Artificial Intelligence and Law
https://doi.org/10.1007/s10506-023-09363-w

ORIGINAL RESEARCH

An approach to temporalised legal revision through addition of literals

Martín O. Moguillansky[1] · Diego C. Martinez[1] · Luciano H. Tamargo[1] · Antonino Rotolo[2]

Accepted: 16 May 2023

Abstract

As lawmakers produce norms, the underlying normative system is affected showing the intrinsic dynamism of law. Through undertaken actions of legal change, the normative system is continuously modified. In a usual legislative practice, the time for an enacted legal provision to be in force may differ from that of its inclusion to the legal system, or from that in which it produces legal effects. Even more, some provisions can produce effects retroactively in time. In this article we study a simulation of such process through the formalisation of a temporalised logical framework upon which a novel belief revision model tackles the dynamic nature of law. Represented through intervals, the temporalisation of sentences allows differentiating the temporal parameters of norms. In addition, a proposed revision operator allows assessing change to the legal system by including a new temporalised literal while preserving the time-based consistency. This can be achieved either by pushing out conflictive pieces of pre-existing norms or through the modification of intervals in which such norms can be either in force, or produce effects. Finally, the construction of the temporalised revision operator is axiomatically characterised and its rational behavior proved through a corresponding representation theorem.

✉ Martín O. Moguillansky
mom@cs.uns.edu.ar

Diego C. Martinez
dcm@cs.uns.edu.ar

Luciano H. Tamargo
lt@cs.uns.edu.ar

Antonino Rotolo
antonino.rotolo@unibo.it

1 Socio-juridical and AI Research (JuriSIA), Institute for Research in Computer Science and Engineering (ICIC), National Council of Scientific and Technical Research (CONICET), Universidad Nacional del Sur (UNS), Bahía Blanca, Argentina

2 Centro Interdipartimentale di Ricerca in Storia del Diritto, Filosofia e Sociologia del Diritto e Informatica Giuridica (CIRSFID), University of Bologna (UNIBO), Bologna, Italy

Published online: 08 June 2023

Keywords Legal revision · Norm change · Belief revision · Temporal reasoning

1 Introduction and motivation

According to Kelsen (1991) and Hart (1994), the law can be seen as a dynamic normative system. As a natural action of lawmakers, norms are produced or changed showing the intrinsic dynamism of law. Such actions stand for legal change operations like enactment, derogation, annulment and abrogation, among others (Governatori et al. 2005a; Governatori and Rotolo 2015). Through such change operations the legal system is modified. For instance, rules are introduced through enactment, whereas they are eliminated, made inapplicable or no longer in force through derogation, annulment and abrogation rules. Any process of legal change is potentially conflictive with regards to pre-existing rules in the system, and may cause some re-interpretation with the objective to observe the general principles of law and legal doctrine.

For instance, in the Argentine legislation according to Law 24.449, a driver that provoked a car accident might eventually require a reduction of charges if it has been proved that his own driving capacity was affected by alcohol consumption. Under such circumstances, the defendant could be considered not fully responsible for the negligent act, since according to the original law, there was nothing against the interpretation which considered that driving under the influence of alcohol could be taken as a mitigating factor in favor of a lesser sentence. Law 24.449 was effective until 2017, when it was amended specifically for considering such circumstances as an undeniable aggravating factor. Of course, traffic crimes committed beforehand, previous to 2017, would still be affected by the former law due to the legal dogmatics' principle of non-retroactivity that forbids the retrospective application of laws. On the other hand, Thailand legislation states in Criminal Code 2499 that "a person shall be criminally punished only when the act done by such person is provided to be an offence (...) by the law in force at the time of the doing of such act". Nevertheless, it explicitly states that "if, according to the law as provided afterwards, such act is no more an offence, the person doing such act shall be relieved from being an offender" and therefore the corresponding punishment, if any, shall terminate. This shows a different approach in which retroactive application of the law is allowed only when it goes in favor of the defendant. Such retrospective application is supported in legal dogmatics by the principles of equality and punishment necessity. The former principle holds that human beings, despite their differences, are to be regarded as equals, whereas punishment is justified only for guiding responses to deviant behavior, and thus, punishment meant to get even with the criminal (retribution), discourage the decision to commit crime (deterrence), limit the ability to commit crime (incapacitation), or minimize the propensity to commit crime (rehabilitation). In addition, the retroactive application of the law in favor of the defendant can be seen as an application of the principle of *in dubio pro reo* enforcing the principle of innocence and preserving the principle of legality. Here, since culpability can no longer be proved, innocence prevails and thus legality may imply reducing the punishment or even absolving the defendant. The Argentinian and Thailand examples are common to other legal systems around the globe, which make clear how time is relevant for the applicability of norms and their dynamics.

Belief revision appears as a suitable alternative to restore consistency after a legal change operation has been performed. Research upon legal dynamics was for long time underdeveloped. A pioneering research by Alchourrón et al. (1985) develop a logical model (AGM) with the intention to deal with norm change. The well-known AGM model presents three change operations over theories—i.e., logically closed sets. *Expansion* is the simplest of the three operations which incorporates a sentence to a theory, *contraction* is the operation that removes a specified sentence from the theory, and finally, *revision* is the most complex operation by which a sentence can be incorporated to the theory in a consistent manner, probably implying the removal of other conflicting sentences. Each operation is guided by a set of rationality principles, such is the case of minimal change, which ensures that the resulting theory is the smallest logically closed set satisfying the objective of the applied operation. Alchourrón, Gärdenfors and Makinson claimed that, when the mentioned theory stands for a code of legal norms, expansion corresponds to enactment, contraction to derogation and revision to amendment. The AGM model is founded upon theories of logical assertions. In that sense, such an abstraction serves for dealing with basic circumstances of the dynamics of legal systems, such as change to obligations and permissions (Boella et al. 2009; Governatori and Rotolo 2010).

Posterior work developed by Hansson (1991, 1999) studied the application of the AGM change operations upon belief bases: a computable alternative to theories. Thus, if $\mathbf{T}$ is a theory, a set A of sentences is a belief base if and only if $\mathbf{T} = Cn(A)$. Although not required by definition to be finite, in most realistic applications belief bases will be so. Elements in a belief base are basic beliefs and those in its logical closure which are not elements of the base itself, are know as derived beliefs. In computer science, belief bases were adopted as a natural approach when considering computability matters such as process termination. This is the main reason by which belief bases gained so much attention among philosophers. In our approach, changes will be performed upon belief bases, and derived beliefs will be changed only as a consequence of changes in the base.

Novel research has been developed thereafter with the intention to reformulate AGM to apply upon extended rule-based logical systems (Stolpe 2010; Rotolo 2010). Nevertheless, such attempts fail to handle the temporal features of norm change. Observe that legal norms are circumscribed by temporal properties: when the norm comes into existence and when the norm is in force. Failing to consider such temporal aspects unveils a serious limitation for an appropriate modeling of the of legal dynamics.

In Tamargo et al. (2019) a belief revision operator was proposed for a logical temporalised framework that considers *time intervals* for modelling norm change in the law. There, a temporalised belief base is defined with the corresponding timed derivation. This has been proved to be a useful approach towards legal reasoning about temporalised information. However, the level of abstraction adopted there has some limitations. For instance, the antecedent in temporalised rules is a single literal and then the corresponding temporalised derivation is simple, restraining a more detailed study of the application domain. A new, more refined belief revision operator is desirable, in order to properly capture other legal situations. Also, the characterization of interval operations in that work is partially engaged, due to simpler derivations. By introducing a more detailed underlying logic, further elaborations about interval decorations can be addressed. As an improvement of the same line of work, the present article proposes a complete and more advanced temporalised logical framework upon which a fresh new model of change is constructed and axiomatised. The logical framework we present here is capable of understanding and reason upon alternative representations of time-labeled sentences. This new temporalised framework harnesses the observation of rationality principles of change like the aforementioned principle of minimality, providing a more flexible and expressive theory towards future more ambitious models of legal revision. Further proposals build on top of the theory here presented will formalise richer revision operators without prioritization on the incoming sentence—for more realistic amendment—or more powerful multiple revision operators capable of revising the normative system as a consequence of the enactment of a whole piece of legislation—usually constituted by an extensive set of norms.

The layout of the paper is as follows. Section 2 shows a legal example to motivate the main ideas of our framework. Section 3 proposes the fundamentals for a complete temporalised logical framework allowing to construct the temporalised belief base and temporalised derivation upon it. Section 4 proposes a formal reconstruction of the legal example introduced in Sect. 2 upon the temporalised logical framework presented in Sect. 3. Here, a legal revision process is proposed, introducing the reader to further discussions on rationality and appropriate construction of the legal revision operator. Section 5 presents a set of properties that the temporalised belief revision operators should satisfy, by means of postulates. Section 6 introduces both a complete construction for legal revision operator based on temporalised belief base and their characterization regarding the presented postulates through a representation theorem. Section 7 reports on related work. Finally, in Sect. 8 conclusions are offered and ideas for future work are given. The proof for the representation theorem can be found in "Appendix".

2 The problem and motivating legal example

Belief revision, and specifically the AGM paradigm, has been claimed to be a suitable, but abstract model for legal change. Standard techniques of change are not suitable for dealing with the following aspects of the law (Governatori and Rotolo 2010; Tamargo et al. 2019):

1. the law usually regulate its own changes by setting specific norms whose peculiar objective is to change the system by stating what and how other existing norms should be modified;
2. since legal modifications are derived from these peculiar norms, they can be in conflict and so are defeasible;

3. legal norms are qualified by temporal properties, such as the time when the norm comes into existence and belongs to the legal system or the time when the norm is in force.

The general temporal model assumes that all legal norms are qualified by different temporal parameters:

- the time when the norm comes into existence and belongs to the legal system,
- the time when the norm is in force,
- the time when the norm produces legal effects (it is applicable), and
- the time when the normative effects (conclusions) hold.

In a usual legislative practice, the force of a recently enacted legal provision can be postponed for different reasons. Similarly, a provision in force can make effective some specific fragment at a subsequent time. Even more, some provisions can produce effects retroactively in time. In this paper we focus on subject 3, the integration of belief revision to time in the law. Regarding 2, we do not develop defeasible reasoning, but a revision operator that may restrict intervals when clashing of norms appears in the system: for instance, if n is effective from 2001 to 2008 and we incorporate a contradictory norm which is effective in 2006, we know that n is still effective from 2001 to 2005 and from 2007 to 2008, but not applicable/valid in 2006.

Example 1 Consider the following pieces of information regarding a legislative attempt to ease tax pressure for people that have been unemployed.

(a) A citizen was unemployed from 1980 to 1985.
(b) If unemployed for a full year then a tax exemption applies for that year, in order to increase individual savings. The citizen would be exempted to pay taxes from 1980 to 1985. However, due to economical crisis, a new norm is promulgated for suspending such benefit for a restricted time-lapse:
(c) Due to economical crisis, new authorities in government revoked tax exemption for years 1985 and 1986, in an exceptional manner.

Now, due to the exceptional suspension of the taxation law validity during 1985 and 1986, the citizen cannot be granted for tax exemption in 1985. This renders the exemption in favor of the citizen, only from 1980 to 1984.

Here some statements are produced and, as it happens in legislative bodies, norms change later according to the political and economical context. Statement (a) provides time-bounded information: only between 1980 and 1985 the status of being unemployed holds for a given citizen. Statement (b) states that if some property (unemployed) holds in a given year, then other property (tax exemption) holds in that year. Statement (c) establishes that this is no longer valid for a certain interval of time. This means that, from now on, statement (b) of tax exemption should not be applied in its original text unrestrictedly. In other words, the intervals of statement (b) need to be *revised* according to the new political positions upon the economical crisis. Such revision is actually about the moments in which this benefit can be applied. In fact, statement (c) demands a revision of the interval for tax exemption, which ends up affecting the validity of such taxation law. Hence, it cannot be the case that there is a rule in the normative system that entails a tax exemption for 1985 and 1986. From (c) and (b), it can be concluded that the benefit is only applied until 1984 and from 1987. Therefore, (b) should not be used literally anymore and new rules should appear in its place to represent such changes. This is naturally a process of belief revision. It is important to mention that there are a number of alternatives to model such a situation. In this work, we will be interested in the inclusion of a new rule for restricting the applicability of (b). This will be made clearer afterwards. Our interest is the formalization of a belief revision operator that can address the evaluation of *temporalised rules* representing legal norms. Technical aspects of temporalised knowledge, and a complete logical framework for dealing with them, are developed in the following section. Afterwards, in Sect. 4 we consider the legal example just introduced once again, providing a formal reinterpretation upon the logical framework described in Sect. 3, and applying some intuitions on that which we consider would be an appropriate legal revision operator for such purpose. This will serve for discussing afterwards, in Sect. 5, the rationality aspects of such a legal revision operator which will be formalised thereafter, in Sect. 6, and mathematically demonstrated in the “Appendix” of this work.

3 A temporalised logical framework for the legal domain

For reasoning upon temporal knowledge it is necessary to determine a primitive to represent time and its corresponding metric relations. Two traditional approaches can be distinguished: point based approaches (Governatori and Rotolo 2010) and interval based approaches (Allen 1984; Augusto and Simari 2001; Budán et al. 2017). The former approach handles *instants* of time, or timestamps, and a precedence relation, whereas for latter approaches, time stands as sets of instants. Intervals are constructed upon starting and ending instants of time. In this work, we consider the richer alternative of time intervals since it seems more appropriate for modeling the legal dynamics. Inspired by the semantics of the temporalised rules proposed in Governatori and Rotolo (2010), in the present work, the revision operator may modify intervals to restore consistency after change has been performed. The aforementioned temporal machinery is able to explicitly model two temporal dimensions: the time of norm effectiveness—i.e. when a norm can produce legal effects—and the time when the norm effects hold.

3.1 Preliminaries and notation

We will adopt a propositional language $\mathbb{L}$ with a set of boolean connectives $\neg$, $\wedge$, and $\rightarrow$. *Sentences* in $\mathbb{L}$ can be either literals or rules. A *literal* is either a *proposition* or a *negated proposition*. A *rule* in $\mathbb{L}$ is constructed as a *conjunction of literals* as the left-hand side and a literal as the right-hand side. Greek letters, sometimes sub-indexed with natural numbers, will be used for referring to elements in $\mathbb{L}$. Concretely, we say that a literal α is the complement of the literal $\neg\alpha$ and vice versa. A simple rule in $\mathbb{L}$ is constructed as $\alpha \rightarrow \beta$, where both α and β, stand for a literal. A complex rule, is constructed in $\mathbb{L}$ as $\alpha_1 \wedge \ldots \wedge \alpha_n \rightarrow \beta$, where, once again, each letter α_i (with $1 \leq i \leq n$) and β, stands for a literal. Greek letters φ and ψ, will be reserved only for referring to a *sentence* in $\mathbb{L}$, in an abstract manner, without specifying whether it refers to a rule or a literal—or a conjuntion of literals as left-hand side of a rule. Thus, when we write $\varphi \in \mathbb{L}$, we say that φ is a sentence and thus, it can be a literal or a rule. When we write $\varphi \rightarrow \alpha \in \mathbb{L}$, we say φ can be a literal or a conjunction of literals, while α is restricted only to a literal. It is important to say that the Greek letter "σ" will be reserved for a different specific use throughout this article.

We also use a consequence operator, denoted $Cn(\cdot)$, that takes sets of sentences in $\mathbb{L}$ and produces new sets of sentences. This operator $Cn(\cdot)$ satisfies *inclusion* ($A \subseteq Cn(A)$), *idempotence* ($Cn(A) = Cn(Cn(A))$), and *monotony* (if $A \subseteq B$ then $Cn(A) \subseteq Cn(B)$). We will assume that the consequence operator includes classical consequences and verifies the standard properties of *supraclassicality*[1] (if α can be derived from A by deduction in classical logic, then $\alpha \in Cn(A)$), *deduction* ($\beta \in Cn(A \cup \{\alpha\})$ if and only if $(\alpha \rightarrow \beta) \in Cn(A)$) and *compactness* (if $\alpha \in Cn(A)$ then $\alpha \in Cn(A')$ for some finite subset A' of A). In general, we will write $\alpha \in Cn(A)$ as $A \vdash \alpha$. We will recognise consistency of a set A by verifying $A \nvdash \bot$, for instance the set $\{\alpha, \neg\alpha\}$ is inconsistent.

As aforementioned, sentences in $\mathbb{L}$ are either literals or rules. For simplicity, we assume every rule $\alpha_1 \wedge \ldots \wedge \alpha_{n-1} \rightarrow \alpha_n$, with $n > 1$, is normalized to its minimal form, that is, for any pair α_i and α_j ($1 \leq i,j \leq n$) it holds $\alpha_i \wedge \alpha_j \neq \alpha_i$ and $\{\alpha_i, \alpha_j\} \nvdash \bot$. That means that a rule like $p \wedge q \wedge \neg p \rightarrow q$ is not in $\mathbb{L}$. Observe also that $\mathbb{L}$ does not include individual contradictory sentences, thus for any $\varphi \in \mathbb{L}$ it holds $\varphi \nvdash \bot$.

The AGM model (Alchourrón et al. 1985) represents epistemic states by means of belief sets, that is, sets of sentences closed under logical consequence. Other models use belief bases, i.e., arbitrary sets of sentences (Fuhrmann 1991; Hansson 1992; Wassermann 2000). Our epistemic model is based on an adapted version of belief bases which additionally incorporates time intervals. The usage of belief bases makes the representation of the legal system more natural and computationally tractable. That is, following (Hansson 1999; Wassermann 2000), we consider that legal systems' sentences could be represented by a finite number of sentences that correspond to the explicit beliefs on the legal system. That norm change captured by base revision was also discussed by Governatori and Rotolo (2010).

3.2 Time interval

We will consider a *universal set* $\mathbb{T}$ *of time labels*, where each label $t \in \mathbb{T}$ represents a unique time instant according to a unique time measurement. For instance, $2010 \in \mathbb{T}$ holds, being 2010 interpreted as a time instant, whenever *years* is the

[1] Note that supraclassicality is restricted within the scope of the adopted language. That is, since we do not admit disjunction, disjunctive terms will not be part of the consequence operator.

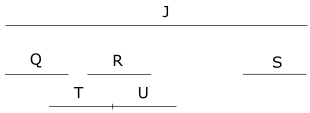

Fig. 1 Interval classification

adopted time measurement. In this article, we expand the notion of temporalised literals (Governatori et al. 2005c) by relying upon time intervals. Informally speaking, *time intervals* $[t_i, t_j]$ are a notational simplification for referring through a pair of time labels $t_i, t_j \in \mathbb{T}$ to an incrementally ordered sequence of time labels $t_i, \dots, t_j \in \mathbb{T}$. A time interval represents a continuous time-lapse through the instances of time or *timepoints* compounding the underlying sequence. Generally speaking $t_i \leq \dots \leq t_{k-1} \leq t_k \leq t_{k+1} \leq \dots \leq t_j$, where either $t_i = t_j$, representing an instant in accordance to the adopted time measurement, or t_i is the instant immediately preceding instant t_j, or t_{k-1} is the immediately previous instant to the instant t_k and t_{k+1} is the immediately posterior instant to the instant t_k. We will accept infinite time intervals for the specific case in which the rightmost interval boundary t_j corresponds to the special symbol ∞ such that for any $t \in \mathbb{T}$, $t < \infty$ holds. For such special case, we refer to the domain $\mathbb{T}^\infty = \mathbb{T} \cup \{\infty\}$. The *universal set of time intervals* $\mathbb{I}$ will stand for referring to the domain $\mathbb{T} \times \mathbb{T}^\infty$. Thus, for any interval $J \in \mathbb{I}$ it holds that $J \in \mathbb{T} \times \mathbb{T}^\infty$. Intervals in $\mathbb{I}$ will be denoted by uppercase Latin characters $A, B, C, \dots, Z$.

Definition 1 (*Interval*) A pair $[t_i, t_j]$ is a time interval if and only if for its bound timepoints $t_i \in \mathbb{T}$ and $t_j \in \mathbb{T}^\infty$ it holds $t_i \leq t_j$. In such case, $[t_i, t_j] \in \mathbb{I}$ holds.

For the exceptional case where $t_i = t_j$ and just for simplicity, we will accept a notational overload by assuming $[t_i]$, $[t_j]$ and $[t_i, t_j]$ are three different notations for referring to the same time interval. Intervals are formally classified as follows.

Definition 2 (*Interval Classification*) Let $R, S \in \mathbb{I}$ be two intervals where $R = [r_i, r_j]$ and $S = [s_i, s_j]$. The following classification identifies the way intervals may interrelate according to their boundaries:

(Contained) R is contained in S, denoted $R \subseteq S$ if and only if $r_i \geq s_i$ and $r_j \leq s_j$.
(Disjoint) R and S are disjoint, denoted $R \bowtie S$ if and only if $(r_j) + 1 < s_i$ or $(s_j) + 1 < r_i$.
(Contiguous) R is contiguous to S, denoted $R \triangleright S$ if and only if $(r_j) + 1 = s_i$.
(Overlapped) R and S are overlapped, denoted $R \top S$ if and only if there exists $t_i \in R$ such that $t_i \in S$.

Example 2 According to the graphical representation of intervals shown in Fig. 1, we have:

- *Contained* intervals $Q \subseteq J$, $R \subseteq J$, $S \subseteq J$, $T \subseteq J$, and $U \subseteq J$,
- *Disjoint* intervals $Q \bowtie R$, $R \bowtie S$, $Q \bowtie S$, $U \bowtie S$, $T \bowtie S$, and $Q \bowtie U$,

- *Contiguous* intervals $T \rhd U$, and
- *Overlapped* intervals $Q \top T$, $R \top T$, and $R \top U$, and in particular, the interval J is overlapping all the intervals in the figure.

Observe that when an interval is contained in a second interval it will also mean that both are overlapping, however, the reverse is not true. That is the case, for instance, of Q and T in Fig. 1. Also note that, if two intervals are contiguous, they are not disjoint, as is the case of T and U, since they share a timepoint.

Remark 1 Let $I, J \in \mathbb{I}$ be two intervals:

- if $I \subseteq J$ then $I \top J$.
- if $I \rhd J$ then $I \bowtie J$ does not hold.

Intervals can be joined together to construct bigger intervals. We refer to that action as *interval composition*, formally defined as follows.

Definition 3 (*Maximising Interval Composition*) Let $\mathbf{I} \subseteq \mathbb{I}$ be a set of time intervals such that $\mathbf{R} = \{R_k = [r_i^k, r_j^k] \mid \text{with } 1 \leq k \leq n\} \subseteq \mathbf{I}$, where the n intervals in $\mathbf{R}$ are pairwise overlapped, contiguous, or contained. That is, there is no $R \in \mathbf{R}$ such that for every $R' \in \mathbf{R}$ with $R \neq R'$ it holds $R \bowtie R'$. An interval $Q = [q_i, q_j] \in \mathbb{I}$ is a maximising interval composition in $\mathbf{I}$ if and only if it follows:

- $[q_i, q_j] = [min_{k=1}^n(r_i^k), max_{k=1}^n(r_j^k)]$ and
- for any interval $S = [s_i, s_j] \in \mathbf{I}$ either:
  - $S \subseteq Q$, that is $q_i \leq s_i$ and $s_j \leq q_j$, or
  - $S \bowtie Q$, that is either $(s_j) + 1 < q_i$ or $(q_j) + 1 < s_i$.

We refer to $\mathbf{R}$ as an overlapping set and Q is the maximising interval composition in $\mathbf{R}$.

From the definition above, observe that given an arbitrary set $\mathbf{I}$ of time intervals, several disjoint maximising interval compositions may be recognised from $\mathbf{I}$, whereas for a given overlapping set $\mathbf{R}$, there is only one possible maximising interval composition. The following example illustrates its usage.

Example 3 Assume from Fig. 1 two sets $\mathbf{R} = \{T, R, U\}$ and $\mathbf{I} = \{T, R, U, S\}$. Observe that $\mathbf{R}$ is an overlapping set and intervals $T = [t_i, t_j]$ and $U = [u_i, u_j]$ compose an interval $V = [t_i, u_j] \in \mathbb{I}$ which is a maximising interval composition in $\mathbf{I}$, and V is the maximising interval composition in $\mathbf{R}$. Observe also that, along with V, interval S is time maximising in $\mathbf{I}$ as well.

3.3 Temporalised belief base

The discreteness of the flow of time is appropriate for modelling the dynamics of norms since norms usually refer to time in the spectrum of hours, days, months and years. Thus, given a literal $\alpha \in \mathbb{L}$, we will refer to expressions like α^J, with $J \in \mathbb{I}$ such that, either:

- $J = [t_i]$, thus α holds at time t_i. In this case, according to Governatori and Rotolo (2010), we say α is *transient*, meaning that it holds at precisely one instant of time.
- $J = [t_i, \infty]$, thus α holds from t_i. In this case, according to Governatori and Rotolo (2010), we say α is (indefinitely) *persistent* from t_i.
- $J = [t_i, t_j]$, thus α holds from time t_i to t_j (inclusive) with $t_i \leq t_j$. When $t_i = t_j$, for simplicity we usually write $[t_i]$.

Throughout this work we will say that α^J is a *temporalised literal* meaning that the literal α stands for a normative statement (or a fragment of it) that is *effective*, *i.e*, has both *validity* and *applicability* assured in the time lapse covered by J. Therefore, we preserve the semantics of classical propositional logic applied to a time context: a temporalised literal $\alpha^{[t_a,t_b]}$ holds when its non-temporalised literal α is true in every time point within $[t_i, t_j]$. That also implies that, $\alpha^{[t_a,t_b]}$ holds if and only if $\alpha^{[t_i]}$ holds for every $t_a \leq t_i \leq t_b$.

As rules are part of the language $\mathbb{L}$, they are subject of temporal effectiveness too. That is, we have also the representation of *temporalised rules* as $\varphi^Q \rightarrow \alpha^R$, where φ is a conjunction of literals in $\mathbb{L}$. Observe that a temporalised rule is a temporalised conjuntion of literals as left-hand side and a temporalised literal as right-hand side. In this perspective we can have expressions like

$$(\alpha_1 \wedge \ldots \wedge \alpha_n)^{[t_a,t_b]} \rightarrow \beta^{[t_c,t_d]}$$

or equivalently,

$$\alpha_1^{[t_a,t_b]} \wedge \ldots \wedge \alpha_n^{[t_a,t_b]} \rightarrow \beta^{[t_c,t_d]}$$

meaning that the rule can derive that β holds from time t_c to t_d if we can prove that each α_i (with $1 \leq i \leq n$) holds from time t_a to t_b. In the same way a conclusion can persist, this applies as well to rules and then to derivations. Note that the implication itself is not decorated with intervals. Thus, if we have such an implication and its antecedent holds, that is, $\alpha_1 \wedge \ldots \wedge \alpha_n$ holds at $[t_a, t_b]$, then β also holds at $[t_c, t_d]$.

Observe that the expressivity of our temporalised language restricts the construction of rules to a single temporalisation in the antecedent, i.e., in the left-hand side. Thus, two different intervals need to be present for representing rules: one for the left-hand side and another one for the right-hand side—being possible both to coincide. Although a more expressive temporalised language can be easily adopted—like, for instance, admitting independent intervals for each conjunctive literal in the antecedent of rules—it falls beyond the scope of this article. This decision has been taken in favor of the simplicity of the framework and subsequent dynamic model.

We will define the notion of *temporalised belief base* which will contain temporalised sentences, that is, temporalised literals and temporalised rules. This base represents a legal system in which each temporalised sentence defines a norm whose time interval determines its effectiveness in the context of time.

Definition 4 (*Temporalised Belief Base*) Let $\mathbb{L}$ be a propositional language, and $\mathbb{I}$ the universal set of time intervals. A set $\mathbb{K}$ is a temporalised belief base if and only if for any temporalised sentence $\varphi \in \mathbb{K}$ it holds:

- $\varphi = \alpha^Q$, with $Q \in \mathbb{I}$ and $\alpha \in \mathbb{L}$, or
- $\varphi = (\alpha_1 \wedge \ldots \wedge \alpha_{n-1})^Q \rightarrow \alpha_n^R$, with $Q, R \in \mathbb{I}$ and $\alpha_1 \wedge \ldots \wedge \alpha_{n-1} \rightarrow \alpha_n \in \mathbb{L}$, $n > 1$

Every referred α stands for a literal. We will write $\mathbb{K} \in \mathcal{U}_{\mathbb{L}}^{\mathbb{I}}$ to formally specify that the temporalised belief base $\mathbb{K}$ corresponds to the universal set $\mathcal{U}_{\mathbb{L}}^{\mathbb{I}}$ of $\mathbb{I}$-temporalised belief bases over $\mathbb{L}$.

Example 4 We have the normative provision: "If unemployed from 1980 to 1983 and the taxation law is valid in that time-lapse, then a tax exemption applies from 1984 to 1986". It is possible to formalise such temporalised norm as follows:

$$(unemployed \wedge taxation\_law\_validity)^{[1980,1983]} \rightarrow exemption^{[1984,1986]}$$

The previous example shows a way in which the language defined admits the configuration of temporalised norms. The following is an abstract and more complex example which will serve for illustrating further notions of our theory.

Example 5 (Running example) The set $\mathbb{K}$ is a valid temporalised belief base.

$$\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_2,t_6]}, \beta^{[t_4,t_5]}, \beta^{[t_5,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]}, \\ \delta^{[t_1,t_3]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}, \\ \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}$$

Example 5 shows that some sentences can be represented in different manners, appearing more than once with different intervals. For instance, we say that β is *intermittent* given that it holds from t_4 to t_6 and from t_8 to t_{10}, skipping instant t_7. Observe also that interval overlapping is possible, as is the case of $\beta^{[t_4,t_5]}$ and $\beta^{[t_5,t_6]}$, which implies that β holds in $[t_4, t_6]$. Such situations show that a temporalised belief base can have redundancies and alternative representations. The definition of *Temporal-Normalized Set* allows constructing a unified version of any temporalised belief base $\mathbb{K}$, maximising intervals and reducing the base cardinality.

Definition 5 (*Temporal-Normalised Set*) Let $\mathbb{K} \in \mathcal{U}_{\mathbb{L}}^{\mathbb{I}}$ be a temporalised belief base and let Q be the maximising interval in the overlapping set $\{R_1, \ldots, R_n\}$, with $n \geq 1$. The set $\widehat{\mathbb{K}}$ identifies the temporal-normalised set of $\mathbb{K}$ such that $\varphi \in \widehat{\mathbb{K}}$ is a temporal-normalised sentence if and only if, either

1. φ is a temporalised literal α^Q such that:

(a) $\alpha^Q \in \widehat{\mathbb{K}}$ if and only if $\alpha^{R_1}, \ldots, \alpha^{R_n} \in \mathbb{K}$, and
(b) if $\alpha^S, \alpha^T \in \widehat{\mathbb{K}}$ then $S \bowtie T$

2. φ is a temporalised rule $\varphi_1 \rightarrow \alpha^Q$ such that:

(a) $\varphi_1 \rightarrow \alpha^Q \in \widehat{\mathbb{K}}$ if and only if $\varphi_1 \rightarrow \alpha^{R_1} \in \mathbb{K}$ and $\cdots$ and $\varphi_1 \rightarrow \alpha^{R_n} \in \mathbb{K}$, and
(b) if $\varphi_1 \rightarrow \alpha^S \in \widehat{\mathbb{K}}$ and $\varphi_1 \rightarrow \alpha^T \in \widehat{\mathbb{K}}$ then $S \bowtie T$

For the cases in which $n = 1$, there is no overlapping set and thus $Q = R_1$.

The following example shows the *temporal-normalisation* of a belief base as characterized in Definition 5. Observe that, as a consequence, intervals are *time-maximised*. For instance, α has been unified in a single time-maximising interval $[t_1, t_4]$. A similar situation occurs with the temporalised rule $\delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{14}]}$.

Example 6 Considering the temporalised base $\mathbb{K}$ from Example 5, according to Definition 5, the resulting temporal-normalised set follows:

$$\widehat{\mathbb{K}} = \{\alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_2,t_6]}, \beta^{[t_4,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]}, \delta^{[t_1,t_3]},$$
$$\delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{14}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}$$

The process of temporal-normalisation is a fixed-point construction modifying the incoming belief base. That means that a temporal-normalised set cannot produce a different result if it is "re-normalised". This is shown in the following proposition.

Proposition 1 $\widehat{\mathbb{K}} = \widehat{\widehat{\mathbb{K}}}$[2]

Next we will propose some set-theoretic operators and other additional notions for defining the complete inference machinery. We will show afterwards, in Proposition 19, the equivalence of a belief base and its resulting temporal-normalised set.

3.4 Temporalised set operations

In this section we will develop the inference machinery for our temporalised theory. The *temporalised in-operator* defined next allows recognising the validity of a temporalised sentence independently of its specific representation. In such a way, if the temporalised belief base contains a sentence like $\alpha^{[t_1,t_4]}$ we expect to recognise, for instance, that $\alpha^{[t_1,t_2]}$ is implicitly valid in the belief base, although not explicitly contained. In a similar manner, we can recognise that $\alpha^{[t_4,t_6]}$ is implicitly valid, if in addition to $\alpha^{[t_1,t_4]}$, we also have $\alpha^{[t_5,t_8]}$ explicitly contained in the belief base.

[2] The interested reader can follow this, and all the proofs for the properties proposed in this article, in the "Appendix" starting in page 22.

Definition 6 (*Temporalised In-Operator*) Let $\mathbb{K}$ be a set of temporalised sentences. The symbol $\in^t$ identifies the set in-operator such that a temporalised sentence $\varphi \in^t \mathbb{K}$ if and only if, either:

- $\varphi = (\alpha_1 \wedge \ldots \wedge \alpha_n)^Q$ (with $n \geq 1$), and thus, for each α_i there is J_i such that $\alpha_i^{J_i} \in \widehat{\mathbb{K}}$ and $Q \subseteq J_i$, or
- $\varphi = \varphi_1 \rightarrow \alpha^Q$, and thus, $\varphi_1 \rightarrow \varphi_2 \in \widehat{\mathbb{K}}$ and $\alpha^Q \in^t \{\varphi_2\}$.

For simplicity, we will overload the temporalised in-operator by writing $\varphi_1 \in^t \varphi_2$ whenever $\varphi_1 \in^t \{\varphi_2\}$.

Example 7 Given the belief base $\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_7,t_9]}\}$, the following expressions hold:

- $\alpha^{[t_2]} \in^t \mathbb{K}$
- $\alpha^{[t_1,t_4]} \rightarrow \beta^{[t_5]} \in^t \mathbb{K}$
- $\alpha^{[t_3,t_5]} \in^t \mathbb{K}$
- $\alpha^{[t_1,t_4]} \rightarrow \beta^{[t_5,t_8]} \in^t \mathbb{K}$

The following proposition states the base-case for the temporalised in-operator: any temporalised sentence that is part of a belief base is also recognised by the temporalised in-operator. In other words, it is reasonable to state that the temporalised in-operator recognises implicit temporalised sentences from a belief base, and that includes the explicit sentences from the original base.

Proposition 2 *if* $\varphi \in \mathbb{K}$ *then* $\varphi \in^t \mathbb{K}$

Given the definition of a temporalised in-operator, we formalise the following definitions upon the concept of *temporalised inclusion-operator*.

Definition 7 (*Temporalised Inclusion-Operator*) Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two sets of temporalised sentences and φ be a temporalised sentence. The symbol $\subseteq^t$ (resp., $\subset^t$) identifies the temporalised set (resp., strict-set) inclusion-operator if and only if:

- $\mathbb{K}_1 \subseteq^t \mathbb{K}_2$ if and only if $\varphi \in^t \mathbb{K}_2$, for every $\varphi \in \mathbb{K}_1$.
- $\mathbb{K}_1 \subset^t \mathbb{K}_2$ if and only $\mathbb{K}_1 \subseteq^t \mathbb{K}_2$ and there is some $\varphi \in^t \mathbb{K}_2$ such that $\varphi \notin^t \mathbb{K}_1$.

Definition 8 (*Temporalised Equivalence-Operator*) Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two sets of temporalised sentences and φ be a temporalised sentence. The symbol $\equiv^t$ identifies the temporalised equivalence-operator if and only if:

- $\mathbb{K}_1 \equiv^t \mathbb{K}_2$ if and only if $\mathbb{K}_1 \subseteq^t \mathbb{K}_2$ and $\mathbb{K}_2 \subseteq^t \mathbb{K}_1$.

Example 8 (Exapmle 7 continued) Given the following temporalised belief bases:

- $\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_7,t_9]}\}$
- $\mathbb{K}' = \{\alpha^{[t_2]}, \alpha^{[t_3,t_5]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5,t_8]}\}$
- $\mathbb{K}'' = \{\alpha^{[t_1,t_2]}, \alpha^{[t_3,t_5]}, \alpha^{[t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5,t_9]}\}$

Both the expressions $\mathbb{K}' \subset^t \mathbb{K}$ and $\mathbb{K} \equiv^t \mathbb{K}''$, hold.

The following definition studies the interaction among the previous temporalised operators and the notion of normalised set.

Definition 9 (*Temporal-Normalised Inclusion-Operator*) Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two sets of temporalised sentences and φ be a temporalised sentence. The symbol $\sqsubseteq^t$ (resp., $\sqsubset^t$) identifies the temporal-normalised (strict) set inclusion-operator if and only if:

- $\mathbb{K}_1 \sqsubseteq^t \mathbb{K}_2$ if and only if $\mathbb{K}_1 \subseteq^t \mathbb{K}_2$ and $\mathbb{K}_1 = \widehat{\mathbb{K}_1}$.
- $\mathbb{K}_1 \sqsubset^t \mathbb{K}_2$ if and only if $\mathbb{K}_1 \subset^t \mathbb{K}_2$ and $\mathbb{K}_1 = \widehat{\mathbb{K}_1}$.

Example 9 Observe from Exapmle 8, that $\mathbb{K}' \not\sqsubseteq^t \mathbb{K}$ given that $\mathbb{K}'$ is not normalised, that is: $\mathbb{K}' \neq \widehat{\mathbb{K}'}$. The following examples illustrate the usage of the operator:

- $\{\alpha^{[t_2,t_3]}\} \sqsubset^t \{\alpha^{[t_1,t_5]}\}$
- $\{\alpha^{[t_2,t_3]}\} \sqsubset^t \{\alpha^{[t_1,t_2]}, \alpha^{[t_3,t_5]}\}$
- $\{\alpha^{[t_1,t_5]}, \alpha^{[t_1,t_4]} \to \beta^{[t_4,t_8]}\} \sqsubset^t \{\alpha^{[t_1,t_3]}, \alpha^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_7,t_9]}\}$

The following propositions follow straightforwardly from the definitions formalised in this section.

Proposition 3 $\widehat{\mathbb{K}} \equiv^t \mathbb{K}$

Proposition 4 $\widehat{\mathbb{K}} \sqsubseteq^t \mathbb{K}$

Proposition 5 $\mathbb{K} \sqsubseteq^t \widehat{\mathbb{K}}$ *if and only if* $\mathbb{K} = \widehat{\mathbb{K}}$

The following definition formalises the temporalised set intersection operator. Intuitively, a temporalised sentence that is part of the intersection-operator is a subsentence with maximal time interval from one of both sets whose subsentences belongs—through temporalised in-operator—to the other set.

Definition 10 (*Temporalised Intersection-Operator*) Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two sets of temporalised sentences. The symbol $\cap^t$ identifies the temporalised set intersection-operator such that a temporalised sentence $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ if and only if:

1. $\forall \varphi' \in^t \varphi$ it holds $\varphi' \in^t \mathbb{K}_1$ and $\varphi' \in^t \mathbb{K}_2$, and
2. $\exists \varphi'' \in \mathbb{K}_1$ or $\exists \varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, and
   (a) $\forall \varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \notin^t \mathbb{K}_1$ or $\varphi''' \notin^t \mathbb{K}_2$

Observe that, the first condition stands for ensuring that any temporalised fragment in φ is included in both sets $\mathbb{K}_1$ and $\mathbb{K}_2$. For the other way around, the second condition stands for ensuring that in the case that φ is a fragment itself of a temporalised sentence φ'' belonging to some of the sets $\mathbb{K}_1$ or $\mathbb{K}_2$, the part of φ'' that is not in φ is not contained in some of the sets $\mathbb{K}_1$ or $\mathbb{K}_2$.

Example 10 Given the temporalised belief bases from Exapmle 8:

- $\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_4,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_7,t_9]}\}$
- $\mathbb{K}' = \{\alpha^{[t_2]}, \alpha^{[t_3,t_5]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5,t_8]}\}$

We know $\mathbb{K}' \subset^t \mathbb{K}$ holds. Observe that $\mathbb{K}' \cap^t \mathbb{K} = \mathbb{K}'$. Consider the following temporalised belief base:

- $\mathbb{K}'' = \{\alpha^{[t_3,t_7]}\}$.

Observe that $\mathbb{K}'' \cap^t \mathbb{K} = \{\alpha^{[t_3,t_6]}, \alpha^{[t_3]}, \alpha^{[t_4,t_6]}\}$.

As is natural for the standard set intersection, the temporalised intersection is a commutative operator.

Proposition 6 $\mathbb{K}_1 \cap^t \mathbb{K}_2 = \mathbb{K}_2 \cap^t \mathbb{K}_1$

The following additional propositions explore some important properties that are satisfied by the temporalised intersection-operator.

Proposition 7 $\varphi \in^t (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ *if and only if* $\varphi \in^t \mathbb{K}_1$ *and* $\varphi \in^t \mathbb{K}_2$

Proposition 8 $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ *if and only if* $\varphi_1 \in \mathbb{K}_1$ *and* $\varphi \in \{\varphi_1\} \cap^t \mathbb{K}_2$

The following operation allows removing from a temporalised base all the temporalised instants appearing in a second temporalised base. Intuitively, a temporalised sentence that is part of the minus-operator is a subsentence with maximal time interval from the first sets whose subsentences does not belong—through temporalised in-operator—to the second set.

Definition 11 (*Temporalised Minus-Operator*) Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two sets of temporalised sentences. The symbol $\setminus^t$ identifies the temporalised set minus-operator such that a temporalised sentence $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ if and only if:

1. $\forall \varphi' \in^t \varphi$ it holds $\varphi' \notin^t \mathbb{K}_1$, and
2. $\exists \varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, and
   (a) $\forall \varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \in^t \mathbb{K}_1$

We will refer to $\mathbb{K}_2$ as the objective set and to $\mathbb{K}_1$ as the differential set.

Observe that, the first condition stands for ensuring that every temporalised fragment in φ is not included in set $\mathbb{K}_1$. For the other way around, the second condition stands for ensuring that in the case that φ is a fragment itself of a temporalised sentence φ'' belonging to the set $\mathbb{K}_2$, the part of φ'' that is not in φ is necessarily contained in the set $\mathbb{K}_1$.

Example 11 Consider two temporalised belief bases $\mathbb{K}_1 = \{\alpha^{[t_3,t_5]}, \alpha^{[t_8,t_{11}]}, \alpha^{[t_{15},\infty]}\}$ and $\mathbb{K}_2 = \{\alpha^{[t_1,\infty]}\}$. According to Definition 11, the resulting temporalised belief base would be: $\mathbb{K}_2 \setminus^t \mathbb{K}_1 = \{\alpha^{[t_1,t_2]}, \alpha^{[t_6,t_7]}, \alpha^{[t_{12},t_{14}]}\}$. For instance, for the case of $\alpha^{[t_1,t_2]}$ we have that for any instant t_1 or t_2, α does not hold in $\mathbb{K}_1$ verifying condition 1, that is: for $\alpha^{[t_1]} \in^t \alpha^{[t_1,t_2]}$ it follows $\alpha^{[t_1]} \notin^t \mathbb{K}_1$ and for $\alpha^{[t_2]} \in^t \alpha^{[t_1,t_2]}$ it follows $\alpha^{[t_2]} \notin^t \mathbb{K}_1$. However, being $\varphi'' = \alpha^{[t_1,\infty]} \in \mathbb{K}_2$ the only temporalised sentence in $\mathbb{K}_2$, we have that for every $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ (a) in condition 2 holds. That is, $\alpha^{[t_1,t_2]} \in^t \alpha^{[t_1,\infty]}$, $\alpha^{[t_6,t_7]} \in^t \alpha^{[t_1,\infty]}$ and $\alpha^{[t_{12},t_{14}]} \in^t \alpha^{[t_1,\infty]}$ hold, and for any other timestamp beyond we have that α holds in $\mathbb{K}_1$. That is, for instance, $\alpha^{[t_3,t_4]} \in^t \alpha^{[t_1,\infty]}$ and $\alpha^{[t_3,t_4]} \notin^t \alpha^{[t_1,t_2]}$ and also $\alpha^{[t_3,t_4]} \in^t \mathbb{K}_1$.

Note that a sentence like $\alpha^{[t_2,t_3]} \notin \mathbb{K}_2 \setminus^t \mathbb{K}_1$ given that $\alpha^{[t_3]} \in^t \alpha^{[t_2,t_3]}$ but $\alpha^{[t_3]} \in^t \mathbb{K}_1$ which is therefore violating condition 1, and also given that $\alpha^{[t_2,t_3]} \in \alpha^{[t_1,\infty]}$ but $\alpha^{[t_2]} \notin^t \mathbb{K}_1$ which is therefore violating (a) in condition 2.

Note that a simple condition like $\varphi \in^t \mathbb{K}_2$ and $\varphi \notin^t \mathbb{K}_1$ does not ensure that $\varphi \in^t \mathbb{K}_2 \setminus^t \mathbb{K}_1$. Observe that, in the example above, a sentence like $\alpha^{[t_2,t_4]}$ would conform such a condition but clearly, it should not be included in $\mathbb{K}_2 \setminus^t \mathbb{K}_1$. Nevertheless, this is a property that can be verified only as a consequence from the application of Definition 11.

Proposition 9 *If* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\varphi \in^t \mathbb{K}_2$ *and* $\varphi \notin^t \mathbb{K}_1$

Proposition 10 *If* $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\varphi \in^t \mathbb{K}_2$ *and* $\varphi \notin^t \mathbb{K}_1$

The following are important properties of the temporalised minus-operator (and its relation to the temporalised intersection, in some cases). Those and other properties will serve afterwards for proving the representation theorem of the dynamic theory proposed in this work.

Proposition 11 *If* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$

Proposition 12 *If* $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$

Proposition 13 *If* $\varphi \in \mathbb{K}_2$ *and* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$ *then* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$

Proposition 14 $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *if and only if* $\varphi \in^t \mathbb{K}_2$ *and* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$

Proposition 15 *If* $\mathbb{K} \cap^t \mathbb{K}_1 = \mathbb{K} \cap^t \mathbb{K}_2$ *then* $\mathbb{K} \setminus^t \mathbb{K}_1 = \mathbb{K} \setminus^t \mathbb{K}_2$

Proposition 16 $(\mathbb{K}_2 \setminus^t \mathbb{K}_1) \subseteq (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$

Proposition 17 *If* $\mathbb{K} \cap^t \mathbb{K}_1 = \emptyset$ *then* $(\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K} = (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$

It is worth to mention that both temporalised set-operators, intersection (Definition 10) and minus (Definition 11), trigger in its respective resulting set sentences that are valid in a subinterval regarding the original sentence contained in the interacting sets. For instance, if $\varphi \rightarrow \alpha^J \in \mathbb{K}_1 \cap^t \mathbb{K}_2$ then either $\varphi \rightarrow \alpha^Q \in \mathbb{K}_1$ or $\varphi \rightarrow \alpha^Q \in \mathbb{K}_2$, with $J \subseteq Q$. A similar situation occurs with literals and with respect of the temporalised set minus-operation. That is, no base normalisation is performed to obtain the result of the required operation, but a simple alteration of time intervals. This decision serves for rendering the sentences representation and redundancies as close as they are in the original sets, and is important for the observation of the principle of minimal change for constructing a rational belief revision model.

3.5 Temporalised derivation

As is usual in a complete logical framework, beliefs can be recognised simply because they are contained in a belief base or due to their derivation from the application of an inference rule. The temporalised sentences in our temporalised logical framework can be recognised given they are explicitly contained in a temporalised belief base $\mathbb{K}$, or either by being implied by $\mathbb{K}$. For instance, from Example 5 it can be explicitly recognised the existence of the temporalised literal $\alpha^{[t_4]} \in \mathbb{K}$. Nevertheless, a temporalised literal $\beta^{[t_9,t_{11}]}$ can be implicitly identified by following the subset $\{\omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}\}$. Observe that this is reasonable by applying a sort of temporalised variation of the *modus ponens* inference rule, where the antecedent of the temporalised rule is satisfied by the explicit temporalised sentence $\omega^{[t_3]}$. Next, we formalise the notion of *temporalised derivation* which allows capturing such a dynamic behavior of our temporalised logical framework.

Definition 12 (*Temporalised Derivation*) Let $\mathbb{K}$ be a set of temporalised sentences. The operator $Cn^t(\mathbb{K})$ is the *set of temporalised consequences* of $\mathbb{K}$ such that a temporalised sentence $\varphi \in Cn^t(\mathbb{K})$ if and only if it follows:

1. $\varphi \in^t \mathbb{K}$, or
2. $(\psi \rightarrow \varphi) \in Cn^t(\mathbb{K})$ and $\psi \in Cn^t(\mathbb{K})$.

A sentence φ is a temporalised derivation from $\mathbb{K}$, by writing $\mathbb{K} \vdash^t \varphi$ if and only if $\varphi \in Cn^t(\mathbb{K})$.

The definition of temporalised derivation is contructed by relying upon the usage of the temporalised in-operator from Definition 6, along with the application of the *modus ponens* inference rule.

Example 12 Consider the temporalised belief base from Example 5:
$\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_2,t_6]}, \beta^{[t_4,t_5]}, \beta^{[t_5,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]}, \delta^{[t_1,t_3]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}$

Observe that $\beta^{[t_8,t_{12}]}$ is a temporalised derivation from $\mathbb{K}$. This is so since, $\beta^{[t_8,t_{10}]}$ $\beta^{[t_8,t_{10}]} \in \mathbb{K}$, which implies $\beta^{[t_8,t_{10}]} \in^t \mathbb{K}$, and thus $\beta^{[t_8,t_{10}]} \in Cn^t(\mathbb{K})$ (first condition); and we also know $\beta^{[t_{11},t_{12}]} \in Cn^t(\mathbb{K})$ given that, from the second condition, $\delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]} \in Cn^t(\mathbb{K})$ and $\delta^{[t_2]} \in Cn^t(\mathbb{K})$ holds since $\delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]} \in \mathbb{K}$ and thus $\delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]} \in^t \mathbb{K}$, and $\delta^{[t_2]} \in^t \mathbb{K}$ since $\delta^{[t_1,t_3]} \in \mathbb{K}$.

The following expressions also hold:

- $\mathbb{K} \vdash^t \beta^{[t_8,t_{14}]}$, or equivalently $\beta^{[t_8,t_{14}]} \in Cn^t(\mathbb{K})$,
- $\mathbb{K} \vdash^t \alpha^{[t_1,t_4]}$, or equivalently $\alpha^{[t_1,t_4]} \in Cn^t(\mathbb{K})$, and
- $\mathbb{K} \vdash^t \{\alpha^{[t_1,t_4]}, \beta^{[t_8,t_{14}]}\}$, or equivalently $\{\alpha^{[t_1,t_4]}, \beta^{[t_8,t_{14}]}\} \subseteq Cn^t(\mathbb{K})$.

Similar to the property shown in Proposition 2, the following proposition states the base-case for the temporalised derivation: any temporalised sentence that is part of a belief base is also part of its temporalised consequences.

Proposition 18 *if* $\varphi \in \mathbb{K}$ *then* $\varphi \in Cn^t(\mathbb{K})$

It is natural to have that the set of temporalised consequences of a belief base coincides with the set of temporalised consequences of its normalised set. This shows the logical equivalence between both temporalised belief bases.

Proposition 19 $Cn^t(\mathbb{K}) = Cn^t(\widehat{\mathbb{K}})$

The following property is important for the characterization of the revision operator proposed in the following sections. Intuitively, it verifies that the temporalised minus-operator will trigger identical results when applied upon the same objective set regarding two differential temporalised bases which are known to be logically equivalent, i.e., two differential temporalised bases with identical temporalised derivation sets.

Proposition 20 *The temporalised minus-operator* " $\setminus^t$ " *is a well defined function*[3] *for a fixed objective set.*

In our temporalised logical framework, a *contradiction* arises when two complementary sentences with overlapping intervals can be derived. For instance, let $\mathbb{K} = \{\alpha^{[t_2,t_9]}, \neg\alpha^{[t_1,t_3]}\}$, here it is easy to recognise the existence of a contradiction in time instants t_2 and t_3. This implies that $\mathbb{K}$ is an inconsistent temporalised belief base.

Definition 13 (*Temporalised Consistency*) Let $\mathbb{K}$ be a temporalised belief base. We say $\mathbb{K}$ is temporally consistent if and only if for every pair $\alpha^{[t_i]} \in Cn^t(\mathbb{K})$ and $\beta^{[t_i]} \in Cn^t(\mathbb{K})$, it holds $\{\alpha, \beta\} \nvdash \bot$. We write $\mathbb{K} \nvdash^t \bot$ (or, $\mathbb{K} \vdash^t \bot$) for stating that $\mathbb{K}$ is temporalised consistent (respectively of, inconsistent).

[3] A function is said to be well defined (or unambiguous) if for each input provided it corresponds a unique output value.

The dynamic model for legal revision that we are proposing here pursues the consistency protection of the temporalised belief base when a revision is needed. This is important for the context to which our theory is intended to be situated. In the following section we show an applied example upon the legal domain.

4 Applied legal revision

In this section we consider again the legal example previously introduced in Sect. 2. Upon the logical framework developed in Sect. 3, we illustrate the application of a legal revision operator to deal with the legal problem introduced. This applied legal example clarifies the goals of our revision process, and it leads to the discussion of rational behavior and the subsequent formalisation of the legal revision operator.

Example 13 (Continues from Exapmle 1) It is possible to formalize the system as follows:

$$
\begin{aligned}
\mathbb{K} = \{ & \\
& unemployed^{[1980,1985]}, \\
& taxation\_law\_validity^{[1980,\infty]}, \\
& unemployed^{[1980]} \wedge taxation\_law\_validity^{[1980]} \rightarrow \quad exemption^{[1980]} \\
& unemployed^{[1981]} \wedge taxation\_law\_validity^{[1981]} \rightarrow \quad exemption^{[1981]} \\
& \vdots \\
& unemployed^{[1985]} \wedge taxation\_law\_validity^{[1985]} \rightarrow \quad exemption^{[1985]} \\
\} &
\end{aligned}
$$

Hence, we can infer exemption through 1980 to 1985:

$$
\begin{gathered}
exemption^{[1980]} \in Cn^t(\mathbb{K}), \\
\vdots \\
exemption^{[1985]} \in Cn^t(\mathbb{K}),
\end{gathered}
$$

Or equivalently:

$$exemption^{[1980,1985]} \in Cn^t(\mathbb{K}).$$

Nevertheless, due to economical crisis, application of taxation law was exceptionally suspended for years 1985 and 1986. This means that a new temporalised sentence like $\varphi = \neg taxation\_law\_validity^{[1985,1986]}$ needs to be incorporated to the base. To that end, we will assume a simple (prioritized) revision operator "$*$". This would render the following revised belief base:

$$
\begin{aligned}
\mathbb{K} * \varphi = \{ & \\
& unemployed^{[1980,1985]}, \\
& taxation\_law\_validity^{[1980,1984]}, \\
& \neg taxation\_law\_validity^{[1985,1986]}, \\
& taxation\_law\_validity^{[1987,\infty]}, \\
& unemployed^{[1980]} \wedge taxation\_law\_validity^{[1980]} \rightarrow \quad exemption^{[1980]} \\
& unemployed^{[1981]} \wedge taxation\_law\_validity^{[1981]} \rightarrow \quad exemption^{[1981]} \\
& \vdots \\
& unemployed^{[1985]} \wedge taxation\_law\_validity^{[1985]} \rightarrow \quad exemption^{[1985]} \\
\} &
\end{aligned}
$$

Finally, it is clear that the tax exemption cannot be applied for 1985. Hence:

$$exemption^{[1980,1984]} \in Cn^t(\mathbb{K} * \varphi).$$

5 Temporal belief revision postulates

In this section, we will propose a basic set of rationality, and temporalised, postulates upon which we can characterize any prioritized legal revision defined upon the temporalised language that we have defined in Sect. 3. In general, the postulates we are going to propose here, follow the intuitions discussed in articles like (Hansson 1994, 1999; Wassermann 2000; Moguillansky et al. 2012; Moguillansky and Tamargo 2021).

The simplest way to introduce a new sentence into a belief base is to add it set theoretically. A non-closing expansion can be represented through an operation $\mathbb{K} \cup \{\varphi\}$. However, this is suitable only when the new belief does not contradict the original belief base. For instance, for a belief base containing $\neg\varphi$, the outcome of the expansion by φ is a belief base containing both $\neg\varphi$ and φ. To avoid such inconsistent results it is necessary to construct a more complex revision operator. We will denote through the symbol "$*$" to a general temporalised revision operator.

Just like the expansion, the revision operator should guarantee that the incoming temporalised literal ends up being part of the resulting belief base.

(Success) $\quad \varphi \in \mathbb{K}*\varphi$

However, contrary to expansion, the revision operation should avoid inconsistencies. It is important to keep in mind that postulate serves for characterising the revision operator by interacting with other postulates. Hence, we need to ensure that the postulates will serve all together to that end. Consequently, we need to ensure that the incoming temporalised sentence is not contradictory by itself since this would violate (Success).

(Consistency) if $\varphi \nvdash^t \bot$ then $\mathbb{K} * \varphi \nvdash^t \bot$

The revised belief base should consist of the incoming temporalised sentence along with those temporalised sentences from the original belief base that have not been excluded in order to make room for the incoming belief in a consistent manner. It follows that the revision should be *temporally* included in the expansion, *id est*, through the time instants of the temporalised sentences. Observe that $\mathbb{K}$ is assumed consistent by definition.

(Inclusion) $\mathbb{K} * \varphi \subseteq^t \mathbb{K} \cup \{\varphi\}$

However, none of the three postulates introduced so far serves to prevent unnecessarily large losses from the original belief base $\mathbb{K}$. Indeed, an operation such that $\mathbb{K} * \varphi = \{\varphi\}$ for all φ, is compatible with the three postulates. Therefore, we need to ensure that nothing is lost from $\mathbb{K}$ unless its exclusion serves to make room for φ. For this purpose we can use the following postulate.

(Core-Retainment) if $\psi \in \mathbb{K} \setminus^t \mathbb{K} * \alpha^J$ then there is some $\mathbb{K}' \subseteq^t \mathbb{K}$ such that $\bot \notin Cn^t(\mathbb{K}' \cup \{\alpha^J\})$ and $\bot \in Cn^t(\mathbb{K}' \cup \{\alpha^J, \psi\})$

As usual, revision operators are intended to incorporate a belief that contradicts the objective belief base. However, from a formal viewpoint, it is important to define these operators for all sentences. We need a postulate to model such a situation. That is, no previous beliefs had to be removed in order to accept the new incoming belief preserving the *(Consistency)* postulate.

(Vacuity) If $\mathbb{K} \nvdash^t \neg\varphi$ then $\mathbb{K} * \varphi = \mathbb{K} \cup \{\varphi\}$

Nevertheless, *(Vacuity)* can be shown to follow from the previous postulates, therefore, it is natural to exclude it from the basic set of postulates upon which we are going to rely towards the rationality characterization of the revision model presented in this work.

Proposition 21 *If an operator "$*$" for $\mathbb{K}$ satisfies (Success)., (Inclusion), and (Core-Retainment), then it satisfies (Vacuity).*

Once again, we are interested in modelling a revision operator for an entirely temporalised logical framework. Thus, the decision regarding which elements of the original belief base $\mathbb{K}$ to retain will depend on their logical relations to the incoming information. Therefore, if two sentences are temporally inconsistent in the same time interval with the same subsets of $\mathbb{K}$ then they should push out the same temporalised sentences from $\mathbb{K}$.

(Uniformity) For all $\mathbb{K}' \subseteq^t \mathbb{K}$, if $(\mathbb{K}' \cup \{\alpha^J\}) \vdash^t \bot$ *iff* $(\mathbb{K}' \cup \{\beta^J\}) \vdash^t \bot$ then $\mathbb{K} \cap^t (\mathbb{K}*\alpha^J) = \mathbb{K} \cap^t (\mathbb{K}*\beta^J)$

This closes the basic set of temporalised postulates we propose for this work. In the following section we define the construction of the temporalised belief revision operator and afterwards, the corresponding representation theorem will follow by relying upon the basic set of postulates for ensuring the rationality characterization of our model.

6 Legal belief revision

From a rational viewpoint, a legal system should preserve consistency or either should deal with inconsistencies to reason in a rational manner. In this work, we pursue a fully consistent logical framework. Since we are dealing with time modalities, consistency preservation implies that a belief base after being modified by a revision operator cannot contain contradictory norms at any time. To that end, we propose a *prioritised legal revision operator*, that is, a legal revision operator that prioritises the full acceptance of the new incoming belief.

Our legal revision operator is applied upon the temporalised logical framework developed in Sect. 3. Following the concept of temporalised consistency, the revision operator should push out the possibility of derivation of every temporalised sentence in all the specific instants in which otherwise, the incorporation of the incoming temporalised literal would provoke contradictions. Observe that the revision operator that we formalise in this article, pursues only the consistent incorporation of a temporalised literal. Such a design decision carries out an expressivity reduction in order to control the complexity of our theory. Ongoing work defines a more expressive revision operator to deal with the incorporation of full temporalised sentences and even with a multiple revision operator to incorporate complete normative proposals. This will be discussed later in Sect. 8.

In order to incorporate a temporalised literal $\neg\alpha^J$ into a legal system, it is necessary to break the derivation of the complementary literal α^J. Observe that it is enough to derive α in a single time instant in the interval J to trigger inconsistency. Thus, we need some operation for allowing the recognition of all the sentences that are valid inside the given time interval. Therefore, we are not really looking for the temporalised literal α^J but for all the contained intervals in J validating the literal α. To that purpose, we formalise the concept of *time-maximising sub-sentence of a time interval* as follows.

Definition 14 (*Time-Maximising sub-sentences*) Let $\mathbb{K}$ be a temporalised belief base and α^J a temporalised sentence. A time-maximising sub-sentence of α^J in $\mathbb{K}$ is a temporalised sentence α^Q if and only if it holds:

1. $\alpha^Q \in Cn^t(\mathbb{K})$ with $Q \subseteq J$, and
2. there is no $\alpha^R \in Cn^t(\mathbb{K})$ such that $Q \subset R \subseteq J$.

The set $^{\sqsubseteq}(\alpha^J, \mathbb{K})$ returns the set of all the time-maximising sub-sentences of α^J in $\mathbb{K}$.

The following example illustrates in an intuitive manner the concept of time-maximising sub-sentences defined before.

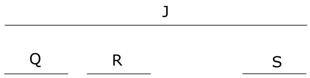

Fig. 2 Time-maximising intervals for a set of time-maximising sub-sentences of α^J

Example 14 Let us assume we have a temporalised belief base $\mathbb{K}$ where a sentence α is valid in different instants of time within an interval J, however, $\alpha^J \notin Cn^t(\mathbb{K})$. Assume that such small pieces of time trigger, through interval composition, three time-maximising intervals within J, say intervals Q, R, and S (see Fig. 2). Thus, $\alpha^Q, \alpha^R, \alpha^S \in Cn^t(\mathbb{K})$. For recognizing such time-maximising intervals where α is valid within an arbitrary interval J, we refer to the notion of time-maximising sub-sentences of α^J, that is, $^{\sqsubseteq}(\alpha^J, \mathbb{K}) = \{\alpha^Q, \alpha^R, \alpha^S\}$.

Regarding interval composition, suppose that $Q = [u_i, t_f]$, and if $\alpha^U \in \mathbb{K}$ and $\alpha^T \in \mathbb{K}$, where $U = [u_i, u_j]$ and $T = [u_j, t_f]$, that is, $U \rhd T$, and $U, T \subseteq J$, then $\alpha^Q \in {}^{\sqsubseteq}(\alpha^J, \mathbb{K})$, but $\alpha^U, \alpha^T \notin {}^{\sqsubseteq}(\alpha^J, \mathbb{K})$ since that would violate the time-maximality condition (see cond. 2, from Definition 14).

In the following example we apply the definition of time-maximising sub-sentences in the formal context of a concrete temporalised belief base.

Example 15 Consider the temporalised belief base from Example 5:

$$\begin{aligned}\mathbb{K} = \{&\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_2,t_6]},\\ &\beta^{[t_4,t_5]}, \beta^{[t_5,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]}, \delta^{[t_1,t_3]}, \delta^{[t_2]}\\ &\to \beta^{[t_{11},t_{12}]}, \delta^{[t_2]} \to \beta^{[t_{12},t_{14}]}, \omega^{[t_3]},\\ &\omega^{[t_3]} \to \beta^{[t_9,t_{11}]}, \delta^{[t_3]} \to \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}\end{aligned}$$

According to Definition 14, the set of time-miximising sub-sentences of $\beta^{[t_5,t_{30}]}$ is:

$$^{\sqsubseteq}(\beta^{[t_5,t_{30}]}, \mathbb{K}) = \{\beta^{[t_5,t_6]}, \beta^{[t_8,t_{14}]}, \beta^{[t_{16},t_{21}]}\}.$$

The next propositions follow straightforward from Definition 14.

Proposition 22 $\alpha^J \in Cn^t(\mathbb{K})$ *if and only if* $^{\sqsubseteq}(\alpha^J, \mathbb{K}) = \{\alpha^J\}$

Proposition 23 *If* $\{\alpha^R, \alpha^Q\} \subseteq {}^{\sqsubseteq}(\alpha^J, \mathbb{K})$ *then* $R \bowtie Q$

Observe that since a set of time-maximising sub-sentences contains only non-overlapping temporalised literals, it satisfies the conditions of a temporal-normalised set in accordance to Definition 5.

Proposition 24 $^{\sqsubseteq}(\alpha^J, \mathbb{K}) = \widehat{^{\sqsubseteq}(\alpha^J, \mathbb{K})}$

As being commented before, the purpose of recognising the set of time-maximising sub-sentences of the complementary literal to be incorporated by the revision operator, is to break all the possible derivations of each sub-sentence towards consistency preservation. Thus, before we can formalise an operator that allows performing such derivation breakage, we need to recognise the portions in a belief base that allows derivating a temporalised literal. To that end, we propose the following construction for covering the concept of *minimal proof*.

Definition 15 (*Minimal Proof*) Let $\mathbb{K}$ be a temporalised belief base and α^J a temporalised sentence. The set $\mathbb{H}$ is a minimal proof of α^J, if and only if it holds:

1 $\mathbb{H} \sqsubseteq^t \mathbb{K}$, and
2 $\alpha^J \in Cn^t(\mathbb{H})$, and
3 if $\mathbb{H}' \sqsubset^t \mathbb{H}$ then for any $\alpha^R \in Cn^t(\mathbb{H}')$, it holds $R \subset J$.

Given a temporalised sentence α^J, the function $\vDash(\alpha^J, \mathbb{K})$ returns the set of all the minimal proofs for α^J from $\mathbb{K}$.

The previous concept of minimal proof is inspired by the notion of *kernel sets*, first introduced by Hansson (1994) who was inspired beforehand by the notion of *entailment sets* introduced by Fuhrmann (1991). The following example illustrates their construction upon our running example.

Example 16 Consider the temporalised belief base from Example 5:

$$\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_2,t_6]}, \beta^{[t_4,t_5]}, \beta^{[t_5,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]}, \delta^{[t_1,t_3]}, \delta^{[t_2]} \\ \to \beta^{[t_{11},t_{12}]}, \delta^{[t_2]} \to \beta^{[t_{12},t_{14}]}, \omega^{[t_3]}, \omega^{[t_3]} \to \beta^{[t_9,t_{11}]}, \delta^{[t_3]} \to \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}$$

Applying Definition 15, the set of minimal proofs for $\beta^{[t_5,t_6]}$ is $\vDash(\beta^{[t_5,t_6]}, \mathbb{K}) = \{\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3, \mathbb{H}_4\}$ where:

- $\mathbb{H}_1 = \{\alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5,t_6]}\}$
- $\mathbb{H}_2 = \{\beta^{[t_5,t_6]}\}$
- $\mathbb{H}_3 = \{\alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5]}, \beta^{[t_6]}\}$
- $\mathbb{H}_4 = \{\beta^{[t_5]}, \alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \to \beta^{[t_6]}\}$

In what follows, the *principle of minimality* will be underlying each piece of theory defining our model for legal dynamics. Minimality is one of the most important principles followed in belief revision for ensuring that as little as possible changes will be applied to the original belief base in order to achieve a successful change operation. This sort of principles is usually followed in order to provide an appropriate construction of the dynamic model which behaves in a rational manner. Such rationality is concretised by providing a basic set of postulates—proposed in Sect. 5 for our theory. Hence, in the case of the theory here proposed, minimality provides a controlled tool for ensuring that only the necessary temporalised sentences, in only the precise time intervals, will be finally pushed out from the resulting revised belief base.

The following definition formalises the construction of a set containing all the minimal proofs of every time-maximising sub-sentence of a given temporalised literal α^J. This set will contain all the minimal proofs that we need to break in order to incorporate the complementary literal in a consistent manner: the *zero tolerance set* for α within J.

Definition 16 (*Zero Tolerance Set*) Let $\mathbb{K}$ be a temporalised belief base and α^J a temporalised sentence. The set $\Pi(\alpha^J, \mathbb{K})$ is the zero tolerance set for α within J if and only if it holds:

$$\Pi(\alpha^J, \mathbb{K}) = \{\mathbb{H} \in {}^{\models}(\varphi, \mathbb{K}) \mid \varphi \in {}^{\sqsubseteq}(\alpha^J, \mathbb{K})\}$$

The following example shows in a very simple and trivial manner the construction of the set of minimal proofs for a persistent literal, that is, a literal which is valid in an infinite time interval. Afterwards, Example 16 shows its application on the more complex running example of this section.

Example 17 Consider the temporalised belief base $\mathbb{K} = \{\alpha^{[t_1,t_5]}, \alpha^{[t_8,t_{11}]}, \alpha^{[t_{15},\infty]}\}$. The zero tolerance set for α within $[t_3, \infty]$ is $\Pi(\alpha^{[t_3,\infty]}, \mathbb{K}) = \{\{\alpha^{[t_3,t_5]}\}, \{\alpha^{[t_8,t_{11}]}\}, \{\alpha^{[t_{15},\infty]}\}\}$. Note that, according to Definition 14, the corresponding set of time-maximising sub-sentences would be: ${}^{\sqsubseteq}(\alpha^{[t_3,\infty]}, \mathbb{K}) = \{\alpha^{[t_3,t_5]}, \alpha^{[t_8,t_{11}]}, \alpha^{[t_{15},\infty]}\}$. Afterwards, the construction of each proof in $\Pi(\alpha^J, \mathbb{K})$ is trivial. Observe that a possible proof for $\alpha^{[t_3,t_5]}$ can be the set $\mathbb{H}' = \{\alpha^{[t_3,t_4]}, \alpha^{[t_5]}\}$, however it is not minimal time-maximising: although $\mathbb{H}' \subseteq^t \mathbb{K}$ holds, it is not normalised, and thus $\mathbb{H} \sqsubseteq^t \mathbb{K}$ does not hold, violating condition 1 in Definition 15.

Example 18 Consider the temporalised belief base from Example 5:

$$\mathbb{K} = \{\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_2,t_6]}, \beta^{[t_4,t_5]}, \beta^{[t_5,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]}, \delta^{[t_1,t_3]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}$$

From Example 15 we have ${}^{\sqsubseteq}(\beta^{[t_5,t_{30}]}, \mathbb{K}) = \{\beta^{[t_5,t_6]}, \beta^{[t_8,t_{14}]}, \beta^{[t_{16},t_{21}]}\}$. Therefore, according to Definition 16, the zero tolerance set for β within $[t_5, t_{30}]$ is $\Pi(\beta^{[t_5,t_{30}]}, \mathbb{K}) = {}^{\models}(\beta^{[t_5,t_6]}, \mathbb{K}) \cup {}^{\models}(\beta^{[t_8,t_{14}]}, \mathbb{K}) \cup {}^{\models}(\beta^{[t_{16},t_{21}]}, \mathbb{K}) = \{\mathbb{H}_1, \dots, \mathbb{H}_4\} \cup \{\mathbb{H}_5, \dots, \mathbb{H}_{12}\} \cup \{\mathbb{H}_{13}, \dots, \mathbb{H}_{20}\} = \{\mathbb{H}_1, \dots, \mathbb{H}_{20}\}$ where:

- $\mathbb{H}_1 = \{\alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_5,t_6]}\}$
- $\mathbb{H}_2 = \{\beta^{[t_5,t_6]}\}$
- $\mathbb{H}_3 = \{\alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_5]}, \beta^{[t_6]}\}$
- $\mathbb{H}_4 = \{\beta^{[t_5]}, \alpha^{[t_1,t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_6]}\}$
- $\mathbb{H}_5 = \{\beta^{[t_8,t_{10}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{14}]}\}$
- $\mathbb{H}_6 = \{\beta^{[t_8,t_{10}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_{11}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}\}$
- $\mathbb{H}_7 = \{\beta^{[t_8,t_9]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_{10}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{14}]}\}$
- $\mathbb{H}_8 = \{\beta^{[t_8,t_9]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_{10},t_{11}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}\}$
- $\mathbb{H}_9 = \{\beta^{[t_8]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{10}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{14}]}\}$

- $\mathbb{H}_{10} = \{\beta^{[t_8]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}\}$
- $\mathbb{H}_{11} = \{\beta^{[t_8]}, \beta^{[t_{10}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{14}]}\}$
- $\mathbb{H}_{12} = \{\beta^{[t_8]}, \beta^{[t_{10}]}, \omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9]}, \omega^{[t_3]} \rightarrow \beta^{[t_{11}]}, \delta^{[t_2]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]}\}$
- $\mathbb{H}_{13} = \{\beta^{[t_{16},t_{19}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{20},t_{21}]}\}$
- $\mathbb{H}_{14} = \{\beta^{[t_{16},t_{18}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{19},t_{21}]}\}$
- $\mathbb{H}_{15} = \{\beta^{[t_{16},t_{17}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{18},t_{21}]}\}$
- $\mathbb{H}_{16} = \{\beta^{[t_{16}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{21}]}\}$
- $\mathbb{H}_{17} = \{\beta^{[t_{16}]}, \beta^{[t_{18}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{19},t_{21}]}\}$
- $\mathbb{H}_{18} = \{\beta^{[t_{16}]}, \beta^{[t_{18},t_{19}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{20},t_{21}]}\}$
- $\mathbb{H}_{19} = \{\beta^{[t_{16}]}, \beta^{[t_{19}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{18}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{20},t_{21}]}\}$
- $\mathbb{H}_{20} = \{\beta^{[t_{16},t_{17}]}, \beta^{[t_{19}]}, \delta^{[t_3]}, \delta^{[t_3]} \rightarrow \beta^{[t_{18}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{20},t_{21}]}\}$

The following property is of utmost relevance for recognising if there are temporalised literals triggering inconsistencies in the same time intervals with the same subsets of the belief base. This is useful to ensure that the dynamic model will push out the same temporalised sentences from the base in such situations. This property allows interconnecting the model of legal revision here proposed and the postulate of *(Uniformity)* which will be referred afterwards for showing the representation theorem of our model.

Proposition 25 *The following conditions are equivalent*:

1. $\Pi(\alpha^J, \mathbb{K}) = \Pi(\beta^J, \mathbb{K})$
2. *for all* $\mathbb{K}' \subseteq^t \mathbb{K}, \mathbb{K}' \cup \{\neg\alpha^J\} \vdash^t \perp$ *iff* $\mathbb{K}' \cup \{\neg\beta^J\} \vdash^t \perp$

The approach we follow in this model is the selection of sentences that contribute to the violation of consistency when incorporating the new incoming belief. The objective of such selection is to discard such sentences from the resulting belief base. This approach was originally proposed by Alchourrón and Makinson (1985) for introducing the operation of *safe contraction*. A more general variant of the same approach, *kernel contraction*, was introduced much later by Hansson (1994). There, the notion of *incision function* was originally coined given that it makes an incision into each kernel set in order to break each minimal proof. For that purpose, the sentences being selected by the incision function have to be discarded afterwards.

In our approach, the notion of incision is slightly different. Each proof—or kernel—is minimal with regards to a temporalised literal in a time-maximising interval. That means that, in order to break—or to make an appropriate incision to—each kernel, we need to make sure that *each instant of the interval* corresponding to each kernel, has been incised by the function.

Definition 17 (*Instant Incision function*) Let $\mathbb{K}$ be a temporalised belief base and α^J a temporalised sentence. An operator σ is a α^J-incision function for $\mathbb{K}$ if and only if the following conditions hold:

1. $\sigma(\Pi(\alpha^J, \mathbb{K})) \subseteq \bigcup(\Pi(\alpha^J, \mathbb{K}))$

2. if $\mathbb{H} \in \Pi(\alpha^J, \mathbb{K})$ then there is no $t_i \in J$ such that $\alpha^{[t_i]} \in Cn^t(\mathbb{H} \setminus^t \sigma(\Pi(\alpha^J, \mathbb{K})))$
3. if $\mathbb{K}' \subset^t \sigma(\Pi(\alpha^J, \mathbb{K}))$ then there is some $t_i \in J$ such that $\alpha^{[t_i]} \in Cn^t(\mathbb{K} \setminus^t \mathbb{K}')$

Lets explore the three conditions in Definition 17. The first one just give the context, it serves for indicating that the incision will not output sentences that are not included in any minimal proof. The second condition has the objective to ensure that every proof $\mathbb{H}$ will be cut off. That means that there will be no instant inside J where α can be derived from the remainder of $\mathbb{H}$—after retiring from it the output sentences of the incision. Finally, the third condition stands for observing minimality of the construction of the incision function. That is, every sentence inside the incision is mandatory to preserve the second condition. Hence, if instead of the incision we consider some strict-subset of it, there will appear an instant inside J where α can be derived from the remainder of the base—after retiring $\mathbb{K}'$ from it—thus violating the second condition. The following example illustrates the application of the instant incision function upon the running example in this section.

Example 19 Considering Example 18, the following set is a valid construction of an instant incision for $\mathbb{K}$:

$$\begin{aligned}\sigma(\Pi(\beta^{[t_5,t_{30}]}, \mathbb{K})) = \{&\beta^{[t_5,t_6]}, \alpha^{[t_1,t_4]} \to \beta^{[t_5,t_6]},\\ &\beta^{[t_8,t_{10}]}, \delta^{[t_2]} \to \beta^{[t_{11},t_{14}]}, \omega^{[t_3]} \to \beta^{[t_9,t_{11}]},\\ &\beta^{[t_{16},t_{19}]}, \delta^{[t_3]} \to \beta^{[t_{17},t_{21}]}\}\end{aligned}$$

The two main sub-operations of a revision are, firstly the addition of the new incoming belief to the belief base, and secondly, to ensure that the resulting belief base ends up in a consistent epistemic state—this is so, unless the new belief to incorporate is inconsistent. The first sub-operation can be accomplished through the expansion commented before, in Sect. 5. The second sub-operation can be accomplished through the contraction by the complement of the incoming belief. This follows from this reasoning: an epistemic state that does not imply a certain belief can consistently accept the addition of its complementary belief. The operator of revision can therefore be constructed out of such two sub-operations. This composition of sub-operations is expressed by the *Levi identity* (Levi 1977), which specifies that $\mathbb{K} * \alpha = (\mathbb{K} - \neg\alpha) \cup \alpha$. For the model of change presented in this article, the operation of contraction can be accomplished by applying to the belief base, the temporalised minus-operator by the instant incision function. Thus, the Levi identity can be translated into our temporalised framework as is specified next.

Definition 18 Let $\mathbb{K}$ be a consistent temporalised belief base and α^J be a temporalised sentence. The operator "$\otimes$" is a prioritised legal revision operator if and only if:

$$\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$$

Where the operator "σ" stands for an instant incision function.

Technically speaking, the contraction applied through the incision of proofs, allows discarding temporalised literals. Observe however, that in our theory, a different interpretation of this sub-operation is that the incision of proofs may just end up modifying some time intervals. (Of course, the reduction of intervals implies pushing out beliefs from the base.) This is made clearer in the following example, for instance regarding the literal β, while it was valid in the interval $[t_4, t_6]$, that is $\beta^{[t_4,t_6]} \in^t \mathbb{K}$, its interval ends up reduced to an instant t_4 after the revision is applied, that is $\beta^{[t_4]} \in^t \mathbb{K} \otimes \neg\beta^{[t_5,t_{30}]}$.

Example 20 Considering the temporalised belief base $\mathbb{K}$ from Example 5:

$$\begin{aligned}\mathbb{K} =\{&\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_2,t_6]},\\ &\beta^{[t_4,t_5]}, \beta^{[t_5,t_6]}, \beta^{[t_8,t_{10}]}, \beta^{[t_{16},t_{19}]},\\ &\delta^{[t_1,t_3]}, \delta^{[t_2]} \rightarrow \beta^{[t_{11},t_{12}]}, \delta^{[t_2]} \rightarrow \beta^{[t_{12},t_{14}]},\\ &\omega^{[t_3]}, \omega^{[t_3]} \rightarrow \beta^{[t_9,t_{11}]}, \delta^{[t_3]} \rightarrow \beta^{[t_{17},t_{21}]}, \epsilon^{[t_1,\infty]}\}\end{aligned}$$

and continuing Example 19, suppose that we need to incorporate a new temporalised literal $\neg\beta^{[t_5,t_{30}]}$. The revision operation $\mathbb{K} \otimes \neg\beta^{[t_5,t_{30}]}$ is applied:

$$\begin{aligned}\mathbb{K} \otimes \neg\beta^{[t_5,t_{30}]} =&(\mathbb{K} \setminus^t \sigma(\varPi(\beta^{[t_5,t_{30}]}, \mathbb{K}))) \cup \{\neg\beta^{[t_5,t_{30}]}\}\\ =&\{\alpha^{[t_1,t_3]}, \alpha^{[t_4]}, \alpha^{[t_1,t_4]} \rightarrow \beta^{[t_2,t_4]},\\ &\beta^{[t_4]}, \delta^{[t_1,t_3]},\\ &\omega^{[t_3]}, \epsilon^{[t_1,\infty]}, \neg\beta^{[t_5,t_{30}]}\}\end{aligned}$$

As an alternative to the Levi identity, it is possible to accomplish the revision operation by inverting the order of both sub-operations. This alternative has as side effect the fact that the revision operation passes through an inconsistent intermediate epistemic state. This is so, since the addition of the incoming belief takes place before contracting the derivation of its complement. For a hypothetical implementation of our theory, and for the legal domain in general, such situation may not be desirable, mostly taking into account the possibility of concurrency of the legal system. Nevertheless, for theoretical reasons, the well know *reversed Levi identity* (Hansson 1993) is particularly useful for showing the representation theorem. The reversed identity in general terms is an operation $\mathbb{K} * \alpha = (\mathbb{K} \cup \alpha) - \neg\alpha$. The following proposition formally shows that the reversed identity is equivalent to the revision operator defined before.

Proposition 26 $\mathbb{K} \otimes \alpha^J = (\mathbb{K} \cup \{\alpha^J\}) \setminus^t \sigma(\varPi(\neg\alpha^J, \mathbb{K}))$

The completion of the legal revision operator requires an appropriate characterisation of its behaviour by relying upon the proposed temporalised postulates presented in Sect. 5. Next we present the representation theorem that accomplish with such objective of rationality.

Representation Theorem 1 An operator "$\otimes$ " is a prioritised legal revision for $\mathbb{K}$ if and only if it satisfies the postulates of *(Success)*., *(Consistency)*, *(Inclusion)*, *(Core-Retainment)*, and *(Uniformity)*.

The reader not-well-acquainted with belief revision theories may feel confused about this theorem. A representation theorem has two main objectives. On one direction, it serves for ensuring that the proposal of the revision model behaves in accordance to the given set of postulates. On the other direction, it ensures that for any other construction of a revision operator satisfying the given postulates, its output will coincide with the output of the original revision operator being proposed. In other words, the representation theorem serves for ensuring that satisfying the nominated set of postulates is a necessary and sufficient condition for achieving the objective revision operation.

7 Related work

As stated in the introduction, the main antecedent of this work is Tamargo et al. (2019), where a belief revision operator was proposed using a logical temporalised framework introducing *time intervals* as decorations for literals. In that article, an operator was introduced for modeling norm change in the law, which is considered appropriate for certain level of abstraction. That formalization has, however, some reasonable limitations, as pointed out before. Since the underlying logic uses temporalised rules with a single literal as antecedent, the corresponding temporalised derivation is much simpler than the one presented in this paper. However, such reduced expressivity in Tamargo et al. (2019) is not suitable to represent more realistic legal provisions, like the one presented in Sect. 4. As a consequence, in the present work a more exhaustive temporalised logical framework is studied. This brings the advantage of understanding multiple representations of temporalised belief bases and even to normalise temporalised bases as a way to achieve temporal-equivalence. Upon those notions, temporalised set-operations were presented which are of utmost importance for studying well suitable rationality postulates and a corresponding revision operation. This allows the development of a change model which conforms the original representation for developing the dynamic process towards a revised belief base. The refined theory presented in this work, is better suited for further and more expressive models of legal change.

Some previous works relate to this line of research. Alchourrón, Makinson and Bulygin were probably the first researchers to propose the study of changes in legal codes upon a logical basis (Alchourrón and Makinson 1981, 1982; Alchourrón and Bulygin 1981). They identified three different types of change operations. The *enactment* as the incorporation of a new norm n that provokes the expansion of the code and the implicit derivation of those new regulations appearing from the interaction with n. The *amendment* of the code that results from the process of consistency restitution after the incorporation of a conflicting norm, which may imply rejecting pre-existing norms. Finally, the *derogation* that is managed by removing a norm along with whatever part of the legal code implying such norm. Our scope is different. In our approach, we model cases where the in-force time for an enacted legal provision may differ from its inclusion in the legal system, or from that in which it produces legal effects. Even more, we model cases where a provisions can produce effects in the past, leading to some form of retroactivity.

Founded upon the aforementioned articles, the well known AGM model (Alchourrón et al. 1985) for revision of logical theories was proposed. Besides the fact that in that paper there is no model of time, is that proposal a satisfactory framework for the problem of norm revision? Controversies arised with the AGM operation of contraction. For instance, regarding the recovery postulate as being related to legal changes such as derogations and repeals, which are meant to contract legal effects upon quite different perspectives (Governatori and Rotolo 2010; Wheeler and Alberti 2011). In this regard, standard AGM seems to be surpassed due to its abstract nature: it works with theories of logical assertions which is perhaps more suitable to model the general dynamics of obligations and permissions over the specific one of legal norms. A clear drawback arises for AGM to represent the different forms to contract legal effects in relation of the manner in which norms has changed. To that matter, alternative hybrid models (Governatori et al. 2005b, 2007; Governatori and Rotolo 2010) developed temporal reasoning upon rule-based systems.

Additionally, the AGM model relies on sets of sentences closed under logical consequence (belief sets). Other models use belief bases, i.e., arbitrary sets of sentences (Fuhrmann 1991; Hansson 1992; Wassermann 2000). Our epistemic model is based on an adapted version of belief bases which additionally incorporates time intervals. This leads to a best representation of the legal system and, even more, a computationally tractable one. Following the work of Hansson and Wassermann, we consider that legal systems sentences could be represented by a finite number of sentences that correspond to the explicit beliefs on the legal system. That norm change captured by base revision was also discussed by Governatori and Rotolo (2010).

Other alternatives has been proposed to reframe AGM into more expressive rule-based logical systems. For instance, incorporating defeasible logic (Rotolo 2010; Governatori et al. 2013), input/output logic (Boella et al. 2009; Stolpe 2010) and argumentation (Moguillansky et al. 2012, 2019; Moguillansky and Tamargo 2021). Another alternative was to extend the AGM framework towards iterated belief change, two-dimensional belief change, and weakened contraction (Wheeler and Alberti 2011). In this paper we aimed to extend base revision with temporal reasoning, and, in particular, through the use of time intervals. Intervals were first introduced by Allen (1983), and later applied to alternative scenarios like modal logics (Halpern and Shoham 1991; Della Monica et al. 2011), defeasible logics (Augusto and Simari 2001; Governatori and Terenziani 2007) and abstract argumentation (Budán et al. 2017). While (Governatori and Terenziani 2007) is founded upon duration, periodicity, and several forms of causality, (Augusto and Simari 2001) proposed an hybrid interaction between defeasible reasoning and temporal reasoning. Our approach is able to deal with constituents holding in an interval of time, thus an expression $\alpha^{[t_1,t_2]}$ meaning that α holds between t_1 and t_2 can be seen as a shorthand of the pair of rules from (Governatori and Rotolo 2010) (defeasible and defeater) $\Longrightarrow \alpha^{[t_1,t_2]}$ and $\rightsquigarrow \neg\alpha$. We have made such design decision in order to simplify the construction of the revision operator and thus, the revision operator may sometimes just modify the intervals for restoring consistency.

In relation with belief revision and temporal reasoning there are two prominent lines of investigation: (Bonanno 2007, 2009) and (Shapiro et al. 2011). The former line contrast ours in that they consider sets of sentences closed under logical consequence. Their main purpose is to adapt the AGM postulates to a modal language. Belief revision is therefore proposed for handling information upon time and thus temporal logic is being applied. Branching-time frames are considered to represent alternative evolution of beliefs, which introduces temporalisation to possible worlds. The second mentioned line (Shapiro et al. 2011) is based on an extended theory of action in situation calculus. A notion of plausibility on situations is incorporated, handling nested belief, belief introspection, mistaken belief, belief revision and belief update along with iterated belief change.

8 Conclusions and future work

Beneath the notion of law, a dynamic system of rules can be recognised. In order to provide new norms for society, rules are incorporated to that system. As a consequence, this dynamic process may trigger unexpected conflicts with preexisting rules. This demands an appropriate revision of the system that needs to be capable of restoring its consistency. Some dynamic features of legal reasoning can be captured by considering a temporal dimension applied to the normative elements. However, since the normative system is revised as a consequence of the incorporation of new temporalised sentences, two dynamic aspects of the law must be considered: the change of the set of sentences, and the ability to reason about temporalised knowledge. To that end, we study novel formal models of belief revision under timed sentences as an alternative way to capture this intrinsic dynamism of law.

Following that idea, we introduced a complete interval-based logic framework along with a time-based belief revision operator for legal systems. Intervals allow for the representation of legal knowledge to be effective or relevant in specific periods of time. On these interval-decorated sentences a detailed temporalised logical framework allows reasoning through different timed set operations. Moreover, the consideration of time requires an adaptation of the notions of contradiction and inconsistency in the classical sense. We state that temporalised knowledge base is inconsistent only if contradictory information can be derived for the same instant of time. Hence, if new legal knowledge is added to the system causing the derivation of two complementary sentences at a specific moment, a revision of sentences is required to return to a consistent epistemic state. To that end, we defined a novel belief revision operator that allows for the consistent addition of temporalised sentences in a temporalised belief base. If a contradiction arises, the revision operator may either completely remove conflictive temporalised sentences or accordingly modify the intervals of some of them. This last action is made because a given consequence α at interval I may fall in contradiction during a sub-interval of I. Thus, α should be a consequence after the revision, only for the non-conflictive parts of I.

Inspired by the well known AGM model (Alchourrón et al. 1985), change operators were presented. The underlying model for their construction is followed by a related representation theorem. In that manner, we ensure that the revision proposed behaves in a rational manner according to a proposed set of rationality timed-postulates. Our revision operator is based on Hansson's model of kernel revision (Hansson 1994). Upon such, a selection of sentences is made to push out contradictory temporalised information. As aforementioned, this process may involve an update to the intervals of time where previous legal knowledge has been accepted previous to the incorporation of the new piece of temporalised knowledge.

Ongoing and future work involves the study and development of a family of temporalised revision operators with extended capabilities. On one hand, while the operator proposed in this article is capable of performing the revision by a temporalised literal, it is necessary to extend its usage to full temporalised sentences in a sense that a temporalised rule could be consistently incorporated to the base. In addition, it is natural to study the development of a multiple revision operator which would be capable of incorporating a set of temporalised beliefs that would stand for a complete incoming provision to incorporate to the legal system. On the other hand, while the revision operator here proposed prioritises the complete incorporation of the incoming belief, an alternative approach would imply the conditional incorporation of the incoming belief (non-prioritised approach). That means that the incoming belief could possibly be partially accepted in order to preserve consistency of the resulting temporalised belief base. Its potential usage has to do with a more natural process of legal enactment where the new incoming legal provision would itself require an inner-revision process to be adapted to the legal system when clashing norms conform prior-provisions corresponding to higher order strata in the Kelsen's pyramid model. Towards such formal specification of extended revision operators, it is reasonable to have an exhaustive formalisation of other operators of change interacting in background as is the case of a temporalised contraction, for pushing out specific norms, and a temporalised consolidation, for restituting the consistency with regards to non-prioritising models.

Appendix

Proposition 1 $\widehat{\mathbb{K}} = \widehat{\widehat{\mathbb{K}}}$

Proof From Definition 5 we know that if $\alpha^S, \alpha^T \in \widehat{\mathbb{K}}$ then $S \bowtie T$ and also that if $\varphi_1 \rightarrow \alpha^S \in \widehat{\mathbb{K}}$ and $\varphi_1 \rightarrow \alpha^T \in \widehat{\mathbb{K}}$ then $S \bowtie T$. This means that there is no overlapping set in $\widehat{\mathbb{K}}$ and therefore we have that $\varphi \in \widehat{\mathbb{K}}$ if and only if $\varphi \in \widehat{\widehat{\mathbb{K}}}$. □

Proposition 2 If $\varphi \in \mathbb{K}$ then $\varphi \in^t \mathbb{K}$

Proof Assuming that $\varphi \in \mathbb{K}$ holds, we need to show that $\varphi \in^t \mathbb{K}$ also holds. From Definition 6 we have two alternatives, either (1) $\varphi = (\alpha_1 \wedge \ldots \wedge \alpha_n)^Q$ or (2) $\varphi = \varphi_1 \rightarrow \alpha^Q$.

For (1) we have that $(\alpha_1 \wedge \ldots \wedge \alpha_n)^Q \in^t \mathbb{K}$ (with $n \geq 1$) and thus $\forall_{i=1}^n(\alpha_i^{J_i} \in \widehat{\mathbb{K}}$ and $Q \subseteq J_i)$. However, since we know that $\varphi \in \mathbb{K}$, from Definition 4 we have necessarily that $n = 1$, in consequence we only have that $\varphi = \alpha_1^Q$ and then we need to show that $\alpha_1^Q \in^t \mathbb{K}$. By *reductio ad absurdum*, we will assume that $\alpha_1^Q \notin^t \mathbb{K}$, which therefore, from Definition 6, we have that $\alpha_1^J \notin \widehat{\mathbb{K}}$ holds for any $Q \subseteq J$. Afterwards, from Definitions 5 and 3 we have that there is no overlapping set $\{R_1, \ldots, R_m\}$ from which Q results as the maximising interval composition and thus $\alpha_1^Q \notin \mathbb{K}$ which is absurd.

For (2) we have that $\varphi_1 \rightarrow \alpha^Q \in \mathbb{K}$. By *reductio ad absurdum*, we will assume that $\varphi_1 \rightarrow \alpha^Q \notin^t \mathbb{K}$ and thus, from Definition 6, there is no $\varphi_1 \rightarrow \varphi_2 \in \widehat{\mathbb{K}}$ such that $\alpha^Q \in^t \{\varphi_2\}$. Afterwards, from Definitions 5 and 3 we have that there is no overlapping set $\{R_1, \ldots, R_m\}$ from which Q results as the maximising interval composition and thus $\varphi_1 \rightarrow \alpha_1^Q \notin \mathbb{K}$ which is absurd. □

Proposition 3 $\widehat{\mathbb{K}} \equiv^t \mathbb{K}$

Proof From Definition 8 we need to show that (1) $\widehat{\mathbb{K}} \subseteq^t \mathbb{K}$ and (2) $\mathbb{K} \subseteq^t \widehat{\mathbb{K}}$.

For (1) From Definition 7, we need to show that for every $\varphi \in \widehat{\mathbb{K}}$, it follows $\varphi \in^t \mathbb{K}$. *By reductio ad absurdum* we will assume that there is some $\varphi \in \widehat{\mathbb{K}}$ such that $\varphi \notin^t \mathbb{K}$. Since $\varphi \notin^t \mathbb{K}$, from Definition 6 we know there is no temporalised sentence, be it either a temporalised literal $\alpha^J \notin \widehat{\mathbb{K}}$ or a temporalised rule $\psi \rightarrow \alpha^J \notin \widehat{\mathbb{K}}$, whose interval J contains Q, that is $Q \subseteq J$, for either $\alpha^Q \notin^t \mathbb{K}$ or $\psi \rightarrow \alpha^Q \notin^t \mathbb{K}$. But this is absurd, given that $\varphi \in \widehat{\mathbb{K}}$ holds by hypothesis, and thus $\alpha^J \in \widehat{\mathbb{K}}$ or $\psi \rightarrow \alpha^J \in \widehat{\mathbb{K}}$.

For (2) From Definition 7, we need to show that for every $\varphi \in \mathbb{K}$, $\varphi \in^t \widehat{\mathbb{K}}$ holds. By hypothesis we assume $\varphi \in \mathbb{K}$ holds. From Proposition 2 we know $\varphi \in^t \mathbb{K}$. From Definition 6, we know that φ is either a temporalised literal $\alpha^Q \in^t \mathbb{K}$ or a temporalised rule $\psi \rightarrow \alpha^Q \in^t \mathbb{K}$, which therefore means that there is some interval $Q \subseteq J$ such that either $\alpha^J \in \widehat{\mathbb{K}}$ or $\psi \rightarrow \alpha^J \in \widehat{\mathbb{K}}$. From Proposition 1, we have either $\alpha^J \in \widehat{\widehat{\mathbb{K}}}$ or $\psi \rightarrow \alpha^J \in \widehat{\widehat{\mathbb{K}}}$. Once again, from Definition 6 we have that either $\alpha^K \in^t \widehat{\mathbb{K}}$ or $\psi \rightarrow \alpha^K \in^t \widehat{\mathbb{K}}$, for any $K \subseteq J$ and in particular, since $Q \subseteq J$, we have that either $\alpha^Q \in^t \widehat{\mathbb{K}}$ or $\psi \rightarrow \alpha^Q \in^t \widehat{\mathbb{K}}$, which finally means that $\varphi \in^t \widehat{\mathbb{K}}$. □

Corollary 1 $\widehat{\mathbb{K}} \subseteq^t \mathbb{K}$

Proof This corollary arises from Proposition 3. □

Proposition 4 $\widehat{\mathbb{K}} \sqsubseteq^t \mathbb{K}$

Proof Straightforward from Corollary 1 and Definition 9. □

Proposition 5 $\mathbb{K} \sqsubseteq^t \widehat{\mathbb{K}}$ if and only if $\mathbb{K} = \widehat{\mathbb{K}}$

Proof This proof follows by double implication:

⇒) The proof is straightforward. Assuming $\mathbb{K} \sqsubseteq^t \widehat{\mathbb{K}}$ holds, from Definition 9 we know that $\mathbb{K} = \widehat{\mathbb{K}}$.

⇐) We assume $\mathbb{K} = \widehat{\mathbb{K}}$ holds. We need to show that $\mathbb{K} \sqsubseteq^t \widehat{\mathbb{K}}$ holds, or equivalently that $\mathbb{K} \sqsubseteq^t \mathbb{K}$ holds. From Definition 9, $\mathbb{K} \subseteq^t \mathbb{K}$, and therefore, from Definition 7, if $\varphi \in \mathbb{K}$ then $\varphi \in^t \mathbb{K}$, which has been shown to be true in Proposition 2. □

Proposition 6 $\mathbb{K}_1 \cap^t \mathbb{K}_2 = \mathbb{K}_2 \cap^t \mathbb{K}_1$

Proof By double implication $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ if and only if $\varphi \in (\mathbb{K}_2 \cap^t \mathbb{K}_1)$.

⇒) Let $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$. Both conditions in Definition 10 are satisfied. By *reductio ad absurdum*, we have that $\varphi \notin (\mathbb{K}_2 \cap^t \mathbb{K}_1)$. Thus, either condition or condition 2 is violated for φ. The former is trivially absurd. For the latter, and given that condition 1 holds, we necessarily have that $\exists\varphi'' \in \mathbb{K}_1$ or $\exists\varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$. Hence, we need to show that (a) in condition 2 is violated. This is also absurd, given that (a) holds for $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$, which means $\forall\varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \notin^t \mathbb{K}_1$ or $\varphi''' \notin^t \mathbb{K}_2$, and thus the same condition trivially holds for $\varphi \in (\mathbb{K}_2 \cap^t \mathbb{K}_1)$.

⇐) This part of the proof can be similarly shown. □

Proposition 7 $\varphi \in^t (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ *if and only if* $\varphi \in^t \mathbb{K}_1$ *and* $\varphi \in^t \mathbb{K}_2$

Proof This proof follows by double implication:

⇒) Let $\varphi \in^t (\mathbb{K}_1 \cap^t \mathbb{K}_2)$. There is some $\mathbb{K}' \subseteq (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ such that $\varphi \in^t \mathbb{K}'$. Observe that for every $\psi \in \mathbb{K}'$, both conditions in Definition 10 hold. It is easy to see from condition 1 that $\psi \in^t \mathbb{K}_1$ and $\psi \in^t \mathbb{K}_2$ hold. Thus, $\mathbb{K}' \subseteq^t \mathbb{K}_1$ and $\mathbb{K}' \subseteq^t \mathbb{K}_2$. Clearly, $\varphi \in^t \mathbb{K}_1$ and $\varphi \in^t \mathbb{K}_2$ holds.

⇐) Let $\varphi \in^t \mathbb{K}_1$ and $\varphi \in^t \mathbb{K}_2$. There is $\mathbb{K}'_1 \subseteq \mathbb{K}_1$ and $\mathbb{K}'_2 \subseteq \mathbb{K}_2$ such that $\varphi \in^t \mathbb{K}'_1$ and $\varphi \in^t \mathbb{K}'_2$. Every temporalised sentence ψ in $\mathbb{K}'_1$ or $\mathbb{K}'_2$ is either a temporalised literal (say α^J) or a temporalised rule (say $\psi' \rightarrow \alpha^J$). Note φ is either a temporalised literal (say α^Q) or a temporalised rule (say $\psi' \rightarrow \alpha^Q$). Two alternatives regarding the intervals J and Q. Either $Q \subseteq J$ or $J \subseteq Q$. It is easy to see that the former case implies $\varphi \in^t \psi$. The second case implies that Q arises as an interval composition from which J is part. Afterwards, it is clear that for such each ψ, it holds $\psi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$. Finally, no matter if all the cases corresponds to the first case, or just the second one, or even any combination of both cases; we have that $\varphi \in^t (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ holds. □

Proposition 8 $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ *if and only if* $\varphi_1 \in \mathbb{K}_1$ *and* $\varphi \in \{\varphi_1\} \cap^t \mathbb{K}_2$

Proof This proof follows by double implication:

$\Rightarrow$) Let $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$. Then, both conditions from Definition 10 hold. From condition 1, we know that $\varphi \in^t \mathbb{K}_1$ and $\varphi \in^t \mathbb{K}_2$. By considering $\varphi_1 = \varphi''$, from condition 2, we have that there exists $\varphi_1 \in \mathbb{K}_1$ such that $\varphi \in^t \varphi_1$. So far, from Proposition 7, we know that $\varphi \in^t \{\varphi_1\} \cap^t \mathbb{K}_2$. Besides, from (a) in condition 2, for every $\varphi''' \in^t \varphi_1$ such that $\varphi''' \notin^t \varphi$, it holds $\varphi''' \notin^t \mathbb{K}_2$. Hence, both conditions from Definition 10 hold for $\{\varphi_1\} \cap^t \mathbb{K}_2$, and finally, it follows $\varphi \in \{\varphi_1\} \cap^t \mathbb{K}_2$.

$\Leftarrow$) Let $\varphi_1 \in \mathbb{K}_1$ and $\varphi \in \{\varphi_1\} \cap^t \mathbb{K}_2$. From Proposition 7, we know that $\varphi \in^t \varphi_1$ and thus $\varphi \in^t \mathbb{K}_1$. So far, condition 1 from Definition 10 holds. Besides, by considering $\varphi_1 = \varphi''$, condition 2 holds. Thus, we have $\varphi \in (\mathbb{K}_1 \cap^t \mathbb{K}_2)$ holds. □

Proposition 9 *If* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\varphi \in^t \mathbb{K}_2$ *and* $\varphi \notin^t \mathbb{K}_1$

Proof Let $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. Thus, both conditions from Definition 11 hold. Observe that, from condition 1, there is in particular $\varphi' \in^t \varphi$ such that $\varphi' = \varphi$ (from Proposition 2, $\varphi \in^t \varphi$) and thus, $\varphi \notin^t \mathbb{K}_1$ holds. From condition 2, we know there is some $\varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, and in particular (from Definition 6), $\varphi \in^t \mathbb{K}_2$. □

Proposition 10 *If* $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\varphi \in^t \mathbb{K}_2$ *and* $\varphi \notin^t \mathbb{K}_1$

Proof Let $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. There is some $\mathbb{K}' \subseteq (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ such that $\varphi \in^t \mathbb{K}'$. Afterwards, for every $\psi \in \mathbb{K}'$ we know $\psi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ and thus, both conditions in Definition 11 hold. From Proposition 9 $\psi \in^t \mathbb{K}_2$ and $\psi \notin^t \mathbb{K}_1$ holds for every $\psi \in \mathbb{K}'$. Therefore, $\mathbb{K}' \subseteq^t \mathbb{K}_2$ and $\mathbb{K}' \cap^t \mathbb{K}_1 = \emptyset$. It is clear that $\varphi \in^t \mathbb{K}_2$ and $\varphi \notin^t \mathbb{K}_1$. □

Proposition 11 *If* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$

Proof Let $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. The conditions from Definition 11 hold. By *reductio ad absurdum*, let $\psi \in \{\varphi\} \cap^t \mathbb{K}_1$. From Proposition 2, we have $\psi \in^t \{\varphi\} \cap^t \mathbb{K}_1$ and from Proposition 7, $\psi \in^t \{\varphi\}$ and $\psi \in^t \mathbb{K}_1$. This is absurd given that from condition 1 (see Definition 11), if $\psi \in^t \{\varphi\}$ then $\psi \notin^t \mathbb{K}_1$. Hence, $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$. □

Proposition 12 *If* $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *then* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$

Proof Let $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. There is some $\mathbb{K}' \subseteq (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ such that $\varphi \in^t \mathbb{K}'$. Afterwards, for every $\psi \in \mathbb{K}'$ we know $\psi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ and thus, both conditions in Definition 11 hold. From Proposition 11 we know $\{\psi\} \cap^t \mathbb{K}_1 = \emptyset$. Afterwards, $\mathbb{K}' \cap^t \mathbb{K}_1 = \emptyset$. Consequently, $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$ holds. □

Proposition 13 *If* $\varphi \in \mathbb{K}_2$ *and* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$ *then* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$

Proof Straightforward from Definition 11. Condition 1 follows trivially from $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$, whereas, for verifying condition 2, it is enough to consider $\varphi'' = \varphi$. Observe that in this case, the preconditions of (a) in condition 2 cannot be satisfied. □

Proposition 14 $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *if and only if* $\varphi \in^t \mathbb{K}_2$ *and* $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$

Proof This proof follows by double implication:

⇒) Let $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. This implies that there is $\mathbb{K}' \subseteq (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ such that $\varphi \in^t \mathbb{K}'$. It follows that for all $\psi \in \mathbb{K}'$ both conditions in Definition 11 hold. From Proposition 11 we know that $\{\psi\} \cap^t \mathbb{K}_1 = \emptyset$, therefore, it is clear that $\mathbb{K}' \cap^t \mathbb{K}_1 = \emptyset$ and thus, $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$. From condition 2, we know that there is $\psi' \in \mathbb{K}_2$ such that $\psi \in^t \psi''$. Hence, there is $\mathbb{K}'' \subseteq \mathbb{K}_2$ such that $\mathbb{K}' \subseteq^t \mathbb{K}''$. Thus, $\varphi \in^t \mathbb{K}_2$ holds.

⇐) From $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$ it is clear that condition 1 (Definition 11) holds. Let $\varphi \in^t \mathbb{K}_2$, this means that there is some subset $\mathbb{K}' \subseteq \mathbb{K}_2$ such that $\varphi \in^t \mathbb{K}'$. Observe that for all $\psi \in \mathbb{K}'$ it happens that ψ is either a temporalised literal (say α^J) or a temporalised rule (say $\psi' \to \alpha^J$). Observe also that φ is either a temporalised literal (say α^Q) or a temporalised rule (say $\psi' \to \alpha^Q$). Moreover, note that since $\{\varphi\} \cap^t \mathbb{K}_1 = \emptyset$, we have two alternatives regarding the intervals J and Q. Either $Q \subseteq J$ or $J \subseteq Q$. It is easy to see that the former case implies $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. The second case implies that Q arises as an interval composition from which J is part. Afterwards, it is clear that $\psi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. This is implied by all $\psi \in \mathbb{K}'$, thus $\mathbb{K}' \subseteq (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ and therefore, $\varphi \in^t (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. □

Proposition 15 *If* $\mathbb{K} \cap^t \mathbb{K}_1 = \mathbb{K} \cap^t \mathbb{K}_2$ *then* $\mathbb{K} \setminus^t \mathbb{K}_1 = \mathbb{K} \setminus^t \mathbb{K}_2$

Proof Let $\mathbb{K} \cap^t \mathbb{K}_1 = \mathbb{K} \cap^t \mathbb{K}_2$. By *reductio ad absurdum*, let $\mathbb{K}\setminus^t\mathbb{K}_1 \neq \mathbb{K}\setminus^t\mathbb{K}_2$. Hence, there is some $\psi \in \mathbb{K} \setminus^t \mathbb{K}_1$ such that $\psi \notin \mathbb{K} \setminus^t \mathbb{K}_2$. Afterwards, from Proposition 9 we know that $\psi \in^t \mathbb{K}$ and $\psi \notin^t \mathbb{K}_1$, but also $\psi \in^t \mathbb{K}_2$. From Definition 10, it is clear that, $\psi \notin^t \mathbb{K} \cap^t \mathbb{K}_1$ but $\psi \in^t \mathbb{K} \cap^t \mathbb{K}_2$. Hence, $\mathbb{K} \cap^t \mathbb{K}_1 \neq \mathbb{K} \cap^t \mathbb{K}_2$, which is absurd. □

Proposition 16 $(\mathbb{K}_2\setminus^t\mathbb{K}_1) \subseteq (\mathbb{K}_2 \cup \mathbb{K})\setminus^t\mathbb{K}_1$

Proof Let $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. From Definition 11 we know the following conditions hold:

1. $\forall \varphi' \in^t \varphi$ it holds $\varphi' \notin^t \mathbb{K}_1$, and
2. $\exists \varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, and
   (a) $\forall \varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \in^t \mathbb{K}_1$

Let $\mathbb{K}' = \mathbb{K}_2 \cup \mathbb{K}$. Observe that both conditions 1 and 2 hold for $\mathbb{K}' \setminus^t \mathbb{K}_1$. That is, since we know that if there is some $\varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, then (a) in condition 2 holds, it is easy to see that such for φ'', it happens that $\varphi'' \in \mathbb{K}'$ holds and still both $\varphi \in^t \varphi''$ and condition (a) hold. Hence, $\varphi \in (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$ holds. □

Proposition 17 *If* $\mathbb{K} \cap^t \mathbb{K}_1 = \emptyset$ *then* $(\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K} = (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$

Proof Let $\mathbb{K} \cap^t \mathbb{K}_1 = \emptyset$. By double inclusion, we will show that

$$(\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K} = (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$$

($\subseteq$) Let $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K}$. Two alternatives, either i) $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ or ii) $\varphi \in \mathbb{K}$. For the former case, we know that $\varphi \in (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$ (see Proposition 16). For the second case, by *reductio ad absurdum*, we assume that $\varphi \notin (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$. This means that the conditions in Definition 11 does not hold. That is, either:

1. there is some $\varphi' \in^t \varphi$ such that $\varphi' \in^t \mathbb{K}_1$, or
2. there is no $\varphi'' \in (\mathbb{K}_2 \cup \mathbb{K})$ such that $\varphi \in^t \varphi''$, or
3. $\exists \varphi'' \in (\mathbb{K}_2 \cup \mathbb{K})$ such that $\varphi \in^t \varphi''$, and
   (a) there is some $\varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ and it holds $\varphi''' \notin^t \mathbb{K}_1$

- Condition 1, is an absurd since it is violating the hypothesis $\mathbb{K} \cap^t \mathbb{K}_1 = \emptyset$.
- Condition 2, is an absurd since we know that $\varphi \in \mathbb{K}$, we know $\varphi \in (\mathbb{K}_2 \cup \mathbb{K})$ and thus $\varphi \in^t \varphi$ holds.
- Condition 3, is an absurd by considering that $\varphi'' = \varphi$ and given that there is no $\varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$.

Finally, we know that $\mathbb{K} \subseteq (\mathbb{K}_2 \cup \mathbb{K})$ and from Proposition 16, $(\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K} \subseteq (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$ holds.

($\supseteq$) Let $\varphi \in (\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1$. From Definition 11, the first condition is covered by hypothesis given that $\mathbb{K} \cap^t \mathbb{K}_1 = \emptyset$ holds. More interestingly, the case of the second condition ensures that there is some $\varphi'' \in (\mathbb{K}_2 \cup \mathbb{K})$ such that $\varphi \in^t \varphi''$ and $\forall \varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \in^t \mathbb{K}_1$. Two alternatives, either $\varphi'' \in \mathbb{K}_2$ or $\varphi'' \in \mathbb{K}$. For the second case, since $\mathbb{K} \cap^t \mathbb{K}_1 = \emptyset$ holds, we know that there is no $\varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ if it is the case that we are considering $\varphi'' = \varphi$. Therefore, it is clear that $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K}$. For the former case, in which $\varphi'' \in \mathbb{K}_2$, the conditions of Definition 11 are verified and therefore, we know $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ and clearly, $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K}$. Finally, we know that $(\mathbb{K}_2 \cup \mathbb{K}) \setminus^t \mathbb{K}_1 \subseteq (\mathbb{K}_2 \setminus^t \mathbb{K}_1) \cup \mathbb{K}$.

□

Proposition 18 *If* $\varphi \in \mathbb{K}$ *then* $\varphi \in Cn^t(\mathbb{K})$

Proof Straightforward from Proposition 2 and Definition 12. □

Proposition 19 $Cn^t(\mathbb{K}) = Cn^t(\widehat{\mathbb{K}})$

Proof For this proof we use double inclusion:

⇒) For showing that $Cn^t(\mathbb{K}) \subseteq Cn^t(\widehat{\mathbb{K}})$ holds, let $\varphi \in Cn^t(\mathbb{K})$. From Definition 12 we have that either (1) $\varphi \in^t \mathbb{K}$ or (2) $(\psi \to \varphi) \in Cn^t(\mathbb{K})$ and $\psi \in Cn^t(\mathbb{K})$.

For (1), from Definition 6 we have that φ is either $(\alpha_1 \wedge \ldots \wedge \alpha_n)^Q$ or $\varphi_1 \to \alpha^Q$:

- $(\alpha_1 \wedge \ldots \wedge \alpha_n)^Q \in^t \mathbb{K}$ (with $n \geq 1$) and thus $\forall_{i=1}^n(\alpha_i^{J_i} \in \widehat{\mathbb{K}}$ and $Q \subseteq J_i)$. Since $\widehat{\mathbb{K}}$ is a temporalised belief base, let us assume $\widehat{\mathbb{K}} = \mathbb{K}'$. This means that each $\alpha_i^{J_i} \in \mathbb{K}'$ and from Definition 5, we have that each $\alpha_i^{J_i} \in \widehat{\mathbb{K}'}$. Afterwards, from Definition 6 we have that each $\alpha_i^{J_i} \in^t \mathbb{K}'$, and in particular $(\alpha_1 \wedge \ldots \wedge \alpha_n)^Q \in^t \mathbb{K}'$. Finally, from Definition 12, $(\alpha_1 \wedge \ldots \wedge \alpha_n)^Q \in Cn^t(\mathbb{K}')$ which is equivalent to $(\alpha_1 \wedge \ldots \wedge \alpha_n)^Q \in Cn^t(\widehat{\mathbb{K}})$ and $\varphi \in Cn^t(\widehat{\mathbb{K}})$. Hence, $Cn^t(\mathbb{K}) \subseteq Cn^t(\widehat{\mathbb{K}})$.
- $\varphi_1 \to \alpha^Q \in^t \mathbb{K}$ if and only if $\varphi_1 \to \varphi_2 \in \widehat{\mathbb{K}}$ and $\alpha^Q \in^t \{\varphi_2\}$. By *reductio ad absurdum* we assume $\varphi_1 \to \alpha^Q \notin Cn^t(\widehat{\mathbb{K}})$ which from Definition 12 implies that $\varphi_1 \to \alpha^Q \notin^t \widehat{\mathbb{K}}$ and from Definition 6 we have that $\varphi_1 \to \varphi_2 \notin \widehat{\widehat{\mathbb{K}}}$, since the normalised base is a fix point (see Proposition 1) we have that $\varphi_1 \to \varphi_2 \notin \widehat{\mathbb{K}}$ which is absurd. Thus, we know $\varphi_1 \to \alpha^Q \in Cn^t(\widehat{\mathbb{K}})$ and $\varphi \in Cn^t(\widehat{\mathbb{K}})$. Hence, $Cn^t(\mathbb{K}) \subseteq Cn^t(\widehat{\mathbb{K}})$.

For (2) Since $(\psi \to \varphi') \in Cn^t(\mathbb{K})$ and $\psi \in Cn^t(\mathbb{K})$, from Definition 12 we know that $(\psi \to \varphi') \in^t \mathbb{K}$ and $\psi \in^t \mathbb{K}$ hold. From Definition 6 we have that $\varphi' = \alpha^Q$ and then $(\psi \to \alpha^Q) \in \widehat{\mathbb{K}}$ and $\psi \in \widehat{\mathbb{K}}$. From Proposition 2, we have $(\psi \to \alpha^Q) \in^t \widehat{\mathbb{K}}$ and $\psi \in^t \widehat{\mathbb{K}}$. Finally, once again from Definition 12, we have $(\psi \to \alpha^Q) \in Cn^t(\widehat{\mathbb{K}})$ and $\psi \in Cn^t(\widehat{\mathbb{K}})$, and therefore, $\alpha^Q \in Cn^t(\widehat{\mathbb{K}})$ which is equivalent to $\varphi \in Cn^t(\widehat{\mathbb{K}})$. Hence, $Cn^t(\mathbb{K}) \subseteq Cn^t(\widehat{\mathbb{K}})$.

⇐) For showing that $Cn^t(\widehat{\mathbb{K}}) \subseteq Cn^t(\mathbb{K})$ holds, let $\varphi \in Cn^t(\widehat{\mathbb{K}})$. From Definition12, φ arises either from (1) $\varphi \in^t \widehat{\mathbb{K}}$ or from a derivation having (2) $(\psi \to \varphi) \in Cn^t(\widehat{\mathbb{K}})$ and $\psi \in Cn^t(\widehat{\mathbb{K}})$, and thus $(\psi \to \varphi) \in^t \widehat{\mathbb{K}}$ and $\psi \in^t \widehat{\mathbb{K}}$. From Definition 6 we have $(\psi \to \varphi') \in \widehat{\widehat{\mathbb{K}}}$ and $\varphi \in^t \{\varphi'\}$ and $\psi \in \widehat{\widehat{\mathbb{K}}}$. From Proposition 1, we have $(\psi \to \varphi') \in \widehat{\mathbb{K}}$ and $\psi \in \widehat{\mathbb{K}}$, which therefore means $(\psi \to \varphi) \in^t \mathbb{K}$ and $\psi \in^t \mathbb{K}$ (from Definition 6). Afterwards, from Definition 12 we know $(\psi \to \varphi) \in Cn^t(\mathbb{K})$ and

$\psi \in Cn^t(\mathbb{K})$ and thus $\varphi \in Cn^t(\mathbb{K})$ holds. Note that case (1) can be shown similarly. Finally, it is clear that $Cn^t(\widehat{\mathbb{K}}) \subseteq Cn^t(\mathbb{K})$. □

Proposition 20 *The temporalised minus-operator "$\setminus^t$" is a well defined function for a fixed objective set.*

Proof For this proof we will reformulate the proposition upon the following construction. Let $\mathcal{P}^t$ be the function $\mathcal{P}^t : \mathcal{U}_{\mathbb{L}}^{\mathbb{I}} \rightarrow 2^{\mathcal{U}_{\mathbb{L}}^{\mathbb{I}}}$ of temporal partitions such that $\mathcal{P}^t(\mathbb{K}_2)$ is the set $\{\mathbb{K} \mid$ for all $\mathbb{K} \subseteq^t \mathbb{K}_2$, where $\mathbb{K}_2 \in \mathcal{U}_{\mathbb{L}}^{\mathbb{I}}\}$ of temporal partitions of $\mathbb{K}_2$. Let $f_{\mathbb{K}_2}$ be a function $f_{\mathbb{K}_2} : \mathcal{U}_{\mathbb{L}}^{\mathbb{I}} \rightarrow \mathcal{P}^t(\mathbb{K}_2)$ of relative complement with respect to $\mathbb{K}_2$ such that given a temporalised belief base $\mathbb{K}_1 \in \mathcal{U}_{\mathbb{L}}^{\mathbb{I}}$, the expression $f_{\mathbb{K}_2}(\mathbb{K}_1)$ is the set $\mathbb{K}_2 \setminus^t \mathbb{K}_1$. We need to show that $f_{\mathbb{K}_2}$ is a well defined function. That is, given two temporalised belief bases $\mathbb{K}_1, \mathbb{K}_1' \in \mathcal{U}_{\mathbb{L}}^{\mathbb{I}}$, if $Cn^t(\mathbb{K}_1) = Cn^t(\mathbb{K}_1')$ then $f_{\mathbb{K}_2}(\mathbb{K}_1) = f_{\mathbb{K}_2}(\mathbb{K}_1')$. Thus, assuming $Cn^t(\mathbb{K}_1) = Cn^t(\mathbb{K}_1')$ holds, we need to show $\mathbb{K}_2 \setminus^t \mathbb{K}_1 = \mathbb{K}_2 \setminus^t \mathbb{K}_1'$. Let $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$. From Definition 11, we have:

1. $\forall \varphi' \in^t \varphi$ it holds $\varphi' \notin^t \mathbb{K}_1$, and
2. $\exists \varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, and
   (a) $\forall \varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \in^t \mathbb{K}_1$

Since $Cn^t(\mathbb{K}_1) = Cn^t(\mathbb{K}_1')$, it is easy to see that $\psi \in^t \mathbb{K}_1$ *iff* $\psi \in^t \mathbb{K}_1'$. Thus, from condition 1 we know that $\forall \varphi' \in^t \varphi$ it holds $\varphi' \notin^t \mathbb{K}_1'$, and from (a) in condition 2 we know that $\exists \varphi'' \in \mathbb{K}_2$ such that $\varphi \in^t \varphi''$, and $\forall \varphi''' \in^t \varphi''$ such that $\varphi''' \notin^t \varphi$ it holds $\varphi''' \in^t \mathbb{K}_1'$. Hence, $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1)$ *iff* $\varphi \in (\mathbb{K}_2 \setminus^t \mathbb{K}_1')$, and thus $\mathbb{K}_2 \setminus^t \mathbb{K}_1 = \mathbb{K}_2 \setminus^t \mathbb{K}_1'$. This means that $f_{\mathbb{K}_2}(\mathbb{K}_1) = f_{\mathbb{K}_2}(\mathbb{K}_1')$ holds, and that implies that $f_{\mathbb{K}_2}$ is a well defined function. This completes the proof. □

Proposition 21 *If an operator "$*$" for $\mathbb{K}$ satisfies (Success)., (Inclusion), and (Core-Retainment), then it satisfies (Vacuity).*

Proof Let $\neg\varphi \notin Cn^t(\mathbb{K})$. This proof follows by double inclusion:

($\subseteq$) The proof for $\mathbb{K} * \varphi \subseteq^t \mathbb{K} \cup \{\varphi\}$ follows straightforwardly from *(Inclusion)*.

($\supseteq$) Let $\psi \in \mathbb{K} \cup \{\varphi\}$. Two alternatives, either $\psi \in \mathbb{K}$ or $\psi \in \{\varphi\}$. From the latter case, we have that $\psi = \varphi$. Therefore, the proof is completed straightforwardly since by (Success)., we know that $\varphi \in \mathbb{K} * \varphi$. From the former case, let $\psi \in \mathbb{K}$. From *(Cor e-Retainment)*, if $\psi \in \mathbb{K} \setminus^t \mathbb{K} * \varphi$ then there is some $\mathbb{K}' \subseteq^t \mathbb{K}$ such that $\bot \notin Cn^t(\mathbb{K}' \cup \{\varphi\})$ and $\bot \in Cn^t(\mathbb{K}' \cup \{\varphi, \psi\})$. Let us assume $\psi \in \mathbb{K} \setminus^t \mathbb{K} * \varphi$. From Proposition 9 we know that

$\psi \notin^t \mathbb{K}*\varphi$, and from Proposition 2, we know that $\psi \notin \mathbb{K}*\varphi$. However, since $\neg\varphi \notin Cn^t(\mathbb{K})$, we know that $\neg\varphi \notin Cn^t(\mathbb{K}')$ and certainly, $\bot \notin Cn^t(\mathbb{K}' \cup \{\varphi\})$, but since $\psi \in \mathbb{K}$ and also $\psi \in^t \mathbb{K}$ (see Proposition 2), it follows $\psi \cap^t \neg\varphi = \emptyset$. It is clear that $\bot \notin Cn^t(\mathbb{K}' \cup \{\varphi, \psi\})$, which is absurd. This implies that, there is no $\psi \in \mathbb{K}\setminus^t\mathbb{K}*\varphi$, and thus, we necessarily have that $\psi \in \mathbb{K}*\varphi$. Finally, $\mathbb{K} \cup \{\varphi\} \subseteq \mathbb{K}*\varphi$ holds.

□

Proposition 22 $\alpha^J \in Cn^t(\mathbb{K})$ *if and only if* ${}^{\sqsubseteq}(\alpha^J, \mathbb{K}) = \{\alpha^J\}$

Proof Straightforward from Definition 14. Observe that, condition 1 is satisfied by assuming $Q = J$ and thus it cannot appear any other time maximising sentence satisfying condition 2. □

Proposition 23 *If* $\{\alpha^R, \alpha^Q\} \subseteq {}^{\sqsubseteq}(\alpha^J, \mathbb{K})$ *then* $R \bowtie Q$

Proof Straightforward from Definition 14. Observe that, *by reductio ad absurdum*, if we assume $\{\alpha^R, \alpha^Q\} \subseteq {}^{\sqsubseteq}(\alpha^J, \mathbb{K})$ holds, but $R \bowtie Q$ does not hold, hence condition 2 would be violated. □

Proposition 24 ${}^{\sqsubseteq}(\alpha^J, \mathbb{K}) = {}^{\sqsubseteq}\widehat{(\alpha^J, \mathbb{K})}$

Proof Straightforward from Proposition 23, Definition 14, and in particular from Definition 5, condition 11b. □

Proposition 25 *The following conditions are equivalent*:

1. $\Pi(\alpha^J, \mathbb{K}) = \Pi(\beta^J, \mathbb{K})$
2. *for all* $\mathbb{K}' \subseteq^t \mathbb{K}, \mathbb{K}' \cup \{\neg\alpha^J\} \vdash^t \bot$ *iff* $\mathbb{K}' \cup \{\neg\beta^J\} \vdash^t \bot$

Proof This proof follows by double implication:

- (1 ⇒ 2) By *reductio ad absurdum*, suppose 2 does not hold, without loss of generality, we can assume that there is some set $\mathbb{K}' \subseteq^t \mathbb{K}$ such that $\mathbb{K}' \cup \{\neg\alpha^J\} \vdash^t \bot$ and $\mathbb{K}' \cup \{\neg\beta^J\} \nvdash^t \bot$. It is clear that $\mathbb{K}' \vdash^t \alpha^J$. We can assume there is some set $\mathbb{H} \subseteq^t \mathbb{K}'$ such that $\alpha^J \in Cn^t(\mathbb{H})$ and $\mathbb{H} \in \Pi(\alpha^J, \mathbb{K})$. Besides, since $\mathbb{K}' \cup \{\neg\beta^J\} \nvdash^t \bot$, we have $\mathbb{K}' \nvdash^t \beta^J$, or equivalently, $\beta^J \notin Cn^t(\mathbb{K}')$. Given that $\mathbb{H} \subseteq^t \mathbb{K}'$, we know that $\beta^J \notin Cn^t(\mathbb{H})$ and thus $\mathbb{H} \notin \Pi(\beta^J, \mathbb{K})$. This implies $\Pi(\alpha^J, \mathbb{K}) \neq \Pi(\beta^J, \mathbb{K})$ which is absurd since it violates 1.

- $(2 \Rightarrow 1)$ By *reductio ad absurdum*, suppose 1 does not hold, that is, $\Pi(\alpha^J, \mathbb{K}) \neq \Pi(\beta^J, \mathbb{K})$. Without loss of generality, we can assume that there is some set $\mathbb{H} \in \Pi(\alpha^J, \mathbb{K})$ such that $\mathbb{H} \notin \Pi(\beta^J, \mathbb{K})$. Two alternatives, either i) $\beta^J \notin Cn^t(\mathbb{H})$ or ii) $\beta^J \in Cn^t(\mathbb{H})$.
  - For the former case, observe that $\mathbb{H} \nvdash^t \beta^J$ and thus, $\mathbb{H} \cup \{\neg\beta^J\} \nvdash^t \bot$. However, since $\mathbb{H} \in \Pi(\alpha^J, \mathbb{K})$, we have $\mathbb{H} \cup \{\neg\alpha^J\} \vdash^t \bot$. This is absurd since 2 is being violated.
  - For the second case, since $\mathbb{H} \notin \Pi(\beta^J, \mathbb{K})$, we necessarily have some $\mathbb{H}' \subset^t \mathbb{H}$ such that $\beta^J \in Cn^t(\mathbb{H}')$ and thus $\mathbb{H}' \in \Pi(\beta^J, \mathbb{K})$. Besides, we have $\alpha^J \notin Cn^t(\mathbb{H}')$. This means that $\mathbb{H}' \cup \{\neg\alpha^J\} \nvdash^t \bot$ and $\mathbb{H}' \cup \{\neg\beta^J\} \vdash^t \bot$. This is absurd since 2 is being violated.

□

Proposition 26 $\mathbb{K} \otimes \alpha^J = (\mathbb{K} \cup \{\alpha^J\}) \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$

Proof We need to show that

$$(\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\} = (\mathbb{K} \cup \{\alpha^J\}) \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$$

From Definitions 17, 16 and 15, we know that $\alpha^{[t_i]} \notin \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$ for any $t_i \in J$. It is easy to see that $\{\alpha^J\} \cap^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})) = \emptyset$. Thereafter, from Proposition 17, the proof follows straightforwardly. □

Representation Theorem 1 An operator "$\otimes$ " is a prioritised legal revision for $\mathbb{K}$ if and only if it satisfies the postulates of *(Success)*., *(Consistency)*, *(Inclusion)*, *(Core-Retainment)*, and *(Uniformity)*.

Proof This proof begins by showing that the model of legal revision proposed in this article satisfies the set of postulates discussed in Sect. 5. Afterwards, we do the prove in the opposite direction: the set of postulates is assumed by hypothesis in order to show that they constitute a full set for characterizing the construction of the proposed model of legal revision. Finally, in order to pursue the full characterization of a legal revision operator, we prove that the operator characterized through the postulates is in fact the legal revision proposed in our model.

$\Rightarrow$) Construction to postulates:

Let "σ" be an instant incision function, the operator "$\otimes$" a prioritised legal revision operator, $\mathbb{K}$ a temporalised belief base. Then, for all temporalised literal α^J, by Definition 18:

$$\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$$

We prove that the five postulates hold for the given construction, as follows.

- The proof for *(Success)*. follows straightforwardly from Definition 18.
- The proof for *(Inclusion)* follows from Definition 18. Let $\varphi \in \mathbb{K} \otimes \alpha^J$, then $\varphi \in (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$. This means either $\varphi = \alpha^J$ or $\varphi \in \mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$. For the latter, from Proposition 9 we know $\varphi \notin^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$ and $\varphi \in^t \mathbb{K}$. Hence, $\varphi \in^t \mathbb{K} \cup \{\alpha^J\}$ holds. Finally, $\mathbb{K} * \alpha^J \subseteq^t \mathbb{K} \cup \{\alpha^J\}$.
- For *(Consistency)*, we assume $\mathbb{K}$ is consistent, that is $\mathbb{K} \nvdash^t \bot$. Observe that the incoming temporalised literal α^J is trivially consistent. We need to show that $\mathbb{K} \otimes \alpha^J \nvdash^t \bot$. From Definition 18 we know $\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$. It is clear that $(\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})))$ is consistent given that so it is $\mathbb{K}$. The potential inconsistency could appear while incorporating α^J. So we need to make sure that $(\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})))$ can accept α^J without triggering inconsistencies. To that end, from Definition 17, we know that

  (1) $\sigma(\Pi(\neg\alpha^J, \mathbb{K})) \subseteq \bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$, and
  (2) if $\mathbb{H} \in \Pi(\neg\alpha^J, \mathbb{K})$ then there is no $t_i \in J$ such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{H} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})))$.

  From conditions (1) and (2), we know consistency is ensured given that $\neg\alpha^{[t_i]} \notin Cn^t(\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})))$, for any $t_i \in J$. This is so, given that $\sigma(\Pi(\neg\alpha^J, \mathbb{K}))$ is included in the union of all the minimal proofs $\mathbb{H}$ for time-maximising subsentences of $\neg\alpha^J$ and all such minimal proofs end up cut given that $\neg\alpha^{[t_i]} \notin Cn^t(\mathbb{H} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})))$ holds for every such $\mathbb{H}$. Therefore, in accordance to Definition 13, we know there is no $t_i \in J$ such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{K} \otimes \alpha^J)$ and $\alpha^{[t_i]} \in Cn^t(\mathbb{K} \otimes \alpha^J)$. Hence, we know *(Consistency)* is preserved, thus $\mathbb{K} \otimes \alpha^J \nvdash^t \bot$. This completes the proof.
- For *(Uniformity)*, we assume an interval J and two temporalised literals α^J and β^J, such that for all $\mathbb{K}' \subseteq^t \mathbb{K}$, we know that $(\mathbb{K}' \cup \{\alpha^J\}) \vdash^t \bot$ *iff* $(\mathbb{K}' \cup \{\beta^J\}) \vdash^t \bot$. From Proposition 25, we have $\Pi(\neg\alpha^J, \mathbb{K}) = \Pi(\neg\beta^J, \mathbb{K})$, and thus $\sigma(\Pi(\neg\alpha^J, \mathbb{K})) = \sigma(\Pi(\neg\beta^J, \mathbb{K}))$ (see Definition 17). Therefore, $(\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\beta^J, \mathbb{K})))$. From Definition 18 we know $\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$ and $\mathbb{K} \otimes \beta^J = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\beta^J, \mathbb{K}))) \cup \{\beta^J\}$. Finally, $\mathbb{K} \cap^t (\mathbb{K} \otimes \alpha^J) = \mathbb{K} \cap^t (\mathbb{K} \otimes \beta^J)$ holds, which completes the proof.
- For *(Core-Retainment)*, let $\psi \in \mathbb{K} \setminus^t \mathbb{K} \otimes \alpha^J$. From Proposition 9, $\psi \in^t \mathbb{K}$ and $\psi \notin^t \mathbb{K} \otimes \alpha^J$. From Definition 18, $\mathbb{K} \otimes \alpha^J = (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$. Thus, $\psi \notin^t \{\alpha^J\}$ and $\psi \notin^t (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K})))$. Since $\psi \in^t \mathbb{K}$ we necessarily have that $\psi \in^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$. Since *(Consistency)* was proved before, we know that $\mathbb{K} \otimes \alpha^J \nvdash^t \bot$, and thus $\bot \notin Cn^t((\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\})$. Let $\mathbb{K}'' = \sigma(\Pi(\neg\alpha^J, \mathbb{K})) \setminus^t \{\psi\}$. Clearly, $\mathbb{K}'' \subset^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$. From Definition 17 condition 3, we know that there is some $t_i \in J$ such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{K} \setminus^t \mathbb{K}'')$. It is

clear that $\psi \in^t \mathbb{K} \setminus^t \mathbb{K}''$. This means that $\neg\alpha^{[t_i]} \in Cn^t((\mathbb{K}\setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\psi\})$, and thus $\bot \in Cn^t((\mathbb{K}\setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J, \psi\})$. This completes the proof.

$\Leftarrow$) Postulates to construction:

Let $*$ be an operator that satisfies the five postulates proposed in Sect. 5. We have to show that $*$ is a legal revision operator *à la* $\otimes$ (see Definition 18). This means that we need to show:

(I) $\mathbb{K} * \alpha^J = \mathbb{K} \otimes \alpha^J$

We will assume the construction of the operator * by relying upon an incision-like function. To that end, let σ^c be a function such that for every temporalised base $\mathbb{K}$ and for every temporalised literal α^J it holds:

(H.1) $\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})) = \mathbb{K}\setminus^t \mathbb{K} * \alpha^J$

Consequently, we firstly need to show that σ^c is an instant incision function. To do this we show that σ^c is a well-defined function and that the conditions in Definition 17 are satisfied by σ^c; that is:

(1) σ^c is a well-defined function
(2) $\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})) \subseteq \bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$
(3) if $\mathbb{H} \in \Pi(\neg\alpha^J, \mathbb{K})$ then there is no $t_i \in J$ such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{H}\setminus^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K})))$
(4) if $\mathbb{K}' \subset^t \sigma^c(\Pi(\alpha^J, \mathbb{K}))$ then there is some $t_i \in J$ such that $\alpha^{[t_i]} \in Cn^t(\mathbb{K}\setminus^t \mathbb{K}')$

- To prove (1), let us assume we have two temporalised sentences $\neg\alpha^J$ and $\neg\beta^J$ such that $\Pi(\neg\alpha^J, \mathbb{K}) = \Pi(\neg\beta^J, \mathbb{K})$. From Proposition 25 we know that for all set $\mathbb{K}' \subseteq^t \mathbb{K}$, $(\mathbb{K}' \cup \{\alpha^J\}) \vdash^t \bot$ *iff* $(\mathbb{K}' \cup \{\beta^J\}) \vdash^t \bot$. This satisfies the precondition of *(Uniformity)*. Hence, $\mathbb{K} \cap^t (\mathbb{K} * \alpha^J) = \mathbb{K} \cap^t (\mathbb{K} * \beta^J)$. Therefore, from Proposition 15, $\mathbb{K}\setminus^t \mathbb{K} * \alpha^J = \mathbb{K}\setminus^t \mathbb{K} * \beta^J$. From (H.1), we know $\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})) = \sigma^c(\Pi(\neg\beta^J, \mathbb{K}))$ holds. Therefore, σ^c is a well-defined function. The proof for (1) is complete.
- To prove (2) we need to show that for every termporalised sentence in $\sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ also is in $\bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$. Let $\psi \in \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. By (H.1) we know that $\psi \in \mathbb{K}\setminus^t \mathbb{K} * \alpha^J$. By *(Core-Retainment)*, we have that there is some $\mathbb{K}' \subseteq^t \mathbb{K}$ such that $\bot \notin Cn^t(\mathbb{K}' \cup \{\alpha^J\})$ and $\bot \in Cn^t(\mathbb{K}' \cup \{\alpha^J, \psi\})$. Observe that $\neg\alpha^{t_i} \in Cn^t(\mathbb{K}' \cup \{\psi\})$, for some $t_i \in J$. Thus, it is easy to see that there is an instant in J clashing with regards to α in $\mathbb{K}' \cup \{\alpha^J, \psi\}$. This means that, in particular, there is some set $\mathbb{H}$ which is a minimal proof for a time-maximising subsentence $\neg\alpha^Q$ of $\neg\alpha^J$, where $Q \subseteq J$ and $t_i \in Q$, such that $\mathbb{K}' \cup \{\psi\} \subseteq \mathbb{H} \subseteq \bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$. Afterwards, since $\psi \in \mathbb{H}$, we know

$\psi \in \bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$, and thus $\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})) \subseteq \bigcup(\Pi(\neg\alpha^J, \mathbb{K}))$. This completes the proof for (2).

- To prove (3), for every $\mathbb{H} \in \Pi(\neg\alpha^J, \mathbb{K})$ we need to show that there is no $t_i \in J$ such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{H}\setminus^t\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})))$. By *reductio ad absurdum* we assume there is $t_i \in J$, such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{H}\setminus^t\sigma^c(\Pi(\neg\alpha^J, \mathbb{K})))$. This means that there is some $\mathbb{H}' \subseteq^t \mathbb{H}$ such that $\mathbb{H}' \nsubseteq^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ and $\neg\alpha^{[t_i]} \in^t \mathbb{H}'$. Afterwards, from (H.1), we know $\mathbb{H}' \nsubseteq^t \mathbb{K}\setminus^t\mathbb{K} * \alpha^J$. We know $\mathbb{H}$ is a minimal proof for a time-maximising sub-sentence of $\neg\alpha^J$. It is clear that $\mathbb{H} \subseteq^t \mathbb{K}$. Thus, we know $\mathbb{H}' \subseteq^t \mathbb{K}$ holds, and thus $\mathbb{H}' \subseteq^t \mathbb{K} * \alpha^J$, which means that $\neg\alpha^{[t_i]} \in^t \mathbb{K} * \alpha^J$. From (Success). we know that $\alpha^J \in \mathbb{K} * \alpha^J$. This implies that $\{\alpha^J, \neg\alpha^{[t_i]}\} \subseteq^t \mathbb{K} * \alpha^J$ and since $t_i \in J$, we have that $\bot \in^t \mathbb{K} * \alpha^J$. This is absurd given that from *(Consistency)* we know that $\bot \notin^t \mathbb{K} * \alpha^J$ unless $\bot \in^t \alpha^J$, which does not hold by definition of the language. This completes the proof.
- To prove (4), we need to show that if $\mathbb{K}' \subset^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ then there is some $t_i \in J$ such that $\neg\alpha^{[t_i]} \in Cn^t(\mathbb{K}\setminus^t\mathbb{K}')$. Let $\mathbb{K}' \subset^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. By *reductio ad absurdum*, we assume that $\neg\alpha^{[t_i]} \notin Cn^t(\mathbb{K}\setminus^t\mathbb{K}')$ holds for every $t_i \in J$. Thus, $\bot \notin Cn^t((\mathbb{K}\setminus^t\mathbb{K}') \cup \{\alpha^J\})$. Let us assume, without loss of generality, that $\psi \in^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$ and $\psi \notin^t \mathbb{K}'$. From (H.1), $\psi \in^t \mathbb{K}\setminus^t\mathbb{K}*\alpha^J$. Hence, $\psi \in^t \mathbb{K}$. Therefore, $\psi \in Cn^t((\mathbb{K}\setminus^t\mathbb{K}') \cup \{\alpha^J\})$ holds, and thus $\bot \notin Cn^t((\mathbb{K}\setminus^t\mathbb{K}') \cup \{\alpha^J, \psi\})$. It is clear that $(\mathbb{K} \setminus^t \mathbb{K}') \subseteq^t \mathbb{K}$. This is absurd since it is violating *(Core-Retainment)*. The proof for (4) is completed.

This completes the first part of the proof: σ^c is an instant incision function. Thereafter, we need to show (I), that is $\mathbb{K}*\alpha^J = \mathbb{K}\otimes\alpha^J$.

($\subseteq$)

Let $\varphi \in \mathbb{K}*\alpha^J$. From Proposition 2 we know that $\varphi \in^t \mathbb{K}*\alpha^J$. From Propositon 9, we know that $\varphi \notin \mathbb{K}\setminus^t\mathbb{K}*\alpha^J$, no matter the relation between φ and $\mathbb{K}$. Thus, by (H.1), $\varphi \notin \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. From *(Inclusion)* we known that $\mathbb{K}*\alpha^J \subseteq^t \mathbb{K} \cup \{\alpha^J\}$, hence $\varphi \in^t \mathbb{K} \cup \{\alpha^J\}$. From Proposition 26, we know that $\mathbb{K}\otimes\alpha^J = (\mathbb{K} \cup \{\alpha^J\})\setminus^t\sigma(\Pi(\neg\alpha^J, \mathbb{K}))$, and since we have shown that $\sigma(\Pi(\neg\alpha^J, \mathbb{K})) = \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$, we know that $\varphi \notin \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$ and thus $\varphi \notin^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$. Finally, we know $\varphi \in^t \mathbb{K}\otimes\alpha^J$. This shows that $\mathbb{K}*\alpha^J \subseteq^t \mathbb{K}\otimes\alpha^J$.

($\supseteq$)

Let $\varphi \in \mathbb{K}\otimes\alpha^J$. By Definition 18, $\varphi \in (\mathbb{K}\setminus^t\sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$. We have two alternatives, $\varphi \in \{\alpha^J\}$ or $\varphi \in (\mathbb{K}\setminus^t\sigma(\Pi(\neg\alpha^J, \mathbb{K})))$. For the former case, from (Success). we know $\varphi \in \mathbb{K} * \alpha^J$. For the latter case, from Proposition 9 we know that $\varphi \in^t \mathbb{K}$ and $\varphi \notin^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))$. We have shown before that $\sigma(\Pi(\neg\alpha^J, \mathbb{K})) = \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$, hence we also know that $\varphi \notin^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$. Thus, by (H.1), $\varphi \notin^t \mathbb{K}\setminus^t\mathbb{K}*\alpha^J$. From Definition 11 and Propositon 9, we know that $\varphi \in^t \mathbb{K}*\alpha^J$. This shows that $\mathbb{K}\otimes\alpha^J \subseteq^t \mathbb{K}*\alpha^J$.

So far, we have shown that $\mathbb{K} * \alpha^J \equiv^t \mathbb{K} \otimes \alpha^J$. That is, $(\mathbb{K} \setminus^t \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\} \equiv^t (\mathbb{K} \setminus^t \sigma(\Pi(\neg\alpha^J, \mathbb{K}))) \cup \{\alpha^J\}$. As we have shown before that $\sigma(\Pi(\neg\alpha^J, \mathbb{K})) = \sigma^c(\Pi(\neg\alpha^J, \mathbb{K}))$, and in particular that they are well defined functions, the proof is reduced to show that " $\setminus^t$ " is a well defined function. As that is shown in Proposition 20, we finally know that $\mathbb{K} * \alpha^J = \mathbb{K} \otimes \alpha^J$. □

Acknowledgements This work was partially supported by PGI-UNS (Grant PGI 24/ZN37).

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