

Steady asymptotic equilibria in conformal relativistic fluids

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When one considers a shock wave in the frame where the shock is at rest, on either side one has a steady flow which converges to equilibrium away from the shock. However, hydrodynamics is unable to describe this flow if the asymptotic velocity is higher than the characteristic speed of the theory. We obtain an exact solution for the decay rate to equilibrium for a conformal fluid in kinetic theory under the relaxation time approximation, and compare it to two hydrodynamic schemes, one accounting for the second moments of the distribution function and thus equivalent, in the small deviations from equilibrium limit, to an Israel-Stewart framework, and another accounting for both second and third moments. While still having a finite characteristic speed, the second model is a significant improvement on the first.

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I. INTRODUCTION

Shock waves are one of the most interesting phenomena in relativistic hydrodynamics [1] and as such have spawned a significant literature [2–18]. In this work we want to focus on one aspect of the problem, which has been long considered critical for the development of viscous relativistic hydrodynamics [19].

When one has a stationary shock wave, on both sides of the shock there is a steady flow which converges to equilibrium as we move away from the shock; the flow is supersonic on one side (which we shall define to be the left side) and subsonic on the other. This would seem to be a very simple configuration, nevertheless relativistic hydrodynamics is incapable to describe it unless the asymptotic velocity, on the supersonic side, is below some threshold. The reason is that viscous relativistic hydrodynamics is built to transmit signals at a definite characteristic speed strictly less than that of light [20,21]; we show this explicitly in Appendix E. We will comment further on why the characteristic velocity sets a limit for the existence of smooth solutions in Sec. V. Similar problems arise already when studying shocks in nonrelativistic dilute gases [22–26], see [27].

As a way out of this situation, we shall endorse the view that viscous relativistic hydrodynamics must be regarded not as a single theory but rather as a hierarchy of theories of increasing complexity. The more complex theories allow for faster signal propagation than the simpler ones, and so, although every single theory has a finite threshold, any shock wave in nature may be described by a sophisticated enough theory [28,29].

In models whose fundamental description is kinetic theory, a particular way of building this theoretical hierarchy is by parametrizing the one particle distribution function in such a way that the parametrized distribution function reproduces the evolution of N moments of the actual distribution function [30–33]. In this class of models it may be proved that the fastest speed of propagation increases with N and tends to the speed of light as $N \rightarrow \infty$ [34].

In this work we shall demonstrate a particular realization of this scenario. We shall consider a conformal fluid [35] and we shall assume that its first principles description is given by kinetic theory under the relaxation time or Anderson-Witting approximation [36–40]. We shall derive an exact expression for the decay constant of the solution away to equilibrium, and compare it with two hydrodynamic models of the divergence type theory (DTT) class [41–48]. The first is built to match the second moments of the distribution function, and the second is an improved version that also matches the third moments.

It should be noted that both these theories have some interest on their own. The first one has been used to analyze flows on Bjorken and Gubser backgrounds [49] and also the interaction between viscous fluids and gravitational waves in the early Universe [50,51]. It has been extended to include thermal [52] and turbulent [53] fluctuations. It has also been extended to charged plasmas to study the amplification of magnetic fields in the early Universe [54]. By adding also the third moments, one obtains a theory which reproduces the propagators of the energy momentum tensor as derived from kinetic theory [40]; it also recovers the dynamics of the spin 2 degrees of freedom in the fluid as wave like, and not simply relaxational.

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The paper is organized as follows. In next section we fix our notation by considering shock waves in ideal [1], Landau-Lifshitz [12,55], and Israel-Stewart [12,56–60] fluids. The Landau-Lifshitz framework does yield a finite decay rate for any asymptotic speed, but it seems to be an artifact beyond the limit of weak shocks. When one regards hydrodynamics as rooted in kinetic theory, the Chapman-Enskog framework leads to the Landau-Lifshitz theory, and the Grad approximation to the Israel-Stewart one [56]. We shall show this connection in Appendix A. This means that the Israel-Stewart decay rate (with its limitations) will obtain in any theory that reduces to Grad's in the small deviations from equilibrium limit, such as anisotropic hydrodynamics [61–63] or our first DTT.

In Sec. III we analyze the same problem within kinetic theory with an Anderson-Witting collision term. We show that there is a finite decay rate for any value of the asymptotic fluid velocity in the shock frame. That settles the issue that the problem of theory breakdown for strong shocks lies entirely within hydrodynamics. The dependence of the decay rate on the asymptotic velocity resembles that derived from holography [12] but the divergence of the decay rate as the asymptotic velocity approaches light speed is stronger.

In Sec. IV we consider the decay rate in our DTT. Since we already know the first DTT will revert to Israel-Stewart, the emphasis is on the second one, including third moments. This theory still has a highest propagation speed strictly less than light, and therefore also breaks down for a finite asymptotic velocity, but nevertheless it is a significant improvement on the Israel-Stewart result, both on the left and right sides of the shock.

We summarize our results and conclusions in the final section.

We have left some further details for the Appendixes. Appendix A shows the connection of the approaches in Sec. II to kinetic theory. The following two appendixes have purely technical details. Appendix D shows that consideration of the entropy current [64–66] makes the dynamics of viscous relativistic fluids essentially unique. In Appendix E we compute the speed of signal propagation in both DTTs, thus allowing to check directly that it is the speed of propagation that defines the maximum asymptotic velocity the theory can handle [20], and finally in Appendix F we shall discuss the straightforward modifications of our argument to compute the decay rates in the subsonic side of the shock.

II. COMMON APPROACHES TO RELATIVISTIC FLUIDS

A. Shocks in ideal fluids

An ideal fluid may be at equilibrium at each side (L , R) of the shock, with a discontinuity in temperature and velocity across the shock. We assume the shock lies at

the $z = 0$ plane and is isotropic and translation invariant in this plane, and that all quantities depend only on the distance to the shock z . The discontinuity is restricted by energy momentum tensor (EMT) conservation $T^{\mu z}_{;\mu} = 0$, so we must have

$$\begin{aligned} T_L^{0z} &= T_R^{0z} \\ T_L^{zz} &= T_R^{zz} \\ T_L^{az} &= T_R^{az} \end{aligned} \quad (1)$$

$a = x, y$, where (L) refers to the half space $z < 0$ and R to $z > 0$. The fluid is characterized by its temperature T and its four velocity u^μ with $u^2 = -1$, which may be further parametrized:

$$\begin{aligned} u^0 &= \frac{1}{\sqrt{1-v^2}} \\ u^z &= \frac{v}{\sqrt{1-v^2}} \\ u^a &= 0. \end{aligned} \quad (2)$$

The energy-momentum tensor has the ideal form for a conformal fluid (for simplicity we assume Maxwell-Jüttner statistics)

$$T^{\mu\nu}_{id} = \frac{1}{\pi^2} T^4 [4u^\mu u^\nu + \eta^{\mu\nu}], \quad (3)$$

where $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. We then get

$$\begin{aligned} K &= \frac{4}{\pi^2} T_L^4 \frac{v_L}{1-v_L^2} = \frac{4}{\pi^2} T_R^4 \frac{v_R}{1-v_R^2} \\ K' &= \frac{1}{\pi^2} T_L^4 \frac{1+3v_L^2}{1-v_L^2} = \frac{1}{\pi^2} T_R^4 \frac{1+3v_R^2}{1-v_R^2}. \end{aligned} \quad (4)$$

K is a constant which expresses the common value of T^{0z} on both sides of the shock, similarly K' represents the common value of T^{zz} . In the more complex theories to be considered below, temperature and velocity will no longer be constant on either side, but as long as energy momentum is conserved, T^{0z} and T^{zz} will be constant, and K and K' will still represent them, respectively. Their actual value is defined by the asymptotic temperature and velocity, which we call T_L and v_L in all the models we shall consider.

Eliminating $T_{L,R}$ from Eq. (4)

$$3v_L^2 - 4Cv_L + 1 = 0, \quad (5)$$

where $C = K'/K$, so

$$v_{L,R} = \frac{2}{3} \left\{ C \pm \sqrt{C^2 - \frac{3}{4}} \right\}. \quad (6)$$

There is a nontrivial shock when both roots are real and ≤ 1 . In the allowed range we have

$$v_L v_R = \frac{1}{3}. \quad (7)$$

We shall call v_L the root such that $1/\sqrt{3} \leq v_L \leq 1$, and then $1/\sqrt{3} \geq v_R \geq 1/3$. Then

$$\left(\frac{T_R}{T_L}\right)^4 = \frac{v_L(1-v_R^2)}{v_R(1-v_L^2)} = \frac{(3v_L^2 - \frac{1}{3})}{(1-v_L^2)}. \quad (8)$$

Observe that $T_R \geq T_L$ and so the entropy density behind the shock is greater than in front of it, in agreement with the second law.

B. Shocks in Landau-Lifshitz theory

A viscous fluid cannot sustain a discontinuity, but for a solution which depends only on z , integrating EMT conservation from $z = -\infty$ to $z = \infty$, we see that the relations (1) hold for the asymptotic values. In particular, we may assume that $u^{x,y} \rightarrow 0$ asymptotically. We shall make the stronger assumption that the solution is axially symmetric around the z direction everywhere. Thus we are seeking a solution depending only on the z coordinate and axially symmetric which asymptotically reduces to an ideal fluid when $z \rightarrow \pm\infty$, with boundary conditions being the junction conditions (1) for an ideal fluid, namely conditions (4).

It is interesting to see the shock structure in Landau-Lifshitz theory, where

$$T^{\mu\nu} = T_{id}^{\mu\nu} - \frac{1}{\pi^2} \eta_0 T^3 \sigma^{\mu\nu}, \quad (9)$$

where η_0 is a dimensionless constant, essentially the viscosity to entropy ratio, and

$$\begin{aligned} \sigma^{\mu\nu} &= \Delta^{\mu\rho} \Delta^{\nu\sigma} \left[u_{\rho,\sigma} + u_{\sigma,\rho} - \frac{2}{3} \Delta_{\rho\sigma} u_{,\lambda}^{\lambda} \right] \\ &= \Delta^{\nu\sigma} u_{,\sigma}^{\mu} + \Delta^{\mu\rho} u_{,\rho}^{\nu} - \frac{2}{3} \Delta^{\mu\nu} u_{,\lambda}^{\lambda}, \end{aligned} \quad (10)$$

$\Delta^{\mu\rho} = \eta^{\mu\rho} + u^{\mu} u^{\rho}$. Since by definition $\sigma_{\mu\nu} u^{\mu} = 0$, we must have

$$\begin{aligned} \sigma^{z0} &= v \sigma^{zz} \\ \sigma^{00} &= v^2 \sigma^{zz} \end{aligned} \quad (11)$$

and

$$\sigma^{zz} = \frac{4}{3} \frac{v_{,z}}{(1-v^2)^{5/2}}. \quad (12)$$

Now the constancy of T^{0z} and T^{zz} yields two equations:

$$\begin{aligned} \frac{1}{\pi^2} T^4 \frac{4v}{1-v^2} - \frac{1}{\pi^2} \eta_0 T^3 \frac{4}{3} \frac{v v_{,z}}{(1-v^2)^{5/2}} &= K \\ \frac{1}{\pi^2} T^4 \left[\frac{1+3v^2}{1-v^2} \right] - \frac{1}{\pi^2} \eta_0 T^3 \frac{4}{3} \frac{v_{,z}}{(1-v^2)^{5/2}} &= K' \end{aligned} \quad (13)$$

with $K, K' = \text{constant}$. With the boundary conditions that $v \rightarrow v_L$ and $T \rightarrow T_L$ as $z \rightarrow -\infty$ they are the same constants as in Eq. (4); then $v \rightarrow v_R$ and $T \rightarrow T_R$ as $z \rightarrow \infty$. We may write

$$\frac{4}{3} \eta_0 \frac{v_{,z}}{(1-v^2)^{3/2}} = -3T \frac{(v_L - v)(v - \frac{1}{3v_L})}{[1 - \frac{v}{4v_L}(3v_L^2 + 1)]}. \quad (14)$$

We see that v_L and $v_R = 1/(3v_L)$ are the only values of v where $v_{,z}$ may vanish. v goes monotonically from one to the other, reaching the limiting values only asymptotically. To find the speed of approach to the asymptotic value, we write $v = v_L - \vartheta$, with $\vartheta \propto e^{\lambda_{LL} z}$. Then to first order in ϑ we get [12]

$$\lambda_{LL} = 3T_L \frac{(1-v_L^2)^{1/2}}{\eta_0 v_L} \left(v_L^2 - \frac{1}{3} \right). \quad (15)$$

After solving the equation for v , we may find the temperature from

$$\frac{4vT^4}{1-v^2} \left[1 + \frac{3(v_L - v)(v - \frac{1}{3v_L})}{4(1 - \frac{v}{4v_L}(3v_L^2 + 1))} \right] = \frac{4v_L T_L^4}{1-v_L^2}. \quad (16)$$

C. Israel-Stewart fluids

In the Israel-Stewart or extended thermodynamics approach, the viscous part of $T^{\mu\nu}$ is left undetermined:

$$T^{\mu\nu} = T_{id}^{\mu\nu} + \Pi^{\mu\nu}. \quad (17)$$

Since $\Pi^{\mu\nu} u_{\nu} = 0$, we have $\Pi^{0z} = v \Pi^{zz}$. Thus calling $\Pi^{zz} = \Pi$, we find

$$\begin{aligned} K &= T^{0z} = \frac{4}{\pi^2} \frac{v_L T_L^4}{1-v_L^2} = \frac{4}{\pi^2} \frac{v T^4}{1-v^2} + v \Pi \\ K' &= T^{zz} = \frac{3}{\pi^2} \frac{T_L^4}{1-v_L^2} \left[v_L^2 + \frac{1}{3} \right] = \frac{3}{\pi^2} \frac{T^4}{1-v^2} \left[v^2 + \frac{1}{3} \right] + \Pi. \end{aligned} \quad (18)$$

Then

$$\frac{\pi^2}{3}(1-v^2)T^{-4}\Pi = \frac{(v_L-v)(v-\frac{1}{3v_L})}{[1-\frac{3v}{4v_L}(v_L^2+\frac{1}{3})]}. \quad (19)$$

The system is closed by asking that $\Pi^{\mu\nu}$ relaxes to its Landau-Lifshitz form on a timescale $\tau = \tau_0/T$

$$\frac{\tau_0}{T}u^\rho\Pi_{,\rho}^{\mu\nu} + \Pi^{\mu\nu} = -\frac{1}{\pi^2}\eta_0T^3\sigma^{\mu\nu}. \quad (20)$$

Taking the zz component we get

$$\begin{aligned} & \frac{3T^4}{\pi^2} \frac{(v_L-v)(v-\frac{1}{3v_L})}{(1-v^2)[1-\frac{3v}{4v_L}(v_L^2+\frac{1}{3})]} \\ & + \frac{\tau_0 v}{T\sqrt{1-v^2}} \frac{d}{dz} \frac{3T^4}{\pi^2} \frac{(v_L-v)(v-\frac{1}{3v_L})}{(1-v^2)[1-\frac{3v}{4v_L}(v_L^2+\frac{1}{3})]} \\ & = -\frac{4}{3\pi^2} \frac{\eta_0 T^3 v_{,z}}{(1-v^2)^{5/2}}. \end{aligned} \quad (21)$$

When $z \rightarrow -\infty$ we write $v = v_L - \vartheta e^{\lambda_{IS}z}$ and linearize on ϑ :

$$\lambda_{IS} = 3T_L \frac{(v_L^2 - \frac{1}{3})(1 - v_L^2)^{1/2}}{v_L[\eta_0 - 3\tau_0(v_L^2 - \frac{1}{3})]}. \quad (22)$$

A suitable model must satisfy

$$v_L^2 - \frac{1}{3} < \frac{1}{3} \frac{\eta_0}{\tau_0}. \quad (23)$$

Causality requires the right-hand side to be strictly less than $2/3$ (see Appendixes A and E), and so this sets an upper bound for v_L which is strictly less than 1, otherwise there is no solution smoothly approaching equilibrium. For example, AdS-CFT yields a value $\eta_0 = 1$, $\tau_0 = 1 - (\ln 2/2)$, and the criterion Eq. (23) becomes $v_L^2 \leq 0.84$ [12]. Both the Chapman-Enskog and Grad approaches yield $\eta_0 = (4/5)\tau_0$, and so the theory breaks down when $v_L^2 \geq 3/5$.

III. KINETIC THEORY

In this section we will show that, in kinetic theory under the relaxation time approximation, there are solutions smoothly approaching equilibrium regardless of the asymptotic velocity v_L .

Under the relaxation time or Anderson-Witting approximation, the kinetic equation reads

$$p^\mu f_{,;\mu} = \frac{T}{\tau_0} (u_{\text{eq}\mu} p^\mu) [f - f_{\text{eq}}], \quad (24)$$

where f_{eq} is the Maxwell-Jüttner distribution with parameters T_{eq} , u_{eq}^μ defined by the consistency condition

$$T^{\mu\nu} u_{\text{eq}\nu} = -\frac{3}{\pi^2} T_{\text{eq}}^4 u_{\text{eq}}^\mu. \quad (25)$$

For simplicity, we shall call $T_{\text{eq}} = T$ and $v_{\text{eq}} = v$. Given the symmetries of the shock solution, the Boltzmann equation (24) reduces to

$$f_{,z} + \Lambda(z)f = \Lambda(z)f_{\text{eq}}, \quad (26)$$

or else

$$[f - f_{\text{eq}}]_{,z} + \Lambda[f - f_{\text{eq}}] = \phi_{,z} f_{\text{eq}}, \quad (27)$$

where

$$\begin{aligned} \Lambda(z) &= \frac{T[-u_{\text{eq}\mu} p^\mu]}{(p^z \tau_0)} \\ f_{\text{eq}} &= e^{-\phi(z)} \\ \phi(z) &= \frac{[-u_{\text{eq}\mu} p^\mu]}{T}. \end{aligned} \quad (28)$$

We are seeking the solution which reaches asymptotically equilibrium values as $z \rightarrow -\infty$. This is

$$f(z) = f_{\text{eq}}(z) + \int_{-\infty}^z dz' e^{-\int_{z'}^z dz'' \Lambda(z'')} [\phi_{,z} f_{\text{eq}}](z'). \quad (29)$$

When we use this to compute the EMT, we find

$$T^{\mu\nu} = T_{\text{eq}}^{\mu\nu} + t^{\mu\nu}. \quad (30)$$

The consistency condition (25) becomes

$$t^{\mu\nu} u_{\text{eq}\nu} = 0. \quad (31)$$

This condition implies energy momentum tensor conservation, so we also get that T^{0z} and T^{zz} remain constant.

Let us analyze $t^{\mu\nu}$ more closely:

$$t^{\mu\nu} = \int_{-\infty}^z dz' \int Dp p^\mu p^\nu e^{-\int_{z'}^z dz'' \Lambda(z'')} [\phi_{,z} f_{\text{eq}}](z'). \quad (32)$$

$Dp = 2\delta(-p^2)\theta(p^0)d^4p/(2\pi)^3$ is the covariant momentum space volume element. For each fixed z' introduce new variables

$$\begin{aligned} q^0 &= \frac{p^0 - v(z')p^z}{\sqrt{1 - v(z')^2}} \\ q^z &= \frac{p^z - v(z')p^0}{\sqrt{1 - v(z')^2}} \\ q^{x,y} &= p^{x,y} \end{aligned} \quad (33)$$

($q_x^2 = q_y^2 = q_0^2 - q_z^2 = p_0^2 - p_z^2$), then we may write

$$\phi[p^\mu, z'] = \frac{q^0}{T}. \quad (34) \quad \text{Now}$$

Observe that we regard the transformation (33) as a change of variables within the momentum integral at a given point in space, rather than as a global coordinate change.

The z derivative in Eq. (29) is taken at p held constant, so upon the change of variables we must write

$$\left. \frac{\partial \phi}{\partial z} \right|_p = \left. \frac{\partial \phi}{\partial z} \right|_q + \frac{\partial \phi}{\partial q^\mu} \left. \frac{\partial q^\mu}{\partial z} \right|_p. \quad (35)$$

Actually the q^μ derivatives are nonzero for $\mu = 0$ and z only, so

$$\left. \frac{\partial \phi}{\partial z} \right|_p = \left. \frac{\partial \phi}{\partial z} \right|_q + \frac{\partial \phi}{\partial q^0} \left. \frac{\partial q^0}{\partial z} \right|_p + \frac{\partial \phi}{\partial q^z} \left. \frac{\partial q^z}{\partial z} \right|_p. \quad (36)$$

$$\begin{pmatrix} q^0 \\ q^z \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^z \end{pmatrix} \quad (37)$$

$$\frac{\partial}{\partial z} \begin{pmatrix} q^0 \\ q^z \end{pmatrix} = \frac{v'}{(1-v^2)} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q^0 \\ q^z \end{pmatrix} \quad (38)$$

so

$$\left. \frac{\partial \phi}{\partial z} \right|_p = \left. \frac{\partial \phi}{\partial z} \right|_q - \frac{v_{,z}}{(1-v^2)} \left[q^z \frac{\partial \phi}{\partial q^0} + q^0 \frac{\partial \phi}{\partial q^z} \right]. \quad (39)$$

Now the conditions (31) become

$$\begin{aligned} 0 &= \int_{-\infty}^z \frac{dz'}{T(z')} \int Dq(q^0 + v(z')q^z)q^0 e^{-\int_{z'}^z dz'' \Lambda(z', z'')} \left[\frac{q^0 T_{,z'}(z')}{T(z')} + \frac{q^z v_{,z'}(z')}{(1-v^2(z'))} \right] e^{-q^0/T(z')} \\ 0 &= \int_{-\infty}^z \frac{dz'}{T(z')} \int Dq(q^z + v(z')q^0)q^0 e^{-\int_{z'}^z dz'' \Lambda(z', z'')} \left[\frac{q^0 T_{,z'}(z')}{T(z')} + \frac{q^z v_{,z'}(z')}{(1-v^2(z'))} \right] e^{-q^0/T(z')}, \end{aligned} \quad (40)$$

where

$$\Lambda(z', z'') = \frac{T(z'')[(1-v(z')v(z''))q^0 + (v(z')-v(z''))q^z]}{(q^z + v(z')q^0)\tau_0 \sqrt{1-v^2(z'')}}. \quad (41)$$

We define dimensionless momenta $q^\mu = T(z')r^\mu$ and go to polar coordinates to get

$$\begin{aligned} 0 &= \frac{6}{\pi^2} \int_{-\infty}^z dz' T^4(z') \int_{-1}^1 dx (1+v(z')x) e^{-\int_{z'}^z dz'' \Lambda(x; z', z'')} \left[\frac{T_{,z'}(z')}{T(z')} + \frac{xv_{,z'}(z')}{(1-v^2(z'))} \right] \\ 0 &= \frac{6}{\pi^2} \int_{-\infty}^z dz' T^4(z') \int_{-1}^1 dx (x+v(z')) e^{-\int_{z'}^z dz'' \Lambda(x; z', z'')} \left[\frac{T_{,z'}(z')}{T(z')} + \frac{xv_{,z'}(z')}{(1-v^2(z'))} \right], \end{aligned} \quad (42)$$

where now

$$\Lambda(x; z', z'') = \frac{T(z'')[1-v(z')v(z'') + (v(z')-v(z''))x]}{(x+v(z'))\tau_0 \sqrt{1-v^2(z'')}}. \quad (43)$$

When $z \rightarrow -\infty$ we expect a solution where $T = T_L(1 + te^{\lambda_{\text{AW}}z})$, $v = v_L - \vartheta e^{\lambda_{\text{AW}}z}$. Linearizing we get

$$\begin{aligned} At - B \frac{\vartheta}{(1-v_L^2)} &= 0 \\ Ct - D \frac{\vartheta}{(1-v_L^2)} &= 0, \end{aligned} \quad (44)$$

where

$$\kappa = v_L + \frac{T_L}{\lambda_{\text{AW}}\tau_0} \sqrt{1-v_L^2} \quad (45)$$

$$\begin{aligned} A &= \int_{-1}^1 dx \frac{(1+v_Lx)(x+v_L)}{x+\kappa} \\ B &= \int_{-1}^1 dx x \frac{(1+v_Lx)(x+v_L)}{x+\kappa} \\ C &= \int_{-1}^1 dx \frac{(x+v_L)^2}{x+\kappa} \\ D &= \int_{-1}^1 dx x \frac{(x+v_L)^2}{x+\kappa}. \end{aligned} \quad (46)$$

The dispersion relation $AD - BC = 0$ reduces to (see Appendix C)

$$0 = v_L^2 - \frac{1}{3} - v_L G[\kappa], \quad (47)$$

where

$$G[\kappa] = \kappa - \frac{1}{3} \frac{J_0[\kappa]}{\kappa J_0[\kappa] - 2} \quad (48)$$

$$J_0[k] = \ln \frac{k+1}{k-1}. \quad (49)$$

Equations (45) and (47) define parametrically λ_{AW} as a function of v_L .

Let us analyze the limiting cases. When $k \rightarrow \infty$, $G(k) \approx 4/(15k)$, so $v^2 \rightarrow 1/3$,

$$\kappa \approx \frac{4}{15} \frac{v_L}{v_L^2 - \frac{1}{3}} \quad (50)$$

$$\lambda_{\text{AW}} \approx \frac{T_L}{\tau_0} \frac{\sqrt{1-v_L^2}}{v_L(\frac{3}{5}-v_L^2)} \left(v_L^2 - \frac{1}{3}\right) \approx \frac{15}{4} \sqrt{2} \frac{T_L}{\tau_0} \left(v_L^2 - \frac{1}{3}\right). \quad (51)$$

When $k \rightarrow 1$, $J_0(k) \approx -\ln(k-1) + \ln 2 \rightarrow \infty$, $v_L \rightarrow 1$,

$$G(\kappa) \approx \frac{2}{3} \left(1 - \frac{1}{\ln(k-1)}\right) \quad (52)$$

$$\kappa \approx 1 + e^{-1/(2(1-v_L))} \quad (53)$$

$$\lambda_{\text{AW}} \approx \frac{T_L}{\tau_0} \sqrt{\frac{2}{1-v_L}}. \quad (54)$$

We can check that these analytical asymptotic forms match very well the exact solution in Eqs. (45) and (47) in their respective regimes, see Fig. 1.

We conclude that kinetic theory may describe the approach to equilibrium regardless of the limiting velocity, as long as $v_L < 1$. The difference in behavior between kinetic theory and hydrodynamics may be traced back to the fact that, in kinetic theory, the speed of signal propagation is the maximum speed of the particles for which the distribution function is not zero [19]. For the near equilibrium distribution functions we are considering, which cover the full range, the asymptotic velocity v_L shall always be below the speed of signal propagation.

IV. CAUSAL FLUIDS

We reduce kinetic theory to hydrodynamics by making the ansatz

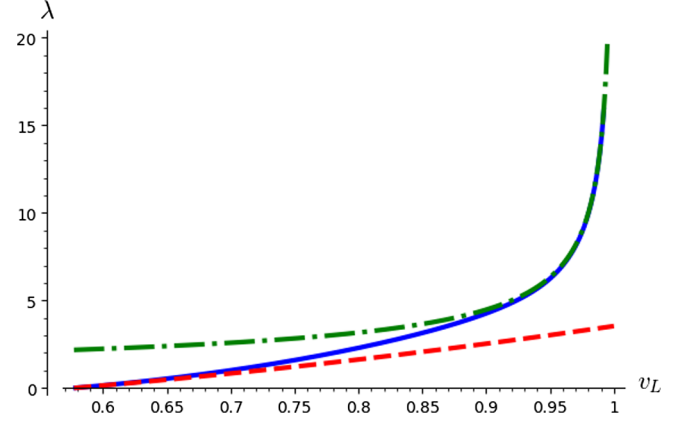


FIG. 1. Exact decay rate derived from kinetic theory with an Anderson-Witting collision term and its asymptotic forms: (blue, full line) exact decay rate, defined parametrically by Eqs. (45) and (47); (red, dashes) asymptotic behavior for $v_L \rightarrow 1/\sqrt{3}$, Eq. (51); (green, dots and dashes) asymptotic behavior for $v_L \rightarrow 1$, Eq. (54). The divergence in the decay rate is stronger than predicted by holography [12].

$$f = \exp \left\{ \frac{1}{T} (u_\mu p^\mu) + \frac{\zeta_{\mu\nu} p^\mu p^\nu}{(-u_\mu p^\mu)} + \frac{\xi_{\mu\nu\rho} p^\mu p^\nu p^\rho}{(-u_\mu p^\mu)^2} \right\} \quad (55)$$

for the distribution function. The tensors $\zeta_{\mu\nu}$ and $\xi_{\mu\nu\rho}$ are totally symmetric, transverse to u^μ , and traceless on any pair of indexes. The equations for the coefficients are derived by taking moments of the kinetic equation, for which we assume the Anderson-Witting form (24); see Appendix D.

Under our symmetry assumptions u^μ is characterized by the single velocity v in the z direction. Likewise, $\zeta_{\mu\nu}$ and $\xi_{\mu\nu\rho}$ contribute a single degree of freedom each. To see this, observe that in the local rest frame of the fluid, all components with a 0 index must vanish, while symmetry implies that components with an odd number of x, y components also vanish, $\zeta_{xx} = \zeta_{yy}$ and $\xi_{zxx} = \xi_{zyy}$. Now tracelessness implies that $\zeta_{xx} = (-1/2)\zeta_{zz}$ and $\xi_{zxx} = (-1/2)\xi_{zzz}$. Henceforth we shall call ζ and ξ the single nontrivial component of these tensors in the local rest frame. Introducing the momenta in the local rest frame as in Eq. (33) we may write

$$f = \exp \left\{ -\frac{1}{T} q^0 + \zeta H_\zeta + \xi H_\xi \right\}, \quad (56)$$

where

$$H_\zeta = \frac{(3q_z^2 - q_0^2)}{2q^0} \\ H_\xi = \frac{q^z(5q_z^2 - 3q_0^2)}{2q_0^2}. \quad (57)$$

We see that if $\xi \neq 0$, f is not even in q^z , and for this reason $v_{\text{eq}} \neq 0$ in the local rest frame either. For example, let us consider again the definition of T_{eq} and u_{eq} . After performing the change of variables (33), which of course has unit Jacobian, we get, in the local rest frame of the fluid

$$\begin{aligned} \int Dq q^0 (u_{\text{eq}}^0 q^0 - u_{\text{eq}}^z q^z) f &= \frac{3}{\pi^2} T_{\text{eq}}^4 u_{\text{eq}}^0 \\ \int Dq q^z (u_{\text{eq}}^0 q^0 - u_{\text{eq}}^z q^z) f &= \frac{3}{\pi^2} T_{\text{eq}}^4 u_{\text{eq}}^z, \end{aligned} \quad (58)$$

or in terms of the velocity v_{eq}

$$\begin{aligned} \int Dq q^0 (q^0 - v_{\text{eq}} q^z) f &= \frac{3}{\pi^2} T_{\text{eq}}^4 \\ \int Dq q^z (q^0 - v_{\text{eq}} q^z) f &= \frac{3}{\pi^2} T_{\text{eq}}^4 v_{\text{eq}}. \end{aligned} \quad (59)$$

It is easy to see that $v_{\text{eq}} = 0$ and $T_{\text{eq}} = T$ to first order in ζ and ξ , but not to higher order. This is related to the possibility of building vector fields out of $\zeta_{\mu\nu}$ and $\xi_{\mu\nu\rho}$, such as $\xi_{\mu\nu\rho} \zeta^{\nu\rho}$ or $\xi_{\mu\nu\rho} \xi^{\lambda\sigma} \xi_{\lambda\sigma}^{\rho}$.

A. Energy momentum tensor

Insofar as the energy momentum conservation conditions are still exact equations of the theory, and the energy momentum tensor in the rest frame of the shock depends only on z , we still have the identities

$$\begin{aligned} K &= T^{0z} = \int Dp p^0 p^z f = \text{constant} \\ K' &= T^{zz} = \int Dp (p^z)^2 f = \text{constant} \end{aligned} \quad (60)$$

with K and K' depending only on the asymptotic state as for an ideal fluid, see Eq. (4). Performing the change of variables (33) and linearizing on ζ and ξ we get

$$\begin{aligned} T^{0z} &= \frac{1}{1-v^2} \int Dq (q^0 + v q^z) (q^z + v q^0) f \\ &= \frac{4T^4}{\pi^2} \frac{v}{(1-v^2)} \left\{ 1 + \frac{2}{5} \zeta T \right\} \\ T^{zz} &= \frac{1}{1-v^2} \int Dq (q^z + v q^0)^2 f \\ &= \frac{3T^4}{\pi^2} \frac{1}{(1-v^2)} \left\{ v^2 + \frac{1}{3} + \frac{8}{15} \zeta T \right\}. \end{aligned} \quad (61)$$

This is equivalent to the Israel-Stewart energy momentum tensor identifying

$$\Pi = \frac{8}{5\pi^2} \frac{T^4}{(1-v^2)} \zeta T. \quad (62)$$

We thus obtain two relations among T , v and the dimensionless combination ζT . Eliminating T we get

$$\zeta T = \frac{15(v_L - v)(v - \frac{1}{3v_L})}{8[1 - \frac{3}{4} \frac{v}{v_L}(v_L^2 + \frac{1}{3})]} \quad (63)$$

which is equivalent to Eq. (19), and further writing $v = v_L - \vartheta e^{\lambda_{\text{DIT}} z}$ and linearizing on ϑ

$$\zeta T = \frac{5}{2} \frac{(v_L^2 - \frac{1}{3})}{v_L(1 - v_L^2)} \vartheta e^{\lambda_{\text{DIT}} z}. \quad (64)$$

B. Equations of motion

The equations of motion will have the form

$$\int Dp H_\alpha(z, p) \left[p^z \frac{\partial f}{\partial z} \Big|_p - I_{\text{col}}[z, p] \right] = 0 \quad (65)$$

for suitable functions H_α [30–33].

When we perform the transformation (33) we must take into account that z is not transformed (this is a change of variables, not a change of coordinates). So even if the function H_α is a particular component of a tensor, we do not transform it as such, but only as a given function of z and p . The same argument may be used to transform the collision integral, so

$$\int Dq H_\alpha(z, q) \left[\frac{q^z + v q^0}{\sqrt{1-v^2}} \frac{\partial f}{\partial z} \Big|_p - I_{\text{col}}[z, q] \right] = 0. \quad (66)$$

The f derivative is transformed as in Eq. (39). It is convenient to move the derivatives out of the integral, observing that

$$\frac{\partial}{\partial z} \delta(p^2) \Big|_p = \frac{\partial}{\partial z} p^z \Big|_p = 0 \quad (67)$$

in either the $-p$ or $-q$ representation. Therefore we get

$$0 = \frac{d}{dz} A_\alpha - B_\alpha - I_\alpha, \quad (68)$$

where

$$\begin{aligned}
A_\alpha &= \int Dq H_\alpha(q) \frac{q^z + vq^0}{\sqrt{1-v^2}} f \\
B_\alpha &= \int Dq \left\{ \left[\frac{\partial}{\partial z} - \frac{v'}{(1-v^2)} \left[q^z \frac{\partial}{\partial q^0} + q^0 \frac{\partial}{\partial q^z} \right] \right] H_\alpha \right\} \\
&\quad \times \frac{q^z + vq^0}{\sqrt{1-v^2}} f \\
I_\alpha &= \int Dq H_\alpha(q) I_{\text{col}}[z, q].
\end{aligned} \tag{69}$$

If we choose $H_0 = p^0$ and $H_z = p^z$, $B_{0,z}$ and $I_{0,z}$ vanish and we obtain once again the constancy of the EMT. Entropy considerations, discussed further in Appendix D, suggest choosing the remaining functions as H_ζ and H_ξ in the equations of (57). Then, linearizing on ζ , ξ , and v'

$$\begin{aligned}
A_\zeta &= \frac{12}{5\pi^2} \frac{v_L \zeta}{\sqrt{1-v_L^2}} T_L^5 + \frac{36}{35\pi^2} \frac{\xi}{\sqrt{1-v_L^2}} T_L^5 \\
B_\zeta &= -\frac{8}{5\pi^2} \frac{v'}{(1-v_L^2)^{3/2}} T_L^4 \\
I_\zeta &= -\frac{12}{5\pi^2} \frac{\zeta}{\tau_0} T_L^6 \\
A_\xi &= \frac{36}{35\pi^2} \frac{\zeta T_L^5}{\sqrt{1-v_L^2}} + \frac{12}{7\pi^2} \frac{v_L \xi T_L^5}{\sqrt{1-v_L^2}} \\
B_\xi &= 0 \\
I_\xi &= -\frac{12}{7\pi^2} \frac{\xi T_L^6}{\tau_0}.
\end{aligned} \tag{70}$$

If we do not include the ξ term, we revert to the equations derived from Grad's ansatz, identifying $C^{\mu\nu} = \zeta^{\mu\nu}$.

Assuming that all variables depend on z as $e^{\lambda_{\text{DTT}} z}$, the set of equations (64) and (68) becomes

$$\begin{pmatrix} -\frac{2}{3}\lambda_{\text{DTT}} & \lambda_{\text{DTT}} v_L + \frac{T_L}{\tau_0} \sqrt{1-v_L^2} & \frac{3}{7}\lambda_{\text{DTT}} \\ v_L^2 - \frac{1}{3} & -\frac{2}{5}v_L & 0 \\ 0 & \frac{3}{5}\lambda_{\text{DTT}} & \lambda_{\text{DTT}} v_L + \frac{T_L}{\tau_0} \sqrt{1-v_L^2} \end{pmatrix} \begin{pmatrix} \frac{\theta}{1-v_L^2} \\ \zeta T \\ \xi T \end{pmatrix} = 0 \tag{71}$$

writing

$$\lambda_{\text{DTT}} = \frac{\alpha T_L}{v_L \tau_0} \sqrt{1-v_L^2}. \tag{72}$$

The allowed values of α are the roots of

$$\frac{\alpha^2}{v_L^2} \left[\frac{6}{7} v_L^2 - v_L^4 - \frac{3}{35} \right] + 2\alpha \left[\frac{7}{15} - v_L^2 \right] + \frac{1}{3} - v_L^2 = 0. \tag{73}$$

There will be a positive root as long as the coefficient of α^2 is positive, which ceases to be true when $v_L^2 \approx 0.74$; see Appendix E.

V. RESULTS AND FINAL REMARKS

In this paper we have computed the decay rate of the solution toward equilibrium at velocity v_L as $z \rightarrow -\infty$ for several models, namely ideal fluids (where there is no decay), Landau-Lifshitz [Eq. (15)], Israel-Stewart [Eq. (22)], which actually holds for any theory which reduces to the Grad ansatz in the linear regime, kinetic theory with a relaxation time or Anderson-Witting collision term [Eqs. (45) and (47)], and finally for a DTT including third moments of the distribution function [Eqs. (72) and (73)]. The results are summarized in Fig. 2.

The ‘‘exact’’ calculation yields a decay rate which resembles the one derived from AdS-CFT correspondence

[12] but with a stronger divergence in the upper limit; it diverges as $\gamma = (1-v_L^2)^{-1/2}$ while the result from holography diverges as $\gamma^{1/2}$ [12].

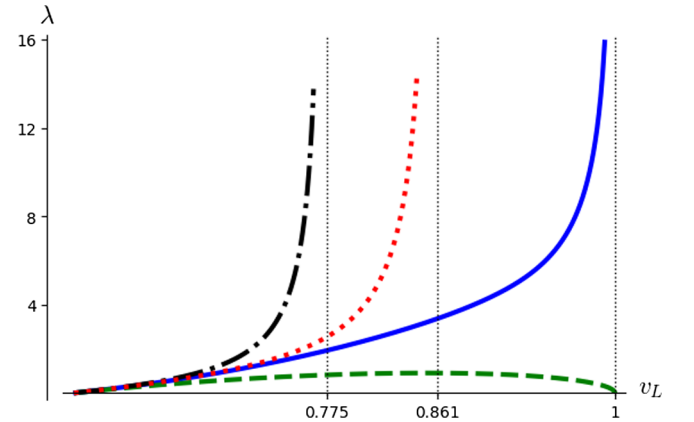


FIG. 2. The decay rates for the different theories discussed in this work: (blue, full line) exact decay rate, defined parametrically by Eqs. (45) and (47); (green, dashes) the decay rate from Landau-Lifshitz theory, Eq. (15); (black, dots and dashes) decay rate for an Israel-Stewart fluid with the constitutive relation derived from Grad's ansatz, $\eta_0 = 4\tau_0/5$, Eq. (22); (red, dots) the decay rate derived from the DTT including the third momentum of the distribution function, Eqs. (72) and (73). The vertical grid lines show the characteristic speeds of the Israel-Stewart and DTT models, see Appendix E.

Landau-Lifshitz provides a regular solution for any v_L , but the quantitative agreement to the exact result is not satisfactory beyond weak shocks.

Both the Israel-Stewart and DTT decay rates blow up at a finite value of v_L set up by the highest speed of signal propagation (as we show in Appendix E). This exercise therefore provides a concrete example of the scenario discussed in [20,29].

It is remarkable that if we extended the kinetic theory analysis to complex values of the asymptotic velocity v_L , then the decay rate λ would be an analytic function of v_L with a cut in the complex plane, signaled by the appearance of a logarithm in Eq. (49). These nonanalyticities are a generic feature of kinetic theories that very much define the limit of validity of hydrodynamics [39].

If the limiting factor for hydrodynamics is that it cannot handle fast asymptotic velocities, then there should be no problem in the right-hand side of the shock, where velocities are subsonic throughout. The decay rate can be computed with a straightforward adaptation of the arguments above (see Appendix F); we show the result in Fig. 3. As expected, there is no divergence in any of the models we are considering, but again the DTT outperforms the Landau-Lifshitz and Israel-Stewart schemes as a quantitative match to kinetic theory.

The fact that the speed of signal propagation sets the upper asymptotic velocity for which a regular solution exists may be easier to understand if we regard the time-independent configurations we have analyzed in this paper as the long time limit of the actual process by which the shock is formed.

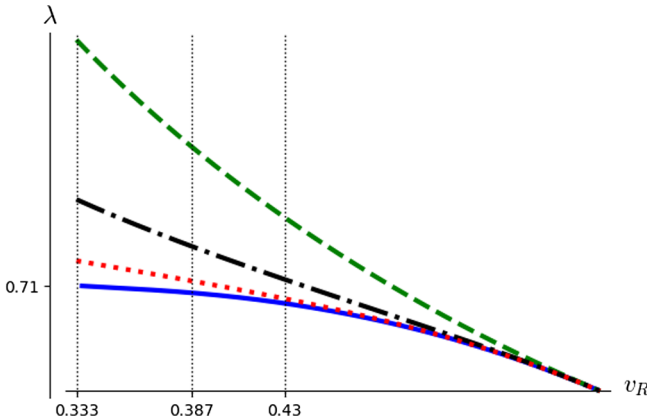


FIG. 3. The decay rates for the different theories discussed in this work on the right-hand side of the shock: (blue, full line) exact decay rate, defined parametrically by Eq. (F9); (green, dashes) the decay rate from Landau-Lifshitz theory, Eq. (F1); (black, dots and dashes) decay rate for an Israel-Stewart fluid with the constitutive relation derived from Grad's ansatz, $\eta_0 = 4\tau_0/5$, Eq. (F2); (red, dots) the decay rate derived from the DTT including the third momentum of the distribution function, Eqs. (F12) and (F16). The vertical grid lines show the asymptotic right side velocities corresponding to the characteristic speeds of the Israel-Stewart and DTT models, see Appendix E, which mark their applicability limit.

Remember that we are describing the fluid in the frame where the shock is at rest and the fluid advances from the left at velocity v_L . We may as well use the frame where the fluid is at rest and the shock advances to the left at velocity $-v_L$. Now picture the shock as a piston which materializes at $t = 0$ at the position $z = 0$, and then starts moving against fluid at rest. If $v_L < c$ the speed of signal propagation (see Appendix E), then the influence of the piston will outrun the piston itself. At time t , the fluid will remain at rest for all $z \leq -ct$, and there will be a buffer zone between $z = -ct$ and the piston at $z = -v_L t$. At long times and finite distances from the piston, in the frame where the piston is at rest, the fluid in the buffer zone will settle to a steady flow; this is the configuration we have described in this paper. However, if $v_L \geq c$ this is not possible; the piston keeps pushing against fluid at rest, and the hydrodynamic solution, if it exists at all, must be discontinuous [20].

In spite of its limitations, Figs. 2 and 3 show that including third moments in the DTT allows for a much more accurate description of the convergence to equilibrium. This should be considered along with the results of [40] in choosing the correct hydrodynamic framework for a concrete application.

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APPENDIX A: CHAPMAN-ENSKOG AND GRAD

The Chapman-Enskog and Grad approaches attempt to anchor hydrodynamics on kinetic theory.

Under the Chapman-Enskog approach, we seek a solution of the kinetic equation (24) of the form

$$f = e^{u_\mu p^\mu/T} [1 + \delta f]. \quad (\text{A1})$$

Then, see Appendix B,

$$\delta f = -\frac{\tau_0}{2T^2 |u_\rho p^\rho|} p^\mu p^\nu \sigma_{\mu\nu}, \quad (\text{A2})$$

where $\sigma_{\mu\nu}$ is defined in Eq. (10), leading to

$$\Pi^{\mu\nu} = -\frac{4}{5\pi^2} \tau_0 T^3 \sigma^{\mu\nu} \quad (\text{A3})$$

which is the Landau-Lifshitz ansatz under the identification

$$\eta_0 = \frac{4}{5} \tau_0. \quad (\text{A4})$$

If we use this value of η_0 in the equations from the Israel-Stewart approach, we find the theory becomes singular when $v_L^2 = 3/5$.

In the Grad approach, we write a decomposition (A1) but with a less constrained perturbation

$$\delta f = \frac{p^\mu p^\nu}{|u_\rho p^\rho|} C_{\mu\nu}, \quad (\text{A5})$$

where $u^\mu C_{\mu\nu} = C_\mu^\mu = 0$. This satisfies the constraints (B1) and leads to

$$\Pi^{\mu\nu} = \frac{8}{5\pi^2} T^5 C^{\mu\nu} \quad (\text{A6})$$

[compare to Eq. (A3)]. To determine $C^{\mu\nu}$ we ask that some second moment of the Boltzmann equation is satisfied, or, using the linearity of the kinetic equation, simply substitute Eq. (A5) into Eq. (24) [67], getting

$$u^\rho C_{,\rho}^{\mu\nu} + \frac{T}{\tau_0} C^{\mu\nu} + \frac{1}{2T} \sigma^{\mu\nu} = 0. \quad (\text{A7})$$

We may use Eq. (A6) to transform this to an equation for $\Pi^{\mu\nu}$, which turns out to be Eq. (20) with the same τ_0 and η_0 given by Eq. (A4). As we already know, this leads to a theory breakdown when $v_L^2 \geq 3/5$, see Eq. (23).

APPENDIX B: DERIVATION OF EQ. (A2)

To make the decomposition (A1) unique, we assume the constraints

$$\int Dp p^\mu (-u_\nu p^\nu) e^{u_\mu p^\mu / T} \delta f = 0. \quad (\text{B1})$$

This means the $e^{u_\mu p^\mu / T} = f_{\text{eq}}$. Then, assuming that the derivatives of β^μ / T are ‘‘small,’’ we solve Eq. (24) to first order to get

$$\delta f = -\frac{\tau_0}{T|u_\rho p^\rho|} p^\mu p^\nu \beta_{\mu,\nu}. \quad (\text{B2})$$

or else

$$0 = (1 - v_L^2)(v_L^2(J_0 J_2 - J_1^2) + v_L(J_0 J_3 - J_1 J_2) + (J_1 J_3 - J_2^2)). \quad (\text{C4})$$

J_0 is defined in Eq. (49). The remaining J functions obey the recursion relations

The constraints (B1) become the ideal hydrodynamic equations

$$\begin{aligned} \frac{\dot{T}}{T} + \frac{1}{3} u_{,\lambda}^\lambda &= 0 \\ \dot{u}^\mu + \Delta^{\mu\nu} \frac{T_{,\nu}}{T} &= 0. \end{aligned} \quad (\text{B3})$$

For fields T and u^μ satisfying Eq. (B3) we may simplify

$$\beta_{\mu,\nu} + \beta_{\nu,\mu} = \frac{1}{T} \left[u_{\mu,\nu} + u_{\nu,\mu} + u_\mu \dot{u}_\nu + u_\nu \dot{u}_\mu - \frac{2}{3} u_\mu u_\nu u_{,\lambda}^\lambda \right] \quad (\text{B4})$$

and subtracting a term proportional to $\eta_{\mu\nu}$, which does not contribute to f because $p^2 = 0$, we may substitute

$$\beta_{\mu,\nu} + \beta_{\nu,\mu} \rightarrow \frac{1}{T} \sigma_{\mu\nu}, \quad (\text{B5})$$

where $\sigma_{\mu\nu}$ is the shear tensor (10), in Eq. (B2).

APPENDIX C: DERIVATION OF EQ. (47)

Call

$$J_n = \int_{-1}^1 dx \frac{x^n}{x + \kappa}. \quad (\text{C1})$$

Then the functions A , B , C and D from Eq. (46)

$$\begin{aligned} A &= J_1 + v_L(J_0 + J_2) + v_L^2 J_1 \\ B &= J_2 + v_L(J_1 + J_3) + v_L^2 J_2 \\ C &= J_2 + 2v_L J_1 + v_L^2 J_0 \\ D &= J_3 + 2v_L J_2 + v_L^2 J_1 \end{aligned} \quad (\text{C2})$$

and the dispersion relation is

$$\begin{aligned} 0 &= v_L^4 J_1^2 + v_L^3 J_1(J_0 + 3J_2) + v_L^2(J_1(J_1 + J_3) + 2J_2(J_0 + J_2)) + v_L(J_3(J_0 + J_2) + 2J_1 J_2) + J_1 J_3 \\ &\quad - [v_L^4 J_0 J_2 + v_L^3(2J_1 J_2 + J_0(J_1 + J_3)) + v_L^2(J_2^2 + J_0 J_2 + 2J_1(J_1 + J_3)) + v_L J_2(3J_1 + J_3) + J_2^2] \end{aligned} \quad (\text{C3})$$

$$\begin{aligned} J_1 &= 2 - \kappa J_0 \\ J_2 &= -\kappa J_1 = \kappa^2 J_0 - 2\kappa \\ J_3 &= \frac{2}{3} - \kappa J_2 = \frac{2}{3} + 2\kappa^2 - \kappa^3 J_0 \end{aligned} \quad (\text{C5})$$

so we get

$$0 = v_L^2(\kappa J_0 - 2) + v_L \left(\frac{1}{3} J_0 - \kappa^2 J_0 + 2\kappa \right) + \frac{1}{3} (2 - \kappa J_0) \quad (\text{C6})$$

which yields Eq. (47) immediately.

APPENDIX D: ENTROPY AND THE EQUATIONS OF MOTION

Recall the entropy flux from kinetic theory [1,19]

$$S^\mu = \int Dp p^\mu f [1 - \ln f] \\ = \Phi^\mu - \beta_\nu T^{\mu\nu} - \zeta_{\nu\rho} A^{\mu\nu\rho}, \quad (\text{D1})$$

where the Massieu function current

$$\Phi^\mu = \int Dp p^\mu f \quad (\text{D2})$$

is the potential for the hydrodynamic tensors, for example

$$T^{\mu\nu} = \frac{\partial \Phi^\mu}{\partial \beta_\nu}. \quad (\text{D3})$$

We have made use of the symmetry of the shock wave problem to reduce the number of unknowns to just scalar variables. Moreover, we have seen that we may write $f = \exp(-\phi)$. To set up the hydrodynamic formulation, we assume ϕ is a homogeneous function of the rest frame momenta q^μ of degree 1, namely

$$\phi = \frac{q^0}{T} \varphi[z, x], \quad (\text{D4})$$

where $x = \cos\theta = q^z/q^0$. The function φ may be expanded in Legendre polynomials of the variable x [33]:

$$\varphi = 1 - \sum_{\ell \geq 1} Z_\ell[z] P_\ell(x). \quad (\text{D5})$$

We move from kinetic theory to hydrodynamics when we truncate this series [30–33]: ideal hydrodynamics keeps only $\ell = 0$ and 1, but assumes that $Z_1 = 0$ in the local rest frame; Israel-Stewart keeps $\ell = 0, 1$ and 2, but linearizes on Z_2 , once again forcing $Z_1 = 0$. The DTT presented above keeps $Z_0 = 1$, $Z_2 = \zeta T$ and $Z_3 = \xi T$ (up to normalization of the Legendre polynomials), with $Z_1 = 0$ to linear order in Z_2 and Z_3 . We introduce a dimensionless momentum $r^\mu = q^\mu/T$. The relevant components of $T^{\mu\nu}$ are

$$K = T^{0z} = \frac{T^4}{1-v^2} \int Dr (r^0 + vr^z) (r^z + vr^0) e^{-r^0(1-\sum_\ell Z_\ell P_\ell)} \\ K' = T^{zz} = \frac{T^4}{1-v^2} \int Dr (r^z + vr^0)^2 e^{-r^0(1-\sum_\ell Z_\ell P_\ell)}. \quad (\text{D6})$$

The second law reads

$$S_{;z}^z = \sigma \geq 0 \quad (\text{D7})$$

so we only need the z component of the entropy flux

$$S = S^z = \frac{T^3}{\sqrt{1-v^2}} \int Dr (r^z + vr^0) \\ \times \left[1 + r^0 \left(1 - \sum_\ell Z_\ell P_\ell \right) \right] e^{-r^0(1-\sum_\ell Z_\ell P_\ell)}. \quad (\text{D8})$$

Write

$$r^0 = \frac{(r^0 + vr^z) - v(r^z + vr^0)}{1-v^2} \quad (\text{D9})$$

to get

$$S = \Phi + \frac{1}{\sqrt{1-v^2}T} [T^{0z} - vT^{zz}] - \frac{1}{T} \sum_\ell Z_\ell A_\ell, \quad (\text{D10})$$

where

$$\Phi = \frac{T^3}{\sqrt{1-v^2}} \int Dr (r^z + vr^0) e^{-r^0(1-\sum_\ell Z_\ell P_\ell)} \\ A_\ell = \frac{T^4}{\sqrt{1-v^2}} \int Dr (r^z + vr^0) r^0 P_\ell[x] e^{-r^0(1-\sum_\ell Z_\ell P_\ell)}. \quad (\text{D11})$$

Then

$$S' = \frac{T'}{T} \left[3\Phi - \frac{1}{\sqrt{1-v^2}T} [T^{0z} - vT^{zz}] + \frac{1}{T} \sum_\ell Z_\ell A_\ell \right] \\ + v' \left[\frac{\partial \Phi}{\partial v} + \frac{vT^{0z} - T^{zz}}{(1-v^2)^{3/2}T} \right] \\ + \sum_\ell \left[Z'_\ell \left[\frac{\partial \Phi}{\partial Z_\ell} - \frac{1}{T} A_\ell \right] - \frac{1}{T} Z_\ell A'_\ell \right]. \quad (\text{D12})$$

It is clear that the coefficient of Z'_ℓ vanishes. Now compute

$$\frac{\partial \Phi}{\partial v} = \frac{T^3}{(1-v^2)^{3/2}} \int \frac{d^3r}{(2\pi)^3 r^0} (vr^z + r^0) e^{-r^0(1-\sum_\ell Z_\ell P_\ell)}. \quad (\text{D13})$$

On the other hand

$$\frac{1}{T} [vT^{0z} - T^{zz}] = -T^3 \int \frac{d^3r}{(2\pi)^3 r^0} r^z (r^z + vr^0) e^{-r^0(1-\sum_\ell Z_\ell P_\ell)} \quad (\text{D14})$$

but

$$-\frac{r^z}{r^0} e^{-r^0} = \frac{\partial}{\partial r^z} e^{-r^0} \quad (\text{D15})$$

so integrating by parts

$$\begin{aligned} \frac{1}{T} [vT^{0z} - T^{zz}] &= -T^3 \int \frac{d^3 r}{(2\pi)^3 r^0} \left(r^0 + vr^z + (r^z + vr^0) \right) \\ &\times \sum_{\ell} Z_{\ell} r^0 \frac{\partial}{\partial r^z} [r^0 P_{\ell}] e^{-r^0(1-\sum_{\ell} Z_{\ell} P_{\ell})}, \end{aligned} \quad (\text{D16})$$

where we are regarding r^0 as $\sqrt{r^{z2} + r^{x2} + r^{y2}}$ rather than as the independent 0 component of the r^{μ} vector; the relationship among the two approaches is

$$r^0 \frac{\partial}{\partial r^z} = r^0 \frac{\partial}{\partial r^z} \Big|_{r^0} + r^z \frac{\partial}{\partial r^0} \Big|_{r^z}. \quad (\text{D17})$$

Next consider

$$\begin{aligned} \frac{1}{\sqrt{1-v^2}T} [T^{0z} - vT^{zz}] &= \frac{T^3}{\sqrt{1-v^2}} \int \frac{d^3 r}{(2\pi)^3 r^0} (r^z + vr^0) r^0 e^{-r^0(1-\sum_{\ell} Z_{\ell} P_{\ell})} \\ &= \frac{T^3}{\sqrt{1-v^2}} \int \frac{d^3 r}{(2\pi)^3 r^0} (r^z + vr^0) e^{r^0 \sum_{\ell} Z_{\ell} P_{\ell}} (-\vec{r} \cdot \nabla) e^{-r^0} \\ &= 3\Phi + \frac{T^3}{\sqrt{1-v^2}} \int \frac{d^3 r}{(2\pi)^3 r^0} (r^z + vr^0) e^{-r^0(1-\sum_{\ell} Z_{\ell} P_{\ell})} \sum_{\ell} Z_{\ell} (\vec{r} \cdot \nabla) r^0 P_{\ell}. \end{aligned} \quad (\text{D18})$$

If H is a homogeneous function of degree n , then $(\vec{r} \cdot \nabla)H = nH$. In our case $n = 1$, and we get

$$\frac{1}{\sqrt{1-v^2}T} [T^{0z} - vT^{zz}] = 3\Phi + \frac{1}{T} \sum_{\ell} Z_{\ell} A_{\ell}. \quad (\text{D19})$$

Therefore

$$S' = -\frac{1}{T} \sum_{\ell} Z_{\ell} [A'_{\ell} - B_{\ell}], \quad (\text{D20})$$

where

$$\begin{aligned} B_{\ell} &= -\frac{v'T^4}{(1-v^2)^{3/2}} \int \frac{d^3 r}{(2\pi)^3 r^0} (r^z + vr^0) \\ &\times \left(r^0 \frac{\partial r^0 P_{\ell}}{\partial r^z} \right) e^{-r^0(1-\sum_{\ell} Z_{\ell} P_{\ell})} \end{aligned} \quad (\text{D21})$$

which agrees with our result above. To enforce the second law we need equations of motion of the form

$$A'_{\ell} - B_{\ell} = I_{\ell} \quad (\text{D22})$$

such that $\sum_{\ell} Z_{\ell} I_{\ell} \leq 0$. The natural choice is

$$I_{\ell} = \int Dq q^0 P_{\ell} I_{\text{col}} \quad (\text{D23})$$

since then the nonpositivity is enforced by the H theorem.

APPENDIX E: DTT CHARACTERISTICS

We shall investigate the characteristics of a DTT. We need to reinstate the time dependence, but we shall only consider linearized deviations from rest. The equations are

$$\begin{aligned} \dot{T}^{00} + T^{0z}_{,z} &= 0 \\ \dot{T}^{z0} + T^{zz}_{,z} &= 0 \\ \dot{A}_{\zeta}^0 + A_{\zeta,z}^z - B_{\zeta}^0 \frac{\dot{v}}{(1-v^2)} - B_{\zeta}^z \frac{v_{,z}}{(1-v^2)} &= I_{\zeta} \\ \dot{A}_{\xi}^0 + A_{\xi,z}^z - B_{\xi}^0 \frac{\dot{v}}{(1-v^2)} - B_{\xi}^z \frac{v_{,z}}{(1-v^2)} &= I_{\xi}. \end{aligned} \quad (\text{E1})$$

$A_{\zeta}^z, B_{\zeta}^z, A_{\xi}^z$ and B_{ξ}^z have already been computed in the main text (where we omitted the z superscript), the same as I_{ζ} and I_{ξ} . We introduce dimensionless variables $Z = \zeta T$ and $X = \xi T$, and further write $T = T_0 e^t$, where t is the linear deviation from equilibrium. Then

$$\begin{aligned} T^{00} &= \frac{T_0^4}{\pi^2} 3(1+4t); & T^{0z} &= \frac{T_0^4}{\pi^2} 4v; & T^{zz} &= \frac{T_0^4}{\pi^2} \left[1 + 4t + \frac{8}{5} Z \right] \\ A_{\zeta}^0 &= \frac{T_0^4}{\pi^2} \frac{12}{5} Z; & A_{\zeta}^z &= \frac{T_0^4}{\pi^2} \frac{36}{35} X; & B_{\zeta}^0 &= 0; & B_{\zeta}^z &= \frac{T_0^4}{\pi^2} \frac{8}{5}; & I_{\zeta} &= -\frac{T_0^5}{\pi^2} \frac{1}{\tau_0} \frac{12}{5} Z \\ A_{\xi}^0 &= \frac{T_0^4}{\pi^2} \frac{12}{7} X; & A_{\xi}^z &= \frac{T_0^4}{\pi^2} \frac{36}{35} Z; & B_{\xi}^0 &= B_{\xi}^z = 0; & I_{\xi} &= -\frac{T_0^5}{\pi^2} \frac{1}{\tau_0} \frac{12}{7} X. \end{aligned} \quad (\text{E2})$$

If we call $X^a = (t, v, Z, X)$, we get equations of the form $\dot{X}^a + \Gamma_b^a X'^b + \Lambda_b^a X^b = 0$. We are interested in the penetration of a front into fluid at rest. The variables $X^a = 0$ at the front and are continuous across the front, but the first derivatives X'^a are not. Since the X^a remain constant as we move along with the front with speed c , at the front $\dot{X}^a + cX'^a = 0$. From the equations of motion this means that $[\Gamma_b^a - c\delta_b^a]X'^b = 0$. We thereby get the dispersion relation as

$$\det \begin{pmatrix} c & -\frac{1}{3} & 0 & 0 \\ -1 & c & -\frac{2}{5} & 0 \\ 0 & -\frac{2}{3} & c & -\frac{3}{7} \\ 0 & 0 & -\frac{3}{5} & c \end{pmatrix} = 0. \quad (\text{E3})$$

Not including either Z or X is equivalent to considering only the upper left 2×2 block; we thus get the usual result $c^2 = 1/3$. For any $v_L > c$ the ideal fluid solution is discontinuous.

Including Z but not X means considering only the upper left 3×3 block. We then get $c^2 = 3/5$, which we recognize as the upper value of v_L for an Israel-Stewart fluid under the constitutive relation $\eta_0 = 4\tau_0/5$ as demanded by the Grad approximation.

Finally, the characteristic velocity for the full theory is

$$c^4 - \frac{6}{7}c^2 + \frac{3}{35} = 0 \quad (\text{E4})$$

with roots

$$c^2 = \frac{3}{7} \left[1 + \sqrt{\frac{8}{15}} \right] \approx 0.74 \quad (\text{E5})$$

and $c'^2 = 3/(35c) \approx 0.11$. We recognize that the coefficient of the leading term in Eq. (73) may be written as

$$\frac{6}{7}v_L^2 - v_L^4 - \frac{3}{35} = (c^2 - v_L^2)(v_L^2 - c'^2) \quad (\text{E6})$$

and since $v_L \geq 1/\sqrt{3} > c'$, positivity of this coefficient, and therefore existence of a solution, requires $v_L \leq c$.

A similar calculation yields the characteristic speed of an Israel-Stewart model. Linearizing the equations of motion around a static equilibrium ($v = \Pi = 0$) we get

$$\begin{aligned} 3\frac{\dot{T}}{T} + v' &= 0 \\ \frac{T'}{T} + \dot{v} + \frac{\pi^2}{4T^4}\Pi' &= 0 \\ \frac{\eta_0}{\tau_0}v' + \frac{\pi^2}{4T^4}\dot{\Pi} &= 0. \end{aligned} \quad (\text{E7})$$

On the front we have $\dot{v} = -Vv'$ and likewise for T and Π , so we get the characteristic velocities as $V = 0$ or

$$V^2 = \frac{1}{3} \left[1 + \frac{\eta_0}{\tau_0} \right] \quad (\text{E8})$$

which shows that the Israel-Stewart model must break down at a finite velocity, see Eq. (23).

APPENDIX F: THE RIGHT-HAND SIDE OF THE SHOCK

If the drawback of hydrodynamics is not being able to handle fast asymptotic velocities, then the approach to equilibrium on the right side, where speeds are subsonic throughout, should pose no problems.

Let us start with Landau-Lifshitz fluids. Equation (14) is still valid, observe that we can write indistinctly v_L or $v_R = 1/3v_L$. Now we write $v = v_R + \vartheta e^{-\lambda_L z}$ and linearize, getting the equivalent to Eq. (15):

$$\lambda_{LL}^R = T_R \frac{(1 - v_R^2)^{1/2}}{\eta_0 v_R} (1 - 3v_R^2). \quad (\text{F1})$$

We now move to Israel-Stewart fluids. Up to Eq. (21) nothing changes, then we write $v = v_R + \vartheta e^{-\lambda_{IS} z}$ and linearize, getting, instead of Eq. (22),

$$\lambda_{IS}^R = T_R \frac{(1 - 3v_R^2)(1 - v_R^2)^{1/2}}{v_R[\eta_0 + \tau_0(1 - 3v_R^2)]}, \quad (\text{F2})$$

where we further set $\eta_0 = 4/5\tau_0$ as derived from the Grad approximation.

In kinetic theory, the solution that goes to equilibrium as $z \rightarrow \infty$ is [cf. Eq. (29)]

$$f(z) = f_{\text{eq}}(z) - \int_z^\infty dz' e^{\int_z^{z'} dz'' \Lambda(z'')} [\phi_{,z} f_{\text{eq}}](z'). \quad (\text{F3})$$

Once again we find $T^{\mu\nu} = T_{id}^{\mu\nu} - t^{\mu\nu}$ with $t^{\mu\nu} u_{eq\nu} = 0$. For large z , $T = T_R(1 - te^{-\lambda z})$ and $v = v_R + \vartheta e^{-\lambda z}$. The analysis carries on as in the text, and we get the dispersion relations

$$\begin{aligned} A_R t - B_R \frac{\vartheta}{(1 - v_R^2)} &= 0 \\ C_R t - D_R \frac{\vartheta}{(1 - v_R^2)} &= 0, \end{aligned} \quad (\text{F4})$$

where [cf. Eqs. (45) and (46)]

$$\kappa_R = \frac{T_R}{\lambda_{AW}^R \tau_0} \sqrt{1 - v_R^2} - v_R \quad (\text{F5})$$

$$\begin{aligned}
A_R &= \int_{-1}^1 dx \frac{(1 + v_R x)(x + v_R)}{x - \kappa_R} \\
B_R &= \int_{-1}^1 dx x \frac{(1 + v_R x)(x + v_R)}{x - \kappa_R} \\
C_R &= \int_{-1}^1 dx \frac{(x + v_R)^2}{x - \kappa_R} \\
D_R &= \int_{-1}^1 dx x \frac{(x + v_R)^2}{x - \kappa_R}
\end{aligned} \tag{F6}$$

to the effect that instead of Eq. (47) we now get

$$0 = v_R^2 - \frac{1}{3} + v_R G_R[\kappa_R], \tag{F7}$$

where

$$G_R[\kappa_R] = \kappa_R - \frac{1}{3} \frac{J_0[\kappa_R]}{\kappa_R J_0[\kappa_R] - 2} \tag{F8}$$

J_0 as in Eq. (49). The final parametric relationship between v_R and λ_{AW}^R is

$$\begin{aligned}
v_R &= \frac{1}{2} \left[\sqrt{G_R^2[\kappa_R] + \frac{4}{3}} - G_R[\kappa_R] \right] \\
\lambda_{\text{AW}}^R &= \frac{T_R}{\tau_0(v_R + \kappa_R)} \sqrt{1 - v_R^2}.
\end{aligned} \tag{F9}$$

Let us analyze the limiting cases. When $\kappa \rightarrow \infty$, $G_R(\kappa) \approx 4/(15\kappa)$, so $v_R^2 \rightarrow 1/3$ and $\lambda_{\text{AW}}^R \rightarrow 0$. When $\kappa \rightarrow 1$, $J_0 \rightarrow \infty$, $v_R \rightarrow 1/3$,

$$\lambda_{\text{AW}}^R \approx \frac{T_R \sqrt{2}}{\tau_0 2}. \tag{F10}$$

Finally, let us consider the DTT. The analysis in the main text goes unchanged until Eq. (63), which, after linearization $v = v_R + \vartheta e^{-\lambda_{\text{DTT}}^R z}$, becomes [cf. Eq. (64)]

$$\zeta T_R = \frac{5}{6} \frac{(1 - 3v_R^2)}{v_R(1 - v_R^2)} \vartheta e^{-\lambda_{\text{DTT}}^R z}. \tag{F11}$$

Also the calculation of the A , B and I scalars in Eq. (70) goes unchanged, except that now we linearize around an equilibrium with temperature T_R and velocity v_R . Considering that now ζT and $\xi T \propto \exp\{-\lambda_{\text{DTT}}^R z\}$, and writing [cf. Eq. (72)]

$$\lambda_{\text{DTT}}^R = \frac{\alpha^R T_R}{v_R \tau_0} \sqrt{1 - v_R^2} \tag{F12}$$

we get the set of equations

$$\begin{pmatrix} \frac{2}{3} \alpha^R & v_R(\alpha^R - 1) & \frac{3}{7} \alpha^R \\ -(1 - 3v_R^2) & \frac{6}{5} v_R & 0 \\ 0 & \frac{3}{5} \alpha^R & v_R(\alpha^R - 1) \end{pmatrix} \begin{pmatrix} \frac{\vartheta}{1 - v_R^2} \\ \zeta T \\ \xi T \end{pmatrix} = 0. \tag{F13}$$

The allowed values of α are the roots of

$$a \alpha^{R2} - 2b v_R^2 \alpha^R + c v_R^2 = 0, \tag{F14}$$

where

$$\begin{aligned}
a &= \frac{4}{5} v_R^2 + (1 - 3v_R^2) \left(v_R^2 - \frac{9}{35} \right) \\
b &= \frac{7}{5} - 3v_R^2 \\
c &= 1 - 3v_R^2
\end{aligned} \tag{F15}$$

namely

$$\alpha = \frac{v_R^2}{a} \left[b - \sqrt{b^2 - \frac{ac}{v_R^2}} \right]. \tag{F16}$$

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