}_+)$ of $\mathcal{H}_+$ iff $\xi \neq 0$. In this case $\psi_+^*(U_+, 0, \theta, \varphi_+, \xi, \eta, \sigma_\theta, \sigma_\varphi)$ is future pointing iff $\xi > 0$.*

Proof. On $\mathcal{H}_+$, one has $V_+ = 0$ and hence the metric in the +-Kruskal coordinates (2.4.15a) reduces to

$$g|_{\mathcal{H}_+} = (f_1^+ + f_2^+)U_+^2 \, \mathrm{d}V_+^2 + f_3^+ \, \mathrm{d}U_+ \, \mathrm{d}V_+ - f_4^+ U_+ \, \mathrm{d}\varphi_+ \, \mathrm{d}V_+ + \frac{\rho^2}{\Delta_\theta} \, \mathrm{d}\theta^2 + g_{\varphi\varphi} \, \mathrm{d}\varphi_+^2 \, .$$

Of these functions, $f_3^+ < 0$ on $\mathcal{H}_+$ since $G_+ < 0$, $\rho^2/\Delta_\theta = g_{\theta\theta} > 0$ and $g_{\varphi\varphi} > 0$ away from the axis. Let us first consider points away from the axis. Then, for a covector $k = (\xi, \eta, \sigma_\theta, \sigma_\varphi)$ on $\mathcal{H}_+$, we have

$$\begin{aligned} g^{-1}(k,k) = & \frac{1}{f_3^{+2} g_{\varphi\varphi}} \left(f_4^{+2} - 4(f_1^+ + f_2^+) g_{\varphi\varphi} \right) \xi^2 + \frac{4\xi\eta}{f_3^+} - \frac{2 f_4^+ \xi \sigma_\varphi}{f_3^+ g_{\varphi\varphi}} \\ & + \frac{\sigma_\theta^2}{g_{\theta\theta}} + \frac{\sigma_\varphi^2}{g_{\varphi\varphi}} \, . \end{aligned} \tag{5.1.10}$$

If $\xi = 0$, then the covector can only be null if also $\sigma_\theta = \sigma_\epsilon = 0$. In this case, the covector is of the form $k = (0, \eta, 0, 0)$ with arbitrary $\eta \in \mathbb{R}$. However, this implies $k \in N^*(\mathcal{H}_+)$.

If $\xi \neq 0$, $g^{-1}(k,k)$ is a linear function of η and thus has a unique root, i.e. there is a unique $\eta(\xi, \sigma_\theta, \sigma_\varphi)$ such that $g^{-1}(k,k) = 0$. This proves the first claim off the axis.

To consider the axis, we first note that on $r = r_+$,

$$\begin{aligned} \frac{\sigma_\theta^2}{g_{\theta\theta}} + \frac{\sigma_\varphi^2}{g_{\varphi\varphi}} = & \frac{\rho_+^2 \chi^2}{\Delta_\theta (r_+^2 + a^2)^2} \left(\frac{\sigma_\varphi^2}{\sin^2\theta} + \sigma_\theta^2 \right) \\ & + \frac{2\chi(2r_+ - a^2) + a^2 \sin^2\theta \left[\lambda^2 (r_+^2 + a^2)^2 - \chi^2 \right]}{\rho^2 \Delta_\theta (r^2 + a^2)^2} a^2 \sin^2\theta \sigma_\theta^2 \, . \end{aligned} \tag{5.1.11}$$

As $\sin\theta \to 0$, $\rho_+^2 \chi^2 / \Delta_\theta (r_+^2 + a^2)^2 \to 1/(r_+^2 + a^2)$. One can introduce new coordinates in a neighbourhood of one half of the axis, for example $\theta \sim 0$, by setting

$$y = \sin\theta \cos\varphi_+ \, , \quad z = \sin\theta \sin\varphi_+ \, ,$$

see [75], [39, Rem. 3.3]. In these coordinates, the axis is located at $y = z = 0$. If we denote the corresponding covector elements by σ_y and σ_z, one obtains

$$\begin{aligned} \left(\frac{\sigma_\varphi^2}{\sin^2\theta} + \sigma_\theta^2 \right) &= \sigma_y^2 + \sigma_z^2 - (y\sigma_y + z\sigma_z)^2 \\ \sin^2\theta \sigma_\theta^2 &= (1 - y^2 - z^2)(y\sigma_y + z\sigma_z)^2 \, . \end{aligned}$$

Hence, for $y, z \to 0$, (5.1.11) approaches

$$\frac{\sigma_\theta^2}{g_{\theta\theta}} + \frac{\sigma_\varphi^2}{g_{\varphi\varphi}} = \frac{\sigma_y^2 + \sigma_z^2}{r_+^2 + a^2} + \mathcal{O}(y, z).$$

Therefore, the discussion above also holds on the axis.

For the second claim, we now restrict to the case $\xi \neq 0$. We note that on the horizon $\mathcal{H}_+$, ∂_{U_+} is a future-pointing null vector and $k(\partial_{U_+}) = \xi \neq 0$. As a direct consequence, $\xi \geq 0$ if k is future-pointing. Since $\xi = 0$ has been excluded, this shows the first direction. For the other direction, assume that $\xi = k(\partial_{U_+}) > 0$. Then, one can show that $k(v) > 0$ for any future-pointing time-like vector by going to Gaussian normal coordinates. This concludes the proof. □

5.2 The Unruh state on Kerr-de Sitter

In this section, we define the two-point function of the Unruh state for the free, massive scalar field theory on Kerr-de Sitter and show that it leads to a well-defined state. In particular, following the discussion in Section 2.2, we show that the two-point function is a well-defined bi-distribution, which is positive, a bi-solution to the Klein-Gordon equation on Kerr-de Sitter and satisfies the commutator property. The Hadamard property of this state will be shown in a subsequent section.

Going forward, we will take $j \in \{+, c\}$. We will identify $\mathcal{H}_j = \mathbb{R}_{L_j} \times \mathbb{S}^2_{\theta,\varphi_j}$ and $\mathcal{H}_j^- = \mathbb{R}_{l_j} \times \mathbb{S}^2_{\theta,\varphi_j}$ unless stated otherwise. Here, $L_+ = U_+$, $L_c = V_c$, $l_+ = u$, $l_c = v$ and $\Omega_j = (\theta, \varphi_j)$. $\mathrm{d}^2\Omega_j$ is the usual volume element of $\mathbb{S}^2_{\theta,\varphi_j}$.

Now, let us define the two-point function for the Unruh state. As discussed in [137] and Section 2.5, the physically-motivated definition of the Unruh state is given in terms of a mode sum. These modes are determined by their asymptotic behaviour at $\mathcal{H}_+ \cup \mathcal{H}_c$. In particular, one has one set of modes which vanish at $\mathcal{H}_+$ and have positive frequency with respect to the affine parameter of the null geodesics, V_c, on $\mathcal{H}_c$, and one set of modes which vanish at $\mathcal{H}_c$ and have positive frequency with respect to U_+ at $\mathcal{H}_+$. As noted in Section 2.5, it is more convenient for the proof of the well-definedness and the proof of the Hadamard property to use a different formulation for the Unurh state:

Definition 5.2.1. For $\phi, \psi \in C_0^\infty(\mathcal{H}_j)$, let

$$A_j(\phi, \psi) = -\lim_{\epsilon \to 0} \frac{r_j^2 + a^2}{\chi \pi} \int \frac{\phi(L_j, \Omega_j)\psi(L_j', \Omega_j)}{(L_j - L_j' - i\epsilon)^2} \, \mathrm{d}L_j \, \mathrm{d}L_j' \, \mathrm{d}^2\Omega_j \, . \tag{5.2.1}$$

Then, the two-point function for the real scalar field on the Kerr-de Sitter spacetime $\mathcal{M}$ is given by

$$\begin{aligned} w(f,h) =& w_+(f,h) + w_c(f,h) \\ =& A_+(E(f)|_{\mathcal{H}_+}, E(h)|_{\mathcal{H}_+}) + A_c(E(f)|_{\mathcal{H}_c}, E(h)|_{\mathcal{H}_c}) \\ =& -\lim_{\epsilon\to 0^+} \frac{r_+^2+a^2}{\chi\pi} \int \frac{E(f)|_{\mathcal{H}_+}(U_+,\Omega_+)E(h)|_{\mathcal{H}_+}(U'_+,\Omega_+)}{(U_+-U'_+-i\epsilon)^2} \,\mathrm{d}U_+\,\mathrm{d}U'_+\,\mathrm{d}^2\Omega_+ \\ & -\lim_{\epsilon\to 0^+} \frac{r_c^2+a^2}{\chi\pi} \int \frac{E(f)|_{\mathcal{H}_c}(V_c,\Omega_c)E(h)|_{\mathcal{H}_c}(V'_c,\Omega_c)}{(V_c-V'_c-i\epsilon)^2} \,\mathrm{d}V_c\,\mathrm{d}V'_c\,\mathrm{d}^2\Omega_c \end{aligned} \tag{5.2.2}$$

for any pair of test functions $f, h \in C_0^\infty(\mathcal{M})$.

Since both $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are globally hyperbolic, and $\mathcal{M}$ is embedded in $\tilde{\mathcal{M}}$ in a causality-preserving manner, the unique commutator function on $\mathcal{M}$ is a restriction of the unique commutator function on $\tilde{\mathcal{M}}$ to $C_0^\infty(\mathcal{M})$. As a result, Lemma 5.1.5 together with [68, Thm. 8.2.4] implies that the map $E|_{\mathcal{H}_j} : C_0^\infty(\mathcal{M}) \to C^\infty(\mathcal{H}_j)$ is well-defined. However, $E(f)|_{\mathcal{H}_j}$ is in general not compactly supported and hence the convergence of the integrals in (5.2.2) is not automatically given. To put it briefly, if we want to show that (5.2.2) is the two-point function of a well-defined state, we first have to show that the integrals in (5.2.2) converge for any $f, h \in C_0^\infty(\mathcal{M})$. We show

Proposition 5.2.1. *If $0 < a \ll 1$ or $0 < a < 1$ and $\lambda \ll 1/27$, then (5.2.2) is a well-defined bi-distribution $w \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$.*

In the proof of this proposition, we will use the decay results for solutions of the Klein-Gordon equation obtained in [39]. Their results rely on mode stability, in particular the existence of a spectral gap $\alpha > 0$ for the quasi-normal modes of the (massive) Klein-Gordon equation on Kerr-de Sitter. While this is expected to hold for all sub-extremal Kerr-de Sitter spacetimes, it has only been shown so far by perturbation of corresponding results on Schwarzschild-de Sitter ($a = 0$) [140] and Kerr ($\lambda = 0$) [141]. This is the reason for the restriction to either small λ or small a. The region in which mode stability has been established is sketched in Fig. 5.1.

Proof. One key ingredient in the proof are the results in [39]. Applying the discrete symmetry $(t, \varphi) \to -(t, \varphi)$ and Sobolev embedding to the results in [39], one obtains the estimate

$$|\partial^N E(f)|(t_*, r, \theta, \varphi_*) \leq C e^{\alpha t_*}\,, \quad \partial \in \{\partial_{t_*}, \partial_r, \partial_\theta, \partial_{\varphi_*}\} \tag{5.2.3}$$

for points sufficiently close to i^- with r contained in some compact interval bounded away from r_- and for arbitrarily large $N \in \mathbb{N}$. In this estimate, α is the spectral gap discussed above. φ_* corresponds to φ^* (${}^*\varphi$), i.e. the azimuthal coordinate in the $KdS*$- ($*KdS$-) coordinate system near r_+ (r_c). t_* is equal to the coordinate t in $\{r_+ + \delta < r < r_c - \delta\} \cap \mathrm{I}$ for some small $\delta > 0$, and approaches u near $\mathcal{H}_+^-$ and v near $\mathcal{H}_c^-$ up to finite terms. Thus, for points sufficiently close to i^-, one can find $0 < \delta' < \delta$ and a constant $c > 0$, depending on δ, δ' and the concrete implementation of t_*, such that

$$e^{\alpha t_*} \leq \begin{cases} \tilde{C}(\delta, \delta') e^{\alpha t} & r \in (r_+ + \delta', r_c - \delta') \\ \tilde{C}(\delta, \delta') e^{\alpha u_+} & r \in [r_+, r_+ + \delta'] \\ \tilde{C}(\delta, \delta') e^{\alpha v_-} & r \in [r_c - \delta', r_c] \end{cases} . \tag{5.2.4}$$

The constant C in (5.2.3) still depends on f. However, let us assume that $\operatorname{supp}(f) \subset K$, with $K \subset \mathcal{M}$ compact. For V_i, $i = 1, \ldots 4$ four linearly independent, smooth vector fields on K and $\beta \in \mathbb{N}^4$ a multi-index, we set

$$\|f\|_{C^m} = \max_{|\beta| \leq m} \sup_{x \in K} \left| V^\beta f(x) \right| . \tag{5.2.5}$$

Then, as discussed in [16], one can estimate the constant C in (5.2.3) by $C' \|f\|_{C^{m(N)}}$, with C' only dependent on K.

By changing from $(t_*, r, \theta, \varphi_*)$ to $(u, v, \theta, \varphi_j)$, one can see that near $\mathcal{H}_+^-$, one has

$$\partial_{t_*} = \partial_{u_+} + \mathcal{O}(r - r_+) ,$$

while near $\mathcal{H}_c^-$,

$$\partial_{t_*} = \partial_{v_c} + \mathcal{O}(r - r_c) .$$

Here, the j-subscript indicates the azimuthal coordinate used. Combining this with the relation between u and v and the Kruskal-type coordinates, one can write the estimate (5.2.3) in the form

$$|\partial^N E(f)| \leq C' \|f\|_{C^{m(N)}} e^{\alpha t} \text{ on } \{r_+ + \delta' < r < r_c - \delta'\} \tag{5.2.6a}$$
$$|\partial^N_{U_+} E(f)| \leq C' \|f\|_{C^{m(N)}} |U_+|^{-(N+\alpha/\kappa_+)} \text{ on } \{r_+ \leq r \leq r_+ + \delta'\} \tag{5.2.6b}$$
$$|\partial^N_{V_c} E(f)| \leq C' \|f\|_{C^{m(N)}} |V_c|^{-(N+\alpha/\kappa_c)} \text{ on } \{r_c - \delta' \leq r \leq r_c\} , \tag{5.2.6c}$$

with $N \in \mathbb{N}$ and $\partial \in \{\partial_t, \partial_r, \partial_\theta, \partial_{\varphi_*}\}$, for any $f \in C_0^\infty(\mathcal{M})$ and for points sufficiently close to i^-.

In fact, these estimates may be used on the whole horizon: from the support properties of E and our study of the null geodesics, we can conclude that for any $f \in C_0^\infty(\mathcal{M})$, we can find U_f, $V_f < \infty$ depending only on $\operatorname{supp}(f)$, so that $E(f)|_{\mathcal{H}_+} \subset \{U_+ \leq U_f\}$ and $E(f)|_{\mathcal{H}_c} \subset \{V_c \leq V_f\}$.

With these estimates we can now show the convergence of the integrals in (5.2.2). We focus on w_+, the corresponding results for w_c can be obtained analogously by interchanging $U \leftrightarrow V$ and $+ \leftrightarrow c$.

Hence, let us consider $|w_+(f, h)|$ for some test functions $f, h \in C_0^\infty(\mathcal{M})$. Using

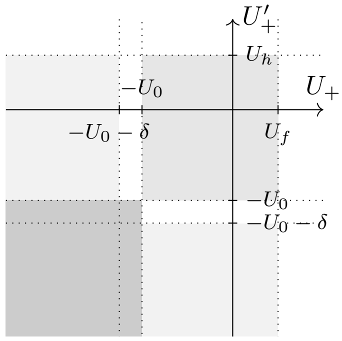


Figure 5.2: The integration regions in the (U_+, U'_+)-plane. The upper-right corner shows the support of the integrand in D_1. The lower-left corner indicates D_4. The light-gray region and the white stripe above and to the left of D_4 are D_2 and D_3.

(5.2.6b) to integrate by part twice, one obtains

$$|w_+(f,h)| = \lim_{\epsilon\to 0}\left|\frac{r_+^2+a^2}{\chi}\int\limits_{\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2} \partial_{U_+}E(f)|_{\mathcal{H}_+}(U_+,\Omega_+)\partial_{U'_+}E(h)|_{\mathcal{H}_+}(U'_+,\Omega_+)\right.$$
$$\left.\times\log(U_+-U'_+-i\epsilon)\,\mathrm{d}U_+\,\mathrm{d}U'_+\,\mathrm{d}^2\Omega_+\right| .$$

To show the convergence of this integral, take $U_0>0$ a constant such that the estimate (5.2.6b) holds for all $U_+\le -U_0$. We then split the integral into integrals A_i, $i=1,\cdots,4$, over the domains

$$D_1 = I\times I\times\mathbb{S}^2\,,\qquad D_2 = I\times I^{\mathrm{c}}\times\mathbb{S}^2\,,$$
$$D_3 = I^{\mathrm{c}}\times I\times\mathbb{S}^2\,,\qquad D_4 = I^{\mathrm{c}}\times I^{\mathrm{c}}\times\mathbb{S}^2\,. \tag{5.2.7}$$

Here, $I=[-U_0,\infty)$ and $I^c=\mathbb{R}\backslash I$. If $\mathrm{supp}(E(f)|_{\mathcal{H}_+})\cap I=\emptyset$ or $\mathrm{supp}(E(h)|_{\mathcal{H}_+})\cap I=\emptyset$, then the integrals over D_1 and D_2 or D_1 and D_3 (or all three if both intersections are empty) vanish and can be neglected. Therefore, in the rest we assume that $U_f, U_h > -U_0$. The integration regions are sketched in Fig. 5.2.

Let us start with the integral over D_1. On D_1, the integrand is compactly supported on $[-U_0,U_f]\times[-U_0,U_h]\times\mathbb{S}^2$. Thus, we can estimate

$$|A_1| \le C_1\sup_{I\times\mathbb{S}^2}\left|\partial_{U_+}E(f)|_{\mathcal{H}_+}\right|\sup_{I\times\mathbb{S}^2}\left|\partial_{U_+}E(h)|_{\mathcal{H}_+}\right|$$
$$\times\left|[-2U_0,U_f+U_h]\right|\left\|\log(y-i\epsilon)\right\|_{L^1([-U_0-U_h,U_0+U_f])}$$

for some $C_1>0$. The suprema can be estimated by some C^k-norm of f and h by the continuity of the causal propagator. Moreover, $\log(\cdot - i\epsilon)$ is in $L^1_{loc}(\mathbb{R})$ and converges to some $l \in L^1_{loc}(\mathbb{R})$ for $\epsilon \to 0$.

Next, we estimate the integral over D_2. For this estimate, it is more convenient to split the integration region again, into $D_2^a = I \times [-U_0 - \delta, -U_0) \times \mathbb{S}^2$ and $D_2^b = D_2 \backslash D_2^a$.

If we replace the half-open integral $[-U_0 - \delta, -U_0)$ in D_2^a with a closed one, the integrand is compactly supported on D_2^a, and the estimate follows along the same lines as for A_1:

$$\begin{aligned} |A_2^a| \leq & C_2^a \sup_{I \times \mathbb{S}^2} \left|\partial_{U_+} E(f)|_{\mathcal{H}_+}\right| \sup_{[-U_0-\delta,-U_0]\times\mathbb{S}^2} \left|\partial_{U_+} E(h)|_{\mathcal{H}_+}\right| \\ & \times |[-U_0 - \delta, U_f]| \, \|\log(y - i\epsilon)\|_{L^1([0,U_f+U_0+\delta])} \, . \end{aligned}$$

In this estimate, we have used that $|[-2U_0 - \delta, U_f - U_0]|$ is the same as $|[-U_0 - \delta, U_f]|$, and $C_2^a > 0$ is some constant.

For the estimate of A_2^b, we make use of (5.2.6b). In addition, we note that by the construction of the D_2^b, $U_+ - U_+' > \delta > 0$ on this domain. We can then use that for any $c > 0$, $\beta > 0$, there is a constant $C_{c,\beta} > 0$ so that $|\log(y - i\epsilon)| \leq C_{c,\beta} |y|^\beta$ for all $|y| > c$. Together with the coordinate transformation $U_+' \to -U_+'$, there is a constant $\tilde{C}_2^b > 0$ so that

$$\begin{aligned} \left|A_2^b\right| &\leq \tilde{C}_2^b \, \|h\|_{C^{m(1)}} \sup_{I \times \mathbb{S}^2} \left|\partial_{U_+} E(f)|_{\mathcal{H}_+}\right| \int\limits_{[-U_0,U_f]\times(U_0+\delta,\infty)} \left|U_+'\right|^{-1-\frac{\alpha}{\kappa_+}} \left|U_+ + U_+'\right|^\beta \, \mathrm{d}U_+ \, \mathrm{d}U_+' \\ &\leq \tilde{\tilde{C}}_2^b \, \|h\|_{C^{m(1)}} \sup_{I \times \mathbb{S}^2} \left|\partial_{U_+} E(f)|_{\mathcal{H}_+}\right| \, |[-U_0, U_f]| \left(1 + \frac{|U_f|}{U_0}\right)^{\frac{\alpha}{2\kappa_+}} \int\limits_{U_0+\delta}^{\infty} \left|U_+'\right|^{-1-\frac{\alpha}{2\kappa_+}} \, \mathrm{d}U_+' \\ &\leq C_2^b \, \|h\|_{C^{m(1)}} \sup_{I \times \mathbb{S}^2} \left|\partial_{U_+} E(f)|_{\mathcal{H}_+}\right| \, |[-U_0, U_f]| \left(1 + \frac{|U_f|}{U_0}\right)^{\frac{\alpha}{2\kappa_+}} \, . \end{aligned}$$

Here, in the second step, we have picked $\beta = \alpha/2\kappa_+$ and utilized

$$\left|U_+ + U_+'\right| \leq \left|U_+'\right| (1 + |U_f| / U_0) \, .$$

The estimate for A_3 works the same way, with the roles of f and h as well as U_+ and U_+' interchanged.

It remains to estimate A_4. Performing a sign flip in both variables, and employing (5.2.6b), $|A_4|$ can be estimated by

$$|A_4| \leq \tilde{C}_4 \, \|f\|_{C^{m(1)}} \, \|h\|_{C^{m(1)}} \int\limits_{(U_0,\infty)\times(U_0,\infty)} (U_+ U_+')^{-1-\frac{\alpha}{\kappa_+}} \left|\log(U_+' - U_+ - i\epsilon)\right| \, \mathrm{d}U_+ \, \mathrm{d}U_+' \, .$$

In [142, Lemma 6.3], it has been shown that the integral in the above estimate is finite and converges to some finite constant for $\epsilon \to 0$.

Collecting all the results above, let $K \subset M$ be a compact set such that $\mathrm{supp}(f) \subset K$

and $\mathrm{supp}(h) \subset K$. Then there is a $m \in \mathbb{N}$, given by the maximum of the m in the above estimates, so that

$$|w_+(f,h)| \leq C(K) \, \|f\|_{C^m} \, \|h\|_{C^m} \ . \tag{5.2.8}$$

As discussed above, the same also holds for w_c by similar arguments. Thus, $w = w_+ + w_c$ is a well-defined bi-distribution satisfying the estimate

$$|w(f,h)| \leq C(K) \, \|f\|_{C^m} \, \|h\|_{C^m} \tag{5.2.9}$$

for some $m \in \mathbb{N}$ and any $f, h \in C_0^\infty(K)$. By the Schwartz kernel theorem, $w(x,y)$ is in $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$. $\square$

An additional benefit of the decay estimates (5.2.6b) and (5.2.6c) is that we can now identify the function spaces in which the (asymptotic) initial data for the Klein-Gordon equation on the horizon lies. In particular, for any $f \in C_0^\infty(\mathcal{M})$,

$$E(f)|_{\mathcal{H}_j} \in S(\mathcal{H}_j) \equiv \Big\{ \phi \in C^\infty(\mathcal{H}_j) : \exists L_\phi, C_{\phi,N}, N = 0,1 : \tag{5.2.10}$$
$$\phi(L_j, \Omega_j) = 0 \, \forall L_j \geq L_\phi \ \text{ and } |\partial_{L_j}^N \phi(L_j, \Omega_j)| \leq C_{\phi,N} (1 + |L_j|)^{-\frac{\alpha}{\kappa_j} - N} \Big\} \ .$$

For $f \in C_0^\infty(\mathrm{I})$, we can infer from our discussion of the null geodesics on Kerr-de Sitter and the support properties, that $\mathrm{supp}(E(f)) \cap \mathcal{H}_+$ is contained in $\{u \leq u_f\}$, and $\mathrm{supp}(E(f)) \cap \mathcal{H}_c$ is contained in $\subset \{v \leq v_f\}$ for some $u_f, v_f < \infty$. Therefore

$$E(f)|_{\mathcal{H}_j} \in S(\mathcal{H}_j^-) \equiv \Big\{ \phi \in C^\infty(\mathcal{H}_j^-) : \exists l_\phi, C_{\phi,N}, N = 0,1 : \tag{5.2.11}$$
$$\phi(l_j, \Omega_j) = 0 \, \forall l_j \geq l_\phi \text{ and } |\partial_{l_j}^N \phi(l_j, \Omega_j)| \leq C_{\phi,N} e^{-\alpha |l_j|} \Big\} \ .$$

In the next step, we have to test that $w(f,h)$ is a weak bi-solution to the Klein-Gordon equation that satisfies positivity and the commutator property.

Since $E(\mathcal{K}f) = 0$ for all $f \in C_0^\infty(\mathcal{M})$, $w(f,h)$ is a weak bi-solution by construction.

For the proof of positivity, we can use that the function spaces $S(\mathcal{H}_j)$ and $S(\mathcal{H}_j^-)$ allow to transfer the results in [43, Sec.3] to the present case by a change of the appearing constant which are related to the spacetime metric. In particular,

Proposition 5.2.2. *1. Equipping the space $C_0^\infty(\mathcal{H}_j)$ with the Hermitian sesquilinear form $A_j(\bar{\cdot}, \cdot)$, the map*

$$F_j : C_0^\infty(\mathcal{H}_j) \to L^2(\mathbb{R}_+ \times \mathbb{S}^2; \, \nu_j(\eta) \, d\eta \, d^2\Omega_j) \tag{5.2.12a}$$
$$\phi \mapsto F_j(\phi) = (2\pi)^{-\frac{1}{2}} \int e^{i\eta L_j} \phi(L_j, \theta, \varphi_j) \, dL_j \Big|_{\{\eta \geq 0\}} \ , \tag{5.2.12b}$$

with $\nu_j(\eta) = 2\eta(r_j^2 + a^2)\chi^{-1}$, is an isometry and by continuity and linearity extends to a Hilbert space isomorphism mapping $\overline{(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot}, \cdot))}$, the Hilbert completion of $(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot}, \cdot))$, onto $L^2(\mathbb{R}_+ \times \mathbb{S}^2; \, \nu_j(\eta) \, d\eta \, d^2\Omega_j)$ [43, Prop. 3.2 a)].

2. *When $C_0^\infty(\mathcal{H}_j^-)$ is equipped with the Hermitian sesquilinear form $A_j(\bar{\cdot}, \cdot)$, the map*

$$\tilde{F}_j : C_0^\infty(\mathcal{H}_j^-) \to L^2(\mathbb{R} \times \mathbb{S}^2;\, \mu_j(\omega)\, d\omega\, d^2\Omega_j) \tag{5.2.13a}$$

$$\phi \mapsto \tilde{F}_j(\phi) = (2\pi)^{-\frac{1}{2}} \int e^{i\omega l_j} \phi(l_j, \theta, \varphi_j)\, dl_j\,, \tag{5.2.13b}$$

$$\mu_j(\omega) = \frac{r_j^2 + a^2}{\chi} \frac{\omega e^{\pi\omega/\kappa_j}}{\sinh(\pi\omega/\kappa_j)}\,, \tag{5.2.13c}$$

is an isometry. $\tilde{F}_j$ uniquely extends to a Hilbert space isomorphism from $\overline{C_0^\infty(\mathcal{H}_j^-)}$, as a Hilbert subspace of $\overline{(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot}, \cdot))}$, to $L^2(\mathbb{R} \times \mathbb{S}^2; \mu_j(\omega)\, d\omega\, d^2\Omega_j)$ [43, Prop. 3.3 a)].

3. *Any $\phi \in S(\mathcal{H}_j^-)$ can be identified with an element in $\overline{(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot}, \cdot))}$ as described in [43, Prop. 3.3 b)], i.e. let $\{\psi_n\}_{n\in\mathbb{N}}$, $\{\psi'_n\}_{n\in\mathbb{N}}$ sequences in $C_0^\infty(\mathcal{H}_j^-)$ that both converge to some $\phi \in S(\mathcal{H}_j^-)$ in the topology of*

$$H^1(\mathcal{H}_j^-) = \{\phi \in L^2(\mathbb{R}_{l_j} \times \mathbb{S}^2_{\Omega_j}; dl_j\, d^2\Omega_j) : \partial_{l_j}\phi \in L^2(\mathbb{R}_{l_j} \times \mathbb{S}^2_{\Omega_j}, dl_j\, d^2\Omega_j)\}\,,$$

compare [43] and [143, App. C]. Then both sequences are of Cauchy type in $\overline{(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot}, \cdot))}$ and the difference $\psi_n - \psi'_n$ converges to zero in this space. The identification of $S(\mathcal{H}_j^-)$ with a subspace of $\overline{(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot}, \cdot))}$ is such that on $S(\mathcal{H}_j^-)$, $\tilde{F}_j$ agrees with the standard Fourier-Plancherel transformation in l_j.

A proof for Proposition 5.2.2 follows along the same lines as in [43]. In fact, the first two claims are exactly the same as in [43, Prop. 3.2a)] and [43, Prop. 3.3a)] up to the following changes of constants related to the spacetime metric: the constant r_S^2 in the definitions of λ_{KW} in [43, Prop. 3.2a)], called A_j in our notation, and in $\mathrm{d}\mu(k)$ in [43, Prop. 3.3a)] is replaced by $(r_j^2 + a^2)/\chi$. Moreover, the constant $(2r_S)^{-1}$, which arises from the connection between the Kruskal-type coordinates and the coordinates u and v in [43, Prop. 3.3a)], is replaced by the corresponding constant κ_j for our coordinates.

The third point can be given in the same way as the proof of [43, Prop. 3.3b)]. One begins by realizing that by definition of $S(\mathcal{H}_j^-)$, any $\phi \in S(\mathcal{H}_j^-)$ is in the Sobolev space $H^1(\mathcal{H}_j^-)$. Employing results on the Fourier-Plancherel transform in one variable on $\mathbb{R}\times\mathbb{S}^2$ which have been worked out in [143, App. C], the Fourier-Plancherel transform of ϕ lies in $L^2(\mathbb{R} \times \mathbb{S}^2; \mu_j(\omega)\, \mathrm{d}\omega\, \mathrm{d}^2\Omega_j)$. Since $C_0^\infty(\mathcal{H}_j^-)$ is dense in $H^1(\mathcal{H}_j^-)$, one can find a sequence $\phi_n \in C_0^\infty(\mathcal{H}_j^-)$ converging to ϕ in $H^1(\mathcal{H}_j^-)$, implying the convergence of the Fourier-Plancherel transformed sequence in $L^2(\mathbb{R} \times \mathbb{S}^2; \mu_j(\omega)\, \mathrm{d}\omega\, \mathrm{d}^2\Omega_j)$. For ϕ_n, the Fourier-Plancherel transform agrees with the map $\tilde{F}_j$, and the isometry property of $\tilde{F}_j$ thus allows to conclude the proof.

With this result, we can now express the two-point function in a way that will make its positivity visible. To this end, let $\xi \in C^\infty(\mathbb{R})$ be a cutoff function which is equal to one for $x > x_0$ and vanishes for $x < x_1$ for some constants $x_1 < x_0 < 0$. With the help of F_j

and E, we can then define the maps

$$K_j : C_0^\infty(\mathcal{M}) \to L^2(\mathbb{R}_+ \times \mathbb{S}^2;\, \nu_j(\eta)\,\mathrm{d}\eta\,\mathrm{d}^2\Omega_j), \tag{5.2.14a}$$
$$K_j(f) = F_j(\xi E(f)|_{\mathcal{H}_j}) + F_j((1-\xi)E(f)|_{\mathcal{H}_j})\,;$$
$$K_j^{\mathrm{I}} : C_0^\infty(\mathrm{I}) \to L^2(\mathbb{R} \times \mathbb{S}^2;\, \mu_j(\omega)\,\mathrm{d}\omega\,\mathrm{d}^2\Omega_j)\,, \tag{5.2.14b}$$
$$K_j^{\mathrm{I}}(f) = \tilde{F}_j(E(f)|_{\mathcal{H}_j})\,.$$

This is well-defined, since $\xi E(f)|_{\mathcal{H}_j}$ is compactly supported on $\mathcal{H}_j$, while the remainder $(1-\xi)E(f)|_{\mathcal{H}_j} \in S(\mathcal{H}_j^-)$ can be identified with an element of $\overline{(C_0^\infty(\mathcal{H}_j), A_j(\bar{\cdot},\cdot))}$ by part 3) of Proposition 5.2.2, see [43]. These maps satisfy

Proposition 5.2.3. *The maps K_j are well-defined in the sense that they are independent of the choice of ξ. They are linear, and we can write*

$$\begin{aligned} w(f,h) =& \left\langle K_+(\overline{f}), K_+(h)\right\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^2;\,\nu_+(\eta)\,d\eta\,d^2\Omega_+)} \\ &+ \left\langle K_c(\overline{f}), K_c(h)\right\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^2;\,\nu_c(\eta)\,d\eta\,d^2\Omega_c)} \\ =& \left\langle K(\overline{f}), K(h)\right\rangle_{L^2(\mathbb{R}_+\times\mathbb{S}^2;\,\nu_+(\eta)\,d\eta\,d^2\Omega_+)\oplus L^2(\mathbb{R}_+\times\mathbb{S}^2;\,\nu_c(\eta)\,d\eta\,d^2\Omega_c)} \end{aligned} \tag{5.2.15}$$

for any $f,h \in C_0^\infty(\mathcal{M})$ and

$$\begin{aligned} w(f,h) =& \left\langle K_+^{\mathrm{I}}(\overline{f}), K_+^{\mathrm{I}}(h)\right\rangle_{L^2(\mathbb{R}\times\mathbb{S}^2;\,\mu_+(\omega)\,d\omega\,d^2\Omega_+)} \\ &+ \left\langle K_c^{\mathrm{I}}(\overline{f}), K_c^{\mathrm{I}}(h)\right\rangle_{L^2(\mathbb{R}\times\mathbb{S}^2;\,\mu_c(\omega)\,d\omega\,d^2\Omega_c)} \end{aligned} \tag{5.2.16}$$

when restricted to $f,h \in C_0^\infty(\mathrm{I})$.

Proof. Let ξ and ξ' be two cutoff functions as in the definition of K_j. Then $\xi - \xi'$ is contained in $C_0^\infty(\mathbb{R})$. Since F_j is a linear map,

$$\begin{aligned} &F_j(\xi E(f)|_{\mathcal{H}_j}) + F_j((1-\xi)E(f)|_{\mathcal{H}_j}) - F_j(\xi' E(f)|_{\mathcal{H}_j}) - F_j((1-\xi')E(f)|_{\mathcal{H}_j}) \\ &= F_j((\xi-\xi')E(f)|_{\mathcal{H}_j}) - F_j((\xi-\xi')E(f)|_{\mathcal{H}_j}) = 0\,. \end{aligned}$$

This shows that the map is independent of the choice of ξ. The linearity of K_j follows directly from the linearity of F_j and E. Since ξ and (the kernel of) E are real, we obtain by the isometry property of F_j, part 1) of Proposition 5.2.2,

$$\begin{aligned} w_j(f,h) =& A_j(E(f)|_{\mathcal{H}_j}, E(h)|_{\mathcal{H}_j}) \\ =& A_j(\xi E(f)|_{\mathcal{H}_j}, \xi E(h)|_{\mathcal{H}_j}) + A_j((1-\xi)E(f)|_{\mathcal{H}_j}, \xi E(h)|_{\mathcal{H}_j}) \\ &+ A_j(\xi E(f)|_{\mathcal{H}_j}, (1-\xi)E(h)|_{\mathcal{H}_j}) + A_j((1-\xi)E(f)|_{\mathcal{H}_j}, (1-\xi)E(h)|_{\mathcal{H}_j}) \\ =& \left\langle F_j(\overline{\xi E(f)|_{\mathcal{H}_j}}), F_j(\xi E(h)|_{\mathcal{H}_j})\right\rangle_{L^2} + \left\langle F_j(\overline{(1-\xi)E(f)|_{\mathcal{H}_j}}), F_j(\xi E(h)|_{\mathcal{H}_j})\right\rangle_{L^2} \\ &+ \left\langle F_j(\overline{\xi E(f)|_{\mathcal{H}_j}}), F_j((1-\xi)E(h)|_{\mathcal{H}_j})\right\rangle_{L^2} \\ &+ \left\langle F_j(\overline{(1-\xi)E(f)|_{\mathcal{H}_j}}), F_j((1-\xi)E(h)|_{\mathcal{H}_j})\right\rangle_{L^2} \end{aligned}$$

$$= \left\langle K_j(\overline{f}), K_j(h) \right\rangle_{L^2}$$

for any $f, h \in C_0^\infty(\mathcal{M})$. Here, we have utilized the short-hand notation L^2 for $L^2(\mathbb{R}_+ \times \mathbb{S}^2; \nu_j(\eta)\, \mathrm{d}\eta\, \mathrm{d}^2\Omega_j)$. Thus, (5.2.15) follows by summing over $j \in \{+, c\}$. For f, $h \in C_0^\infty(\mathrm{I})$, (5.2.16) follows directly from the isometry property of $\widetilde{F}_j$ in part 2) and the identification in part 3) of Proposition 5.2.2. □

From the form of $w(f,h)$ in (5.2.15), one can see immediately that w is positive. Moreover, taking into account that the maps K_j correspond to a Fourier transform in L_j followed by a restriction to positive frequencies, (5.2.15) indicates that the Unruh state defined by (5.2.2) indeed corresponds to a mode expansion in positive frequency modes with respect to U_+ or V_c respectively as in [137].

It remains to show that $w(f,h)$ satisfies the commutator property,

Proposition 5.2.4. *Under the same conditions on (λ, a) as in Proposition 5.2.1, $w(f,h)$ satisfies the commutator property, i.e.*

$$w(f,h) - w(h,f) = iE(f,h) \quad \forall f, h \in C_0^\infty(\mathcal{M})\,. \tag{5.2.17}$$

The proof of this proposition is obtained in a similar way as the proof of [43, Thm. 2.1] and of the commutator property of the Unruh state on RNdS in [16].

Proof. Let us start the proof by considering the right-hand side of (5.2.17). Inserting the definition of w in (5.2.2) and interchanging $L_j \leftrightarrow L'_j$ in $w(h,f)$, one obtains

$$\begin{aligned} w(f,h) - w(h,f) = -\sum_j \lim_{\epsilon\to 0} \frac{r_j^2 + a^2}{\chi\pi} \int\limits_{\mathcal{H}_j} E(f)|_{\mathcal{H}_j}(L_j, \Omega_j) E(h)|_{\mathcal{H}_j}(L'_j, \Omega_j) \\ \times 2i\mathrm{Im}(L_j - L'_j - i\epsilon)^{-2}\, \mathrm{d}L_j\, \mathrm{d}L'_j\, \mathrm{d}^2\Omega_j\,. \end{aligned}$$

Employing the identity $\mathrm{Im}(x - i0^+)^{-2} = -\pi\delta^{(1)}(x)$ and partially integrating, this can be written as

$$\begin{aligned} & w(f,h) - w(h,f) \\ &= \sum_j i\frac{r_j^2 + a^2}{\chi} \int\limits_{\mathcal{H}_j} \left[E(f)|_{\mathcal{H}_j} \partial_{L_j} E(h)|_{\mathcal{H}_j} - E(h)|_{\mathcal{H}_j} \partial_{L_j} E(f)|_{\mathcal{H}_j} \right] (L_j, \Omega_j)\, \mathrm{d}L_j\, \mathrm{d}^2\Omega_j\,. \end{aligned}$$

If we define the current $J : C_0^\infty(\mathcal{M}) \times C_0^\infty(\mathcal{M}) \to \Gamma^*(\mathcal{M})$,

$$J_\nu[f,h] = E(f)\nabla_\nu E(h) - E(h)\nabla_\nu E(f)\,, \tag{5.2.18}$$

compare (2.2.6), this can be simplified to

$$w(f,h) - w(h,f) = \sum_j i\frac{r_j^2 + a^2}{\chi} \int\limits_{\mathcal{H}_j} J_{L_j}[f,h]|_{\mathcal{H}_j}(L_j, \Omega_j)\, \mathrm{d}L_j\, \mathrm{d}^2\Omega_j\,. \tag{5.2.19}$$

We want to compare this to $iE(f,h)$. To do so, let us note that $E(f,h) = \sigma(E(f), E(h))$, as discussed in Section 2.2. Hence, for any Cauchy surface Σ,

$$E(f,h) = \int_{\Sigma} J_\nu[f,h] n^\nu_\Sigma \, \mathrm{d}vol_\gamma \, .$$

Let us choose $\Sigma = \mathcal{H}^L_+ \cup \mathcal{B}_+ \cup \Sigma_{t_0} \cup \mathcal{B}_c \cup \mathcal{H}^R_c$, where $\Sigma_{t_0} = \{t = t_0\} \cap \mathrm{I}$ for some $t_0 < 0$. By the discussion following the proof of Proposition 5.1.3, this is a limit of a sequence of space-like, piecewise smooth Cauchy surfaces of $\mathcal{M}$. Then

$$\begin{aligned} E(f,h) =& \frac{r_+^2 + a^2}{\chi} \int_{\mathcal{H}^L_+} J_{U_+}[f,h]|_{\mathcal{H}_+} \, \mathrm{d}U_+ \, \mathrm{d}^2\Omega_+ \\ &+ \int_{\Sigma_{t_0}} J_\nu[f,h] n^\nu \, \mathrm{d}vol_\gamma + \frac{r_c^2 + a^2}{\chi} \int_{\mathcal{H}^R_c} J_{V_c}[f,h]|_{\mathcal{H}_c} \, \mathrm{d}V_c \, \mathrm{d}^2\Omega_c \, . \end{aligned} \tag{5.2.20}$$

The first and last part of the integral already agree with the corresponding parts in (5.2.19). Therefore, we focus on the integral over Σ_{t_0}. More concretely, we are interested in the limit $t_0 \to -\infty$, as performed in [43].

Keeping in mind that we would like to take this limit, it is easier to further split the integral over Σ_{t_0} into three integrals over the sets $\Sigma_+ = \Sigma_{t_0} \cap \{r_+ < r \leq r_+ + \delta'\}$, $\Sigma_c = \Sigma_{t_0} \cap \{r_c - \delta' \leq r < r_c\}$, and $\Sigma_0 = \Sigma_{t_0} \backslash (\Sigma_+ \cup \Sigma_c)$. Here, $\delta' > 0$ is the same small constant as in the estimate (5.2.4).

The simplest of the three integrals is the one over Σ_0. On this surface, we may use the Boyer-Lindquist coordinates, so that the determinant of the induced metric is given by

$$|\gamma| = |g_{rr} g_{\theta\theta} g_{\varphi\varphi}| = \frac{\rho^2 \sin^2\theta}{\chi^2} \left[\frac{(r^2+a^2)^2}{\Delta_r} - \frac{a^2 \sin^2\theta}{\Delta_\theta} \right]$$

and the future-pointing normal vector is

$$n^a_{\Sigma_0} = (g^{tt})^{\frac{1}{2}} \left((\partial_t)^a - \frac{g_{t\varphi}}{g_{\varphi\varphi}} (\partial_\varphi)^a \right) .$$

Then, the integral over Σ_0 can explicitly be written as

$$\begin{aligned} \int_{\Sigma_0} J_a[f,h] n^a_{\Sigma_0} \, \mathrm{d}vol\gamma = \int_{r_+ + \delta'}^{r_c - \delta'} \int_{\mathbb{S}^2} \Bigg[\left| \frac{(r^2+a^2)^2}{\Delta_r} - \frac{a^2\sin^2\theta}{\Delta_\theta} \right| J_t[f,h] \\ + a \left(\frac{r^2+a^2}{\Delta_r} - \frac{1}{\Delta_\theta} \right) J_\varphi[f,h] \Bigg] \, \mathrm{d}^2\Omega \, \mathrm{d}r \, . \end{aligned} \tag{5.2.21}$$

Since $r - r_+ \geq \delta'$ and $r_c - r \geq \delta'$, one can bound $\left| \frac{(r^2+a^2)^2}{\Delta_r} - \frac{a^2\sin^2\theta}{\Delta_\theta} \right|$ and $\left(\frac{r^2+a^2}{\Delta_r} - \frac{1}{\Delta_\theta} \right)$

by constants of order δ'^{-1}. Moreover, taking into account the definition of $J_\nu[f,h]$ and the estimate (5.2.6a), we can bound $|J_t[f,h]|$ and $|J_\varphi[f,h]|$ by $Ce^{2\alpha t_0}$ for sufficiently small t_0 and some constant $C > 0$, which will depend on f and h, but not on t_0. In combination, we obtain

$$\left| \int\limits_{\Sigma_0} J_\nu[f,h] n^\nu \, \mathrm{d}vol\gamma \right| \leq C(\delta', f, h) e^{2\alpha t_0} \,. \tag{5.2.22}$$

Hence, the contribution of this part of the integral vanishes in the limit $t_0 \to -\infty$.

It remains to analyse the integrals over Σ_+ and Σ_c. Since both of these parts can be handled in the same way, we will focus on Σ_+.

As a first step, we note that in terms of the Kruskal coordinates around r_+, Σ_+ can be expressed as

$$\Sigma_+ = \{V_+ = -e^{-2\kappa_+ t_0} U_+\} \cap \{U_+(t_0, r_+ + \delta') \leq U_+ \leq 0\} \,.$$

In the next step, let us fix some $U_0 < 0$, so that for some $t_{0,\mathrm{max}}$, $U_+(t_0, r_+ + \delta') < U_0$ for all $t_0 < t_{0,\mathrm{max}}$. $\Sigma_+ \cap \{U_+ \geq U_0\}$ can then be interpreted as part of the (piecewise smooth) boundary of the compact region

$$\{0 \leq V_+ \leq -e^{2\kappa_+ t_0} U_+\} \cap \{U_0 \leq U_+ \leq 0\} \subset \tilde{\mathcal{M}} \,.$$

The other parts of the boundary are $\mathcal{H}^- \cap \{U_+ \geq U_0\}$ and

$$S_{t_0} \equiv \{0 \leq V_+ \leq -e^{2\kappa_+ t_0} U_0\} \cap \{U_+ = U_0\} \,.$$

Hence, S_{t_0} corresponds to $[0, -e^{2\kappa_+ t_0} U_0] \times \mathbb{S}^2$ in the $+$-Kruskal coordinates.

Since $E(f)$ and $E(h)$ are solutions to the Klein-Gordon equation (2.2.1) on Kerr-de Sitter, the current $J_\nu[f,h]$ is conserved, $\nabla_\nu J^\nu[f,h] = 0$, see also the discussion in Section 2.2. Hence, by Stoke's theorem

$$\begin{aligned} \int\limits_{\Sigma_+ \cap \{U_1 \leq U_+\}} J_\nu[f,h] n^\nu \, \mathrm{d}vol\gamma &= \int\limits_{\mathcal{H}^-_+ \cap \{U_1 \leq U_+\}} J_\nu[f,h] n^\nu \, \mathrm{d}vol_\gamma + \int\limits_{S_{t_0}} J_\nu[f,h] n^\nu \, \mathrm{d}vol_\gamma \\ &= \frac{r_+^2 + a^2}{\chi} \int\limits_{[U_1,0] \times \mathbb{S}^2} J_{U_+}[f,h]|_{\mathcal{H}_+} \, \mathrm{d}U_+ \, \mathrm{d}^2\Omega_+ + \int\limits_{S_{t_0}} J_\nu[f,h] n^\nu \, \mathrm{d}vol_\gamma \,. \end{aligned} \tag{5.2.23}$$

Taking the form of the metric in the Kruskal-type coordinates (2.4.15a) and the smoothness of $J_\nu[f,h]n^\nu$ on S_{t_0} into account, one can conclude that the contribution of the integral over S_{t_0} vanishes as $t_0 \to -\infty$, compare also the proof of [43, Thm. 2.1]. Consequently,

$$\lim_{t_0\to-\infty}\int\limits_{\Sigma_+\cap\{U_1\le U_+\}} J_\nu[f,h]n^\nu\,\mathrm{d}vol\gamma = \frac{r_+^2+a^2}{\chi}\int\limits_{[U_1,0]\times\mathbb{S}^2} J_{U_+}[f,h]|_{\mathcal{H}_+}\,\mathrm{d}U_+\,\mathrm{d}^2\Omega_+\,. \tag{5.2.24}$$

The remaining piece, the integral over $\Sigma_+\cap\{U_+<U_0\}$, can be performed explicitly in the coordinates (u,v,Ω_+). For this purpose, let us define $u_0=u(U_0)=-\kappa_+^{-1}\log|U_0|$ and

$$\begin{aligned}
F_+(u,v,\theta) &= \frac{r^2(u,v)+a^2}{\chi} - \frac{a^2\sin^2\theta\Delta_r(u,v)}{\chi(r^2(u,v)+a^2)\Delta_\theta}\,,\\
H_+(u,v,\theta) &= \frac{a(r_+^2-r^2(u,v))}{\chi(r_+^2+a^2)} - \frac{a\rho_+^2(u,v,\theta)\Delta_r(u,v)}{\chi(r^2(u,v)+a^2)(r_+^2+a^2)\Delta_\theta}\,.
\end{aligned}$$

Then we can write

$$\begin{aligned}
\int\limits_{\Sigma_+\cap\{U_+\le U_0\}} J_\nu[f,h]n^\nu\,\mathrm{d}vol\gamma = \int\limits_{(-\infty,u_0)\times\mathbb{S}^2} &\mathbf{1}_{(u(t_0,r_++\delta'),\infty)}\left(F_+\left[J_u[f,h]+J_v[f,h]\right]\right.\\
&\left.+\ H_+J_{\varphi_+}[f,h]\right)(u,v,\Omega_+)\big|_{v=2t_0-u}\,\mathrm{d}u\,\mathrm{d}\Omega_+\,.
\end{aligned}\tag{5.2.25}$$

Here, $\mathbf{1}_I$ is the characteristic function of the interval I. We want to use dominated convergence to show that we can interchange the limit $t_0\to-\infty$ with the integration as in [43]. For this, we first note that on Σ_+, we can bound $r_+\le r(u,v)\le r_++\delta'$. This implies also

$$\begin{aligned}
\left|F_+(u,v,\theta)|_{\Sigma_+}\right| &\le \frac{(r_++\delta')^2+a^2}{\chi} + \frac{a^2\Delta_r|_{r_++\delta'}}{\chi(r_+^2+a^2)}\,,\\
\left|H_+(u,v,\theta)|_{\Sigma_+}\right| &\le \frac{|a|((r_++\delta')^2-r_+^2)}{\chi(r_+^2+a^2)} + \frac{|a|\Delta_r|_{r_++\delta'}}{\chi(r_+^2+a^2)}\,.
\end{aligned}$$

The cutoff function is simply bounded by one. A coordinate transform of the estimate (5.2.6b) further allows us to bound $\left|J_\nu[f,h]|_{\Sigma_+}\right|$ for $\nu\in\{u,v,\varphi_+\}$ by $C(f,h,\delta')e^{2\alpha u}$. We can therefore estimate the integrand by a positive function that is independent of t_0 and contained in $L^1((-\infty,u_0)\times\mathbb{S};\,\mathrm{d}u\,\mathrm{d}^2\Omega_+)$. Thus, we may apply dominated convergence and interchange the limit $t_0\to-\infty$ with the integration. This means that

$$r_*(u,v)|_{\Sigma_+} = r_*(u,2t_0-u) = t_0-u\to-\infty\,,$$

or equivalently $r\to r_+$. Noting that H_+ vanishes at $r=r_+$, one obtains

$$\lim_{t_0\to-\infty}\int\limits_{\Sigma_+\cap\{U_+\le U_0\}} J_a[f,h]n^a\,\mathrm{d}vol\gamma = \int\limits_{(-\infty,u_0)\times\mathbb{S}^2}(J_u[f,h]+J_v[f,h])|_{v\to-\infty}\,\frac{r_+^2+a^2}{\chi}\,\mathrm{d}u\,\mathrm{d}\Omega_+\,.$$

It remains to consider the behaviour of $J_v[f,h]$, since v diverges towards the horizon. By switching back to the Kruskal-type coordinates, one finds $J_v[f,h] = \kappa_+ V_+ J_{V_+}[f,h]$. Since the current is a smooth vector field on $\tilde{\mathcal{M}}$ and $V_+ = 0$ on $\mathcal{H}_+^-$, we conclude that

$$\lim_{t_0\to-\infty} \int\limits_{\Sigma_+\cap\{U_+\leq U_0\}} J_\nu[f,h] n^\nu \,\mathrm{d}vol\gamma = \frac{r_+^2+a^2}{\chi} \int\limits_{(-\infty,U_0)\times\mathbb{S}^2} J_{U_+}[f,h]|_{\mathcal{H}_+} \,\mathrm{d}U_+ \,\mathrm{d}^2\Omega_+ \tag{5.2.26}$$

and thus

$$\lim_{t_0\to-\infty} \int\limits_{\Sigma_+} J_\nu[f,h] n^\nu \,\mathrm{d}vol\gamma = \frac{r_+^2+a^2}{\chi} \int\limits_{\mathcal{H}_+^-} J_{U_+}[f,h]|_{\mathcal{H}_+} \,\mathrm{d}U_+ \,\mathrm{d}^2\Omega_+ \,. \tag{5.2.27}$$

This concludes the analysis of the integral over Σ_+, the integral over Σ_c is analysed analogously. Combining all the pieces, we finally obtain that (5.2.19) agrees with (5.2.20) up to a factor of i, finishing the proof. □

With this, we have shown that $w(f,h)$ defined in (5.2.2) is indeed the two-point function of a well-defined quasi-free state on the CCR-algebra $\mathcal{A}(\mathcal{M})$, satisfying all requirements listed in Corollary 2.2.2.

Before we continue to show the Hadamard property, let us remark that the Unruh state constructed in this way is a stationary state in the following sense:

Lemma 5.2.5. *Let $\psi^C_{b*} : \mathbb{R}\times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ with $C\in\mathbb{R}^2$ denote the push-forward along the flow induced by the Killing vector field $v_C = C_1\partial_t + C_2\partial_\varphi$. In Boyer-Lindquist coordinates, it acts on smooth functions on $\mathcal{M}$ by*

$$\psi^C_{b*} f(t,r,\theta,\varphi) = f(t - C_1 b, r, \theta, \varphi - C_2 b) \quad \forall f\in C^\infty(\mathcal{M})\,,\ b\in\mathbb{R}\,.$$

Then for any pair of test functions f, $h\in C_0^\infty(\mathcal{M})$, and for any $b\in\mathbb{R}$ and $C\in\mathbb{R}^2$

$$w(\psi^C_{b*} f, \psi^C_{b*} h) = w(f,h)\,. \tag{5.2.28}$$

Proof. First of all, we notice that $\psi^C_{b*}\circ E = E\circ\psi^C_{b*}$. This follows directly from (2.2.8) and the fact that v_C is a Killing vector field. We obtain that

$$\begin{aligned} E(\psi^C_{b*} f)(U_+,0,\theta,\varphi_+) &= E(f)\left(e^{\kappa_+ C_1 b} U_+, 0, \theta, \varphi_+ + C_+ b\right) \\ E(\psi^C_{b*} f)(0,V_c,\theta,\varphi_c) &= E(f)\left(0, e^{\kappa_c C_1 b} V_c, \theta, \varphi_c + C_c b\right)\,, \end{aligned}$$

where $C_j = a\left(r_j^2 + a^2\right)^{-1} C_1 + C_2$. Let us consider the first integral, $w_+(f,h)$, in (5.2.2); the other integral can be handled analogously.

Then we can introduce the new coordinates $\tilde{U} = e^{\kappa_+ C_1 b} U_+$ and $\tilde{U}' = e^{\kappa_+ C_1 b} U'_+$. Since $e^{\kappa_+ C_1 b}$ is a positive constant, this change of coordinates can be applied to the whole range of integration. Furthermore, applying this change of coordinates to $(U_+ - U'_+ - i\epsilon)^{-2}$, one obtains $e^{2\kappa_+ C_1 b}(\tilde{U} - \tilde{U}' - i\tilde{\epsilon})^{-2}$, where $\tilde{\epsilon} = e^{\kappa_+ C_1 b}\epsilon$ is related to ϵ by a bounded, positive factor and hence converges to zero if and only if ϵ does.

Second, we note that a function $f(\theta, \varphi)$ on $\mathbb{S}^2$ can also be considered as a function on $[-1,1]_x \times \mathbb{R}_\varphi$, $x = \cos\theta$, which is 2π-periodic in φ. Hence, one may change variables to $\tilde{\varphi} = \varphi_+ + C_+ b$. This shifts the integration range from $[0, 2\pi)$ to $[C_+ b, 2\pi + C_+ b)$. However, the result of the integral is not affected by this, since the new integration range still includes one complete period of φ_+.

Putting the pieces together, one concludes

$$
\begin{aligned}
&w_+(\psi^C_{b*} f, \psi^C_{b*} h) \qquad (5.2.29)\\
&= -\lim_{\epsilon \to 0^+} \frac{r_+^2 + a^2}{\chi\pi} \int\limits_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{E(f)|_{\mathcal{H}_+}(\tilde{U}, \theta, \tilde{\varphi}) E(h)|_{\mathcal{H}_+}(\tilde{U}', \theta, \tilde{\varphi})}{(U_+ - U'_+ - i\epsilon)^2} \, \mathrm{d}U_+ \, \mathrm{d}U'_+ \, \mathrm{d}^2\Omega_+ \\
&= -\lim_{\tilde{\epsilon} \to 0^+} \frac{r_+^2 + a^2}{\chi\pi} \int\limits_{\mathbb{R}^2 \times [-1,1] \times [C_+ b, 2\pi + C_+ b)} \frac{E(f)|_{\mathcal{H}_+}(\tilde{U}, x, \tilde{\varphi}) E(h)|_{\mathcal{H}_+}(\tilde{U}', x, \tilde{\varphi})}{e^{-2\kappa_+ C_1 b} (\tilde{U} - \tilde{U}' - i\tilde{\epsilon})^2} e^{-2\kappa_+ C_1 b} \, \mathrm{d}\tilde{U} \, \mathrm{d}\tilde{U}' \, \mathrm{d}x \, \mathrm{d}\tilde{\varphi} \\
&= -\lim_{\tilde{\epsilon} \to 0^+} \frac{r_+^2 + a^2}{\chi\pi} \int\limits_{\mathbb{R}^2 \times \mathbb{S}^2} \frac{E(f)|_{\mathcal{H}}(\tilde{U}, \tilde{\Omega}) E(h)|_{\mathcal{H}}(\tilde{U}', \tilde{\Omega})}{(\tilde{U} - \tilde{U}' - i\tilde{\epsilon})^2} \, \mathrm{d}\tilde{U} \, \mathrm{d}\tilde{U}' \, \mathrm{d}^2\tilde{\Omega} \\
&= w_+(f, h)\,,
\end{aligned}
$$

with $\tilde{\Omega} = (\theta, \tilde{\varphi})$ and $\mathrm{d}^2\tilde{\Omega}$ the infinitesimal volume element of the unit 2-sphere. Combining this with the corresponding result for w_c concludes the proof. $\square$

As shown in this lemma, the two-point function w constructed in (5.2.2) is indeed invariant under the flow generated by any of the Kerr-de Sitter Killing fields, i.e. any linear combination of ∂_t and ∂_φ with constant coefficients. This includes in particular the Killing fields generating the horizons, ∂_{t_+} and ∂_{t_c}. The flow induced by these Killing fields will simply be denoted $\psi^j_b = \psi_b^{(1, a/(r_j^2 + a^2))}$.

5.3 The Hadamard property of the Unruh state

It remains to show that the Unruh state defined in the previous section is indeed a Hadamard state on the Kerr-de Sitter spacetime $\mathcal{M}$.

The proof will proceed in two main steps. In a first step, we will consider a subset of region I, in which we can prove the Hadamard properties following the ideas applied in the black-hole exterior in [43]. After that, we will use a more explicit computation for all remaining cases. This last part is the most novel one.

Before we start, let us make a few remarks. First of all, as discussed in Section 2.3, the two-point function is a distributional bi-solution of the Klein-Gordon equation. Therefore, we have by an application of the Propagation of Singularities Theorem 2.3.4,

Corollary 5.3.1. *Let $w \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ as defined in (5.2.2). If $(x, k; y, l) \in \mathrm{WF}'(w)$, then $g^{-1}(k, k) = g^{-1}(l, l) = 0$ and $B(x, k) \times B(y, l) \subset \mathrm{WF}'(w)$, with $B(p, q)$ the* *bicharacteristic of $\mathcal{K}$ on $\mathcal{M}$ through $(p,q) \in T^*(\mathcal{M})$ as in* (2.3.11).

In other words, instead of considering whether a point $(x,k;y,l) \in T^*(\mathcal{M} \times \mathcal{M})$ is in $\mathrm{WF}(w)$, we can consider any point $(x',k';y',l') \in B(x,k) \times B(y,l)$, and the result will apply to all of $B(x,k) \times B(y,l)$.

As a consequence of [71, Thm. 6.5.3] and also noted in the proof thereof, the support properties of the retarded and advanced Green's operators together with Propagation of Singularities can be used to deduce that the kernel E of the commutator function E satisfies

$$\mathrm{WF}'(E) = \mathcal{C}^+ \cup \mathcal{C}^- , \tag{5.3.1}$$

with $\mathcal{C}^\pm$ as defined in (2.3.13b).

Finally, one can exploit the commutator property and the knowledge on $\mathrm{WF}(E)$ to show the following lemma, which is closely related to [144, Prop. 6.1]:

Lemma 5.3.2. *Let $w \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ as defined in* (5.2.2)*. Assume that*

$$\mathrm{WF}'(w) \cap \Delta_{T^*(\mathcal{M}\times\mathcal{M})} \subset \mathcal{N}^+ \times \mathcal{N}^+ , \tag{5.3.2}$$

where $\Delta_{T^*(\mathcal{M}\times\mathcal{M})} = \{(x,k;x,k) : (x,k) \in T^*\mathcal{M}\} \subset T^*(\mathcal{M} \times \mathcal{M})$ is the diagonal in $T^*(\mathcal{M} \times \mathcal{M})$ and $\mathcal{N}^+$ as defined in (2.1.1)*. Then w satisfies the microlocal spectrum condition,* (2.3.13a).

Proof. The proof makes use of the fact that by Proposition 5.2.3 we can write $w(f,h)$ as an $L^2 \oplus L^2$ inner product of $K(\overline{f})$ and $K(h)$ as in (5.2.15). We want to combine this with the fact that (5.3.2) implies that $(x,-k;x,-k) \notin \mathrm{WF}'(w)$ if $(x,k;x,k) \in \mathrm{WF}'(w)$. For this purpose, we fix a point $(x_0,k_0;y_0,l_0) \in T^*(\mathcal{M} \times \mathcal{M})$ with k_0 and l_0 null or zero. We can now discuss several different cases:

For the first case, assume that either both k_0 and l_0 are non-zero and $B_{\mathcal{M}}(x_0,k_0)$ is not equal to $B_{\mathcal{M}}(y_0,l_0)$, or one of them, say l_0, vanishes and $y_0 \notin B_{\mathcal{M}}(x_0,k_0)$. Recall that $B_{\mathcal{M}}(x,k)$ is the projection of the bicharacteristic $B(x,k)$ to $\mathcal{M}$ and corresponds to the null geodesic defined by (x,k), compare (2.3.12).

In this case, one can find some space-like Cauchy surface Σ which is intersected by $B_{\mathcal{M}}(x_0,k_0)$ and $B_{\mathcal{M}}(y_0,l_0)$ at two distinct points x_1 and y_1. Let $f,h \in C_0^\infty(\mathcal{M};\mathbb{R})$ be real-valued test functions supported in space-like separated neighbourhoods of x_1 and y_1, respectively. Let us fix a coordinate chart covering $\mathrm{supp}(f) \cup \mathrm{supp}(h)$. For any $k \in \mathbb{R}^4$, which we identify with $T^*_x(\mathcal{M})$ in the local trivialization fixed by these coordinates, we then write $f_k(x) = (2\pi)^{-2} e^{ik\cdot x} f(x)$, with $\cdot$ the usual $\mathbb{R}^4$ inner product. Then Proposition 5.2.3 and an application of the Cauchy-Schwarz inequality imply [16]

$$|w(f_k,h_l)|^2 \leq |w(f_k,f_{-k})|\, |w(h_{-l},h_l)| \ .$$

Due to the commutator property at space-like separation, one has similarly

$$|w(f_k,h_l)|^2 = |w(h_l,f_k)|^2 \leq |w(f_{-k},f_k)|\, |w(h_l,h_{-l})| \ .$$

Recalling Definition 2.3.1 of the (primed) wavefront set, this implies that if (5.3.2) holds, f and h may be chosen in such a way that at least one of the two estimates for $|w(f,h)|$ is rapidly decaying in $|(k,l)|$ for (k,l) in some conic neighbourhood of (k_0,l_0) parallel transported to (x_1,y_1). Consequently, these points cannot be in the wavefront set of w if (5.3.2) is satisfied.

The next case we consider is that $l_0 = 0$, but $y_0 \in B_{\mathcal{M}}(x_0,k_0)$. In this case, one cannot apply the previous argument, since none of the points in $B_{\mathcal{M}}(x_0,k_0)$ is space-like separated from y_0. However, one may use that $\mathrm{WF}(E)$, where E is the kernel of the commutator function, does not contain any points of the form $(x,k;y,0)$. If we set $\tilde{w}(f,h) = w(h,f)$, then the commutator property implies $\mathrm{WF}(w-\tilde{w}) = \mathrm{WF}(E)$. Hence, if a point of the form $(x_0,k_0,y_0,0)$ were in $\mathrm{WF}(w)$, it would have to be in $\mathrm{WF}(\tilde{w})$ as well, so that the two singular contributions could cancel out in $iE = w - \tilde{w}$. In other words, if $(x_0,k_0;y,0)$ is in $\mathrm{WF}(w)$, then so must be $(y,0;x_0,k_0)$. Let us assume w.l.o.g. that $y_0 = x_0$, and let $f,h \in C_0^\infty(\mathcal{M};\mathbb{R})$ be real test functions, each supported in a neighbourhood of x_0. We fix some coordinate system covering $\mathrm{supp}(f) \cup \mathrm{supp}(h)$. Then the discussion above implies that if

$$|w(f_k,h)|^2 \leq |w(f_k,f_{-k})|\,|w(h,h)|$$

is not rapidly decreasing in $|k|$ for any choice of f and for k in any conic neighbourhood of k_0, then

$$|w(h,f_k)|^2 \leq |w(f_{-k},f_k)|\,|w(h,h)|$$

must not be rapidly decreasing in $|k|$ either. However, if (5.3.2) holds, one can find test functions f,h and a conic neighbourhood V of $(k_0,0)$, so that at least one of the two estimates is rapidly decreasing in $|k|$ for all $k \in V$. Therefore, if (5.3.2) holds, $\mathrm{WF}(w)$ cannot contain any points of the form $(x,k;y,0)$ or, by the same argument, $(x,0;y,l)$.

The final case that remains to be analysed is $B_{\mathcal{M}}(x_0,k_0) = B_{\mathcal{M}}(y_0,l_0)$. This case can be represented by points of the form $(x_0,k_0;x_0,ck_0) \in T^*(\mathcal{M}\times\mathcal{M})$ for some $0 \neq c \in \mathbb{R}$. Let $f,h \in C_0^\infty(\mathcal{M};\mathbb{R})$ be real test functions as above, supported in a neighbourhood of x_0, and let us fix some coordinate system covering their supports. Then (5.3.2) implies that we can choose f, h, and a conic neighbourhood V of (k_0,ck_0) so that

$$|w(f_k,h_l)|^2 \leq |w(f_k,f_{-k})|\,|w(h_{-l},h_l)|$$

is rapidly decreasing in $|(k,l)|$ for all $(k,l) \in V$, unless (x_0,k_0) is contained in $\mathcal{N}^+$ and (x_0,ck_0) in $\mathcal{N}^-$, i.e. $(x_0,k_0) \in \mathcal{N}^+$ and $c < 0$.

Putting these three cases together, we have shown that (5.3.2) together with (5.2.15) implies that $\mathrm{WF}'(w) \subset \mathcal{N}^+ \times \mathcal{N}^+$. This also implies that $\mathrm{WF}'(\tilde{w}) \subset \mathcal{N}^- \times \mathcal{N}^-$. In particular, the (primed) wavefront sets of w and $\tilde{w}$ do not overlap. As a result, we find

$$\mathcal{C}^+ \cup \mathcal{C}^- = \mathrm{WF}'(E) = \mathrm{WF}'(w-\tilde{w}) = \mathrm{WF}'(w) \cup \mathrm{WF}'(\tilde{w}) \subset \mathcal{N}^+ \cup \mathcal{N}^- ,$$

with the third equality sign due to the fact that the wavefront sets do not overlap, see also the proof in [144, Prop. 6.1]. Since $\mathrm{WF}'(w) \cap \mathcal{C}^- = \mathrm{WF}'(\tilde{w}) \cap \mathcal{C}^+ = \emptyset$, the equation above can only be satisfied if $\mathrm{WF}'(w) = \mathcal{C}^+$. $\square$

Thanks to these results, it is sufficient to show that $\mathrm{WF}'(w) \cap \Delta_{T^*(\mathcal{M}\times\mathcal{M})} \subset \mathcal{N}^+ \times \mathcal{N}^+$ [138], and any result obtained for one point $(x, k; x, k)$ can be propagated to all of $B(x, k) \times B(x, k)$.

5.3.1 The Hadamard condition in $\mathcal{O}$

Following these preliminary considerations, we will now prove the Hadamard property of the Unruh state in a subregion $\mathcal{O}$ of $\mathcal{M}$. In particular, we choose $\mathcal{O} \subset \mathrm{I}$ to be the open set in which both ∂_{t_+} and ∂_{t_c} as defined in (2.4.9) are time-like.

In the light of Lemma 5.3.2, we will show

Proposition 5.3.3. *Let w be as defined in (5.2.2), and let $\mathcal{O} \subset \mathrm{I}$ be such that ∂_{t_+} and ∂_{t_c} are time-like on $\mathcal{O}$. Then for any $x_0 \in \mathcal{O}$,*

$$\mathrm{WF}'(w) \cap T^*_{(x_0,x_0)}(\mathcal{M}\times\mathcal{M}) \cap \Delta_{T^*(\mathcal{M}\times\mathcal{M})} \subset \mathcal{N}^+ \times \mathcal{N}^+ \,. \tag{5.3.3}$$

Using Propagation of Singularities and Lemma 5.3.2, Proposition 5.3.3 implies that

$$\mathrm{WF}'(w) \cap (B(\mathcal{O}) \times B(\mathcal{O})) = \mathcal{C}^+ \cap (B(\mathcal{O}) \times B(\mathcal{O})) \;, \tag{5.3.4}$$

where

$$B(\mathcal{O}) = \{(x, k) \in T^*(\mathcal{M})\backslash o : g^{-1}(x)(k, k) = 0, B_{\mathcal{M}}(x, k) \cap \mathcal{O} \neq \emptyset\} \,.$$

By the results shown in Lemma 5.1.4, this includes all null geodesics that do not end at $\mathcal{H}_+$ or $\mathcal{H}_c$ as long as a and λ are sufficiently small.

For the proof of Proposition 5.3.3, we would like to use the characterisation of the wavefront set in Proposition 2.3.1, originally given in [69, Prop. 2.1]. To do so, we follow largely part 1) and 2) of the proof of the Hadamard property for passive states given in [62, Thm. 5.1]. The remaining parts of the proof of [62, Thm. 5.1] are covered already by the proof of Lemma 5.3.2. This is based on the idea of the proof of [43, Prop. 4.3].

As a first step, we need to prove that the two pieces w_j of w satisfy a "KMS-like"-condition [43] with $\beta = 2\pi\kappa_j^{-1}$ with respect to ∂_{t_j}:

Lemma 5.3.4. *Let $f \in C_0^\infty(\mathrm{I})$, and let*

$$\psi^j_{b*}(f)(u, v, \theta, \varphi_j) = f(u - b, v - b, \theta, \varphi_j)$$

be the push-forward along the flow generated by ∂_{t_j}. Then

$$K^I_j(\psi^j_{b*}f)(\omega, \theta, \varphi_j) = e^{ib\omega} K^I_j(f)(\omega, \theta, \varphi_j) \,. \tag{5.3.5}$$

In addition, w_j is "KMS-like" [43] in the sense that for any $h \in C_0^\infty(\mathbb{R};\mathbb{R})$ and any pair of real-valued test functions on I, $f_{1,2} \in C_0^\infty(\mathrm{I};\mathbb{R})$,

$$\int_{\mathbb{R}} \hat{h}(t) \left\langle K_j^{\mathrm{I}}(f_1), K_j^{\mathrm{I}}(\psi_{t*}^j f_2) \right\rangle_{L^2} dt = \int_{\mathbb{R}} \hat{h}\left(t + \tfrac{2\pi i}{\kappa_j}\right) \left\langle K_j^{\mathrm{I}}(\psi_{t*}^j f_2), K_j^{\mathrm{I}}(f_1) \right\rangle_{L^2} dt\,, \quad (5.3.6)$$

where $\langle\cdot\rangle_{L^2}$ is the usual L^2-inner product of $L^2(\mathbb{R}\times\mathbb{S}^2;\ \mu_j(\omega)\,d\omega\,d^2\Omega_j)$, with $\mu_j(\omega)$ as in (5.2.13c).

Proof. As discussed in the proof of Lemma 5.2.5, since ∂_{t_j}, $j \in \{+, c\}$ are Killing vector fields of the Kerr-de Sitter spacetime $\mathcal{M}$, E commutes with the push-forward along the flow induced by ∂_{t_j}. In other words, one has

$$E(\psi_{b*}^j f)(u, v, \theta, \varphi_j) = E(f)(u-b, v-b, \theta, \varphi_j)\,.$$

Inserting this into K_j^I as defined in (5.2.14b), one obtains the estimate

$$\begin{aligned} K_+^I(\psi_{b*}^+ f)(\omega, \Omega_+) =& (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} E(\psi_{b*}^+ f)(u, v, \Omega_+)|_{v\to-\infty} e^{i\omega u}\, \mathrm{d}u \\ =& (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} E(f)(u-b, v-b, \Omega_+)|_{v\to-\infty} e^{i\omega u}\, \mathrm{d}u \\ =& (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} E(f)(u, v, \Omega_+)|_{v\to-\infty} e^{i\omega(u+b)}\, \mathrm{d}u \\ =& e^{i\omega b} K_+^I(f)(\omega, \Omega_+) \end{aligned}$$

and analogously $K_c^I(\psi_{b*}^c f)(\omega, \Omega_c) = e^{i\omega b} K_c^I(f)(\omega, \Omega_c)$. With this in mind, we can now consider the function

$$\mathbb{R} \ni t \mapsto \left\langle K_j^{\mathrm{I}}(f_1), K_j^{\mathrm{I}}(\psi_{t*}^j f_2) \right\rangle_{L^2} = \left\langle K_j^{\mathrm{I}}(f_1), e^{i\omega t} K_j^{\mathrm{I}}(f_2) \right\rangle_{L^2} \in \mathbb{C}\,,$$

for some pair of real-valued test functions $f_{1,2} \in C_0^\infty(\mathrm{I};\mathbb{R})$, where L^2 is a the short-hand notation for $L^2(\mathbb{R}\times\mathbb{S}^2;\ \mu_j(\omega)\,\mathrm{d}\omega\,\mathrm{d}^2\Omega_j)$. If we replace t by $t + ib$, we obtain

$$\begin{aligned} & \left| \left\langle K_j^{\mathrm{I}}(f_1), e^{i\omega(t+ib)} K_j^{\mathrm{I}}(f_2) \right\rangle_{L^2} \right| \\ =& \left| \int_{\mathbb{R}\times\mathbb{S}^2} \frac{r_j^2 + a^2}{\chi} \frac{\omega e^{i\omega t} e^{\omega\left(\frac{\pi}{\kappa_j} - b\right)}}{\sinh\left(\frac{\pi\omega}{\kappa_j}\right)} \overline{\tilde{F}_j(E(f_1)|_{\mathcal{H}_j})(\omega, \Omega_j)} \tilde{F}_j(E(f_2)|_{\mathcal{H}_j})(\omega, \Omega_j)\, \mathrm{d}\omega\, \mathrm{d}^2\Omega_j \right| \\ \leq& \frac{r_j^2 + a^2}{\chi} \int_{\mathbb{R}\times\mathbb{S}^2} \frac{\omega e^{\omega\left(\frac{\pi}{\kappa_j} - b\right)}}{\sinh\left(\frac{\pi\omega}{\kappa_j}\right)} \left| \tilde{F}_j(E(f_1)|_{\mathcal{H}_j})(\omega, \Omega_j) \right| \left| \tilde{F}_j(E(f_2)|_{\mathcal{H}_j})(\omega, \Omega_j) \right| \mathrm{d}\omega\, \mathrm{d}^2\Omega_j\,. \end{aligned}$$

The decay results (5.2.6b) and (5.2.6c) provide exponential decay in l_j of $\left|\partial_{l_j}^N E(f_{1,2})\right|$ for arbitrarily large $N \in \mathbb{N}$ on the corresponding horizon. Therefore, one obtains the bound

$$\left|\tilde{F}_j(E(f_{1,2})|_{\mathcal{H}_j})(\omega, \Omega_j)\right| \leq c\,(1+\omega^2)^{-\frac{M}{2}}$$

for arbitrarily large M and some constant $c > 0$ depending on $f_{1,2}$ and M. Thus, combining c together with other constants to $C > 0$,

$$\left|\left\langle K_j^{\mathrm{I}}(f_1), e^{i\omega(t+ib)} K_j^{\mathrm{I}}(f_2)\right\rangle_{L^2}\right| \leq C \int\limits_{\mathbb{R}} \frac{\omega e^{\omega\left(\frac{\pi}{\kappa_j}-b\right)}}{(1+\omega^2)^M \sinh\left(\frac{\pi\omega}{\kappa_j}\right)}\,\mathrm{d}\omega\,.$$

This integral is finite as long as $b \in [0, 2\pi/\kappa_j]$ and M is chosen sufficiently large. Similarly, since the integral is absolutely convergent for $0 \leq b \leq 2\pi/\kappa_j$, one can differentiate n times with respect to $z = t + ib$ under the integral, which simply contributes a factor of $i\omega$ to the integrand. Thus, by choosing M sufficiently large depending on the number n of derivatives, one can show that also any number of derivatives of $\left|\left\langle K_j^{\mathrm{I}}(f_1), e^{i\omega(z)} K_j^{\mathrm{I}}(f_2)\right\rangle_{L^2}\right|$ exists and is finite as long as $0 < \mathrm{Im}(z) < 2\pi/\kappa_j$. Thus, this function has an analytic extension to the strip $\{\mathrm{Im}(z) \in (0, 2\pi/\kappa_j)\} \subset \mathbb{C}$.

Let us now get to the second point of the lemma. Let $h \in C_0^\infty(\mathbb{R};\mathbb{R})$, and $f_{1,2} \in C_0^\infty(\mathrm{I};\mathbb{R})$ as before. Then

$$\begin{aligned} &\int\limits_{\mathbb{R}} \hat{h}(t) \left\langle K_j^{\mathrm{I}}(f_1), K_j^{\mathrm{I}}(\psi_{t*}^j f_2)\right\rangle_{L^2} \mathrm{d}t \qquad (5.3.7)\\ &= \int\limits_{\mathbb{R}} \hat{h}(t) \int\limits_{\mathbb{R}\times\mathbb{S}^2} \mu_j(\omega)\overline{K_j^{\mathrm{I}}(f_1)}(\omega,\theta,\varphi_j) e^{i\omega t} K_j^{\mathrm{I}}(f_2)(\omega,\theta,\varphi_j)\,\mathrm{d}\omega\,\mathrm{d}\Omega_j\,\mathrm{d}t\,. \end{aligned}$$

By the definition of K_j^{I}, $\overline{K_j^{\mathrm{I}}(f_{1,2})}(\omega,\theta,\varphi_j) = K_j^{\mathrm{I}}(f_{1,2})(-\omega,\theta,\varphi_j)$. In addition, the measure $\mu_j(\omega)$ satisfies $\mu_j(\omega) = e^{2\pi\omega/\kappa_j}\mu_j(-\omega)$, see (5.2.13c). Moreover, $\hat{h}(t)$ is the Fourier transform of a compactly supported function. Hence, it is entire analytic and vanishes for $\mathrm{Re}(t) \to \pm\infty$ as long as $\mathrm{Im}(t)$ remains finite. As a result, by setting $\tilde{\omega} = -\omega$, one can rewrite (5.3.8) as

$$\begin{aligned} &\int\limits_{\mathbb{R}} \hat{h}(t) \int\limits_{\mathbb{R}\times\mathbb{S}^2} \mu(\tilde{\omega}) e^{-2\pi\frac{\tilde{\omega}}{\kappa_j}} e^{-i\tilde{\omega}t} K_j^{\mathrm{I}}(f_1)(\tilde{\omega},\theta,\varphi_j) K_j^{\mathrm{I}}(f_2)(-\tilde{\omega},\theta,\varphi_j)\,\mathrm{d}\tilde{\omega}\,\mathrm{d}^2\Omega_j\,\mathrm{d}t\\ &= \int\limits_{\mathbb{R}-i\frac{2\pi}{\kappa_j}} \hat{h}\left(t + i\tfrac{2\pi}{\kappa_j}\right) \int\limits_{\mathbb{R}\times\mathbb{S}^2} \mu_j(\tilde{\omega}) e^{-i\tilde{\omega}t}\overline{K_j^{\mathrm{I}}(f_2)}(\tilde{\omega},\theta,\varphi_j) K_j^{\mathrm{I}}(f_1)(\tilde{\omega},\theta,\varphi_j)\,\mathrm{d}\tilde{\omega}\,\mathrm{d}^2\Omega_j\,\mathrm{d}t\\ &= \int\limits_{\mathbb{R}} \hat{h}\left(t + i\tfrac{2\pi}{\kappa_j}\right) \left\langle K_j^{I}(\psi_{t*}^j f_2), K_j^{\mathrm{I}}(f_1)\right\rangle_{L^2} \mathrm{d}t\,. \end{aligned}$$

□

Next, our application of the proof of [62, Thm. 5.1] requires some of the results in [62, Prop. 2.1]. While they were originally proven for passive states, it has been realized in [43], that the "KMS-like" condition (5.3.6) is actually sufficient to prove the relevant parts of [62, Prop. 2.1] by following the original proof step by step (omitting step 3, which deals with the case of ground states). In this way, one obtains

Lemma 5.3.5. *For any $(g_1^\lambda)_{\lambda>0}$, $(g_2^\lambda)_{\lambda>0} \subset C_0^\infty(\mathcal{O};\mathbb{R})$ satisfying $w_j(g_i^\lambda, g_i^\lambda) \leq c\,(1+\lambda^{-1})^s$ for some $c > 0$ and $s > 0$, there exists a $h \in C_0^\infty(\mathbb{R}^2)$ with $\hat{h}(0) = 1$, and for any $(k_0, k_0') \in \mathbb{R}^2\backslash\{0\}$ with $k_0' > 0$, there is an open neighbourhood V_ϵ in $\mathbb{R}^2\backslash\{0\}$ of (k_0, k_0') so that $k_2 > \epsilon > 0\, \forall (k_1, k_2) \in V_\epsilon$ and such that $\forall N \in \mathbb{N}\, \exists C_N > 0,\ \lambda_N > 0$:*

$$\sup_{k\in V_\epsilon} \left| \int e^{i\lambda^{-1}k\cdot t} \hat{h}(t) w_j(\psi^j_{t_1 *} g_1^\lambda, \psi^j_{t_2 *} g_2^\lambda)\, d^2t \right| < C_N \lambda^N \quad \forall 0 < \lambda < \lambda_N\,. \tag{5.3.8}$$

Let us remark that this continues to hold if $\hat{h}$ is replaced by $\phi \cdot \hat{h}$ for some $\phi \in C_0^\infty(\mathbb{R}^2)$ after shrinking V_ϵ if necessary. This follows from an application of [69, Lemma 2.2 b)], see the discussion on the proof of [62, Prop. 2.1]. The above also continues to hold if the functions g_i^λ depend on additional parameters, see the discussion in [62, Rem. 2.2].

With this, we can now prove Proposition 5.3.3.

Proof of Proposition 5.3.3. As mentioned in the preceding remarks, we will follow closely parts 1) and 2) of the proof of [62, Thm. 5.1]. So let us fix some $x_0 \in \mathcal{O}$, and let $j \in \{+, c\}$. As a first step, we define a coordinate chart

$$\psi_j : \mathcal{U}_{x_0} \to \psi_j(\mathcal{U}_{x_0}) \subset \mathbb{R}^4\,, \qquad x \to (t_j(x) = \tfrac{1}{2}(u_j + v_j)(x) - t_{j,0}, \vec{x}(x))$$

on some open neighbourhood $\mathcal{U}_{x_0}$ of x_0. We will choose $t_{j,0}$ and $\vec{x}$ such that $\psi_j(x_0) = 0$. One possible choice would be to take the Cartesian coordinates corresponding to $(r(x), \theta(x), \varphi_j(x))$ and shifting the origin of the coordinates to $\vec{x}(x_0)$.

We require that the coordinate chart is built in such a way that there exists a constant $c > 0$, so that on a sufficiently small, compact neighbourhood $K \subset \mathcal{U}_{x_0}$ of x_0 and for all $|t| < c$, the diffeomorphism ψ_t^j induced by the Killing field ∂_{t_j} can be written as

$$\psi_j \circ \psi_t^j(x) = (t_j(x) + t, \vec{x}(x))\,.$$

Additionally, we define spatial translation by $\vec{y}$ in some sufficiently small, open neighbourhood B of 0 in $\mathbb{R}^3$ acting on K by

$$\psi^j_{\vec{y}}(x) = \psi_j^{-1} \circ \tilde{\psi}^j_{\vec{y}} \circ \psi_j(x)\,, \qquad \tilde{\psi}^j_{\vec{y}}(t_j, \vec{x}) = (t_j, \vec{x} + \vec{y})\,.$$

We will denote the corresponding push-forwards acting on smooth functions on K by ψ^j_{t*}, as in Lemma 5.3.4, and $\psi^j_{\vec{y}*}$.

After the construction of the coordinate system, we will now consider an arbitrary but fixed $(x_0, k_x) \in \mathcal{N}^- \cap T^*_{x_0}\mathcal{M}$ and show that $(x_0, k_x; x_0, -k_x)$ is not in $\mathrm{WF}(w_j)$ by using the description of the wavefront set in Proposition 2.3.1.

To this end, we identify $T^*_{\mathcal{U}_{x_0}}\mathcal{M}$ with its local trivialization $\psi_j(\mathcal{U}_{x_0}) \times \mathbb{R}^4$ induced by the coordinate chart ψ_j. Since $x_0 \in \mathcal{O}$ and ∂_{t_j} is time-like in $\mathcal{O}$, $(x_0, k_x) \in \mathcal{N}^-$ if and only if $k^0 = k(\partial_{t_j}) < 0$. We can then choose an open conic neighbourhood V of $(k_x, -k_x)$ in $\mathbb{R}^4 \times \mathbb{R}^4 \backslash \{0\}$, so that for all $(k, k') \in V$, there is some $\epsilon > 0$ with[2]

$$(k^0, k'^0) = (k(\partial_{t_j}), k'(\partial_{t_j})) \in V_\epsilon \, .$$

Here, $V_\epsilon \subset \mathbb{R}^2 \backslash \{0\}$ is an open neighbourhood of $(k_x^0, -k_x^0)$ so that $k'^0 > \epsilon > 0$ for all $(k^0, k'^0) \in V_\epsilon$ like in Lemma 5.3.5. Note that V and V_ϵ may in general depend on j. However, we suppress this dependence in the notation.

In addition, we define a (j-dependent) function $H \in C_0^\infty(\psi_j(\mathcal{U}_{x_0}) \times \psi_j(\mathcal{U}_{x_0}))$ as

$$H(t_j, \vec{x}, t'_j, \vec{x}') = \phi(t_j, t'_j)\hat{h}(t_j, t'_j)\zeta(\vec{x}, \vec{x}') \, .$$

The function $h \in C_0^\infty(\mathbb{R}^2)$ is chosen as in Lemma 5.3.5. The functions $\phi \in C_0^\infty((-c, c); \mathbb{R})$ and $\zeta \in C_0^\infty(B \times B; \mathbb{R})$ are constructed so that $H(0) = 1$ is satisfied.

Finally, let us pick (j-dependent) functions $g_i \in C_0^\infty(\psi_j(\mathcal{U}_{x_0}); \mathbb{R})$, $i \in \{1, 2\}$ with support in the neighbourhood $\psi_j(K)$ of 0 and with $\widehat{g_1 \otimes g_2}(0, 0) = 1$. We suppress the j-dependence in the notations since it should be clear from the context. Let $p \geq 1$ and set

$$g_i^\lambda(x) = \begin{cases} g_i(\lambda^{-p}(\psi_j(x))) & x \in \mathcal{U}_{x_0} \\ 0 & x \notin \mathcal{U}_{x_0} \end{cases}$$

for $\lambda \leq 1$ and $g_i^\lambda(x) = g_i^1(x)$ for $\lambda > 1$. This is of the same form as the functions in Proposition 2.3.1. By the choice of $\mathrm{supp}(g_i)$ and the construction of ψ_j, we then have $\mathrm{supp}(g_i^\lambda) \subset K$. Hence, push-forwards of g_i^λ by time translations with $t \in (-c, c)$ or by spatial translations with $\vec{y} \in B$ are well-defined.

Furthermore, the g_i^λ satisfy the condition of Lemma 5.3.5: By (5.2.8) or the analogous result for w_c, we have

$$\left|w_j(g_i^\lambda, g_i^\lambda)\right| \leq C \left\|g_i^\lambda\right\|^2_{C^m} \, . \tag{5.3.9}$$

Since the functions g_i^λ are supported in region I within the set on which the coordinate chart ψ_j is defined, we are free to take this norm using the partial derivatives in the ψ_j-coordinate chart as the linearly independent vector fields. This allows us to estimate for

[2]In the rest of this proof, we will work in the coordinate chart ψ_j and we will often omit writing the coordinate chart or the resulting trivialization when no confusion arises.

$\lambda < 1$

$$|w_j(g_i^\lambda, g_i^\lambda)| \leq C \left\|g_i(\lambda^{-p}x)\right\|^2_{C^m(K)} \leq C' \left\|g_i(x)\right\|^2_{C^m(K)} \sum_{|\beta|\leq m} \lambda^{-|\beta|p} \qquad (5.3.10)$$
$$\leq C'' \left\|g_i\right\|^2_{C^m} (1+\lambda^{-1})^{2mp} \,.$$

by applying $\partial_x f(\lambda^{-p}x) = \lambda^{-p}\partial_y f(y)|_{y=\lambda^{-p}x}$. Hence, $g_1^\lambda \otimes g_2^\lambda$ is of the same form as the function in Proposition 2.3.1, while the g_i^λ also satisfy the condition in Lemma 5.3.5.

We can now conclude the proof by estimating

$$\sup_{(k,k')\in V} \left| \int e^{i\lambda^{-1}(k,k')\cdot(x,x')} H(x,x') w_j\left(\psi^j_{t*}\psi^j_{\vec{x}*}g_1^\lambda, \psi^j_{t'*}\psi^j_{\vec{x}'*}g_2^\lambda\right) \mathrm{d}vol_g(x)\,\mathrm{d}vol_g(x')\right|$$
$$= \sup_{(k,k')\in V} \left| \int e^{i\lambda^{-1}(\vec{k}\vec{x}+\vec{k}'\vec{x}')} \zeta(\vec{x},\vec{x}') \left[\int e^{i\lambda^{-1}(k^0 t + k^{0\prime} t')} \phi(t,t')\hat{h}(t,t') \right.\right.$$
$$\left.\left. \times\, w_j\left(\psi^j_{\vec{x}*}g_1^\lambda, \psi^j_{(t'-t)*}\psi^j_{\vec{x}'*}g_2^\lambda\right) \mathrm{d}t\,\mathrm{d}t'\right] \mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{x}'\right| \,,$$

where we have used the coordinates ψ_j and the first part of Lemma 5.3.4. Pulling the absolute value into the $\vec{x}$- and $\vec{x}'$- integral, this can be bounded by

$$\sup_{(k,k')\in V} \int |\zeta(\vec{x},\vec{x}')| \left| \int e^{i\lambda^{-1}(k^0 t + k^{0\prime} t')} \phi(t,t')\hat{h}(t,t') \right.$$
$$\left. \times\, w_j\left(\psi^j_{\vec{x}*}g_1^\lambda, \psi^j_{(t'-t)*}\psi^j_{\vec{x}'*}g_2^\lambda\right) \mathrm{d}t\,\mathrm{d}t'\right| \mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{x}' \,.$$

Using Lemma 5.3.5, for any $N \in \mathbb{N}$ there are $C_N > 0$ and $1 > \lambda_N > 0$ so that the absolute value of the integrals over t and t' is bounded by $C_N\lambda^N$ for all $0 < \lambda < \lambda_N$. This allows us to find a bound for the previous expression of the form

$$\sup_{(k,k')\in V} \int |\zeta(\vec{x},\vec{x}')|\, C_N\lambda^N \,\mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{x}' \leq \tilde{C}_N\lambda^N$$

for all $0 < \lambda < \lambda_N < 1$. The last step follows from the integrability of ζ. By Proposition 2.3.1, this estimate shows that $(x_0, k_x, x_0, -k_x)$ cannot be in $\mathrm{WF}(w_j)$ for $j \in \{+, c\}$ and any $(x_0, k_x) \in \mathcal{N}^- \cap T^*_{\mathcal{O}}\mathcal{M}$. Since $\mathrm{WF}(w_+ + w_c) \subset \mathrm{WF}(w_+) \cup \mathrm{WF}(w_c)$, this concludes the proof of Proposition 5.3.3. □

5.3.2 The Hadamard condition on $M\backslash\mathcal{O}$

In the previous subsection, we have established the Hadamard property in a region $\mathcal{O} \subset \mathrm{I}$, and, by Propagation of Singularities, for all null geodesics intersecting $\mathcal{O}$. This includes all null geodesics that do not intersect $\mathcal{H}_+$ or $\mathcal{H}_c$ if a and λ are sufficiently small by Lemma 5.1.4.

In order to complete the proof of the Hadamard property, we now consider null geodesics intersecting one of the horizons when extended to $\tilde{\mathcal{M}}$. Below, we show

Proposition 5.3.6. *Let $(x_0, k_0) \in \mathcal{N}$, such that $B_{\mathcal{M}}(x_0, k_0) \cap \mathcal{O} = \emptyset$. Assume that λ, a are chosen such that Lemma 5.1.4 and the results of [39] are valid. If $(x_0, k_0; x_0, k_0)$ is in $\mathrm{WF}'(w)$, then $(x_0, k_0) \in \mathcal{N}^+$.*

This result, together with Proposition 5.3.3, Lemma 5.3.2, and Propagation of Singularities as in Corollary 5.3.1 then implies that w indeed satisfies the microlocal spectrum condition (2.3.13a), and is thus the two-point function of a well-defined quasi-free Hadamard state on the CCR-algebra $\mathcal{A}(\mathcal{M})$.

The proof of Proposition 5.3.6 is based on ideas first developed in [142], and applied in similar form in [16, 43, 123]. As a preparation for the proof, let us first prove two lemmata:

Lemma 5.3.7. *Let X, $Y \subset \mathbb{R}^n$. Let $(y_0, k_0) \in Y \times (\mathbb{R}^n \backslash \{0\})$, and let K be any compact neighbourhood of y_0. Let $D \in \mathcal{D}'(X \times Y)$ such that $(x, k; y, 0) \notin \mathrm{WF}(D)$ for all $x \in X$, $y \in Y$ and $k \in \mathbb{R}^n \backslash \{0\}$. Let ${}_X\pi(\mathrm{supp}(D)) \subset X$ be compact, where ${}_X\pi : X \times Y \to X$ is the projection onto X. Assume*

$$(x, l; y_0, k_0) \notin \mathrm{WF}(D) \quad \forall (x, l) \in X \times \mathbb{R}^n \,. \tag{5.3.11}$$

Then we can find a function $f \in C_0^\infty(Y)$ with $f(y_0) = 1$ and support in K, and an open conic neighbourhood $V_{k_0} \subset \mathbb{R}^n \backslash \{0\}$ of k_0 so that for any $N, N' \in \mathbb{N}$ there are positive constants $C_{NN'}$ satisfying

$$|\widehat{(\mathbf{1} \otimes f) \cdot D}|(l, k) \leq \frac{C_{NN'}}{(1 + |l|^N)(1 + |k|^{N'})} \quad \forall l \in \mathbb{R}^4 \,, \; k \in V_{k_0} \,, \tag{5.3.12}$$

where $\mathbf{1}(x) = 1$ for all $x \in X$.

To understand the meaning of this technical lemma, consider some $D \in \mathcal{D}'(X \times Y)$ satisfying the conditions of Lemma 5.3.7. By Definition 2.3.1, for any (x, l), there exist a test function $\Phi_{(x,l)} \in C_0^\infty(X \times Y)$ with $\Phi_{(x,l)}(x, y_0) = 1$, and an open conic neighbourhood $V_{(x,l)} \subset \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ of (l, k_0), so that

$$|\widehat{\Phi_{(x,l)} \cdot D}|(l', k') \leq \frac{C_N^{(x,l)}}{(1 + |(l', k')|)^N} \quad \forall (l', k') \in V_{(x,l)} \,. \tag{5.3.13}$$

for some positive constant $C_N^{(x,l)} > 0$ for any $N \in \mathbb{N}$. Lemma 5.3.7 then shows that for such distributions D, the estimates can be combined to one covering every $l \in \mathbb{R}^n$ and every $x \in {}_X\pi\,\mathrm{supp}(D)$. Moreover, the combined test function Φ can be written as $\Phi(x, y) = \chi(x) f(y)$, where $\chi \in C_0^\infty(x)$ is equal to one on ${}_X\pi\,\mathrm{supp}(D)$ and $f \in C_0^\infty(Y)$ can be chosen such that its support is contained in any fixed but arbitrary compact neighbourhood of y_0.

Proof. As mentioned above, let us assume that (5.3.11) holds for D. By Definition 2.3.1, for any $(x, l) \in {}_X\pi\,\mathrm{supp}(D) \times \mathbb{R}^n$, we can find $\Phi_{(x,l)} \in C_0^\infty$, $V_{(x,l)} \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ and $(C_N^{(x,l)})_{N \in \mathbb{N}}$ as above so that (5.3.13) holds for any $N \in \mathbb{N}$.

Without loss of generality, we may assume that $\Phi_{(x,l)} \geq 0$. If this is not the case, we can multiply by another test function $\chi \in C_0^\infty(X \times Y)$ defined as $\chi = \overline{\Phi}_{(x,l)}$ and obtain an estimate of the same form after possibly shrinking $V_{(x,l)}$ and increasing $\left(C_N^{(x,l)}\right)_{N\in\mathbb{N}}$.

This follows from (the proof of) [68, Lemma 8.1.1], which states that if $v \in \mathcal{E}'(Z)$, $Z \subset \mathbb{R}^m$, and $(x,k) \in Z \times (\mathbb{R}^m \backslash \{0\})$ is a direction of rapid decrease for v, then it is also a direction of rapid decrease for $\phi \cdot v$, where $\phi \in C_0^\infty(Z)$. This result will be used frequently in the rest of the proof.

As a simplification of the following argument, we will label the functions Φ, conic sets V and constants C_N by $\lambda = l/\,|(l,k_0)|$ instead of l. The label λ will then range over the open ball of unit radius around the origin of $\mathbb{R}^n$. In addition, we can consider the projection of the conic sets $V_{(x,\lambda)}$ to the unit sphere

$$\mathbb{S}^{2n-1} = \{(l',k') \in \mathbb{R}^n \times \mathbb{R}^n : |(l',k')| = 1\}$$

in $\mathbb{R}^n \times \mathbb{R}^n$. Since the sets $V_{(x,\lambda)}$ are conic, they are completely described by this projection. The projections $P_{(x,\lambda)} = V_{(x,\lambda)} \cap \mathbb{S}^{2n-1}$ are open neighbourhoods of $\left(\lambda, \frac{\sqrt{1-|\lambda|^2}}{|k_0|} k_0\right)$, and thus form an open cover of

$$P_x = \left\{ \left(\lambda', \frac{\sqrt{1-|\lambda|^2}}{|k_0|} k_0\right) \in \mathbb{S}^{2n-1} : |\lambda'| < 1 \right\} .$$

By the assumption on $\mathrm{WF}(D)$, we know that $(x,l;y_0,0) \notin \mathrm{WF}(D)$. Consequently, we obtain $\Phi_{(x,\lambda)}$, $V_{(x,\lambda)}$ or $P_{(x,\lambda)}$, and $\left(C_N^{(x,\lambda)}\right)_{N\in\mathbb{N}}$ as above for $|\lambda| = 1$ as well. In other words, for any fixed x, the label λ now ranges over the closed unit ball around 0 in $\mathbb{R}^n$, and the $P_{(x,\lambda)}$ cover the compact set

$$\overline{P_x} = \left\{ \left(\lambda', \frac{\sqrt{1-|\lambda|^2}}{|k_0|} k_0\right) \in \mathbb{S}^{2n-1} : |\lambda'| \leq 1 \right\} .$$

Hence, for any fixed but arbitrary x, we may pick a finite open subcover $(P_{(x,\lambda_i)})_{i=1,\ldots,M}$ of $\overline{P_x}$ and define

$$\Phi_x = \prod_{i=1}^{M} \Phi_{(x,\lambda_i)} \in C_0^\infty(X \times Y) .$$

Let us pick some $i \in \{1,\ldots,M\}$. Then one can apply the proof of [68, Lemma 8.1.1] with the identification $\phi = \prod_{j\neq i} \Phi_{(x,\lambda_j)}$ and $v = \Phi_{(x,\lambda_i)} \cdot D$ to show that there are constants $(C_N^x)_{N\in\mathbb{N}}$ with

$$|\widehat{\Phi_x \cdot D}|(l',k') \leq \frac{C_N^x}{(1+|(l',k')|)^N} \quad \forall (l',k') \in V_{(x,\lambda_i)} . \tag{5.3.14}$$

Varying i over $\{1,\ldots,M\}$, this estimates holds for all $(l',k') \in V_x \equiv \bigcup_i V_{(x,\lambda_i)}$. Defining the open conic neighbourhood $V_{k_0}^x$ by

$$V_{k_0}^x = \left\{ k \in \mathbb{R}^n : (l,k) \in \bigcup_i V_{(x,\lambda_i)} \forall l \in \mathbb{R}^n \right\} ,$$

the result implies that the estimate (5.3.14) holds for all $(l',k') \in \mathbb{R}^n \times V_{k_0}^x$.

In the next step, we would like to combine the estimates for different $x \in {}_X\pi\,\mathrm{supp}(D)$. To do so, we fix some small $0 < \epsilon < 1$ and define the open sets

$$\mathcal{U}_x^\epsilon = \{(x',y') \in X \times Y : \Phi_x(x',y') > \epsilon\} \ .$$

Since $(\mathcal{U}_x^\epsilon)_{x \in {}_X\pi\,\mathrm{supp}(D)}$ forms an open cover of the compact subset ${}_X\pi\,\mathrm{supp}(D) \times \{y_0\}$ of $\mathbb{R}^n \times \mathbb{R}^n$, one can again pick a finite open subcover $(\mathcal{U}_{x_i}^\epsilon)_{i=1,\ldots,L}$ of ${}_X\pi\,\mathrm{supp}(D) \times \{y_0\}$. The corresponding functions $\Phi_i \equiv \Phi_{x_i}$ then satisfy

$$\sum_{i=1}^{L} \Phi_i(x',y') \geq \epsilon \quad \forall (x',y') \in {}_X\pi\,\mathrm{supp}(D) \times \mathcal{V} \, .$$

Here, we have defined the set

$$\mathcal{V} = \left\{ y' \in Y : (x',y') \in \bigcup_{i=1}^{L} \mathcal{U}_{x_i}^\epsilon \forall x' \in {}_X\pi\,\mathrm{supp}(D) \right\} .$$

With this in mind, we construct a smooth cutoff function $\chi \in C_0^\infty(X \times Y)$ which satisfies

$$\chi = \begin{cases} \frac{1}{\sum_i \Phi_i} : & \sum_{i=1}^{L} \Phi_i \geq \frac{\epsilon}{2} \\ 0 : & \sum_{i=1}^{L} \Phi_i \leq \frac{\epsilon}{4} \end{cases} .$$

In addition, we pick $f \in C_0^\infty(Y)$ so that $\mathrm{supp}(f) \subset \mathcal{V} \cap K$ and $f(y_0) = 1$. Then $\chi(x,y)f(y) \in C_0^\infty(X \times Y)$, and for any fixed $i \in \{1,\ldots,L\}$ we can apply the proof of [68, Lemma 8.1.1] with $\phi = \chi f$ and $v = \Phi_i \cdot D$ to obtain constants $(C_N^i)_{N \in \mathbb{N}}$ satisfying

$$|\widehat{f\chi\Phi_i \cdot D}|(l,k) \leq \frac{C_N^i}{(1+|(l,k)|)^N} \quad \forall (l,k) \in V_{x_i} \, .$$

Combining the estimates for different i the yields

$$\left|\widehat{(\mathbf{1}\otimes f)\cdot D}\right|(l,k) = \left|\widehat{\sum_{i=1}^{L} f\chi\Phi_i\cdot D}\right|(l,k) \leq \sum_{i=1}^{L}\left|\widehat{f\chi\Phi_i\cdot D}\right|(l,k) \tag{5.3.15}$$
$$\leq \sum_{i=1}^{L}\frac{C_N^i}{(1+|(l,k)|)^N} \leq \frac{\tilde{C}_N}{(1+|(l,k)|)^N}$$

for all $(l,k)\in\bigcap_{i=1}^{L} V_{x_i}\supset\mathbb{R}^n\times V_{k_0}$, with $V_{k_0}=\bigcap_{i=1}^{L} V_{k_0}^{x_i}$.

To obtain (5.3.12), note that there are constants $c_{1,2}>0$ so that

$$c_1(|l|+|k|)\leq|(l,k)|\leq c_2(|l|+|k|)\,.$$

In addition, a simple application of the binomial formula yields that for $a,b>0$

$$(1+a+b)^N\geq 1+a^M+b^{N-M}+a^Mb^{N-M}=(1+a^M)(1+b^{N-M})\,.$$

Applying this to the denominator of (5.3.15) finishes the proof. $\square$

The second lemma we want to show is

Lemma 5.3.8. *Let $(x_0,k_0)\in\mathcal{N}$, with $B_{\mathcal{M}}(x_0,k_0)$ intersecting $\mathcal{H}_+$ (when extended to $\tilde{\mathcal{M}}$), and let us identify k_0 with an element of $\mathbb{R}^4$ under the $+$-Kruskal coordinate chart $\psi_+:\mathcal{M}_+\to\mathbb{R}^2\times\mathbb{S}^2$. Let K be a small compact neighbourhood of x_0 covered by the $+$-Kruskal coordinate chart, and let $V\subset\mathbb{R}^4\backslash\{0\}$ be a sufficiently small conic neighbourhood of k_0, such that $B_{\mathcal{M}}(x,k)$ intersects $\mathcal{H}_+$ in the interior of some compact set $\mathcal{U}\subset\mathcal{H}_+$ for all $x\in K$ and all null covectors $k\in V$. Let $h\in C_0^\infty(\mathcal{H}_+)$ be such that $h=1$ on a neighbourhood of $\mathcal{U}$. Then, there are a function $f\in C_0^\infty(\mathcal{M})$, with $f(x_0)=1$, an open conic neighbourhood $V_{k_0}\subset\mathbb{R}^4\backslash\{0\}$ of k_0, and, $\forall\ N\in\mathbb{N}$ and $n\in\{0,1\}$, positive constant C_{Nn}, $\tilde{C}_{Nn}>0$ such that*

$$\left|\partial_{U_+}^n(1-h)E(f_k)|_{\mathcal{H}_+}\right|\leq|U_+|^{-\alpha/\kappa_+}\frac{C_{Nn}}{1+|k|^N}\quad\forall k\in V_{k_0} \tag{5.3.16}$$

$$\left|\partial_{V_c}^n E(f_k)|_{\mathcal{H}_c}\right|\leq|V_c|^{-\alpha/\kappa_c}\frac{\tilde{C}_N}{1+|k|^N}\quad\forall k\in V_{k_0}\,. \tag{5.3.17}$$

Here, we use the notation $f_k(x)=(2\pi)^{-2}e^{ik\cdot x}f(x)$, where we have fixed the ψ_+-coordinate chart.

Proof. We begin the proof of this lemma with a number of definitions.

First of all, for $K\subset\mathcal{M}$ covered by the coordinate chart ψ_+, identify $T_K^*\mathcal{M}$ with $\psi_+(K)\times\mathbb{R}^4$ in ψ_+. Let $V\subset\mathbb{R}^4\backslash\{0\}$ be an open conic set. Then we set

$$B_{\mathcal{M}}(K,V)=\{x'\in M: x'\in B_{\mathcal{M}}(x,k)\text{ for some } x\in K, k\in V\text{ null}\}\,,$$

and we can continue $B_{\mathcal{M}}(K,V)$ to $\tilde{\mathcal{M}}$ by continuing the individual geodesics. We will refer to this continuation as $B_{\mathcal{M}}(K,V)$ as well.

Second, we define a number of Cauchy surfaces for $\tilde{\mathcal{M}}$: We choose Σ_{k_0} so that it coincides with $\mathcal{H}$ on a neighbourhood of $\mathrm{supp}(h)$. Furthermore, we pick two Cauchy surfaces $\Sigma_\pm$ of $\tilde{\mathcal{M}}$ so that $\Sigma_{k_0} \subset I^+(\Sigma_-) \cap I^-(\Sigma_+)$ and $K \subset I^+(\Sigma_+)$.

Third, we need a number of cutoff functions: Let $\tilde{h} \in C_0^\infty(\tilde{\mathcal{M}})$ and $h' \in C_0^\infty(\tilde{\mathcal{M}})$ be defined such that $\tilde{h} + h' = 1$ on a neighbourhood of $J^-(K) \cap J^+(\Sigma_-) \cap J^-(\Sigma_+)$, $\tilde{h}|_{\mathcal{H}_+} = h$, $\mathrm{supp}(\tilde{h}) \cap \mathcal{H}_c = \emptyset$, and assume that there is an open neighbourhood $\mathcal{V} \subset \tilde{\mathcal{M}}$ of $B_{\mathcal{M}}(K,V)$ satisfying $\mathcal{V} \cap \mathrm{supp}(h') = \emptyset$. Moreover, let $\eta \in C^\infty(\tilde{\mathcal{M}})$ be supported in $\mathcal{V}$, with $\eta = 1$ in a neighbourhood of $B_{\mathcal{M}}(K,V)$. Finally, let $\chi_\pm \in C^\infty(\tilde{\mathcal{M}})$ be a partition of unity, $\chi_+ + \chi_- = 1$, satisfying $\chi_\pm|_{J^\pm(\Sigma_\pm)} = 1$.

We illustrate the Cauchy surfaces and the supports of the various functions in Fig. 5.3.

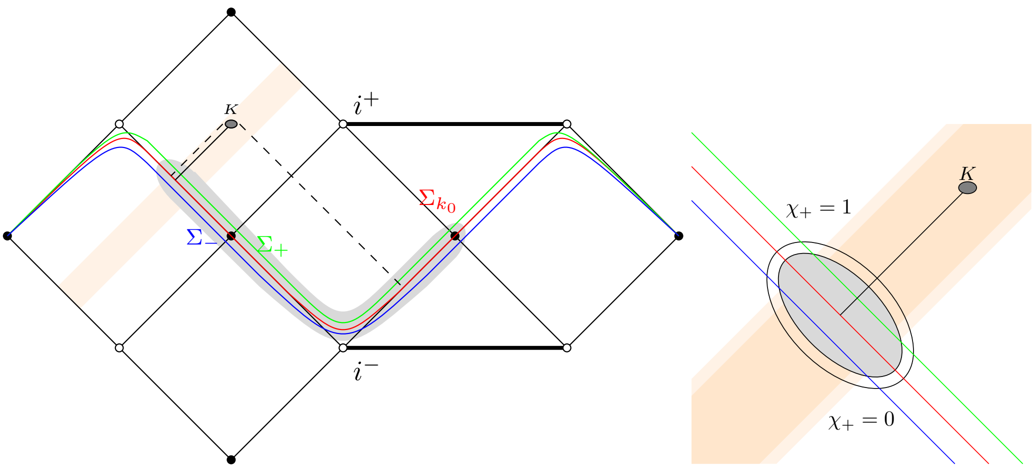


Figure 5.3: Left: The three Cauchy surfaces are, from top to bottom, Σ_+, Σ_{k_0} and Σ_-. The small, dark gray region is K. The solid line joining K and $\mathcal{H}$ indicates the bicharacteristic $B_{\mathcal{M}}(x_0,k_0)$. The dashed lines mark $J^-(K)$. The light orange strip around $B_{\mathcal{M}}(x_0,k_0)$ shows the neighbourhood $\mathcal{V}$ of $B_{\mathcal{M}}(K,V)$, on which $h' = 0$. The light gray region around $J^-(K) \cap J^-(\Sigma_+) \cap J^+(\Sigma_-)$ indicates $\mathrm{supp}(\tilde{h} + h')$. Right: The two ellipses indicate $\tilde{h} = 1$ (inner, shaded ellipse), and $\mathrm{supp}(\tilde{h})$ (outer ellipse). The orange strip indicates $\eta = 1$ (darker shade) and $\mathrm{supp}(\eta)$ (lighter shade). The function χ_+ is equal to one above the green line, which corresponds to Σ_+ and vanishes below the blue one, which corresponds to Σ_-.

Let us now consider an arbitrary test function $g \in C_0^\infty(K)$. Then, for any such test function, set $\tilde{g} \equiv \mathcal{K}(\chi_+ E(g)) \in C_0^\infty(\tilde{\mathcal{M}})$. $\tilde{g}$ is supported in $J^-(K) \cap J^+(\Sigma_-) \cap J^-(\Sigma_+)$ and $E(g) = E(\tilde{g})$, see the discussion in Section 2.2. From the construction of $\tilde{h}$ and h', it also follows that $\tilde{g} = \tilde{h}\tilde{g} + h'\tilde{g}$. Moreover, we can use the linearity of the commutator function together with the properties of the retarded and advanced Green's operators $E^\pm$

to show

$$
\begin{aligned}
E(g) =&E(\tilde{g}) = E(\tilde{h}\tilde{g}) + E(h'\tilde{g}) \qquad (5.3.18)\\
=&E^+(\tilde{h}\mathcal{K}(\chi_+E(g))) + E^-(\tilde{h}\mathcal{K}(\chi_-E(g))) + E(h'\tilde{g})\\
=&E^+(\mathcal{K}(\tilde{h}\chi_+E(g))) + E^-(\mathcal{K}(\tilde{h}\chi_-E(g))) - E^+([\mathcal{K},\tilde{h}]\chi_+E(g))\\
&- E^-([\mathcal{K},\tilde{h}]\chi_-E(g)) + E(h'\tilde{g})\\
=&\tilde{h}E(g) - E^+([\mathcal{K},\tilde{h}]\chi_+E(g)) - E^-([\mathcal{K},\tilde{h}]\chi_-E(g)) + E(h'\tilde{g})\,.
\end{aligned}
$$

Since $\tilde{h}|_{\mathcal{H}_+} = h$ and $\tilde{h}|_{\mathcal{H}_c} = 0$, we can identify $(1-h)E(g)|_{\mathcal{H}_+}$ and $E(g)|_{\mathcal{H}_c}$ with the last three terms in the last line of (5.3.18) restricted to $\mathcal{H}_+$ or $\mathcal{H}_c$ respectively. So let us study these terms further.

First, we note that $h'\tilde{g}$ vanishes in a neighbourhood of $B_\mathcal{M}(K,V)$, while this is not the case for $[\mathcal{K},\tilde{h}]\chi_+E(g)$ or $[\mathcal{K},\tilde{h}]\chi_-E(g)$. However, we note that by construction

$$\mathrm{supp}([\mathcal{K},\tilde{h}]\chi_\pm E(g)) \cap \mathcal{V} \subset J^\pm(\Sigma_\pm) \cap J^\pm(\mathcal{H}_+ \cup \mathcal{H}_c)\,.$$

This allows us to further split the terms involving these expressions as

$$E^\pm([\mathcal{K},\tilde{h}]\chi_\pm E(g)) = E^\pm(\eta[\mathcal{K},\tilde{h}]\chi_\pm E(g)) + E^\pm((1-\eta)[\mathcal{K},\tilde{h}]\chi_\pm E(g))\,.$$

By the definition of η and $E^\pm$, we then have

$$
\begin{aligned}
\mathrm{supp}(E^\pm(\eta[\mathcal{K},\tilde{h}]\chi_\pm E(g))) &\subset J^\pm(\mathrm{supp}([\mathcal{K},\tilde{h}]\chi \pm E(g)) \cap \mathcal{V})\\
&\subset J^\pm(\Sigma_\pm) \cap J^\pm(\mathcal{H}_+ \cup \mathcal{H}_c)\,,
\end{aligned}
$$

and hence these terms vanish on $\mathcal{H}_+ \cup \mathcal{H}_c$ and can be neglected. At the same time, the remaining terms $(1-\eta)[\mathcal{K},\tilde{h}]\chi_\pm E(g)$ vanish in a neighbourhood of $B_\mathcal{M}(K,V)$, just as $h'\tilde{g}$.

This allows us to treat all three of these terms in the same way. Thus, in the following we will focus on the term containing $h'\tilde{g}$, arguments for the other terms can be given along the same lines.

As a next step, we would like to identify the function $f \in C_0^\infty(\mathcal{M})$. To this end, we note that

$$h'\tilde{g} = h'(\Box_g\chi_+ 2\partial_a\chi_+\partial^a)E(g) \equiv h'B(E(g))\,,$$

where we have defined the first-order differential operator $B = \Box_g\chi_+ + 2\partial_a\chi_+\partial^a$.

Let us consider the bi-distribution $h'B(E)$, where B is acting on the first variable of E. This bi-distribution is compactly supported in the first variable due to the compact support of h'. Moreover, since differentiation and the multiplication by a smooth function cannot increase the wavefront set, we obtain

$$(y,l;x_0,k_0) \notin \mathrm{WF}(h'B(E)) \quad \forall (y,l) \in T^*_{\mathrm{supp}(h')}\tilde{\mathcal{M}}\,. \qquad (5.3.19)$$

Let us identify (x_0, k_0) with an element of $\psi_+(K)\times\mathbb{R}^4$ in the chart ψ_+. Let us also fix a coordinate chart ψ covering $\mathrm{supp}(h')$ and identify all (y, l) with points in $\psi(\mathrm{supp}(h')) \times \mathbb{R}^4$. Then we can conclude, alluding to Lemma 5.3.7, that there are a function $f_1 \in C_0^\infty(\mathcal{M})$ with $f_1(x_0) = 1$ and $\mathrm{supp}(f_1) \subset K$, an open conic neighbourhood $\tilde{V}_{k_0} \subset \mathbb{R}^4\setminus\{0\}$ of k_0, and positive constants $(\tilde{C}_{NN'})_{N,N'\in\mathbb{N}}$ so that

$$|\widehat{(h' \otimes f_1) \cdot B(E)}|(l, k) \leq \frac{\tilde{C}_{NN'}}{(1 + |l|^{N'})(1 + |k|^N)} \quad \forall (l, k) \in \mathbb{R}^4 \times \tilde{V}_{k_0} . \tag{5.3.20}$$

As mentioned above, the other two terms can be treated in the same way, and we obtain estimates of the form (5.3.20) for $((1-\eta)[\mathcal{K}, \tilde{h}]\chi_\pm \otimes f_\pm)E$ with different functions $f_\pm \in C_0^\infty(\mathcal{M})$, conic neighbourhoods $V_{k_0}^\pm \subset \mathbb{R}^4\setminus\{0\}$ of k_0 and constants $(C_{NN'}^\pm)_{N,N'\in\mathbb{N}}$.

It then follows from the proof of [68, Lemma 8.1.1] that these estimates continue to hold if f_1 and $f_\pm$ are replaced by

$$f \equiv f_1 \cdot f_+ \cdot f_- \in C_0^\infty(\mathcal{M}) , \tag{5.3.21}$$

and all three estimates hold for $k \in V_{k_0} \equiv \tilde{V}_{k_0} \cap V_{k_0}^+ \cap V_{k_0}^-$.

To reach (5.3.16), let us return to (5.3.20) with f_1 replaced by f. We will use the notation f_k introduced below (5.3.16). In this notation, we can then estimate

$$\begin{aligned}
\left|\partial_x^\alpha h' \tilde{f}_k\right| &= (2\pi)^{-2} \left|\partial_x^\alpha \int\limits_{\mathbb{R}^4} e^{il\cdot x} \widehat{h' \tilde{f}_k}(l)\, \mathrm{d}^4 l\right| \\
&\leq (2\pi)^{-2} \int\limits_{\mathbb{R}^4} |l|^{|\alpha|} \left|\widehat{h' \tilde{f}_k}(l)\right| \, \mathrm{d}^4 l \\
&\leq \tilde{C}'_{NN'} \frac{1}{1 + |k|^N} \int\limits_0^\infty \frac{l^{|\alpha|+3}}{1 + l^{N'}}\, \mathrm{d}l ,
\end{aligned}$$

by using the total convergence of the integral to pull in the differentiation and applying the estimate (5.3.20). In the last step, we have also changed to spherical coordinates and collected all constants in $\tilde{C}'_{NN'}$. We can always choose N' sufficiently large to make the remaining integral finite. Thus, we find that

$$\left\|h' \tilde{f}_k\right\|_{C^m} = \max_{|\alpha|\leq m} \sup_{x\in K'} \left|\partial_x^\alpha h' \tilde{f}_k\right| \leq C_{N,m} \frac{1}{1 + |k|^N} .$$

Here, we have used the partial derivatives in the coordinate system ψ covering $\mathrm{supp}(h')$ as the independent vector fields, and defined $K' \subset \tilde{\mathcal{M}}$ to be a compact region containing $\mathrm{supp}(h')$ and covered by ψ.

Finally, it remains to act on $h' \tilde{f}_k$ with the commutator function E, restrict to $\mathcal{H}_+$ or $\mathcal{H}_c$ and apply the estimate (5.2.6b) or (5.2.6c) from [39]. Since the support of $h' \tilde{f}_k$ will in general not be restricted to $\mathcal{M}$, one has to use these bounds also to determine the decay towards $U_+(V_c) \to \infty$. Since this corresponds to approaching i^+ in a time-reversed Kerr-de Sitter spacetime, and the estimates require only that the initial data for the forward solution is compactly supported away from $r = r_-$, the results apply for this case as well. One obtains

$$|\partial_{U_+}^n E(h'\tilde{f}_k)|_{\mathcal{H}_+}| \lesssim |U_+|^{-\alpha/\kappa_+} \frac{C'_{Nn}}{1+|k|^N} \quad \forall k \in V_{k_0} \tag{5.3.22a}$$

$$|\partial_{V_c}^n E(h'\tilde{f}_k)|_{\mathcal{H}_c}| \lesssim |V_c|^{-\alpha/\kappa_c} \frac{C'_{Nn}}{1+|k|^N} \forall k \in V_{k_0} \tag{5.3.22b}$$

for any $N \in \mathbb{N}$ and $n \in \{0, 1\}$, where the constants C'_{Nn} will depend on $\operatorname{supp}(h')$.

As mentioned above, the other two terms can be treated in the same way, and hence for all $N \in \mathbb{N}$ and $n \in \{0, 1\}$, we find constants C'_{Nn} depending on the support of $\tilde{h}$ so that

$$|\partial_{U_+}^n E^{\pm}((1-\eta)[\mathcal{K}, \tilde{h}]\chi_{\pm} E(f_k))|_{\mathcal{H}_+}| \lesssim |U_+|^{-\alpha/\kappa_+} \frac{C'_{Nn}}{1+|k|^N} \quad \forall k \in V_{k_0} \tag{5.3.23a}$$

$$|\partial_{V_c}^n E^{\pm}((1-\eta)[\mathcal{K}, \tilde{h}]\chi_{\pm} E(f_k))|_{\mathcal{H}_c}| \lesssim |V_c|^{-\alpha/\kappa_c} \frac{C'_{Nn}}{1+|k|^N} \quad \forall k \in V_{k_0} \, . \tag{5.3.23b}$$

Combining the estimates for the three terms then finishes the proof of the lemma. $\square$

We are now finally ready to prove Proposition 5.3.6 and conclude the proof of the Hadamard property.

Proof of Proposition 5.3.6. For this proof, let us fix some $(x_0, k_0) \in \mathcal{N}$ that satisfies $B_{\mathcal{M}}(x_0, k_0) \cap \mathcal{O} = \emptyset$. Without loss of generality, let us assume that $B_{\mathcal{M}}(x_0, k_0)$ (when extended to $\tilde{\mathcal{M}}$) intersects $\mathcal{H}_+$. The case where it intersects $\mathcal{H}_c$ can be handled in the same way.

We can then assume that a compact neighbourhood K of (x_0, k_0) is covered by the +-Kruskal coordinate chart ψ_+ as in Lemma 5.3.8. Let us choose an open conic neighbourhood $V \subset \mathbb{R}^4 \backslash \{0\}$, so that, in the notation introduced in Lemma 5.3.8, $B_{\mathcal{M}}(K, V)$ and $B_{\mathcal{M}}(K, -V)$ both intersect $\mathcal{H}_+$ in the interior of a compact set $\mathcal{U} \subset \mathcal{H}$. Define $h \in C_0^\infty(\mathcal{H}_+)$ as in Lemma 5.3.8.

Let us also introduce an additional function $\zeta \in C_0^\infty(\tilde{\mathcal{M}})$ satisfying $\zeta = 1$ on $\operatorname{supp}(h)$. Then, one can split w as [16, 43, 123, 142]

$$\begin{aligned} w =& (h \cdot A_+ \cdot h) \big(tr_{\mathcal{H}_+} \circ (\zeta \cdot E), tr_{\mathcal{H}_+} \circ (\zeta \cdot E)\big) \\ &+ A_+ \big((1-h) \cdot tr_{\mathcal{H}_+} \circ E, h \cdot tr_{\mathcal{H}_+} \circ E\big) + A_+ \big(h \cdot tr_{\mathcal{H}_+} \circ E, (1-h) \cdot tr_{\mathcal{H}_+} \circ E\big) \\ &+ A_+ \big((1-h) \cdot tr_{\mathcal{H}_+} \circ E, ((1-h) \cdot tr_{\mathcal{H}_+} \circ E\big) + w_c \, , \end{aligned} \tag{5.3.24}$$

where the restriction map to $\mathcal{H}_+$ is called $\mathrm{tr}_{\mathcal{H}_+}$. We can now analyse the different pieces separately. Let us start with the second piece.

We would like to show that it is rapidly decreasing in $(x_0, k_0; x_0, -k_0)$. Therefore, we pick the test function f and conic neighbourhood V_{k_0} constructed in Lemma 5.3.8 and note that, in the notation introduced in this lemma, $\widehat{(f \otimes f)v}(k,l) = v(f_k, f_l)$ for any bi-distribution v in $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$ with the coordinates fixed to be $+$-Kruskal coordinates. We also note that for any test function $f' \in C_0^\infty(\mathcal{M})$, with $\operatorname{supp}(f')$ contained in a compact region $K' \subset \mathcal{M}_+$ which is covered by the $+$-Kruskal coordinates, one obtains a bound

$$\|f'_k\|_{C^m} \leq C(1 + |k|^m)\, \|f'\|_{C^m}$$

for some constant C depending on m. Thus, by (5.2.6b) or (5.2.6c), one can find some fixed L and some constant C, so that

$$\left| hE(f_k)|_{\mathcal{H}_+} \right| \leq C\, |U_+|^{-\alpha/\kappa_+} \left(1 + |k|^L\right) . \tag{5.3.25}$$

We can now follow the same steps as in the proof of Proposition 5.2.1: We write out the integral in the definition of A_+, including the limit $\epsilon \to 0$. Holding this $\epsilon > 0$ fixed, we perform a partial integration and split up the integration into the regions indicated in Fig. 5.2. Choosing $-U_0$ sufficiently small, combined with the compact support of h, the integrals over domains D_3 and D_4 defined in (5.2.7) will not contribute. Following the estimates for the other terms, we find constants $C_{N,L}$ so that $\forall N \in \mathbb{N}$

$$\left| A_+\left((1-h)E(f_k)|_{\mathcal{H}_+}, hE(f_l)|_{\mathcal{H}_+}\right) \right| \leq C_{N,L} \left(1 + |k|^{-N}\right) \left(1 + |l|^L\right) \qquad \forall k \in V_{k_0} .$$

If (k,l) is contained in the conic neighbourhood [43]

$$\{(k,l) \in \mathbb{R}^4 \times \mathbb{R}^4 \backslash \{0\} : 1/2\, |k| < |l| < 2\, |k|\, , k \in V_{k_0}\}$$

of $(k_0, -k_0)$, then the polynomial growth in $|l|$ can always be bounded by the decay in $|k|$ and hence we find that $(x_0, k_0; x_0, -k_0)$ is a direction of rapid decrease for this term.

In the same way, one can use the estimates obtained in Lemma 5.3.8 and (5.3.25) for the other terms in (5.3.24), except for the first one, after possibly replacing k_0 and V by $-k_0$ and $-V$. We apply the bounds (5.3.16) and (5.3.25) to the estimates in the proof of Proposition 5.2.1, and construct an open conic neighbourhood of $(k_0, -k_0)$ in which the decay outweighs the potential polynomial growth.

This shows that if $(x_0, k_0; x_0, -k_0) \in \mathrm{WF}(w)$, then it must be in the wavefront set of the first term in (5.3.24).

Thus, let us now consider the first term in (5.3.24). In order to compute its wavefront set, we start by computing the wavefront sets of $\mathrm{tr}_{\mathcal{H}_+}$ and A_+. In the following, we identify the horizon $\mathcal{H}_+$ with $\mathbb{R} \times \mathbb{S}^2$ in the ψ_+-coordinate chart and $T^*\mathcal{M}$ with its local trivialization in these coordinates. We write (U, Ω, ξ, σ) for points in $T^*(\mathbb{R} \times \mathbb{S}^2)$ where $V_+ = 0$, so $\Omega = (\theta, \varphi_+)$ and $\sigma \in T^*_\Omega \mathbb{S}^2$.

The wavefront set of A_+ can be obtained by a direct computation [16, 142],

$$\begin{aligned} \mathrm{WF}'(A_+) = \big\{ &(U, \Omega, \xi, \sigma; U', \Omega, \xi, \sigma) \in T^*(\mathbb{R} \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{S}^2) \backslash o : \\ &\xi > 0 \text{ if } U = U'\, , \xi = 0 \text{ else } \big\} . \end{aligned} \tag{5.3.26}$$

The wavefront set of the trace map $\mathrm{tr}_{\mathcal{H}_+}$ can be obtained by considering its kernel in the $+$-Kruskal coordinates and then applying [68, Thm. 8.2.4]. It is given by [16, 142]

$$\mathrm{WF}'(tr_{\mathcal{H}_+}) = \Big\{(U,\Omega,\xi,\sigma;x,k) \in T^*(\mathbb{R}\times\mathbb{S}\times\tilde{\mathcal{M}}) : \psi_+(x) = (U,0,\Omega), \\ \psi_+^*(\xi,\eta,\sigma) = k \text{ for some } \eta\in\mathbb{R}\Big\} \;. \tag{5.3.27}$$

As seen before, Lemma 5.1.5, together with [68, Thm. 8.2.4] allow us to make sense of the map $\mathrm{tr}_{\mathcal{H}} \circ E : C_0^\infty(\mathcal{M}) \to C^\infty(\mathbb{R}\times\mathbb{S}^2)$ without an intermediate cutoff function ζ.
However, the advantage of inserting ζ is that the maps $\zeta\cdot E : C_0^\infty(\mathcal{M}) \to C_0^\infty(\tilde{\mathcal{M}})$, $\mathrm{tr}_{\mathcal{H}_+} : C_0^\infty(\tilde{\mathcal{M}}) \to C_0^\infty(\mathbb{R}\times\mathbb{S}^2)$, and $h\cdot A_+\cdot h : C_0^\infty(\mathbb{R}\times\mathbb{S}^2) \to \mathcal{E}'(\mathbb{R}\times\mathbb{S}^2)$ are all properly supported. Therefore, we can apply [68, Thm. 8.2.14] on the wavefront set of operator kernels under the composition of operators. Together with the results above, Lemma 5.1.5 and (5.3.1), we then obtain

$$\begin{aligned}
&\mathrm{WF}'((h\cdot A_+\cdot h)\big(tr_{\mathcal{H}_+}\circ(\zeta\cdot E), tr_{\mathcal{H}_+}\circ(\zeta\cdot E)\big)) \\
&\subset \Big\{(x_1,k_1;x_2,k_2) \in T^*(\mathcal{M}\times\mathcal{M})\backslash o : \exists (y,l) \in T^*_{\mathcal{H}_+}(\tilde{\mathcal{M}}) : \\
&\qquad (x_i,k_i) \sim (y,l)\,,\ i=1,2\,;\ {}^t\mathrm{d}(\psi_+^{-1})(\psi_+(y))l = (\xi,\eta,\sigma) \text{ with } \xi>0\Big\} \\
&\subset \mathcal{N}^+\times\mathcal{N}^+\,.
\end{aligned} \tag{5.3.28}$$

We conclude that $(x_0,k_0;x_0,-k_0)$ can only be in $\mathrm{WF}(w)$ if $(x_0,k_0)\in\mathcal{N}^+$, concluding the proof of the proposition. □

We have thus shown that the two-point function of the Unruh state defined in (5.2.2) indeed satisfies all necessary condition to be the two-point function of a well-defined, quasi-free Hadamard state on the CCR-algebra $\mathcal{A}(\mathcal{M})$ of the free, real scalar field on the Kerr-de Sitter spacetime, as long as the angular momentum a of the black hole and the cosmological constant Λ are sufficiently small.
As a final note, let us try to give an interpretation of the physical situation represented by the Unruh state. Since observers are usually considered to be in the exterior of the black hole, let us restrict our attention to region I. Lemma 5.3.4 reveals that if we restrict to test functions supported in I, the two parts w_+ and w_c of the Unruh state two-point function separately satisfy the KMS-condition at temperatures $T_{+/c} = (2\pi)^{-1}\kappa_{+/c}$ with respect to the respective Killing field $\partial_{t_{+/c}}$ defined in (2.4.9).
One interpretation of this is that there are, in the distant past, thermal populations of in- and out-moving particles at different temperatures, which are also rotating relatively to each other. This makes it apparent that the Unruh state is not an equilibrium state.
In the light of Lemma 5.2.5, and the particular form of the Unruh two-point function, one rather comes to the conclusion that it is a non-equilibrium steady state [145] which is stationary and axisymmetric in the sense that it is invariant under the automorphisms induced by the Killing vector fields ∂_t and ∂_φ on $\mathcal{A}(\mathcal{M})$.

6 Summary and discussion

In this thesis, we have discussed different aspects of free, scalar quantum field theory on asymptotically de Sitter black-hole spacetimes.

We started by trying to answer the question whether quantum effects are able to restore the strong cosmic censorship conjecture on charged, non-rotating, asymptotically de Sitter black holes in the Christodoulou-formulation in cases where it is classically violated. This required numerical computations of solutions to the radial part of the wave equation on these spacetimes in order to compute the quadratically divergent, leading term of the energy flux through the inner horizon in the Unruh state. This quadratic divergence was shown in [16] to be the state-independent leading divergence of the energy flux as long as strong cosmic censorship is classically violated.

The first part of the thesis consisted of developing and implementing a particular ansatz to the solution of the radial part of the wave equation. The results did not only show agreement with existing results of similar calculations [92], but also indicated that strong cosmic censorship can indeed be restored by the quantum effects.

One interesting feature of the leading divergence of the energy flux through the inner horizon is that it can change its sign depending on the charge of the black hole and the mass of the scalar field. This can be interpreted by using the semi-classical Einstein equation in a linearised form to estimate the backreaction effect of the quantum field onto the spacetime geometry. In this estimate, one finds that depending on the sign of the leading divergence of the energy flux at the inner horizon, an observer approaching that horizon will be either infinitely stretched or infinitely squeezed. Thus, while it becomes impossible to travel past the inner horizon in any case, the final fate of an observer approaching it depends on the exact parameters of the spacetime and the scalar field.

Nonetheless, this analysis should be taken with a grain of salt. On the one hand, the analysis of the backreaction made use of the assumption that the backreaction is weak. This clearly ceases to be the case in the last Planck length leading up to the horizon where the quantum effects become sizeable. One potential line of further research on this topic is thus to go beyond the weak backreaction regime. This would require to find the next step towards solving the semi-classical Einstein equation (1.0.1) self-consistently, which we expect to be very difficult.

Beyond that, we found that not only the energy flux, but also its fluctuations diverge towards the horizon. Hence, close to the Cauchy horizon, one does not only leave the domain of validity of the weak backreaction assumption, but of semi-classical gravity in general due to the divergence of curvature and the increasing importance of quantum fluctuations. Addressing this problem in a satisfying way might require a complete theory of quantum gravity.

Despite of these shortcomings, the numerical results obtained in the first part of this thesis show that quantum effects are very important when discussing the causal structure of black hole interiors, and cannot simply be neglected as small or subleading when compared to classical effects.

Throughout this analysis, only real scalar fields have been considered, while the formation of a charged black hole requires the presence of charged matter. Thus, in the next chapter, we considered a charged scalar field on the same spacetime. In contrast to the real scalar field, the charged scalar field can also influence the background electromagnetic field via the semi-classical Maxwell equations which include the expectation value of its charge current. The electromagnetic field, in turn, influences the spacetime. Therefore, for the charged scalar, we are not only interested in the leading divergence of the energy flux as before, but also in the charge current.

As a first step to study this current, a formula for its renormalized expectation value in the Unruh state was obtained utilizing the Hadamard point-split renormalization procedure. It was found that the counterterm for the current is finite and even vanishes on the different horizons. Moreover, it only enters the t-component of the current. This greatly facilitates the numerical computation of the current by the same methods as developed for the real scalar field. Moreover, the state-independence of the leading divergence of the stress-energy tensor at the inner horizon extends from the real to the charged scalar field, and holds not only for the energy flux, but also for the charge current.

These results allowed us to evaluate the charge current numerically at different points in the black-hole spacetime. The numerical results in the exterior and on the event horizon behave as one would expect, and as previously discussed in the literature [130]. However, we see a mismatch with the results one would expect from an application of the Schwinger formula [113] for the computation of the current [131–133]. We attribute this discrepancy to the fact that the Compton wavelength for our scalar field is of the same size as the radius of the cosmological horizon, and the approximation of a flat spacetime is not valid. Nonetheless, the numerical results we find in this region indicate that our formula for the charge current produces the correct results.

In the next step, we considered the current near the inner horizon. There, one can find a parameter range for the spacetime- and scalar field parameters in which the current changes its sign. This finding was very surprising, since from the intuitive particle picture one would expect that quantum effects will always lead to a discharge of the black hole. Therefore, this result is a clear sign that the behaviour of the quantum field near the inner horizon cannot be explained entirely using the particle picture. Thus, first-principle calculations are necessary to determine the behaviour of quantum fields near the Cauchy horizon of a charged black hole.

Nonetheless, since the sign of the current is always positive, indicating a discharge, if the black-hole charge is sufficiently close to its maximal allowed value, the quantum effects seem to be unable to overcharge the black hole and turn it into a naked singularity.

Apart from the current, we also computed the quadratic divergence of the energy flux of the charged scalar field at the inner horizon of a Reissner-Nordström-de Sitter black hole. The results were qualitatively similar to those found for the real scalar field on the same spacetime in an earlier chapter and in [92]. However, for sufficiently large charges of the black hole, the change of sign that was observed for the real scalar field disappears, and the flux remains positive for all black-hole charges considered. This effect seems to be similar to the effect of an increase of mass for the real scalar field.

The change of sign as a function of the black-hole charge does not only disappear for the energy flux if the charge of the scalar field is chosen sufficiently large, but for the charge current, too. All charges tested in this work are very small compared to the electron charge for black holes of at least solar mass. The same applies to the masses of the scalar field tested in this work. Thus, it is conceivable that the interesting quantum effects such as the sign change of the charge current are absent for realistic charges and masses of the scalar field. Nonetheless, the results demonstrate that in principle, the quantum effects near the inner horizon can develop features which elude a description in a simple particle picture.

Since our choice of parameters was mostly due to limitations in our numerical computations, it would be interesting to obtain results for more physical choices of both the scalar-field charge and mass. This would most likely require different numerical methods.

The results up to that point were very interesting and revealed important features of quantum field theory in black-hole interiors. However, they were obtained on charged black-hole spacetimes, while realistically, one would expect astrophysical black holes to be rotating rather than significantly charged. The reason is that any charge would be rapidly lost by absorbing surrounding matter of opposite charge or by quantum effects. Nonetheless, the charged black hole can be considered as a toy model for the rotating one, since it shares many features, such as an inner horizon, while being easier to handle mathematically.

As a next step, we wanted to extend the results from charged to rotating black holes. As a first step in this direction, the third part of the thesis showed how the Unruh state for a real scalar field can be rigorously constructed on a sufficiently slowly rotating Kerr-de Sitter black hole with a sufficiently small cosmological constant. Moreover, it was shown that the state constructed in this way is a Hadamard state not only in the black-hole exterior up to the future cosmological and event horizons, but also beyond them. The proof was based on the decay results for solutions to the classical wave equation on Kerr de-Sitter [39] and combined ideas from different similar proofs obtained in the past [16, 43, 123, 138, 142].

It was shown that the state is invariant under the flows created by any of the Killing vector fields of Kerr-de Sitter. In particular, the state is stationary. However, it is not an equilibrium state. Instead, its structure is that of a non-equilibrium steady state. Restricted to region I, it can be interpreted as the state describing thermal populations of out- and ingoing particles at different temperatures in the distant past, which are rotating relatively to each other.

The Unruh state being well-defined and Hadamard is a prerequisite for its application in the computation of expectation values of observables such as the stress-energy tensor. It is required for the application of Hadamard point-split renormalization and the use of the Unruh state as a comparison state for the regularity analysis of other states. A more specific application of the Unruh state that is only possible thanks to its Hadamard property is the computation of the leading divergence of the stress-energy tensor at the

Cauchy horizon of the Kerr-de Sitter black hole. Two especially interesting components are the energy flux T_{vv} which is expected to give the leading divergence of the stress-energy tensor upon transformation to Kruskal-type coordinates, and $T_{v\varphi}$ which is expected to be connected to the slow-down or speed-up of the rotation of the black hole.

The numerical computation of these components of the stress-energy tensor in the Unruh state is an ongoing project as part of the master thesis of M. Soltani. It would be interesting to know whether the leading divergence of the stress-energy tensor computed in the Unruh state is again the state-independent leading divergence for any state that is Hadamard on the whole Kerr-de Sitter spacetime $\mathcal{M}$. Since there is no classical, linear violation of strong cosmic censorship in Kerr-de Sitter [38], this requires a version of the state-independence result in [16] which holds also for arbitrarily small but positive spectral gaps. This is part of ongoing research.

To summarize, this thesis has discussed the effects of various free quantum fields on the inner horizon of different black holes. One important observation is that the behaviour of the quantum fields near the inner horizon cannot be described by a simple particle-picture estimate, but must be computed using quantum field theory. Another central result is that in many relevant cases the leading divergence of the energy flux and charge current at the inner horizon does not depend on the Hadamard state describing the quantum field up to the inner horizon. In other words, these leading terms are universal in the sense that they only depend on the parameters of the quantum field and the spacetime.

This dissertation has demonstrated that quantum effects can play a large role in relevant questions regarding the geometry of black-hole interiors like the validity of the strong cosmic censorship conjecture. The study of black-hole spacetimes, both classically and semi-classically, remains an important area of research. The results presented in this thesis are merely a glimpse at the exciting interaction between quantum fields and black holes, underlining the significance of a better understanding of the interplay between general relativity and quantum field theory in overcoming the conceptual puzzles of these theories.

This understanding is not only imperative for the accurate interpretation of observational data, but also continues the development of mathematically rigorous approaches to quantum field theory and serves as a guiding post for the development of a theory of quantum gravity. It will be interesting to observe how further work in this direction will shape our apprehension of how the physics at the smallest scales can impact the physics at the largest scales.

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