Extremal Spectral Dynamics and a Fractal Theory for Simplicial Complexes

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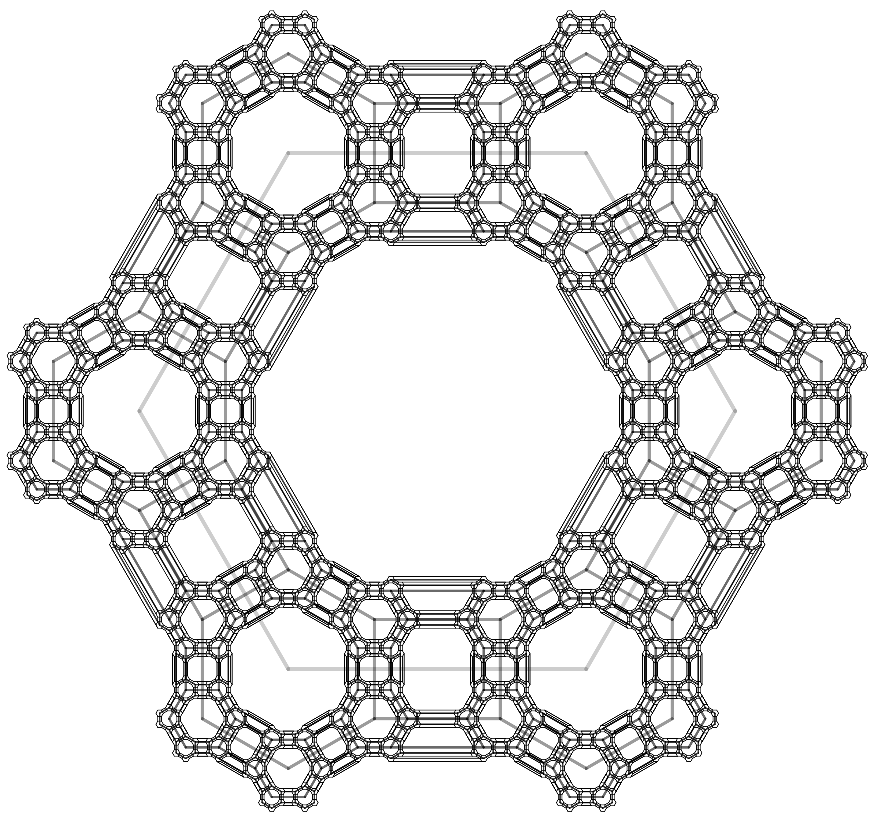

By

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DEDICATION

To my wife Ina.
Thank you for your positivity, honesty and faith in me.

ZUSAMMENFASSUNG

Ziel dieser Arbeit ist die Erforschung spektraler Asymptotiken simplizialer Komplexe verschiedener Geometrie. Der Hauptteil dieser Arbeit beschäftigt sich mit dem kombinatorischen Laplace-Operator eines gegebenen Simplizialkomplexes und dem Einfluss von iterierten Unterteilungen auf dessen Spektrum. Wir betrachten dabei eine Unterklasse der Unterteilungen, welche eine bestimmte Regularitätsannahme erfüllen, die wir "Einbettungsuniformität" nennen. Zunächst entwickeln wir einen universellen Grenzwertsatz für Spektren dieser Art; es wird gezeigt, dass die spektralen Verteilungsfunktionen der Folge der Laplace-Operatoren iterierter Unterteilungen eines beliebigen Ausgangskomplexes gleichmäßig gegen eine Grenzfunktion konvergiert welche bloß von der Dimension des Ausgangskomplexes abhängt. Damit ist dieser Grenzwert universell in dem Sinne, dass es für jede Unterteilungsart div und jede Dimension d eine Grenzfunktion $F_d^{(\mathrm{div})} \in L^\infty(\mathbb{R})$ gibt, welche für jeden d-dimensionalen Komplex K im Grenzwert der Folge $\mathrm{div}^n K$ angenommen wird.
Die Intuition hinter diesem Ergebnis ist, dass sich der Laplace-Operator als Operator welcher nur die lokale Umgebung eines Simplex einbezieht nach Anwendung beliebig vieler lokaler Unterteilungsoperationen einem "lokalen" Spektrum annähert. Um weiter den Weg für Grenzobjekt-Betrachtungen zu ebnen entwickeln wir eine Graph-basierte Fraktaltheorie, welche die Konstruktion iteriert unterteilter Simplizialkomplexe dualisiert.

Klassische Beispiele einbettungsuniformer Unterteilungen sind die baryzentrische Unterteilung in beliebigen Dimensionen oder die sogenannte edgewise-subdivision in Dimension 2. Da die Berechnung des Grenzspektrums für die baryzentrische Unterteilung außerhalb des Graph-Falls schwer ist, berechnen wir anschließend das Grenzspektrum für eine einfachere, aber der baryzentrischen Unterteilung verwandte Unterteilungsart. Hierzu verwenden wir eine Schreiergraph-Approximation der Folge der dualen Graphen. Dazu beschreiben wir eine selbst-ähnliche Untergruppe der Automorphismengruppe eines unendliche tiefen Baumes, welche als ihren Schreier-Graph die Ausgangsfolge bis auf eine asymptotisch kleine Anzahl von Schleifen auf der n-ten Ebene dieses Baumes induziert. Diese Gruppe wird beschrieben durch ein gewähltes, symmetrisches Adress-Schema der Simplizes der Unterteilungsfolge.
Ein analoger Ansatz kann für die baryzentrische Unterteilung gewählt werden, führt dort allerdings nicht zum Erfolg, da die baryzentrische Unterteilung im Gegensatz zu der simplifizierten Unterteilung nicht die Eigenschaft der endlichen Ramifikation hat, welche eine rekursive Berechnung der Determinante per Schur-Komplement möglich macht. Wir geben experimentelle Indizien für eine Selbstähnlichkeit des Grenzwerts für die baryzentrische Unterteilung an. Selbst im vereinfachten Fall müssen wir, um eine volle Rekursion der Spektren zu gewährleisten, zusätzlich zu einer komplizierteren, multi-parametrischen Matrix übergehen, was mit wachsender Komplexität der Unterteilung hochgradig nicht-trivial wird.
Aus diesem Grund präsentieren wir für eine nicht-endlich ramifizierte Unterteilung von hoher Symmetrie ein weiteres Vorgehen zur Berechnung des Grenzspektrums per Floquet-Theorie. Die Symmetrie der Unterteilung geht in diesem Fall derart ein, dass sie eine gegebene Triangulierung des Torus in einem selbst-ähnlichen Sinne verfeinert. Zum Abschluss des ersten Teils präsentieren wir offene Fragen zur Natur des universellen Grenzwerts.

Im zweiten Teil beschäftigen wir uns mit Differential Laplace-Operatoren auf Simplizialkomplexen. Hierzu assoziieren wir zu einem Simplizialkomplex K eine Geometrie, indem jede maximale Seite einen geometrische Simplex im $\mathbb{R}^m$ zugeordnet bekommt. Diese Simplizes müssen sich nicht geometrisch zu einem zu K isomorphen geometrischen Komplex im euklid'schen Raum zusammensetzen sondern dienen bloß als geometrisches Modell für die jeweilige Seite. Insbesondere muss die induzierte Geometrie global auch nicht notwendigerweise euklid'sch sein, während sie lokal flach ist. Diese Zuordnung dient als Verallgemeinerung metrischer Graphen; also Graphen mit festgelegten Kantenlängen, wobei zu einer Länge ℓ das Intervall $[0, \ell]$ als Modell dient. Ähnlich zur Sobolev-Theorie auf metrischen Graphen assoziieren wir zu unserer Geometrie dann lokale Sobolev-Funktionen und ein schwaches äußeres Differential. Dieses Differential erlaubt es uns dann analog zur Riemann'schen Geometrie einen Laplace-Operator dieser Geometrie zu definieren. Es stellt sich heraus, dass der Definitionsbereich dieses Operators die Menge aller stetigen und lokal (das heißt in jedem Simplex) zweifach differenzierbaren Funktionen ist, welche einer höherdimensionalen Kirchhoff-Bedingung längs der Ränder genügen. In Anlehnung an die Theorie der metrischen Graphen werden wir metrische Komplexe zusammen mit einem solchen Laplace-Operator dann Quanten-Simplizialkomplexe nennen.

In diesem Zusammenhang ist eine bereits bekannte spektrale Asymptotik für Riemann'sche Mannigfaltigkeiten welche "Andickungen" eines metrischen Graphen sind besonders interessant. Wir werden die Theorie dieser "Andickungen" auf unsere höherdimensionale Konstruktion verallgemeinern und zeigen Verallgemeinerungen einiger Zwischenresultate welche benötigt werden um die Asymptotik im Graph-Fall zu zeigen. Diese Mannigfaltigkeiten sind lokal als direkte Produkte der geometrischen Simplizes mit einer Mannigfaltigkeit dargestellt; diese zweite Mannigfaltigkeit fassen wir als Faser über dem Simplex auf.
Die verallgemeinerten Zwischenresultate arbeiten darauf hin Operatoren zu definieren welche zwischen den Funktionsräumen der Mannigfaltigkeit und des Komplexes unter Einhaltung einer Normabschätzung vermitteln. Wir vermuten, dass diese Zuordnungen wie im Graph-Fall durch konstante Fortsetzung sowie Mittelung längs der Faser gegeben sind. Man beachte, dass im höherdimensionalen Fall die Ränder weitere nicht-konstante Differentialterme induzieren, welche im Graph-Fall nicht auftauchen.
Auch hier geben wir wieder offene Fragen über dieses Objekt an. Natürlich fragen wir uns nach obiger Betrachtung, ob die spektrale Asymptotik für dünner werdende Mannigfaltigkeiten des beschriebenen Typs weiterhin gilt; genau wie im Graph-Fall lassen Ergebnisse über die Eigenfunktionen des Laplace-Operators des gleichseitigen Dreiecks vermuten, dass es für gleichseitige Komplexe einen Zusammenhang zum Spektrum des kombinatorischen $(d-1)$-up Laplacians gibt. Ebenfalls interessant ist eine Phasentransition im Graph-Fall für Hodge-Laplace-Spektren. Hier ist bekannt, dass für Differentialformen von Grad größer als 1 die Spektren punktweise gegen ∞ divergieren. Die naheliegende Frage ist daher, ob für d-dimensionale Komplexe eine Konvergenz bis Grad d und eine Divergenz für darüberliegende Grade gilt.

ABSTRACT

The aim of this work is the exploration of spectral asymptotics of certain geometries associated to simplicial complexes. We will state how combinatorial and differential Laplacians can be associated to a simplicial complex and describe certain asymptotics linked to their spectra.

First we take under consideration the change of spectrum for the combinatorial Laplacian under a certain class of subdivision procedures and show a universal limit theorem regarding the sequence arising from this construction. Universality in this case means that the limit spectrum carries no spectral information related to the input complex. It is thus only dependent on the dimension of the complex and the subdivision procedure used. We will carry out the explicit calculation of such a limit for one particular example of a subdivision related to barycentric subdivision. Next we point out obstructions to the application of the same procedure to the full barycentric subdivision. It will turn out that the procedure is not favorable if the given subdivision procedure is acting non-trivially on lower dimensional faces. Lastly we give an example of a subdivision procedure of high symmetry, i.e. edgewise subdivision, for which we can determine the spectrum by a group action argument even though it acts non-trivially on lower dimensional faces. Furthermore dual relations to fractal theory are examined and the particular class of fractals arising from subdivision of a complex in the sense of a graph-directed limit construction is formalized. In the end open question regarding the nature of the limit are formulated and initiating thoughts are presented.

Secondly we associate to a simplicial complex a geometry (not necessarily embeddable in euclidean space) and show that there exists a natural differential Laplacian on this geometry. These complexes can be used to model thin structures around their geometry. As this modelling procedure is a higher-dimensional generalization of quantum graphs we will call a complex equipped with this differential structure a quantum complex. Thin structures over such a complex allow for modelling of systems with a larger number of dimensions not constraint by a small diameter. We show generalizations of estimates used in the proof of the spectral asymptotic of these thin structures for the graph case indicating that a general spectral asymptotic might be possible. We formulate open questions on spectral asymptotics and the relation of the combinatorial and differential Laplacian associated to the complex.

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Introduction

Simplicial Complexes and the topological spaces associated to them have a long reaching history in algebraic topology. One outstanding strength of simplicial complexes is that they are a middle ground between flexibility - in the sense that a wide range of spaces is covered - and rigidity - in the sense that the fundamental building blocks are very basic convex sets which can be handled well by computer systems. Simplicial complexes play a central role in computational topology. In particular in the computational determination of homology, cf. [DHSW03] and persistent homology of data point clouds, cf. [EH10], [Zom05]. Another important computational application is the field of finite element method, cf. [BKK20].

Note that in particular the field of finite element exterior calculus, cf. [GHL21], [AFW10], which parallels exterior calculus on simplicial complexes draws a strong link between objects known from differential and simplicial geometry. Especially the work of Dodziuk and Patodi in the 70s can be considered a finite element approach to Riemannian Hodge theory as there have been established approximation results of the Riemannian Hodge-Laplacian of a manifold by a combinatorial Hodge-Laplacian of triangulations becoming progressively finer in [Dod74], [Dod76] and [DP76]. Specifically relevant in the context of this work is a spectral asymptotic shown in [DP76], where a point-wise convergence of the eigenvalues of certain combinatorial Hodge-Laplacians of a sequence of progressively refining triangulations towards the eigenvalues of the Riemannian Hodge-Laplacian was established. This result demonstrates a case where under the correct choice of operator (and inner product) we can win back significant knowledge about the input geometry.

The central objects of this work have already been mentioned above - sequences of iterated subdivisions, i.e. triangulations which are getting progressively finer, and Riemannian manifolds associated to simplicial complexes. Note however the following differences in the approach; the combinatorial Laplacians we analyze are always defined with respect to the standard inner product on chain complexes as no underlying Riemannian geometry is considered in Part I - we thus only study properties intrinsic to the complex at first. In this context we will show that the spectrum of the Laplacian we assign to the intrinsic geometry is universal in the sense that no matter the complex we start with its spectrum will converge (in the appropriate sense) to a universal spectral distribution only depending on the dimension of the complex and the way we refine it.

Furthermore our setting with regards to Riemannian manifolds will be different as we will not start with a triangulation of a manifold but rather with a manifold shrinking onto a simplicial complex. This shrinking procedure will be formalized in Part II. In order to define these thin structures around simplicial complexes we first introduce a generalization of quantum graphs to higher dimensions and assign to it a spectral theory via a differential Laplacian. We then present some results indicating that a generalization of the spectral asymptotic of [EP05] might be possible, though we were not able to fully show it. We present open questions together with first steps in an attempt to answer them.

Part I

Combinatorial Theory

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Introduction

The spectra of k-Laplacians of d-dimensional simplicial complexes, $k \leq d$, encode a variety of combinatorial and topological properties of the respective complex; cf. [HJ13] for an overview of Laplacian operators on simplicial complexes. The case of interest for us is when $k = d$, i.e. the top-dimensional Laplacian of a d-dimensional simplicial complex K which is defined as

$$\mathscr{L}(K) := \mathscr{L}_d(K) := \partial_d^t \partial_d$$

for the simplicial boundary operator ∂_d in dimension d. We are interested in how the spectrum of $\mathscr{L}(K)$ behaves (in the limit) under iterated subdivisions of K. We restrict ourselves to a certain intuitive subclass of geometric subdivisions in the sense of Stanley, [Sta92], which are additionally required to subdivide each face in the same way and independent of orientation. We will call them *inclusion-uniform*. The explicit definition of this class will be given in Section 2. A lot of prominent examples of geometric subdivisions are inclusion-uniform; including the edgewise subdivision of a 2-dimensional complex and barycentric subdivision in arbitrary dimension. For an overview of current research on subdivisions and their algebraic aspects we refer the reader to [Ath16] and the references therein.

We consider the spectrum of a positive-semidefinite self-adjoint operator $L : \mathbb{R}^N \to \mathbb{R}^N$ as the non-decreasing right-continuous bounded (and thus L^1) stair-case function on $[0, 1]$ given as

$$\Lambda(L) := \sum_{j=1}^{N} \lambda_j(L) \mathbf{1}_{[(j-1)/N, j/N)},$$

where $0 \leq \lambda_1(L) \leq \ldots \leq \lambda_N(L)$ are the ordered eigenvalues of L listed with multiplicities and $\mathbf{1}_A$ denotes the indicator function of the set A. This function can be considered as a shift of the quantile function of the *normalized eigenvalue counting function*[1].

Theorem I.1 (Universal Limit Theorem for inclusion-uniform subdivisions)**.** *Let $d \geq 1$ be an integer and let* div *be a inclusion-uniform subdivision acting non-trivially on d-dimensional complexes.*

Then there exists a function $\Lambda_d^{(\mathrm{div})} \in L^1([0,1])$ such that for every d-dimensional complex K it holds

$$\Lambda(\mathscr{L}(\mathrm{div}^n K)) \xrightarrow{n\to\infty}_{L^1} \Lambda_d^{(\mathrm{div})}.$$

We can associate to a complex K a multitude of Laplacians. For $i \in \mathbb{N}$ we might define the i-up Laplacian $\mathscr{L}_i^{\mathrm{up}}(K) := \partial_i \partial_i^t$ and the i-down Laplacian $\mathscr{L}_i^{\mathrm{down}}(K) := \partial_i^t \partial_i$. The i-dimensional Laplacian then is the sum of the i-up and i-down Laplacians

$$\mathscr{L}(K) := \mathscr{L}_i^{\mathrm{up}}(K) + \mathscr{L}_i^{\mathrm{down}}(K).$$

[1]This is sometimes called the integrated density of states or the spectral CDF.

Note that in case $i = 0$ or $i = \dim K$ only one of the operators is non-zero.

The existence of such universal limiting functions for 0-up Laplacians of simplicial complexes has been studied in [Kni15] for the particular case of div being barycentric subdivision. Our main object is the d-down Laplacian which in general - i.e. for $d > 1$ - has no spectral relationship to the 0-up Laplacian. However the i-up Laplacian has a strong spectral correlation with the $(i+1)$-down Laplacian where their spectra are identical including multiplicities except for the eigenvalue $\lambda = 0$. Thus in determining the spectral distribution of top-dimensional Laplacians we have a degree of freedom of whether to choose the $(d-1)$-up or d-down Laplacian to perform spectral analysis on; the choice of the d-down Laplacian will, however, prove to be more suitable as we won't have to compensate for changes in matrix size introduced by gluing (see Section 3 for more details).

The sole dependence on $\dim K$ complements a result by Brenti and Welker, [BW08], showing that the roots of f-polynomials of the sequence of iterated barycentric subdivisions of a complex converge to a universal set of roots only depending on the dimension of K. Effects of this kind can be attributed to the dominance of local features introduced by the repeated subdivision.

Having established the existence of a universal limiting function a natural question to ask is whether we can determine this function for given d and a inclusion-uniform subdivision div. This question can be reduced to one on (signed) graph spectra when considering the d-Laplacian as the graph Laplacian of the d-dual graph of K (as a signed graph). The subdivision operation then induces an operation on the dual graphs by replacing every vertex by a copy of a "fundamental graph" and joining them appropriately by edges. These joining operations in turn depend on the edges of the given graph. We thus seek to analyze the effect a graph operation induced by subdivision has on the spectrum. A variety of such spectral effects of common graph operations is summarized in [BKPS18, Cve75], with one particular example of a unary graph operation being the (barycentric) subdivision of a graph (regarded as a 1-dimensional simplicial complex).

We say that a graph operation $S : G \mapsto S(G)$ admits *"spectral decimation"* if there is a rational function f_S such that the spectrum of $S(G)$ consists of the solutions $\mu \in \mathbb{R}$ of the equations

$$\lambda = f(\mu)$$

for λ in the spectrum of G (with eventual adjustment of multiplicities and up to some "small" exceptional set $\mathscr{E}$). Thus $S(G)$ only carries spectral information stemming from either G or S (up to $\mathscr{E}$). The notion of spectral decimation originates from fractal analysis, e.g. [Jor10]. In Section 4 we will describe how iterated subdivisions fit the framework of spectra of self-similar graph sequences. Graph subdivision is one case for which a spectral decimation holds as long as the input graph is regular, [BKPS18].

In order for spectral decimation to be applicable iteratively we need to assume the initial graph G to be 2-regular. Then $S(G)$ will again be 2-regular. For r-regular graphs G, $r \geq 3$ $S(G)$ is not regular anymore. However as the limiting distribution does not depend on G (as we will see in Theorem I.1) we can pick the initial setting G at will - in particular we might choose it to be 2-regular.

Note that the Laplacian of a 2-regular graph can be written as

$$L(G) = 2 \cdot I - A(G)$$

where $A(G)$ denotes the adjacency matrix of G and I is the identity matrix of proper size. Since 2-regularity is preserved under subdivision so is the relation between Laplacian and adjacency

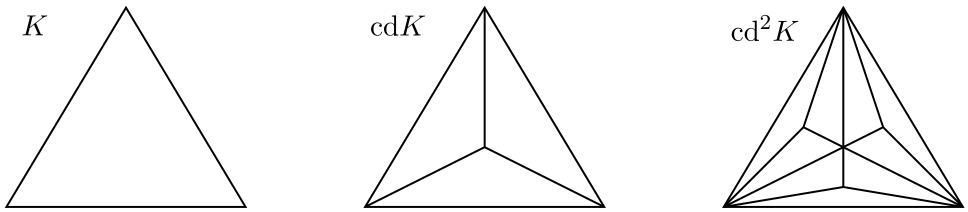


Figure 1.1: The first 3 complexes of the sequence of iterated application of cd for $d = 2$ for the initial complex $K = \Delta^{(2)}$ - the standard-2-simplex.

matrix. As a consequence of this the sequences of spectra of Laplacians and adjacency matrices are related over an affine-linear transformation. In this particular case we obtain that the eigenvalues of the adjacency matrix of $S(G)$ are given by the roots of the polynomial equation

$$f_A(\zeta) = \zeta^2 - 2 = \lambda$$

for λ running over the set of eigenvalues of the adjacency matrix associated to the initial graph G, as shown in [BKPS18] for example. In case such a decimation holds we call f_A the spectral decimation map. Analogously the spectral decimation map for the Laplacian spectrum in the 2-regular graph case is given by

$$f_L(\zeta) = \zeta(4 - \zeta)$$

which can be seen through substituting by the affine-linear transformation of spectra discussed above.

There are many subdivision procedures div which coincide with S on 1-dimensional simplicial complexes. One natural question to ask is which of those generalizes S in a spectral sense. In Section 3 we will find a higher-dimensional analogue of the above decimation for the subdivision operation cd shown in Figure 1.1. For a complex K of dimension d cd$K$ is obtained from the $(d-1)$-skeleton $K^{(d-1)}$ by adding the barycenter v_σ of every facet $\sigma \in F_d(K)$ together with the faces $v_\sigma \cup \tau$ for $\tau < \sigma$. As we will see from this concrete example the determination of an exact spectral decimation is much more involved in this case.

This work is structured as follows: Section 2 gives an introduction to the main objects and frameworks used in the course of this paper. The Universal Limit Theorem, Theorem I.1, is proven in Section 3. The universal limit of the subdivision cd is determined by Theorem I.16 in Section 3. Lastly in Section 4 we point out the strong relation the spectral theory for iterated subdivision has to fractal theory by giving a construction procedure of fractals dualizing subdivision of a complex.

Preliminaries

Basics on simplicial complexes

The following objects are defined in [HJ13] (even though the notation might vary). A thorough introduction to simplicial topology and geometry can be found in [Mun18].

A simplicial complex K on a finite vertex set V is a collection of subsets of V downwards-closed under $\subset$, i.e. if $A \subset B \in K$ then also $A \in K$. We denote by $F_i(K)$ the collection of sets of K of size $i+1$ and call those elements *i-dimensional faces* of K. The dimension of K is the maximum dimension of a face in K.

We call a simplicial complex K oriented if for every face $\tau \in K$ we fix a linear ordering of the vertices of τ. Two orientations of K are said to be equivalent if for every $\tau \in K$ the orderings fixed for the vertices of τ are obtained from each other by applying an even permutation, thus partitioning orientations of τ in two equivalence classes. If the orientation fixed for τ is relevant we emphasize this by writing $[\tau]$ instead of τ. The orientation opposite to $[\tau]$ is denoted by $-[\tau]$. We denote by $C_i(K)$ the $\mathbb{R}$-vector space over the basis elements $\{e_{[\tau]} \mid \tau \in F_i(K)\}$ and call $C_i(K)$ the chain groups of K with coefficients in $\mathbb{R}$. The opposite orientations of elements of $F_i(K)$ are interpreted as elements of $C_i(K)$ by

$$e_{-[\tau]} = -e_{[\tau]}.$$

$C_\bullet(K)$ becomes a chain complex with the usual simplicial boundary operator,

$$\partial_i[v_0, ..., v_i] := \sum_{j=0}^{i} (-1)^j e_{[v_0,...,\hat{v_j},...,v_i]}.$$

Further we equip $C_i(K)$ with the standard inner product and denote by ∂_i^* the operator adjoint to ∂_i with respect to the chosen inner products.

Now we are ready to define the Laplacian operators in different dimensions.

Definition I.2. Let K be an oriented simplicial complex and $i \in \mathbb{N}$ then we define

- the i-up Laplacian to be
$$\mathscr{L}_i^{\mathrm{up}}(K) := \partial_{i+1}\partial_{i+1}^*,$$
- the i-down Laplacian to be
$$\mathscr{L}_i^{\mathrm{down}}(K) := \partial_i^*\partial_i$$
and
- the i-Laplacian to be
$$\mathscr{L}_i(K) := \mathscr{L}_i^{\mathrm{down}}(K) + \mathscr{L}_i^{\mathrm{up}}(K).$$

Note that by definition for a d-dimensional complex K it holds $\mathscr{L}_d^{\text{up}}(K) = 0$ and thus

$$\mathscr{L}_d(K) = \mathscr{L}_d^{\text{down}}(K).$$

We will describe to combinatorics decoded by $\mathscr{L}_d^{\text{down}}(K)$ in the following.

In order to model higher-dimensional adjacencies in K we will say $\tau, \tau' \in F_{i+1}(K)$ are $(i+1)$-*down neighbors* if they share a common i-face, i.e. $\tau \cap \tau' \in F_i(K)$. The i-dual graph $\Gamma^{(i)}(K)$ of a complex K for us then is the graph on vertex set $F_i(K)$ with edge set E modelling the i-down adjacency, i.e. $\{\tau, \tau'\} \in E$ iff $\tau \cap \tau' \in F_{i-1}(K)$.

A signed graph $G = (V, E, \sigma)$ is an undirected graph G with a function $\sigma : E \to \{\pm 1\}$ signing each edge. The degree of a vertex in a signed graph is the degree of a vertex in the underlying undirected graph $G = (V, E)$. Order the vertices of G arbitrarily and denote by $D(G)$ the diagonal matrix of degrees of vertices of G, $D(G)_{ii} = \deg(v_i)$, and $A(G)$ the signed adjacency matrix of G,

$$A(G)_{ij} = \begin{cases} 0 & , \{i,j\} \notin E \\ \sigma(\{i,j\}) & , \{i,j\} \in E \end{cases}.$$

Note that the Laplacian of a simplicial complex then is a natural generalization of the graph Laplacian

$$\mathscr{L}(G) := D(G) + A(G)^1$$

in the following sense:

By Proposition 3.3.3 of [Gol02] we have that for K an oriented simplicial complex it holds that

$$\mathscr{L}_i^{\text{down}}(K) = \mathscr{L}(\Gamma^{(i)}(K), \sigma)$$

where the sign map $\sigma : E \to \{\pm 1\}$ is given by

$$\sigma(\{\tau, \tau'\}) := \delta_\tau(\tau \cap \tau') \cdot \delta_{\tau'}(\tau \cap \tau')$$

for $\delta_\tau : \ F_{i-1}(\tau) \to \{\pm 1\}$ given as

$$\delta_\tau(\nu) := \langle \partial_i e_{[\tau]}, e_{[\nu]} \rangle,$$

i.e. the coefficient of $e_{[\nu]}$ in $\partial_i e_{[\tau]}$. This definition measures if the induced orientation of $[\tau]$ over ∂_i coincides with the orientation $[\nu]$ fixed for ν. Thus if the induced orientations of $[\tau]$ and $[\tau']$ on $\tau \cap \tau'$ are the same we obtain

$$A(\Gamma^{(i)}(K), \sigma)_{\tau,\tau'} = 1$$

and if they differ

$$A(\Gamma^{(i)}(K), \sigma)_{\tau,\tau'} = -1.$$

In case τ and τ' are not even i-down neighbors the adjacency operator is zero in this entry.

A special case where this point of view is particularly interesting is the case of *orientable* complexes. We say a pure d-dimensional simplicial complex K is orientable if there is an orientation of K such that every pair of d-down neighboring faces $\{\tau, \tau'\}$ induces opposing orientations on $\tau \cap \tau'$, i.e. in the above notation

$$\sigma(\{\tau, \tau'\}) = -1.$$

Thus $\sigma \equiv -1$ and the d-dual graph is just an undirected graph with $\mathscr{L}_d(K)$ being its ordinary graph Laplacian. As mentioned above in what follows we will consider the case $i = d = \dim K$ and will denote the top-dimensional Laplacian by $\mathscr{L}(K) := \mathscr{L}_d(K)$.

[1] We obtain the common Laplacian operator for the sign $\sigma \equiv -1$ in this definition.

Asymptotic spectral analysis

Definition I.3. Let L be a Hermitian $N \times N$ matrix. We call the L^1-function

$$\Lambda(L) = \sum_{j=1}^{N-1} \lambda_j(L) \mathbf{1}_{[\frac{j-1}{N}, \frac{j}{N})}$$

the *shifted spectral quantile function of* L.

Note that this notion originates from the fact that $\Lambda(L)$ is a shift of the quantile function of the spectral CDF

$$F_L(x) = \frac{1}{N} \#\{i \in [N] \mid \lambda_i(L) \leq x\}.$$

The quantile function of F_L is given as

$$Q_L(p) = \sum_{j=1}^{N-1} \lambda_j(L) \mathbf{1}_{[j/N,(j+1)/N)} + \lambda_N(L) \mathbf{1}_{\{1\}}$$

and thus $\Lambda(L)$ is the shift

$$\Lambda(L)(p) = Q_L(\min(p + 1/N, 1)).$$

For the rest of this work we will denote by $||\cdot||_1^{\text{norm}}$ the normalized L^1-norm of matrices, i.e. for $A \in \mathbb{C}^{N \times N}$

$$||A||_1^{\text{norm}} := \frac{||A||_1}{N},$$

for the common L^1 matrix-norm.

The following proposition is [LM99, inequality (1.2)]; we refer the reader to the sources mentioned in the introduction therein.

Proposition I.4 (1-Wielandt-Hoffman inequality, [LM99])**.** *Let $L, E \in M_N(\mathbb{C})$ be Hermitian matrices. It holds that*

$$\sum_{j=1}^{N} |\lambda_j(L+E) - \lambda_j(L)| \leq \sum_{j=1}^{k} \sigma_j(E) = ||E||_{S^1},$$

*where $\sigma_j(E)$ denotes the j-th singular value of E and $||\cdot||_{S^1}$ is the Schatten-*1*-norm.*

Together with the fact that $||\cdot||_{S^1} \leq ||\cdot||_1$[2] we obtain the following useful corollary.

Corollary I.5. *Let $L, E \in M_N(\mathbb{C})$ be Hermitian matrices. It holds that*

$$||\Lambda(L+E) - \Lambda(L)||_{L^1} \leq ||E||_1^{\text{norm}},$$

where $||\cdot||_{L^1}$ denotes the $L^1([0,1])$-norm.

We will use this inequality in the proof of Theorem I.1 in a similar manner to how related statements are used for the use of *approximating class of sequences* in GLT matrix theory, cf. [GSC17].

[2]Which can easily be seen from the fact that the Schatten-1-norm is the nuclear norm for 2-tensors as mentioned in [FL18] and the references therein.

Tools for explicit spectral analysis

In order to exactly compute certain determinants or inverses under low-rank perturbations in Section 3 we will use the following two convenient results.

Lemma I.6 (Sherman-Morrison-Woodbury formula, [SM50, Hag89]). *Let $A \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times m}$, $V \in \mathbb{R}^{m\times n}$. Assume A and $I_m - VA^{-1}U$ are invertible. Then the inverse of $A - UV$ is given as*

$$(A - UV)^{-1} = A^{-1} + A^{-1}U(I_m - VA^{-1}U)^{-1}VA^{-1}.$$

In particular for $m = 1$, $U = u \in \mathbb{R}^n, V = v \in \mathbb{R}^n$ we obtain the original Sherman-Morrison formula

$$(A - uv^t)^{-1} = A^{-1} + \frac{A^{-1}uv^tA^{-1}}{1 - v^tA^{-1}u}.$$

Lemma I.7 (Matrix Determinant Lemma, Theorem 18.1.1 of [Har97]). *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $U \in \mathbb{R}^{n\times m}$, $V \in \mathbb{R}^{m\times n}$. It holds that*

$$\det(A + UBV) = \det A \det B \det(B^{-1} + VA^{-1}U).$$

In the particular case of $m = 1$, $B = 1$ and vectors $U = u \in \mathbb{R}^m$, $V = v \in \mathbb{R}^m$ we obtain

$$\det(A + uv^t) = (1 + v^tA^{-1}u)\det A.$$

The following result will help us resolve block matrix determinants.

Lemma I.8 (Schur-Renormalization, Theorem 13.3.8. of [Har97]). *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$. Then it holds that*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} D & C \\ B & A \end{pmatrix} = \det A \det(D - CA^{-1}B).$$

Iterated subdivisions of simplicial complexes

We will be using the notion of geometric subdivisions, cf. [Sta92], [Mun18, p. 83]. To this end we assume every simplicial complex to be a geometric simplicial complex, i.e. be embedded in some euclidean space for the rest of this subsection. This is no obstruction on the simplicial complex as every abstract simplicial complex has a *geometric realization*, cf. [Mun18, Theorem 3.1]. We will thus use the notions of geometric and abstract complexes interchangably - assuming to have fixed some geometric realization of the initial complexes. We assume the standard-d-simplex to be realized as $\mathrm{conv}(e_1, ..., e_{d+1}) \subset \mathbb{R}^{d+1}$ for the standard basis $\{e_1, ..., e_{d+1}\}$.

Furthermore let d be a fixed dimension.

Definition I.9. A procedure div associating to a d-dimensional geometric complex K a geometric complex $\mathrm{div}K$ is called a *subdivision procedure* if the following conditions hold:

(i) Every simplex of $\mathrm{div}K$ is contained in some simplex of K.

(ii) Every simplex of K is the union of finitely many simplices of $\mathrm{div}K$.

It is well-known that every subdivision div$K$ induces a map $s : \mathrm{div}K \to K$ associating to a face $\sigma \in \mathrm{div}K$ the smallest face $\tau \in K$ such that σ is contained in τ. The subcomplexes $\mathrm{div}_K(\tau) := s^{-1}(2^\tau) \leq \mathrm{div}K$ are called *restrictions of divK to* τ for $\tau \in F_i(K)$. $\mathrm{div}_K\tau$ corresponds to the subdivision of τ as a face in K.

Definition I.10. A subdivision procedure div is said to be *inclusion-uniform* if for every d-dimensional complex K and face $\tau \in K$ of dimension i, $i \in \{0, ..., d\}$, every possible identification of τ with Δ_i extends to an isomorphism between $\mathrm{div}_K\tau$ and $\mathrm{div}\Delta_i$, i.e. let $\tau = \mathrm{conv}(v_0, ..., v_i)$ and given a bijection $f : \{v_0, ..., v_i\} \to \{e_1, ..., e_{i+1}\} = F_0(\Delta_i)$ there exists a unique simplicial isomorphism $\tilde{f} : \mathrm{div}_K\tau \to \mathrm{div}\Delta_i$ such that $\tilde{f}_{|\{v_0,...,v_i\}} = f$.

An immediate consequence of the definition is that for two complexes K and L and dedicated faces $\tau \in F_i(K)$, $\sigma \in F_i(L)$ with a bijective vertex map $\pi : F_0(\tau) \to F_0(\sigma)$ there is a unique simplicial isomorphism

$$\tilde{\pi} : \mathrm{div}_K\tau \to \mathrm{div}_L\sigma$$

such that $\tilde{\pi}(v) = \pi(v)$ for $v \in F_0(\tau)$.

Note that the barycentric subdivision - sd defined as the complex of increasing sequences of faces (so called *flags*) in K is itself inclusion-uniform. inclusion-uniform subdivisions are uniquely determined by a sequence of subdivisions $\mathrm{div}\Delta_i$ of Δ_i, $i \in \mathbb{N}$, such that the restriction of $\mathrm{div}\Delta_i$ to σ is isomorphic to $\mathrm{div}\Delta_{i-1}$ for every $\sigma \in F_{i-1}(\Delta_i)$. Such a sequence is called a *subdivision scheme* in the following. As the face number of the subdivided i-simplex is intrinsic to div in what follows we will write

$$f_i(\mathrm{div}) := f_i(\mathrm{div}\Delta_i),$$

i.e. $f_i(\mathrm{div})$ counts the number of facets the standard i-simplex gets subdivided in.

In particular inclusion-uniform subdivisions are a special case of repeatable subdivisions, i.e. subdivisions which can be applied arbitrarily often to any initial complex K. This can be seen by describing the procedure of subdividing according to div in an iterative manner. Let K be a given d-dimensional complex, then the isomorphism type of div$K$ can be obtained from K and a subdivision scheme $\{\mathrm{div}\Delta_i\}_{i=0,...,d}$ by the following inductive construction: Set $K_0 = F_0(K)$.

Now let K_i be constructed for some $0 \leq i < d$. For every $\tau \in F_{i+1}(K)$ let $\tau = \mathrm{conv}(v_0, ..., v_{i+1})$. Identify $\{v_0, ..., v_{i+1}\}$ with $\{e_1, ..., e_{i+2}\}$ arbitarily and let $\tilde{f}$ denote the isomorphism of $\mathrm{div}_K\tau$ and $\mathrm{div}\Delta_{i+1}$ induced by this identification. Add to K_i the pre-image of $\tilde{f}$ and proceed with the next $(i+1)$-face of K. This way we obtain K_{i+1}.

Note that since div is inclusion-uniform the construction does not depend on the chosen identifications and thus K_d is isomorphic to div$K$. It is apparent by this procedure that div is repeatable.

Furthermore in what follows we will call div *finitely ramified* or *of finite ramification* if

$$f_{d-1}(\mathrm{div}) = 1,$$

i.e. if div only acts non-trivially on d-faces. This notion is inspired by the fractal concept underlying the spectral theory we are discussing in the upcoming section, see Section 4 for this connection.

In order to prove the main theorem of this paper we will need another operation on simplicial complexes.

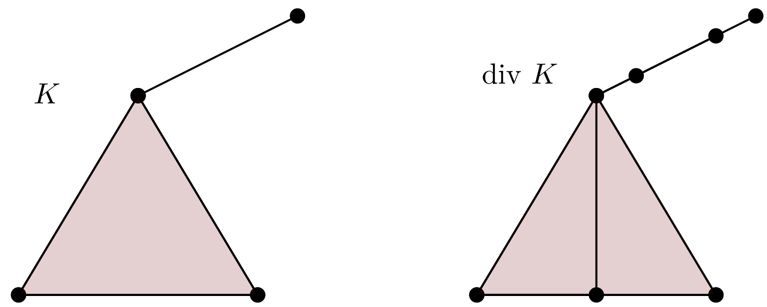


Figure 2.1: Subdivision procedure which is not inclusion-uniform. See how there are edges subdivided by one or two vertices or not even subdivided at all. Obviously those are not isomorphic as simplicial complexes. Note also that the subdivision of the 2-face is not rotational invariant which would be necessary for div to be inclusion-uniform.

Gluing and inclusion-uniform subdivisions

We now consider two formally disjoint d-dimensional complexes K and L. Let $\mathscr{G}$ be a relation on the set $F_0(K) \times F_0(L)$. We write $v\mathscr{G}w$ for $\mathscr{G}(v,w)$.

Definition I.11. We say that $\mathscr{G}$ defines a *gluing* of K and L if the following holds:

- For every vertex $v \in F_0(K)$ there is at most one vertex $w \in F_0(L)$ such that $v\mathscr{G}w$ and vice versa, i.e. let
$$G_0(K) := \{v \in F_0(K) \mid \exists_{w \in F_0(L)} : \ v\mathscr{G}w\}$$
and $G_0(L)$ analogously, then there is a bijection $\varphi : G_0(K) \to G_0(L)$ such that $v\mathscr{G}w$ iff $w = \varphi(v)$.

- φ induces a well-defined simplicial isomorphism between $K_{|_{G_0(K)}}$ and $L_{|_{G_0(L)}}$.

In the following we denote by $G(K)$ and $G(L)$ the vertex-induced subcomplexes $K_{|_{G_0(K)}}$ and $L_{|_{G_0(L)}}$.

Note that since φ induces a well-defined simplicial isomorphism $\tilde{\varphi}$ between $G(K)$ and $G(L)$ the *glued complex*
$$K\mathscr{G}_*L := K \sqcup L \big/ \sim_{\mathscr{G}}$$
is well-defined for $\sim_{\mathscr{G}}$ being the relation on $K \times L$ generated by the relations $\sigma \sim_{\mathscr{G}} \tilde{\varphi}(\sigma)$ for $\sigma \in G(K)$. Denote for a gluing $\mathscr{G}$ by $r_i(\mathscr{G})$ the number of non-trivial relations
$$\tau \sim_{\mathscr{G}} \sigma$$
for $\tau \in F_i(K)$, $\sigma \in F_i(L)$.

Note that gluing procedures of more than two complexes can be defined inductively. In this case we write
$$\mathscr{G}_*(K_1, ..., K_\ell)$$
for the glued complex.

In the following let s_K, s_L, s denote the subdivision maps of K, L and $K\mathscr{G}_*L$, respectively. Given two complexes K and L let $\iota_K :\ \mathrm{div}K \to \mathrm{div}(K\mathscr{G}_*L)$ and $\iota_L :\ \mathrm{div}L \to \mathrm{div}(K\mathscr{G}_*L)$ be the natural geometrical inclusions induced by the inclusions $\iota'_K : K \to K\mathscr{G}_*L$ and $\iota'_L : L \to K\mathscr{G}_*L$

over the isomorphism derived from Definition I.10, i.e. for every face $\tau = \{v_0, ..., v_i\} \in K$ we define

$$\iota_K|_{\mathrm{div}_K \tau} := \widetilde{(\iota'_K)_{|\tau}}.$$

This definition is compatible along boundaries and thus assembles to a well-defined injective function (since $s_K^{-1}(\tau)$ are disjoint sets for distinct τ's).

Obviously two faces in div$K$ and div$L$ can only be mapped onto the same face by ι_K and ι_L in div$(K\mathscr{G}_*L)$ if they lie in some face in $G(K)$ or $G(L)$, respectively. Furthermore the union of images im$\iota_K \cup$ imι_L exhausts div$(K\mathscr{G}_*L)$ and so div$(K\mathscr{G} * L)$ can be obtained as a gluing from div$K$ and div$L$ by identifying faces which are mapped the same face in div$(K\mathscr{G}_*L)$.

This gluing procedure is precisely given by the relation $\mathscr{G}'$ generated by

$$v\mathscr{G}'w$$

for $v \in F_0(\mathrm{div}K)$ and $w \in F_0(\mathrm{div}L)$ if $\iota_K(v) = \iota_L(w)$. Thus

$$G_0(\mathrm{div}K) = F_0(s^{-1}(G(K))), \quad G_0(\mathrm{div}L) = F_0(s^{-1}(G(L)))$$

and the bijection $\varphi' : G_0(\mathrm{div}K) \to G_0(\mathrm{div}L)$ satisfying the two conditions of a gluing is given by

$$\varphi'(v) := \widetilde{(\iota'_L)_{|\sigma}}^{-1} \circ \widetilde{(\iota'_K)_{|\tau}}(v) \tag{2.1}$$

for $\tau := s_K^{-1}(v)$ and $\sigma := \iota_L'^{-1} \circ \iota'_K(\tau)$. By definition the simplicial map defined by φ' is compatible along boundaries and yields an isomorphism of the respective vertex-induced subcomplexes.

By all the above we have

$$\mathrm{div}(K\mathscr{G}_*L) \cong (\mathrm{div}K)\mathscr{G}'_*(\mathrm{div}L).$$

Note that assuming $r_d(\mathscr{G}) = 0$, i.e. $\mathscr{G}$ does not identify facets of K and L with each other, the newly defined gluing $\mathscr{G}'$ satisfies

$$r_{d-1}(\mathscr{G}') = f_{d-1}(\mathrm{div}) \cdot r_{d-1}(\mathscr{G}).$$

We summarize this procedure in the following proposition for later use.

Proposition I.12 (Subdivision gluing)**.** *Let* div *denote a inclusion-uniform subdivision. Given a gluing $\mathscr{G}$ of K and L satisfying $r_d(\mathscr{G}) = 0$ there exists a gluing $\mathscr{G}'$ of* div$K$ *and* div$L$ *so that* div$(K\mathscr{G}_*L) = (\mathrm{div}K)\mathscr{G}'_*(\mathrm{div}L)$ *and*

$$r_{d-1}(\mathscr{G}') = f_{d-1}(\mathrm{div}) \cdot r_{d-1}(\mathscr{G}).$$

The d-Laplacian operator of the glued complex has the form

$$\Delta(K\mathscr{G}_*L) = \begin{pmatrix} \Delta(K) + D_K & G \\ G^t & \Delta(L) + D_L \end{pmatrix},$$

where G maps a d-face τ of K to a sum of d-faces τ' of L (with some signs given by orientations) if there are $\sigma \in F_{d-1}(\tau)$ and $\sigma' \in F_{d-1}(\tau')$ such that $\sigma \sim_{\mathscr{G}} \sigma'$ and D_K, D_L are diagonal matrices counting the $(d-1)$-faces for every d-face which are involved in gluing for K and L, respectively. Thus if we denote by D the maximal down-degree of $K\mathscr{G}_*L$ we have

$$||D_K||_{L^1}, ||D_L||_{L^1} \leq D \cdot \max(f_d(K), f_d(L))$$

and

$$||G||_{L^1} \leq r_{d-1}(\mathscr{G}).$$

The Universal Limit Theorem

Proof of the Universal Limit Theorem

Now that we have all relevant notions from the introductory section at hand we can prove the main result of this paper, Theorem I.1.

The proof works in two steps which we will state in two propositions. The theorem then follows from the combination of Propositions I.13 and I.14.

For the rest of the chapter let d and div as in Theorem I.1 be fixed. Note that the non-triviality of div can be equivalently states as $f_d(\mathrm{div}) > 1$. Further let K be an arbitrary initial d-dimensional complex. $(K_n)_{n\in\mathbb{N}}$ denotes the sequence of complexes generated by iterated application of div to the initial complex K, i.e. $K_n := \mathrm{div}^n K = \mathrm{div} K_{n-1}$, $K_0 = K$. Furthermore by $\mathscr{L}_n$ and Λ_n we denote the corresponding sequence of Laplacians and their shifted spectral quantile functions $\Lambda(\mathscr{L}_n) \in L^1([0,1])$, respectively. The claim is thus that Λ_n converges towards a universal distribution of eigenvalues depending only on d.

Proposition I.13 (Dominance of local spectra)**.** *Let Δ_d denote the standard-d-simplex. Then in the setting of Theorem I.1 it holds that*

$$||\Lambda_n - \Lambda(\mathscr{L}(\mathrm{div}^n \Delta_d))||_{L^1} \xrightarrow{n\to\infty} 0,$$

i.e. the spectral quantile function of K_n is asymptotically L^1-equivalent to the spectral quantile function of the sequence obtained by subdividing Δ_d.

What this means is that global features of the spectrum eventually become dominated by the local features introduced by subdivision of a single simplex.

Proof. The proof esentially uses Corollary I.5 with a counting of non-zero entries which have to be removed in order to transform $\mathscr{L}_n$ in a suitable block-diagonal form. This counting is mainly performed by Proposition I.12.

As K is d-dimensional the only faces relevant for $\mathscr{L}_n$ are the faces in $F_d(K_n)$ and their down-adjacencies (with respect to an arbitrary orientation of K). Thus we can without loss of generality assume K to be pure and consequently K_n to be pure aswell.

Let $N := f_d(K)$. Note that K can be written as a gluing of N standard-d-simplices by purity;

$$K = \mathscr{G}_*(\Delta_d, ..., \Delta_d),$$

where $\mathscr{G}$ is defined by the lower-adjecencies of the facets of K und some arbitrary identification with the N copies of Δ_d. In particular $r_d(\mathscr{G}) = 0$.

Since div is inclusion-uniform the process of subdividing K corresponds to subdividing the copies of Δ_d according to its subdivision scheme $\{\mathrm{div}\Delta_i\}_{i\in\mathbb{N}}$ under induced identification of their faces so that by iterated application of Proposition I.12 we can write K_n as

$$K_n = \mathscr{G}^{(n)}(\mathrm{div}^n \Delta_d, ..., \mathrm{div}^n \Delta_d).$$

Where the number of identifications of $(d-1)$-faces is

$$r_{d-1}(\mathscr{G}^{(n)}) = (f_{d-1}(\mathrm{div}))^n r_{d-1}(\mathscr{G}).$$

Let L_n denote the sequence of Laplacians of $\mathrm{div}^n \Delta_d$. Then the d-Laplacian of K_n is of the form

$$\mathscr{L}_n = \begin{pmatrix} L_n + D_1 & G_{12} & G_{13} & ... & G_{1N} \\ G_{12}^t & L_n + D_2 & G_{23} & ... & G_{2N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & G_{(N-1)N} \\ G_{1N}^t & ... & G_{(N-2)N}^t & G_{(N-1)N}^t & L_n + D_N \end{pmatrix},$$

where D_k corrects the degrees on the diagonal of L_n along the boundary of the k-th copy of $\mathrm{div}^n \Delta_d$. This correction consists of addition by one for every $(d-1)$-face of a d-face involved in the gluing process defined by $\mathscr{G}^{(n)}$. Let D be the maximal down-degree of the facets of K, then

$$||D_i||_1 \leq D \cdot (f_{d-1}(\mathrm{div}))^n.$$

Further G_{ij} are the matrices containing the down-adjecencies added by gluing the copies $\mathrm{div}^n \Delta_d$ according to $\mathscr{G}^{(n)}$. Note that only $r_{d-1}(\mathscr{G})$ of those G_{ij} are non-zero matrices and the non-zero G_{ij}'s have

$$||G_{ij}||_1 \leq (f_{d-1}(\mathrm{div}))^n$$

so that in total by Corollary I.5 we have

$$||\Lambda(\mathscr{L}(\tilde{K}_n)) - \Lambda(\mathscr{L}(K_n))||_{L_1} \leq \frac{(ND + 2 \cdot r_{d-1}(\mathscr{G}))(f_{d-1}(\mathrm{div}))^n}{N \cdot (f_d(\mathrm{div}))^n} \leq (D + 2 \cdot r_{d-1}(\mathscr{G})) \Big(\frac{f_{d-1}(\mathrm{div})}{f_d(\mathrm{div})} \Big)^n.$$

where

$$\tilde{K}_n = \bigsqcup_{j=1}^{N} \mathrm{div}^n \Delta_d$$

with Laplacian matrix

$$\mathscr{L}(\tilde{K}_n) = \mathrm{diag}(L_n, ..., L_n).$$

Note that by this equation it holds that

$$\Lambda(\tilde{K}_n) = \Lambda(\mathrm{div}^n \Delta_d).$$

Thus the claim holds iff

$$f_{d-1}(\mathrm{div}) < f_d(\mathrm{div}).$$

This will be shown in Lemma I.15. □

The above proposition immediately shows universality of a limiting function if it exists. The following proposition shows its existence.

Proposition I.14 (Convergence of local spectra)**.** *Let $K = \Delta_d$ in the setting of Theorem I.1. Then the sequence $(\Lambda_n)_{n\in\mathbb{N}}$ converges in L^1.*

Proof. To this end we show that $(\Lambda_n)_{n\in\mathbb{N}}$ is a Cauchy sequence - showing existence of a limit by completeness of L^1.

The sequence K_n in this case can be obtained as $K_0 = \Delta_d$ and

$$K_n = \mathrm{div}^{n-1}(\mathrm{div}\Delta_d).$$

Note that

$$\mathrm{div}\Delta_d = \mathscr{G}(\Delta_d, ..., \Delta_d)$$

where $\mathscr{G}$ glues $f_d(\mathrm{div})$-many d-faces along at most $\frac{d+1}{2} f_d(\mathrm{div})$ $(d-1)$-faces (note that $\mathrm{div}\Delta_d$ has to be a pseudo-manifold as a triangulation of the d-disk), i.e.

$$r_{d-1}(\mathscr{G}) \leq \frac{d+1}{2} f_d(\mathrm{div})$$

and

$$r_d(\mathscr{G}) = 0.$$

Thus as in the above proposition we have

$$\|\Lambda_n - \underbrace{\Lambda(\bigsqcup_{i=1}^{f_d(\mathrm{div})} K_{n-1})}_{=\Lambda_{n-1}}\|_{L^1} \leq \underbrace{\frac{(d+1+\frac{d+1}{2} f_d(\mathrm{div}))}{f_d(\mathrm{div})}}_{=:c} \Big(\frac{f_{d-1}(\mathrm{div})}{f_d(\mathrm{div})}\Big)^n.$$

We denote by

$$q_d(\mathrm{div}) := \frac{f_{d-1}(\mathrm{div})}{f_d(\mathrm{div})}$$

and will obtain from Lemma I.15

$$q_d(\mathrm{div}) < 1.$$

Denote by $n_m = f_d(K_m)/f_d(K_n) = f_d(\mathrm{div})^{m-n}$. Applying the above inequality $m-n$ times, $m > n$, we obtain by triangle inequality that

$$\|\Lambda_m - \underbrace{\Lambda(\bigsqcup_{i=1}^{n_m} K_n)}_{=\Lambda_n}\|_{L^1} \leq c \sum_{i=n+1}^{\infty} q_d(\mathrm{div})^i \xrightarrow{n\to\infty} 0$$

where the right-hand side is a cut-off of a convergent geometric series. Thus $(\Lambda_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of $L^1([0,1])$ we obtain the claim. □

Lemma I.15. *Let* div *be a non-trivial inclusion-uniform subdivision and* $q_d(\mathrm{div}) := \frac{f_{d-1}(\mathrm{div})}{f_d(\mathrm{div})}$*. Then it holds that*

$$q_d(\mathrm{div}) < 1.$$

Proof. Assume that $F_d(\mathrm{div}\Delta_d)$ has less or equal the amount of elements of $F_{d-1}(\mathrm{div}\Delta_{d-1})$. Recall that $\mathrm{div}\Delta_d$ is a subdivision of Δ_d the standard d-simplex with boundary $F_{d-1}(\Delta_d) = \{\sigma_0, ..., \sigma_d\}$. By definition every restriction $\mathrm{div}_{\Delta_d}(\sigma_i)$ of $\mathrm{div}\Delta_d$ onto σ_i results in a complex isomorphic to $\mathrm{div}\Delta_{d-1}$. But by definition every $\tau \in F_{d-1}(\mathrm{div}_{\Delta_d}\sigma_i)$ is contained in σ_i and thus must be contained in the boundary of some d-face of $\mathrm{div}\Delta_d$ as this complex is homeomorphic to a d-ball. In particular every facet τ of $\mathrm{div}_{\Delta_d}\sigma_i$ is contained in a unique facet $\sigma_\tau \in \mathrm{div}\Delta_d$. Obviously the strict inequality is thus false and equality would need to hold. Assume thus that $f_d(\mathrm{div}\Delta_d) = f_{d-1}(\mathrm{div}\Delta_{d-1})$. Note that it is impossible for $\sigma \in F_d(\mathrm{div}\Delta_d)$ to contain two $(d-1)$-faces in the same σ_i. This is immediate as by definition a inclusion-uniform subdivision has to be geometric and thus every face $\sigma \in F_d(\mathrm{div}\Delta_d)$ has to be spanned by $(d+1)$ affinely independent points. In case two codimension-1-faces of σ are contained in the same σ_i all vertices of σ would be contained in $(d-1)$-dimensional convex hull. A contradiction.

Thus it immediately follows from the above that

$$\#\underbrace{\{\sigma_\tau \mid i \in \{0, ..., d\},\ \tau \in F_{d-1}(\mathrm{div}_{\Delta_d}\sigma_i)\}}_{=:M \subseteq F_d(\mathrm{div}\Delta_d)} = f_{d-1}(\mathrm{div}\Delta_{d-1})$$

because if we had a single unmatched simplex $\sigma \in F_d(\mathrm{div}\Delta_d) \setminus M$ we had

$$f_{d-1}(\mathrm{div}\Delta_{d-1}) \leq \#M < f_d(\mathrm{div}\Delta_d)$$

which contradicts the equality we assumed. Further since it is impossible for $\sigma \in F_d(\mathrm{div}\Delta_d)$ to contain two $(d-1)$-faces in the same σ_i every $\sigma \in F_d(\mathrm{div}\Delta_d)$ needs to be matched by faces $\tau_i \in F_{d-1}(\mathrm{div}_{\Delta_d}\sigma_i)$, $i = 0, ..., d$. However, the only d-simplex $\sigma \subset \Delta_d$ sufficing $\partial\sigma \cap \mathring{\Delta}_d = \emptyset$ is the full simplex itself. Thus $\mathrm{div}\Delta_d \cong \Delta_d$ and the subdivision is trivial. A contradiction to non-triviality of div. $\square$

Example: Universal Limits of Cone Subdivision

The following section is devoted to the calculation of an explicit universal limit of an example of finite ramification, i.e. a inclusion-uniform subdivision div such that $f_{d-1}(\mathrm{div}) = 1$. This property will prove to be convenient in the application of the following method since self-similarity will appear only in one block of our target matrix.

Let d be a given dimension. In the following we calculate the renormalization map for the Cone subdivision which is a special case of *finitely ramified* subdivisions.

Let K be a simplicial complex and for every $\sigma \in F_d(K)$ let v_σ denote its barycenter. The cone subdivision $\mathrm{cd}K$ of K is given by adding to $K^{(d-1)}$ the cone $v_\sigma * \partial\sigma$ for every $\sigma \in F_d(K)$. Here $K^{(d-1)}$ denotes the $(d-1)$-skeleton of K.

Theorem I.16. *Let $d > 1$ and $\mathscr{P}_i$ and $\mathscr{Q}_i$ be the sequences recursively obtained as*

$$\mathscr{P}_i := f^{-i}(d+1), \qquad \mathscr{Q}_i := f^{-i}(d+3)$$

for the polynomial

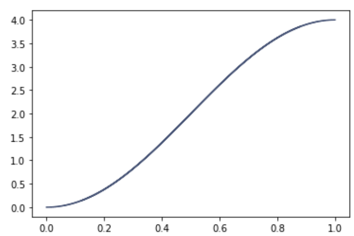
$$f(\zeta) = \zeta(d+3-\zeta).$$

Then $\{\mathscr{P}_i, \mathscr{Q}_i \mid i \in \mathbb{N}\}$ are mutually disjoint and the universal limit $\Lambda_d^{(\mathrm{cd})}$ is the unique increasing step function on $[0,1]$ attaining values in

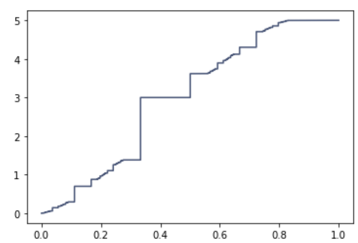
$$\bigcup_{i=0}^{\infty} \mathscr{P}_i \cup \bigcup_{j=0}^{\infty} \mathscr{Q}_j$$

such that $x \in \mathscr{P}_i \cup \mathscr{Q}_i$ is attained on an interval of length

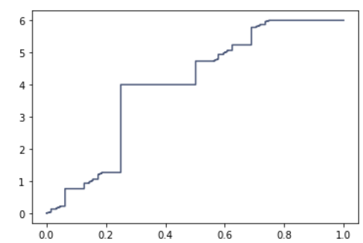
$$\frac{d-1}{2(d+1)^{i+1}}.$$



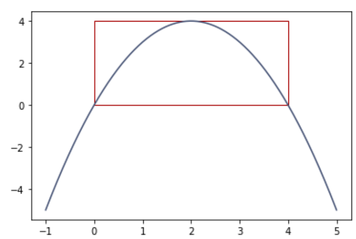
(a) One dimensional limiting distribution. Note that as cd coincides with the barycentric subdivision sd for graphs, i.e. $d = 1$, and the top-dimensional Laplacian is the 1-down Laplacian in this case the limiting distribution $\Lambda_1^{(\mathrm{cd})}(x) = 4\sin^2(\pi x/2)$ as shown in [Kni15].



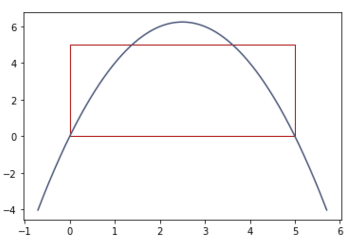
(b) For the two-dimensional limiting distribution note that the continuity of the one-dimensional case does not hold anymore as $\Lambda_2^{(\mathrm{cd})}$ is a step function (Theorem I.16).



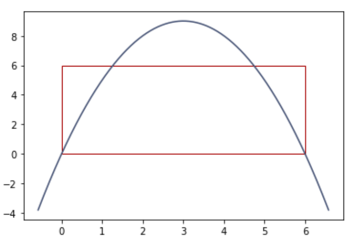
(c) For $d = 3$ and higher values of d the steps of early eigenvalues tend to become larger while the decrease in step length of later eigenvalues enhances (cf. the step lengths of eigenvalues $d+1$ and $d-1$, i.e. $\mathscr{P}_0$ and $\mathscr{Q}_0$).



(d) $f(\zeta) = \zeta(4-\zeta)$.



(e) $f(\zeta) = \zeta(5-\zeta)$.



(f) $f(\zeta) = \zeta(6-\zeta)$.

Figure 3.1: Limiting distributions $\Lambda_d^{(\mathrm{cd})}$ for $d = 1, 2, 3$. Beneath each limit there is a plot of the polynomial f generating the self-similarity of the distributions. The red rectangle shows the range of the feasible values of elements in $\mathscr{P}_i$ and $\mathscr{Q}_i$.

Note that this theorem encodes information about spectral gaps of the limiting distributions (i.e. ranges in which the total number of eigenvalues vanishes compared to the total number of eigenvalues under cd). We can deduce such gaps from the polynomials $f(\zeta) = \zeta(d+3-\zeta)$ as plotted in Figure 3.1. Note that values in the range $f^{-1}([0, d+3]^c)$ are never obtained as a preimage of a value in $\mathscr{P}_i$ or $\mathscr{Q}_i$ under f since $\mathscr{P}_i \cup \mathscr{Q}_i \subset [0, d+3]$. Thus whenever f leaves the range $[0, d+3]$ inside the interval $[0, d+3]$ those values can't be obtained in recursion anymore. Same holds true for the complete backwards orbit of this range under f thus inducing gaps in $\Lambda_d^{(\mathrm{cd})}$ for precisely these ranges.

We show Theorem I.16 by representing $\mathscr{L}_n$ (up to the degrees on the diagonal) as the adjacency operator on the d-dual graph of $\mathrm{cd}^n \Delta_d$ in the following denoted by

$$\Gamma_n := \Gamma^{(d)}(\mathrm{cd}^n \Delta_d).$$

Subsequently we approximate Γ_n by a more convenient graph sequence to work with in terms of asymptotics.

Schreier graph approximation of Γ_n

Let Γ_n denote the d-dual graph of $\mathrm{cd}^n \Delta_d$ as above. In this section we will show in Proposition I.18 that it is isomorphic to a Schreier graph on the n-th level of an action of a particular self-similar group with a slight error. This error is introduced by the Schreier graph approximation; this is due to the fact that Schreier graphs are regular while Γ_n has boundary nodes of degree d though the other (interior) nodes have degree $d+1$. Thus in order to approximate Γ_n by a sequence of Schreier graphs we introduce loops on the boundary to artificially make the graph $(d+1)$-regular. Before we state and prove Proposition I.18 we will need a few definitions and constructions.

To this end we quickly introduce notions of self-similar groups as in [GNŠ15] and [DGL20]. Our aim is to reformulate the setting by a group G acting on a k-ary tree $\mathcal{T}$ so that the Schreier graph of G on the n-th level of the tree is isomorphic to Γ_n. This will prove to be useful since it allows for a recursive block-description of the adjacency operator of Γ_n in terms of a representation of the generators of G.

Since every d-facet of K gets replaced by $(d+1)$ copies of a d-simplex under cd the natural choice is $k = d+1$ and $\mathcal{T}$ is the tree with vertex set X^*, the words of finite length over the alphabet $X = [d+1]$, with root $\emptyset$ (the empty word) and adjacencies given by right-adjunction of a single symbol, i.e. the word w has children of the form wx for $x \in X$. We will further use the notation X^* of the vertex set of $\mathcal{T}$ for $\mathcal{T}$ itself. Note that by this definition the n-th level of X^* is the set X^n of words of length n over X.

Now in order to obtain a self-similar Schreier graph sequence from X^n we define what a self-similar group is by action on X^*. To this end consider the group $\mathrm{Aut}(X^*)$ of all automorphisms of the $(d+1)$-ary tree X^*. Its elements are bijections of the set X^* onto itself which fix the root $\emptyset$ and preserve adjacency relations. Note that for a vertex $v \in X^n$ on level n the subtree $\mathcal{T}_v$ is isomorphic to $\mathcal{T}$ itself by the n-fold left-shift $w_1...w_k \mapsto w_{n+1}...w_k$. Thus every automorphism $\varphi \in \mathrm{Aut}(X^*)$ is given by a permutation $\sigma \in S_X$ of the first level $X^1 = X$ and a tuple of $(d+1)$ elements describing how φ acts on the subtrees $\mathcal{T}_v \cong X^*$ for each $v \in X^1$, i.e.

$$(\varphi_1, ..., \varphi_{d+1}) \in \mathrm{Aut}(X^*)^{d+1}.$$

We now say that a subgroup $G \leq \mathrm{Aut}(X^*)$ is self-similar if for every $\varphi \in G$ the elements $\varphi_1, ..., \varphi_{d+1}$ are themselves elements of G.

Having a self-similar group G and a finite set of generators S the sequence of Schreier graphs defined by G (with respect to S) is given by $G_n := (X^n, E_n)$ where E_n is defined over S by

$$E_n := \{(w, s \cdot w) \mid w \in X^n, s \in S\}.$$

Note that in case $\{s^{-1} \mid s \in S\} = S$ we obtain an undirected graph. Also observe that if S acts such that for every $w \in X^n$ and $s_1, s_2 \in S$ from $s_1 \cdot w = s_2 \cdot w$ it follows that $s_1 = s_2$ the adjacency matrix of G_n is given by

$$A(G_n) = \sum_{s \in S} \rho(s)$$

for the representation $\rho : G \to \mathrm{GL}_{|X^n|}(\mathbb{C})$ defined by the action of G on X^n under some identification of X^n with $[|X^n|]$, i.e. let $\iota : X^n \to [|X^n|]$ be a bijection, then for $\varphi \in G$ let $\rho(\varphi) \cdot e_{\iota(w)} = e_{\iota(\varphi \cdot w)}$. In particular every $\rho(\varphi)$ is a permutation matrix.

Since the graph Γ_n to be approximated does not contain loops we introduce the notion of the *reduced Schreier graph* $\tilde{G}_n$ *defined by* G *(with respect to* S*)* as the graph G_n with loops removed. We say that G_n approximates a graph sequence Γ_n if $\tilde{G}_n$ is isomorphic to Γ_n and for $\ell(G_n)$ the number of loops of G_n it holds that

$$\ell(G_n) \ll v(G_n),$$

i.e. G_n is obtained (up to isomorphism) from Γ_n by adding an asymptotically small number of loops. Note that the motivation for this notion of approximating sequences is due to Corollary I.5 since the addition of loops to Γ_n corresponds to the addition or subtraction of $\ell(G_n)$-many ones along the diagonal of $A(\Gamma_n)$ or $\mathscr{L}(\Gamma_n)$, respectively. Thus

$$||\Lambda_{\mathscr{L}(\Gamma_n)} - \Lambda_{\mathscr{L}(G_n)}||_{L^1} \leq \frac{\ell(G_n)}{v(G_n)} \xrightarrow{n\to\infty} 0$$

so that if we want to describe $\Lambda_d^{(\mathrm{cd})}$ from Theorem I.1 a spectral decimation of G_n suffices which will be more convenient to work with in this manner.

We will now show that the sequence of graphs Γ_n is approximated by the Schreier graph sequence G_n generated by the action of the following group $G \leq \mathrm{Aut}(X^*)$: First consider the cyclic permutation

$$\alpha = ((d+1)\ d\ (d-1)\ \ldots\ 2\ 1) \in S_X$$

and the automorphism a applying α to the last letter of the given word, i.e.

$$a(wx) = w\alpha(x)$$

for $w \in X^{n-1}$, $x \in X$. Note that a is of order $d+1$ and consider the cyclic group A generated by a. Its n-th level Schreier graph with respect to $S = \{a, a^2, ..., a^d\}$ is the graph consisting of $(d+1)^{n-1}$ disjoint copies of K_{d+1}, one for each set of the form

$$\{wx \mid x \in X\}$$

with $w \in X^{n-1}$ fixed. The copies of K_{d+1} here correspond to copies of the dual graph of $\mathrm{cd}\Delta_d$. In order to model the adjacencies between these copies we need to introduce another group generator b.

Let b be given by the following self-similar description

$$b(wx) = \begin{cases} a^{d+1-x}(w) \cdot (d+1-x) & , \ x \neq d+1 \\ b(w) \cdot x & , \ x = d+1 \end{cases}$$

and initial condition $b(i) = i$ for $i \in X$. Here $\cdot$ denotes the concatenation of a word with a letter. Note that the initial condition includes loops in the Schreier graph G_n.

Let G be the group generated by a and b. In order to show that G_n approximates Γ_n we analyze the elementary cell of our subdivision sequence (cf. Figure 3.2 for the case $d = 2$ and $n = 3$).

Lemma I.17. $\Gamma_1 = \Gamma^{(d)}(\mathrm{cd}\Delta_d)$ *is isomorphic to the complete graph* K_{d+1} *where the vertices of* K_{d+1} *are in bijection with the boundary faces* $F_{d-1}(\Delta_d)$ *over the map* $\sigma \mapsto v * \sigma$ *for* v *being the barycenter of* Δ_d*.*

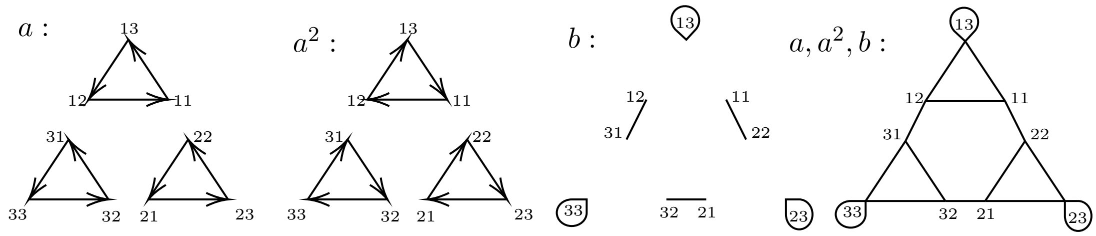


Figure 3.2: Schreier graphs generated by the choices $\{a\}$, $\{a^2\}$, $\{b\}$ and $\{a, a^2, b\}$ of generators S and the group G generated by S.

Proof. To this end note first that by definition every d-face of $\mathrm{cd}\Delta_d$ shares a common $(d-1)$-face with every other d-face. This follows from the fact that $\mathrm{cd}\Delta_d$ is defined as the cone over the boundary of the standard-d-simplex,

$$\mathrm{cd}\Delta_d = v * \partial\Delta_d$$

with v its barycenter. Note that every facet $\sigma \in F_{d-1}(\partial\Delta_d)$ thus corresponds to the unique facet $v \cup \sigma \in F_d(\mathrm{cd}\Delta_d)$ by definition of the cone complex. This correspondence is bijective. Furthermore two facets $v \cup \sigma_1, v \cup \sigma_2 \in F_d(\mathrm{cd}\Delta_d)$ share a common $(d-1)$-face iff σ_1 and σ_2 share a common $(d-2)$-face. But now every two $(d-1)$-faces of $\partial\Delta_d$ share a common $(d-2)$-face. This is due to the fact that every facet $\sigma \in F_{d-1}(\partial\Delta_d)$ has exactly one opposing vertex w_σ. Every other facet τ of $\partial\Delta_d$ can then be obtained as

$$w_\sigma \cup (\sigma \setminus \{w_\tau\}).$$

Note that the common $(d-2)$-face of τ and σ then is

$$\sigma \setminus \{w_\tau\}.$$

□

We will now define a bijection $F_d(\mathrm{cd}^n\Delta_d) \cong X^n$ which will turn out to be a graph isomorphism of Γ_n and $\tilde{G}_n$. This bijection can be thought of as an addressing scheme or a labeling of the facets of $\mathrm{cd}^n\Delta_d$.

Obviously the only facet of $\Delta_d = \mathrm{cd}^0\Delta_d$ gets mapped to the empty word $\emptyset$. Next choose an arbitrary labeling of $F_d(\mathrm{cd}\Delta_d) \cong X$. Let the labeling φ_{n-1} for $\mathrm{cd}^{n-1}\Delta_d$ be defined; let $s : F_d(\mathrm{cd}^n\Delta_d) \to F_d(\mathrm{cd}^{n-1}\Delta_d)$ be the subdivision map restricted to d-faces. Note that under cd every $\nu \in F_d(\mathrm{cd}^{n-1}\Delta_d)$ gets replaced by $d+1$ new d-facets of the form

$$v_\nu * \sigma$$

for $\sigma \in F_{d-1}(\nu)$. Further let p denote the parental map on level n in X^*, i.e.

$$p : X^n \to X^{n-1};\ wx \mapsto w.$$

Given $\tau \in F_d(\mathrm{cd}^n\Delta_d)$ we will define $\varphi_n : F_d(\mathrm{cd}^n\Delta_d) \to X^n$ such that

$$p \circ \varphi_n = \varphi_{n-1} \circ s, \tag{3.1}$$

i.e. the $d+1$ children of $\varphi_{n-1}(\nu)$ in X^* are identified with the $d+1$ facets added for $\nu \in F_d(\mathrm{cd}^{n-1}\Delta_d)$. Thus in order to define φ_n it suffices to give a bijective map $i_\nu : s^{-1}(\nu) \to X$. Consider $v_\nu * \sigma \in s^{-1}(\nu)$, i.e. $\sigma \in F_{d-1}(\nu)$, then we define i_ν depending on a variety of cases for σ:

- In case σ is boundary, i.e. σ has no cofaces besides ν, we set $i_\nu(v_\nu * \sigma) = d+1$.
- Otherwise σ has another unique coface $\nu' \in F_d(\mathrm{cd}^{n-1}\Delta_d)$, $\nu' \neq \nu$. Then we have another two cases;
  - Either $p \circ \varphi_{n-1}(\nu') = p \circ \varphi_{n-1}(\nu)$ then by equation (3.1) there exists $\tau \in F_d(\mathrm{cd}^{n-2}\Delta_d)$ such that
  $$s(\nu) = s(\nu') = \tau.$$
  Let $\ell \in \{1, ..., d\}$ such that
  $$i_\tau(\nu') \equiv i_\tau(\nu) + \ell \pmod{d+1}$$
  then set
  $$i_\nu(v_\nu * \sigma) = \ell.$$
  - or $p \circ \varphi_{n-1}(\nu') \neq p \circ \varphi_{n-1}(\nu)$ then let $i_\nu(v_\nu * \sigma) = d+1$.

Note that this definition of i_ν is a well-defined bijection because there is always only one outwards pointing face of every facet, i.e. a face which is either boundary or has another coface which is not a child node of a common facet in $\mathrm{cd}^{n-2}\Delta_d$. Furthermore when assuming ν fixed every facet ν' which shares a $(d-1)$-face with ν which is not outwards pointing (i.e. $s(\nu) = s(\nu')$) defines a unique value of ℓ since i_τ is a bijection.

Proposition I.18. *φ_n defines an isomorphism of the graphs Γ_n and $\tilde{G}_n$. Furthermore G_n has $d+1$ loops, i.e. G_n approximates Γ_n.*

Proof. We already know that the map $\varphi_n : F_d(\mathrm{cd}\Delta_d) \to X^n$ is a bijection. Note that the respective sets are the vertex sets of Γ_n and G_n, respectively.

Thus in order to obtain an isomorphism we have the show that the edges are in bijection over φ_n aswell.

We proceed by induction. For $n = 1$ the claim is obviously true: $\{a, ..., a^d\}$ introduces the complete $K_{d+1} \cong \Gamma_1$ in G_1 and b acts trivially on X - thus introducing a loop on every vertex in G_1. In particular φ_1 introduces an isomorphism between $\tilde{G}_1$ and Γ_1.

Now we will show that every edge in Γ_n corresponds to the application of b or a power of a on the right-hand side under φ_n (up to loops resulting from application of b). Note that the edges of G_n are precisely the edges of this form. Let $\tau \in F_d(\mathrm{cd}^n\Delta_d)$ be given and let

$$\nu := s(\tau)$$

aswell as

$$\tau = v_\nu * \sigma$$

for some $\sigma \in F_{d-1}(\nu)$. Note that by this as mentioned above τ shares a common $(d-1)$-face with every other face $\tau' \in s^{-1}(\tau)$ of the form

$$\tau' = v_\nu * \sigma'$$

for $\sigma' \in F_{d-1}(\nu)$. Let $i_\nu(\tau)$ and $i_\nu(\tau')$ be as above so that

$$\varphi_n(\tau) = \varphi_{n-1}(\nu) \cdot i_\nu(\tau)$$

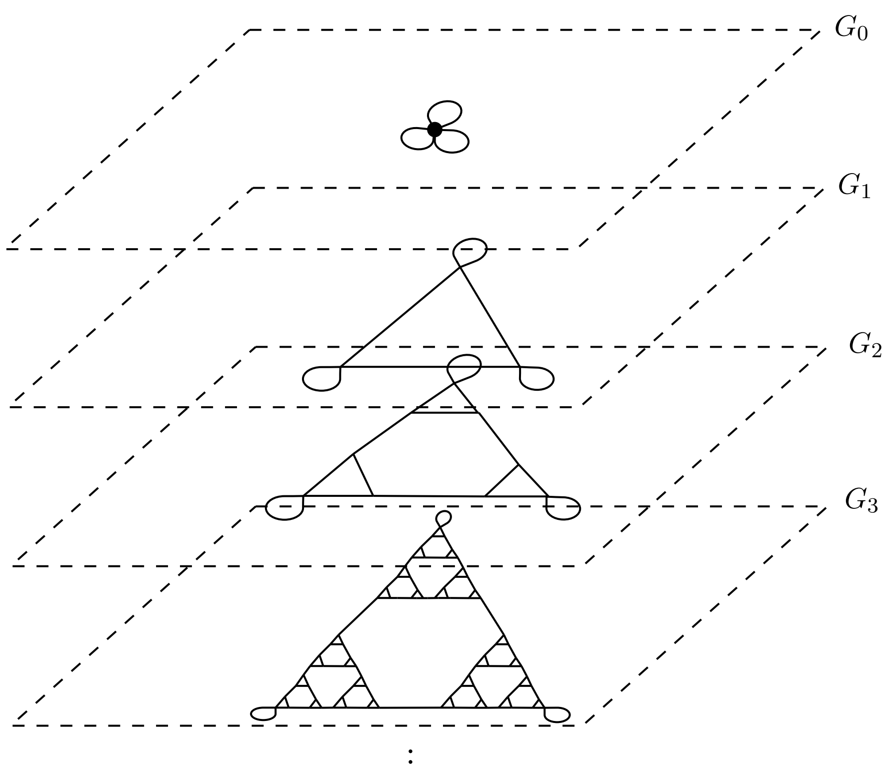


Figure 3.3: The Schreier graph approximation G_n for $n \in \{0, 1, 2\}$ for $d = 2$. Note the structure of the ternary tree indicated by the positions of the triangles K_3 under every node of one layer above.

and

$$\varphi_n(\tau') = \varphi_{n-1}(\nu) \cdot i_\nu(\tau').$$

Further let ℓ be such that

$$i_\nu(\tau') \equiv i_\nu(\tau) + \ell \pmod{d+1}$$

then by definition of a it is immediate that

$$a^\ell(\varphi_n(\tau)) = \varphi_{n-1}(\nu) \cdot \alpha^\ell(i_\nu(\tau)) = \varphi_{n-1}(\nu) \cdot i_\nu(\tau') = \varphi_n(\tau').$$

Thus the edge

$$(\varphi_n(\tau), \varphi_n(\tau'))$$

is contained in G_n for every τ'. Note also that since for fixed τ every value of $\ell \in \{1, ..., d\}$ occurs for τ' and thus all edges introduced by action of a in G_n are of this form.

Thus the only other edge incident to $\varphi_n(\tau)$ in G_n is the edge

$$(\varphi_n(\tau), b(\varphi_n(\tau))).$$

The only other $(d-1)$-face of τ which has a coface that is not interior to ν is $\sigma \leq \tau$. Note that σ is itself a $(d-1)$-face of ν by definition. This $(d-1)$-face is either boundary in which case by definition

$$\varphi_n(\tau) = i(d+1)...(d+1)$$

for arbitrary $i \in X$ and thus b acts on $\varphi_n(\tau)$ as

$$b(\varphi_n(\tau)) = b(i)(d+1)...(d+1)$$

with $b(i) = i$. Thus $b(\varphi_n(\tau)) = \varphi_n(\tau)$ and the corresponding edge in G_n is the loop

$$(\varphi_n(\tau), \varphi_n(\tau))$$

on the boundary face. Note that there are $d+1$-many words of this form $i(d+1)...(d+1)$. Thus $d+1$ loops are included on the boundary faces; those loops are added to Γ_n by the transition to $\tilde{\Gamma_n}$.

In case that there is another coface τ' of σ in $\mathrm{cd}^n \Delta_d$ we apply b to $\varphi_n(\tau)$ and need to differentiate between cases in the definition of b:

In case $i_\nu(\tau) = (d+1)$ we have

$$b(\varphi_n(\tau)) = b(\varphi_{n-1}(\nu))(d+1)$$

Note by definition of φ_n this case corresponds to the case where ν and $\nu' = s(\tau')$ are not interior to a common d-facet in $\mathrm{cd}^{n-2}\Delta_d$. Obviously by symmetry of the fact that $\tau' = v_{\nu'} * \sigma$ and σ being a face of ν' and ν not being interior to a common d-facet in $\mathrm{cd}^{n-2}\Delta_d$ we obtain that $i_{\nu'}(\tau') = d+1$ and thus

$$\varphi_n(\tau') = \varphi_{n-1}(\nu')(d+1).$$

But now since ν and ν' are not interior to a common d-facet of $\mathrm{cd}^{n-2}\Delta_d$ by the induction hypothesis we have

$$\varphi_{n-1}(\nu') = b(\varphi_{n-1}(\nu)$$

and in particular

$$b(\varphi_n(\tau)) = b(\varphi_{n-1}(\nu)) \cdot (d+1) = \varphi_{n-1}(\nu') \cdot (d+1) = \varphi_n(\tau').$$

In particular the edge

$$(\varphi_n(\tau), \varphi_n(\tau'))$$

is in G_n and obviously the corresponding edge (τ, τ') is in Γ_n as τ and τ' are d-down neighbors.

The last case is when $i_\nu(\tau) \neq d+1$. Again let $\nu' = s(\tau')$. By definition of φ_n we then have a d-facet $\mu \in \mathrm{cd}^{n-2}\Delta_d$ such that ν and ν' are in the interior of μ. In particular $i_\nu(\tau) = \ell$ where ℓ is the unique integer in $\{1, ..., d\}$ such that

$$i_\mu(\nu') \equiv i_\mu(\nu) + \ell \pmod{d+1}.$$

By symmetry of this equation we have

$$i_{\nu'}(\tau') = d + 1 - \ell$$

in particular. Application of b gives us

$$b(\varphi_n(\tau)) = a^{d+1-\ell}(\varphi_{n-1}(\nu))(d + 1 - \ell)$$

it thus suffices that $a^{d+1-\ell}(\varphi_{n-1}(\nu)) = \varphi_{n-1}(\nu')$ in order to establish the claim. This is obvious now; $a^{d+1-\ell}$ acts on X^{n-1} by leaving the first $n-2$ letters fixed and sending the last letter x to the unique representative in $\{1, ..., d\}$ of

$$(x + \ell) + (d + 1)\mathbb{Z};$$

in particular it sends $i_\mu(\nu)$ onto $i_\mu(\nu')$ and thus

$$a^{d+1-\ell}(\varphi_{n-1}(\nu)) = \varphi_{n-2}(\mu) \cdot i_\mu(\nu') = \varphi_{n-1}(\nu').$$

Thus

$$b(\varphi_n(\tau)) = \varphi_n(\tau')$$

and the edge

$$(\varphi_n(\tau), \varphi_n(\tau'))$$

is contained in G_n as

$$(\varphi_n(\tau), b(\varphi_n(\tau)).$$

□

Now we have described the sequence Γ_n (up to loops) as a Schreier graph of a self-similar group acting on a $(d+1)$-ary tree in the sense of [GS06]. This viewpoint will be convenient since it gives immediate self-similar descriptions of the Laplacian operator in terms of representations of group elements in a matrix algebra of increasing order.

By all the above it follows that the adjacency matrix of G_n has the form

$$\Xi_n := A(G_n) = \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 1 \end{pmatrix}}_{=:J_n} + \underbrace{\begin{pmatrix} & & a_{n-1} & \\ & \iddots & & \\ a_{n-1}^d & & & \\ & & & b_{n-1} \end{pmatrix}}_{=:b_n} - \mathbf{1}_{(d+1)^n}$$

where $a_n \in M_{(d+1)^n}(\mathbb{C})$ is given as

$$a_n = a_0 \otimes \mathbf{1}_{(d+1)^{n-1}}$$

and

$$a_0 := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}; \; b_0 := 0 \in M_{d+1}(\mathbb{C});$$

though the initial condition b_0 of b is irrelevant for the asymptotic distribution and thus we might also include loops by setting b_0 equal to the identity - obtaining the Schreier graph sequence for the hanoi tower group on 3 pegs in case $d = 2$.

Note that a_n and b_n are the representations of the generators a and b in $\mathrm{GL}_{(d+1)^n}(\mathbb{C})$ as described above. The block structure results from reverse lexicographic ordering, i.e. the i-th column and i-th row correspond to the words of the form $*\dots * i$.

Further we let

$$\Xi_n(\mu, \lambda) = \lambda J_n + b_n - (\lambda + \mu)1_{(d+1)^n}$$

and

$$D_n(\mu, \lambda) = \det \Xi_n(\mu, \lambda).$$

In particular the map $\mu \mapsto D_n(\mu, 1)$ is the characteristic polynomial of the adjacency matrix Ξ_n.

Note that in order to apply Schur-Renormalization we need to determine the determinant of the $d \times d$ upper-left block of Ξ_n which we will denote by X in the following (we drop the subscript n in order to maintain readability).

Note that we have

$$D_n(\mu, \lambda) = \det X \cdot \det(b - \mu \mathbf{1}_{(d+1)^{n-1}} - \lambda^2 \Gamma_X(\mu, \lambda)),$$

where $\Gamma_X(\mu, \lambda)$ denotes the block-coronal of X in this case, i.e.

$$\Gamma_X(\mu, \lambda) = \underline{1}_d^t \cdot X^{-1} \underline{1}_d,$$

where

$$\underline{1}_d = (\underbrace{1_{(d+1)^{n-1}}, \dots, 1_{(d+1)^{n-1}}}_{d-\text{times}})^t.$$

n will always be inferrable from context.

In order to determine Γ_X we will consider X as a matrix over the algebra $\mathscr{A}_n \leq M_{(d+1)^n}(\mathbb{C})$ generated by a_n - which in fact as the group algebra of C_6 is a commutative algebra. How this will help us becomes clear in the following sections.

The procedure applied here was developed by Grigorchuk et al. in order to calculate spectra of Schreier graphs associated to groups acting on k-ary trees, e.g. in [GS06, BG99]. We will use the same approach but from a different viewpoint as our starting point is not the group but rather the graph sequence in a self-similar sense. It is important though that the sequence is representable as a Schreier graph sequence of some group action on the complete k-ary tree in order to determine the adjacency matrix in a simple manner.

Some elementary properties of the algebra $\mathscr{A}$

First note that the algebras $\mathscr{A}_n$ are all isomorphic to $\mathscr{A}_0$ via tensoring by $1_{(d+1)^{n-1}}$. Thus we will denote by $\mathscr{A}$ the generic group algebra of C_6 commonly realized by $\mathscr{A}_0$. The following results thus also hold in an analogous version over $\mathscr{A}_n$.

Proposition I.19. *Let μ, λ be given so that*

$$x := \mu 1_{d+1} + \lambda \sum_{i=1}^{d} a^i \in \mathscr{A}$$

is non-singular, then

$$x^{-1} = \frac{1}{(\mu-\lambda)(\mu+d\lambda)}\Big((\mu+(d-1)\lambda)1_{d+1} - \lambda\sum_{i=1}^{d} a^i\Big).$$

Proof. We decompose x as

$$x = (\mu-\lambda)1_{d+1} + \lambda \underline{1}_{d+1} \cdot \underline{1}_{d+1}^t$$

and apply the Sherman-Morrison formula, Lemma I.6, to yield

$$x^{-1} = \frac{1}{\mu-\lambda}1_{d+1} - \frac{1}{(\mu-\lambda)^2}\frac{\lambda \underline{1}_{d+1}\cdot \underline{1}_{d+1}^t}{1+\frac{\lambda}{\mu-\lambda}\underline{1}_{d+1}^t\underline{1}_{d+1}} = \frac{1}{\mu-\lambda}1_{d+1} - \frac{\lambda}{(\mu-\lambda)(\mu+d\lambda)}\underline{1}_{d+1}\cdot \underline{1}_{d+1}^t.$$

In particular we have

$$x^{-1} = \frac{1}{(\mu-\lambda)(\mu+d\lambda)}\Big((\mu+(d-1)\lambda)1_{d+1} - \lambda\sum_{i=1}^{d} a^i\Big).$$

□

In order to compute determinants of block matrices with blocks in $\mathscr{A}$ we might use the following result relating the usual determinant with the determinant defined in the same way over $\mathscr{A}$, i.e. for $A \in \mathscr{A}^{k\times k}$ let

$$\det_{\mathscr{A}} A := \sum_{\sigma\in S_k} \operatorname{sgn}\sigma A_{1\sigma(1)}...A_{k\sigma(k)} \in \mathscr{A},$$

where $A_{ij} \in \mathscr{A}$ is the block at index (i,j) as usual.

Proposition I.20 ([Sil00])**.** *The usual determinant* det *factorizes over* $\det_{\mathscr{A}}$*, i.e. for any $k\times k$ block matrix $A \in \mathscr{A}^{k\times k}$ with blocks in the commutative matrix algebra $\mathscr{A}$ it holds that*

$$\det A = \det \det_{\mathscr{A}} A.$$

In particular we also can compute the determinant of X from the beginning of the section as

$$\det\det_{\mathscr{A}} X,$$

where $\det_{\mathscr{A}} X$ in this case is a circulant matrix - of which the determinant is readily calculable by general formulae. A formula of this type needed in the subsequent section will be given by the following lemma.

Lemma I.21.

$$\det(\mu \cdot 1_{d+1} + \lambda \sum_{i=1}^{d} a^i) = (\mu + d\lambda)(\mu - \lambda)^d.$$

Proof. This determinant is easily calculated by the Matrix Determinant Lemma, Lemma I.7, after a trivial reparameterization as before;

$$\mu \cdot 1_{d+1} + \lambda \sum_{i=1}^{d} a^i = (\mu - \lambda)1_{d+1} + \lambda \underline{1}_{d+1}\underline{1}_{d+1}^t$$

so that

$$\det(\mu \cdot 1_{d+1} + \lambda \sum_{i=1}^{d} a^i) = \Big(1 + \frac{(d+1)\lambda}{\mu - \lambda}\Big) \det((\mu - \lambda)1_{d+1}) = \frac{\mu + d\lambda}{\mu - \lambda}(\mu - \lambda)^{d+1} = (\mu + d\lambda)(\mu - \lambda)^d.$$

□

Recursion of D_n via renormalization by Γ_X

In order to determine the renormalization maps we need to calculate the matrix coronal of X - which will be given by the following lemma.

Lemma I.22. *The block-linear system*

$$X \cdot v = \underline{1}_d$$

is solved by

$$v = x^{-1} \odot (a^i + \mu + \lambda)_{i=1}^{d}\,{}^{1}$$

iff $x = (\mu + \lambda)(\lambda(d-1) - \mu) + 1 + \lambda \sum_{i=1}^{d} a^i$ *is non-singular.*

Proof. We just check that

$$X \cdot \tilde{v} = x\underline{1}_d$$

for

$$\tilde{v} = (a^i + \mu + \lambda)_{i=1}^{d}.$$

In case $d+1$ is even we need to handle the case $j = \frac{d+1}{2}$ for the following seperately. For all

[1] Here $\odot$ denotes the scalar-multiplication over $\mathscr{A}$; the notation is derived from the common notation $\circ$ for the Hadamard product in combination with $\cdot$ for the scalar multiplication

other cases we have

$$\begin{aligned}
(X\cdot\tilde{v})_j &= -\mu(a^j+\mu+\lambda)+(\lambda+a^j)(a^{d+1-j}+\mu+\lambda)+\sum_{\substack{i=1\\ i\notin\{j,d+1-j\}}}^{d}\lambda(a^i+\mu+\lambda)\\
&= -\mu a^j-\mu(\mu+\lambda)+\lambda(a^{d+1-j}+\mu+\lambda)+a^{d+1}+a^j(\mu+\lambda)+\lambda\sum_{\substack{i=1\\ i\notin\{j,d+1-j\}}}^{d}(a^i+\mu+\lambda)\\
&= -\mu(\mu+\lambda)+a^{d+1}+\lambda a^j+\lambda\sum_{\substack{i=1\\ i\neq j}}^{d}(a^i+\mu+\lambda)\\
&= -\mu(\mu+\lambda)+1+\lambda\sum_{i=1}^{d}a^i+\lambda(d-1)(\mu+\lambda)\\
&= (\lambda(d-1)-\mu)(\mu+\lambda)+\lambda\sum_{i=1}^{d}a^i+1=x.
\end{aligned}$$

In case $d+1$ is even and $j=\frac{d+1}{2}$ we have

$$\begin{aligned}
(X\cdot\tilde{v})_j &= (a^j-\mu)(a^j+\mu+\lambda)+\sum_{\substack{i=1\\ i\neq j}}\lambda(a^i+\mu+\lambda)\\
&= a^{2j}+a^j(\mu+\lambda)-\mu(a^j+\mu+\lambda)+\lambda\sum_{\substack{i=1\\ i\neq j}}^{d}(a^i+\mu+\lambda)\\
&= 1+\lambda a^j-\mu(\mu+\lambda)+\lambda\sum_{\substack{i=1\\ i\neq j}}^{d}(a^i+\mu+\lambda)\\
&= -\mu(\mu+\lambda)+\lambda\sum_{i=1}^{d}a^i+\lambda(d-1)(\mu+\lambda)\\
&= (\lambda(d-1)-\mu)(\mu+\lambda)+\lambda\sum_{i=1}^{d}a^i+1=x.
\end{aligned}$$

Thus in every case we obtain the matrix x and so we have

$$X\tilde{v}=x\odot\underline{1}_d.$$

□

In what follows for $x,y\in\mathscr{A}$ we will simply write $\frac{x}{y}$ for $x^{-1}y$ which is a well-defined fraction since $\mathscr{A}$ is a commutative algebra.

Corollary I.23. *We have*

$$\Gamma_X(\mu,\lambda)=x^{-1}(\sum_{i=1}^{d}a^i+d(\mu+\lambda))=\frac{\sum_{i=1}^{d}a^i+d(\mu+\lambda)}{\lambda\sum_{i=1}^{d}a^i+1+(\lambda(d-1)-\mu)(\mu+\lambda)}.$$

The renormalization is now determined by Γ_X over the coefficients of x^{-1}.

Proposition I.24. *Let $\alpha := (\mu+\lambda)(\lambda(d-1)-\mu)+1-\lambda$, then we have*

$$x^{-1} = \frac{1}{\alpha(\alpha+(d+1)\lambda)}\Big((\alpha+d\lambda)\cdot 1_{(d+1)^{n-1}} - \lambda\cdot\sum_{i=1}^{d} a^i\Big).$$

In particular the matrix coronal of X is given as

$$\Gamma_X(\mu,\lambda) = \frac{1}{\alpha(\alpha+(d+1)\lambda)}\Big(\big((\alpha+d\lambda)d(\mu+\lambda)-d\lambda\big)1_{(d+1)^{n-1}} + \big(\alpha+\lambda-d\lambda(\mu+\lambda)\big)\sum_{i=1}^{d} a^i\Big).$$

Proof. The formula of x^{-1} can easily be inferred from Proposition I.19.

Furthermore note that

$$\begin{aligned}\Gamma_X(\mu,\lambda) &= \frac{1}{\alpha(\alpha+(d+1)\lambda)}((\alpha+d\lambda)1_{(d+1)^{n-1}} - \lambda\sum_{i=1}^{d} a^i)(\sum_{i=1}^{d} a^i + d(\mu+\lambda)) \\ &= \frac{1}{\alpha(\alpha+(d+1)\lambda)}(\alpha+d\lambda-\lambda d(\mu+\lambda))\sum_{i=1}^{d} a^i + (\alpha+d\lambda)d(\mu+\lambda)1_{(d+1)^{n-1}} - \lambda(\sum_{i=1}^{d} a^i)^2\end{aligned}$$

and

$$(\sum_{i=1}^{d} a^i)^2 = d1_{(d+1)^{n-1}} + (d-1)\sum_{i=1}^{d} a^i$$

obtaining the wanted representation of $\Gamma_X(\mu,\lambda)$. ◻

Now let

$$\mu' = \mu + \frac{\lambda^2}{\alpha(\alpha+(d+1)\lambda)}\big((\alpha+d\lambda)d(\mu+\lambda)-d\lambda\big)$$

and

$$\lambda' = -\frac{\lambda^2}{\alpha(\alpha+(d+1)\lambda)}\big(\alpha+\lambda-d\lambda(\mu+\lambda)\big).$$

Corollary I.25. *We have*

$$D_n(\mu,\lambda) = \det X\cdot D_{n-1}(\mu',\lambda').$$

Proof. By Schur Renormalization, Lemma I.8, we have

$$\begin{aligned}D_n(\mu,\lambda) &= \det X\det(b-\mu\cdot 1_{(d+1)^{n-1}} - \lambda^2\Gamma_X(\mu,\lambda)) \\ &= \det X\det\Big(b - \underbrace{\frac{\lambda^2}{\alpha(\alpha+(d+1)\lambda)}\big(\alpha+\lambda-d\lambda(\mu+\lambda)\big)}_{=\lambda'}\sum_{i=1}^{d} a^i \\ &\quad -\Big(\underbrace{\mu+\frac{\lambda^2}{\alpha(\alpha+(d+1)\lambda)}\big((\alpha+d\lambda)d(\mu+\lambda)-d\lambda\big)}_{=\mu'}\Big)1_{(d+1)^{n-1}}\Big)\end{aligned}$$

and thus the formula follows from above proposition. ◻

In order to facilitate computation in what follows we give a factorization of the terms in μ' and λ' from $\Gamma_X(\mu, \lambda)$.

Lemma I.26. *We have*

$$\mu' = \mu + \frac{d\lambda^2((d-1)\lambda^2 + (d-2)\lambda\mu - \mu^2 + \mu)}{((d-1)\lambda - \mu + 1)((d-1)\lambda^2 + (d-2)\lambda\mu - \lambda - \mu^2 + 1)}$$

and

$$\lambda' = \frac{\lambda^2(\lambda + \mu - 1)}{((d-1)\lambda - \mu + 1)((d-1)\lambda^2 + (d-2)\lambda\mu - \lambda - \mu^2 + 1)}$$

Proof. First expand α as

$$\alpha = (d-1)\lambda^2 + (d-2)\lambda\mu - \lambda - \mu^2 + 1.$$

Observe that in our given factorization this is the final form of this term. The degree one term in the denominator stems from $\alpha + (d+1)\lambda$ which expands as

$$\alpha + (d+1)\lambda = (d-1)\lambda^2 + (d-2)\lambda\mu + d\lambda - \mu^2 + 1 = (\lambda + \mu + 1)((d-1)\lambda - \mu + 1).$$

We will show that both numerators are divisible by $(\lambda + \mu + 1)$ - resulting in this term being cancelled.

Let s_0 and s_1 denote the numerators of the quotients in μ' and λ' ignoring λ^2 respectively, i.e.

$$s_0 := (\alpha + d\lambda)d(\mu + \lambda) - d\lambda$$

and

$$s_1 := \alpha + \lambda - d\lambda(\mu + \lambda)$$

with respective expansions

$$s_0 = d\Big((d-1)\lambda^3 + (2d-3)\lambda^2\mu + (d-1)\lambda^2 + (d-3)\lambda\mu^2 + (d-1)\lambda\mu - \mu^3 + \mu\Big)$$

$$s_1 = -\lambda^2 - 2\lambda\mu - \mu^2 + 1.$$

From here the claimed factorization

$$s_0 = d(\lambda + \mu + 1)((d-1)\lambda^2 + (d-2)\lambda\mu - \mu^2 + \mu)$$

$$s_1 = -(\lambda + \mu - 1)(\lambda + \mu + 1)$$

is easily verified. □

Calculation of $\det X$

In this subsection we will drop the subscript from a_0 and will simply denote it by a as every calculation is performed over $\mathscr{A} = \mathscr{A}_0$.

We decompose X_0 as

$$X_0 = \lambda \underline{1}_d \underline{1}_d^t + \underbrace{A - (\mu + \lambda)1_d}_{=:Y}$$

for

$$A = \begin{pmatrix} & & a \\ & \cdot^{\cdot^{\cdot}} & \\ a^d & & \end{pmatrix}.$$

Proposition I.27. *We have*

$$det_{\mathscr{A}} X_0 = (1+\lambda\Gamma_Y(\mu,\lambda))\big((\mu+\lambda-1)(\mu+\lambda+1)\big)^{\lfloor d/2\rfloor} \cdot \begin{cases} 1 & , \ d \ even \\ -\mu-\lambda+a^{\frac{d+1}{2}} & , \ d \ odd \end{cases}.$$

Proof. First we apply the matrix determinant lemma to obtain

$$\det_{\mathscr{A}} X_0 = \det_{\mathscr{A}}(Y+\lambda\underline{1}_d\underline{1}_d^t) = \det_{\mathscr{A}} Y \cdot \det_{\mathscr{A}}(1+\lambda\underline{1}_d^t Y^{-1}\underline{1}_d).$$

Note now that the right-most matrix is a 1×1 block matrix and as such has determinant

$$\det_{\mathscr{A}}(1+\lambda\underline{1}_d^t Y^{-1}\underline{1}_d) = 1+\lambda\Gamma_Y(\mu,\lambda)$$

over $\mathscr{A}$.

We now compute $\det_{\mathscr{A}} Y$ by bringing Y into upper-triangular form. For d even one upper-triangular form is

$$\begin{pmatrix} -\mu-\lambda & & & & & a \\ & \ddots & & & \iddots & \\ & & -\mu-\lambda & a^{d/2} & & \\ & & 0 & \beta & & \\ & \iddots & & & \ddots & \\ 0 & & & & & \beta \end{pmatrix}$$

by elementary transformations, where $\beta = -\mu-\lambda+\frac{a^{d+1}}{\mu+\lambda} = -\mu-\lambda+\frac{1}{\mu+\lambda}$

For d odd a similar upper-triangular form looks like

$$\begin{pmatrix} -\mu-\lambda & & & & & & a \\ & \ddots & & & & \iddots & \\ & & -\mu-\lambda & & a^{\frac{d-1}{2}} & & \\ & & & -\mu-\lambda+a^{\frac{d+1}{2}} & & & \\ & & 0 & & \beta & & \\ & \iddots & & & & \ddots & \\ 0 & & & & & & \beta \end{pmatrix}.$$

Thus

$$\det_{\mathscr{A}} Y = \begin{cases} (-\mu-\lambda)^{d/2}\beta^{d/2} & , \ d \text{ even} \\ (-\mu-\lambda)^{(d-1)/2}\beta^{(d-1)/2}(-\mu-\lambda+a^{\frac{d+1}{2}}) & , \ d \text{ odd} \end{cases}$$

By $\beta = \frac{1-(\mu+\lambda)^2}{\mu+\lambda}$ we obtain

$$(-\mu-\lambda)\beta = (\mu+\lambda)^2-1 = (\mu+\lambda-1)(\mu+\lambda+1)$$

and consequently

$$\det_{\mathscr{A}} Y = \big((\mu+\lambda-1)(\mu+\lambda+1)\big)^{\lfloor d/2\rfloor} \begin{cases} 1 & , \ d \text{ even} \\ -\mu-\lambda+a^{\frac{d+1}{2}} & , \ d \text{ odd} \end{cases}.$$

□

Lemma I.28. *The block-linear system*

$$Y \cdot v = \underline{1}_d$$

is solved by

$$v = \frac{1}{1-(\lambda+\mu)^2}(a^i + \mu + \lambda)_{i=1}^d.$$

In particular it holds that

$$\Gamma_Y(\mu) = \frac{\sum_{i=1}^d a^i + d(\mu+\lambda)}{1-(\mu+\lambda)^2}.$$

Proof. This fact is again easily checked by calculations. Let $\tilde{v} := (a^i + \mu + \lambda)_{i=1}^d$; for every $i \neq (d+1)/2$ it holds that

$$(Y \cdot \tilde{v})_i = -(\mu+\lambda)(a^i+\mu+\lambda)+a^i(a^{d-i+1}+\mu+\lambda) = 1-(\mu+\lambda)a^i-(\mu+\lambda)^2+a^i(\mu+\lambda) = 1-(\mu+\lambda)^2.$$

In case d is odd and $i = (d+1)/2$ we have

$$(Y \cdot \tilde{v})_i = (a^{(d+1)/2} - (\mu+\lambda))(a^{(d+1)/2} + \mu + \lambda) = 1 - (\mu+\lambda)^2$$

thus showing the claim. □

In order to obtain the determinant of X_0 we need to calculate the determinant of $-(\mu + \lambda) + a^{\frac{d+1}{2}}$ for d odd now.

Lemma I.29. *Assume d is odd. Then*

$$\det(-(\mu+\lambda) + a^{\frac{d+1}{2}}) = (\mu+\lambda+1)^{\frac{d+1}{2}}(\mu+\lambda-1)^{\frac{d+1}{2}}.$$

Proof. Similar to the upper-triangular form of Y we might bring this matrix into the upper-triangular form

$$\begin{pmatrix} -(\mu+\lambda) & & & 1 & & \\ & \ddots & & & \ddots & \\ & & -(\mu+\lambda) & & & 1 \\ 0 & & & \beta & & \\ & \ddots & & & \ddots & \\ & & 0 & & & \beta \end{pmatrix}$$

for $\beta = -\mu - \lambda + \frac{1}{\mu+\lambda} = \frac{1-(\mu+\lambda)^2}{\mu+\lambda}$. Consequently

$$\det(-(\mu+\lambda) + a^{\frac{d+1}{2}}) = \big(-(\mu+\lambda)\cdot\beta\big)^{\frac{d+1}{2}} = \big((\mu+\lambda-1)(\mu+\lambda+1)\big)^{\frac{d+1}{2}}$$

showing the claim. □

Thus we are ready to calculate the determinant of X_0.

Proposition I.30. *Let $\phi(\mu,\lambda) := \mu^2 - (d-1)\lambda^2 - (d-2)\lambda\mu - 1$, then*

$$\det X_0 = (\mu - (d-1)\lambda - 1)(\mu+\lambda+1)(\phi(\mu,\lambda)+\lambda)^d((\mu+\lambda)^2-1)^{\binom{d+1}{2}-(d+1)}.$$

Proof. Combining the last two lemmata with Proposition I.27 we first obtain for d odd

$$\begin{aligned}\det \det_{\mathscr{A}} X_0 &= \det(1+\lambda\Gamma_Y(\mu,\lambda))\big((\mu+\lambda-1)(\mu+\lambda+1)\big)^{(d+1)\cdot(d-1)/2}\cdot\big((\mu+\lambda-1)(\mu+\lambda+1)\big)^{(d+1)/2}\\ &= \det(1+\lambda\Gamma_Y(\mu,\lambda))\big((\mu+\lambda-1)(\mu+\lambda+1)\big)^{\binom{d+1}{2}}\\ &= \det(1+\lambda\Gamma_Y(\mu,\lambda))\big((\mu+\lambda)^2-1\big)^{\binom{d+1}{2}},\end{aligned}$$

while for d even we can directly infer this equality from the proposition.

Now note that by Lemma I.28 the left-most term becomes

$$\det(1+\lambda\Gamma_Y(\mu,\lambda)) = \frac{1}{(1-(\mu+\lambda)^2)^{d+1}}\det(1-(\mu+\lambda)^2+d\lambda(\mu+\lambda)+\lambda\sum_{i=1}^{d}a^i).$$

Note that

$$1-(\mu+\lambda)^2+d\lambda(\mu+\lambda) = 1-\mu^2+(d-2)\lambda\mu+(d-1)\lambda^2 = -\phi(\mu,\lambda).$$

This determinant has been determined in Lemma I.21 - yielding

$$\det(\phi(\mu,\lambda)+\lambda\sum_{i=1}^{d}a^i) = (-\phi(\mu,\lambda)+d\lambda)(-\phi(\mu,\lambda)-\lambda)^d.$$

In total we obtain

$$\det \det_{\mathscr{A}} X_0 = (-1)^{d+1}(-\phi(\mu,\lambda)+d\lambda)(-\phi(\mu,\lambda)-\lambda)^d((\mu+\lambda)^2-1)^{\binom{d+1}{2}-(d+1)}$$

and thus the postulated form follows from the easy to verify factorization

$$\phi(\mu,\lambda)-d\lambda = (\mu-(d-1)\lambda-1)(\mu+\lambda+1).$$

□

Unidimensional Spectral Decimation of $\det \Xi_n$

In the preceding section we have deduced a spectral connection between subsequent subdivision steps, i.e. from the recursion presented in Corollary I.25 we are able to compute a factorization of the complete auxiliary spectrum - which is the set of roots of $D_n(\mu,\lambda)$ in $\mathbb{R}^2$.

This spectral set can be decomposed into hyperbolae as we will see which provides us with a way to also deduce the unidimensional spectral decimation stated in Theorem I.16. So what we will show now is that the same procedure as in [GS06] is applicable to the case of arbitrary d, i.e. the coefficient changes and the term $(d-2)\lambda\mu$ which will appears in μ' for $d>2$ does not form an obstruction to spectral decimation.

The main tool for the deduction of unidimensional spectral decimation for $d=2$ in [GS06] is semi-conjugacy of the renormalization F to $f:\mathbb{R}\to\mathbb{R};\ \zeta\mapsto\zeta^2-\zeta-3$. Semi-conjugacy means that there is a suitable way to map $\mathbb{R}^2$ to $\mathbb{R}$ so that this parameter mapping Ψ identifies F with f.

This semi-conjugacy will in fact stay intact for $d>2$, though for the unidimensional map

$$f(\zeta) = \zeta^2-(d-1)\zeta-(d+1)$$

and the parameter mapping

$$\Psi(\mu,\lambda) := \frac{\mu^2 - 1 - (d-1)\lambda\mu - d\lambda^2}{\lambda} =: \frac{\Phi(\mu,\lambda)}{\lambda}$$

as will be shown in the following lemma.

Lemma I.31. *F is semi-conjugate to f over Ψ, i.e.*

$$\Psi \circ F = f \circ \Psi.$$

Proof. This claim can be verified by high-school algebra. $\square$

Now in order to obtain spectral decimation we only need to analyze the behaviour of the factors of $\det X_0$ under renormalization F. To this end we will use the semi-conjugacy of F to the 1-dimensional map f.

Analogously to [GS06] let

$$\Phi_\theta(\mu,\lambda) := \Phi(\mu,\lambda) - \theta\lambda = \lambda(\Psi(\mu,\lambda) - \theta) = \mu^2 - 1 - (d-1)\lambda\mu - d\lambda^2 - \theta\lambda$$

$$L(\mu,\lambda) = \mu - (d-1)\lambda - 1$$

$$K(\mu,\lambda) = \phi(\mu,\lambda) + \lambda = \mu^2 - (d-1)\lambda^2 - (d-2)\lambda\mu + \lambda - 1$$

$$A_1(\mu,\lambda) = \lambda + \mu - 1$$

so that

$$\lambda' = \frac{\lambda^2 A_1(\mu,\lambda)}{L(\mu,\lambda)K(\mu,\lambda)}.$$

Then the semi-conjugacy gives us the following lemma as in [GS06].

Lemma I.32. *Let $\theta \in [-2, d+1]$ and θ_0, θ_1 be the two distinct real roots of $f(x) - \theta$. Then we have*

$$\frac{A_1}{LK}\Phi_{\theta_0}\Phi_{\theta_1} = \Phi_\theta \circ F.$$

Proof. It holds that

$$\Phi_\theta \circ F = \lambda'(\Psi \circ F - \theta) = \lambda'(f \circ \Psi - \theta) = \frac{\lambda^2 A_1}{LK}(\Psi - \theta_0)(\Psi - \theta_1) = \frac{A_1}{LK}\Phi_{\theta_0}\Phi_{\theta_1}.$$

$\square$

One last thing that remains to calculate is the initial polynomial $D_1(\mu,\lambda)$. Note that

$$\Xi_1(\mu,\lambda) = (-\mu+1)1_{d+1} + \lambda\sum_{i=1}^{d} a^i$$

so that

$$D_1(\mu,\lambda) = -(\mu - 1 - d\lambda)(-\mu + 1 - \lambda)^d = \underbrace{(-1)^{d+1}(\mu - 1 - d\lambda)}_{=:D_0(\mu,\lambda)} A_1^d.$$

Now let

$$A_n(\mu,\lambda) = \begin{cases} \mu + \lambda - 1 & , \ n = 1 \\ \prod_{\theta \in f^{-(n-2)}(0)} \Phi_\theta & , n > 1 \end{cases}$$

$$B_n(\mu,\lambda) = \begin{cases} \mu+\lambda+1 & ,n=2 \\ \prod_{\theta\in f^{-(n-3)}(-2)} \Phi_\theta & ,n>2 \end{cases}$$

so that by Proposition I.30 it holds true that

$$\det X_0 = LB_2K^d(A_1B_2)^{\binom{d+1}{2}-(d+1)}.$$

Note here that the quadratic maps Φ_θ are defining the hyperbolae in which the bidimensional auxiliary spectrum can be decomposed, i.e. in which D_n is factorizable (see Proposition I.34). Now in order to factorize D_n into A_i's and B_i's we need to state the behaviour of the factors of $\det X_0$ under composition with F.

Lemma I.33. *The following relations with respect to F hold:*

$$D_0 \circ F = \frac{D_0}{L} A_1$$

$$A_1 \circ F = \frac{A_1}{K} A_2$$

For $n \geq 2$:

$$A_n \circ F = \left(\frac{A_1}{LK}\right)^{2^{n-2}} A_{n+1}$$

$$B_2 \circ F = \frac{B_2}{K} B_3$$

For $n \geq 3$:

$$B_n \circ F = \left(\frac{A_1}{LK}\right)^{2^{n-3}} B_{n+1}$$

Proof. The above lemma allows us to show the claims involving n directly as

$$A_n \circ F = \prod_{\theta\in f^{-(n-2)}(0)} \Phi_\theta \circ F = \left(\frac{A_1}{LK}\right)^{2^{n-2}} \prod_{\theta\in f^{-(n-1)}(0)} \Phi_\theta = \left(\frac{A_1}{LK}\right)^{2^{n-2}} A_{n+1}.$$

The respective claim for B can be shown in a similar fashion. The claims not involving n can again be verified by high-school algebra. It should be noted that $A_2 = \Phi$, $B_3 = \Phi + 2\lambda$ □

Proposition I.34. *The determinant $D_n(\mu,\lambda)$ factorizes as*

$$D_1 = D_0 A_1^d$$

$$D_n = D_0 A_1^{\alpha_n} \ldots A_n^{\alpha_1} B_2^{\beta_n} \ldots B_n^{\beta_2}$$

for $n \geq 2$, where the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 2}$ are given by

$$\alpha_n = \beta_n + d, \qquad \beta_n = \frac{d-1}{2}((d+1)^{n-1} - 1)$$

for $n \geq 2$ and $\alpha_1 = d$.

Proof. For $n = 1$ we have shown the factorization of D_1 above.

We now proceed by induction. In order to ease notation we denote for a function $g(\mu, \lambda)$ the renormalized function $g \circ F$ by g'.

Let $n \geq 2$ and assume the factorization holds for $n - 1$. By the above we have

$$\begin{aligned} D_n &= \left(LB_2K^d(A_1B_2)^{\binom{d+1}{2}-(d+1)}\right)^{(d+1)^{n-2}} D'_{n-1} \\ &= \left(LB_2K^d(A_1B_2)^{\binom{d+1}{2}-(d+1)}\right)^{(d+1)^{n-2}} D'_0 \cdot (A'_1)^{\alpha_{n-1}} \cdot \ldots \cdot (A'_{n-1})^{\alpha_1} \cdot (B'_2)^{\beta_{n-1}} \cdot \ldots \cdot (B'_{n-1})^{\beta_2} \\ &= \left(LB_2K^d(A_1B_2)^{\binom{d+1}{2}-(d+1)}\right)^{(d+1)^{n-2}} \frac{D_0A_1}{L}\left(\frac{A_1A_2}{K}\right)^{\alpha_{n-1}}\left(\frac{A_1}{LK}\right)^{\sigma_n} A_3^{\alpha_{n-2}} \ldots A_n^{\alpha_1}\left(\frac{B_2B_3}{K}\right)^{\beta_{n-1}} B_4^{\beta_{n-2}} \ldots B_n^{\beta_2} \end{aligned}$$

where

$$\sigma_n = (\alpha_{n-2} + 2\alpha_{n-3} + \ldots + 2^{n-3}\alpha_1) + (\beta_{n-2} + 2\beta_{n-3} + \ldots + 2^{n-4}\beta_2) \tag{3.2}$$

for $n > 2$ and $\sigma_2 = 0$. Furthermore for ease of notation let $c_d := \binom{d+1}{2} - (d+1)$.

The above equation for D_n implies that the following equations are necessary for the induction to hold:

$$\alpha_n = c_d(d+1)^{n-2} + \sigma_n + \alpha_{n-1} + 1 \tag{3.3}$$

$$\beta_n = (c_d + 1)(d+1)^{n-2} + \beta_{n-1} \tag{3.4}$$

The values of α, β and σ are determined in by the subsequent Lemma I.35 to be:

$$\beta_n = \frac{c_d + 1}{d}((d+1)^{n-1} - 1)$$

$$\alpha_n = \beta_n + d$$

$$\sigma_n = (d+1)^{n-2} - 1$$

for $n > 2$. Further note that $\frac{c_d+1}{d} = \frac{d-1}{2}$.

Now having the sequences α, β, σ at hand we can verify that these are also sufficient for the induction; this is done by showing that all copies of L and K cancel out. So for the K's it must hold true that

$$d(d+1)^{n-2} = \alpha_{n-1} + \sigma_n + \beta_{n-1} = \left(2\frac{c_d + 1}{d} + 1\right)(d+1)^{n-2} - \left(2\frac{c_d + 1}{d} + 1\right) + d.$$

Taking into consideration that

$$2\frac{c_d + 1}{d} + 1 = d$$

we obtain the validity of the claim.

For the L's we have to check

$$(d+1)^{n-2} = 1 + \sigma_n$$

which already has been verified before.

Thus after the proper reordering and canceling of the terms we obtain

$$D_n = D_0A_1^{\alpha_n} \ldots A_n^{\alpha_1} B_2^{\beta_n} \ldots B_n^{\beta_2}$$

which is the claimed factorization. $\square$

Lemma I.35. *Equations (3.2), (3.3) and (3.4) together with the initial values*

$$\alpha_1 = d, \qquad \beta_1 = 0$$

imply that

$$\sigma_n = (d+1)^{n-2} - 1,$$
$$\beta_n = \frac{c_d+1}{d}((d+1)^{n-1} - 1),$$
$$\alpha_n = \beta_n + d.$$

Proof. Using geometric series it follows that

$$\beta_n = (c_d+1)\sum_{i=0}^{n-2}(d+1)^i = \frac{c_d+1}{d}((d+1)^{n-1} - 1).$$

In order to determine α and σ we need to perform an intertwined induction on both. We claim that

$$\sigma_n = (d+1)^{n-2} - 1$$

for $n \geq 3$,

$$\sigma_1 = \sigma_2 = 0$$

and

$$\alpha_n = \beta_n + d$$

for $n \geq 1$ and $\beta_1 = 0$.

Obviously those claims are easily verified for $n \leq 2$. We will show the claim for $n > 2$ by induction. Assume the claims hold for $n-1$. By (3.2) for n we have

$$\sigma_n = \alpha_{n-2} + \beta_{n-2} + 2\sigma_{n-1} = (d+1)^{n-3}\Big(2\frac{c_d+1}{d} + 2\Big) + d - \Big(2\frac{c_d+1}{d} + 2\Big).$$

Note now by definition of c_d we have

$$2\frac{c_d+1}{d} + 2 = 2\frac{c_d+d+1}{d} = d+1$$

whence

$$\sigma_n = (d+1)^{n-2} - 1.$$

Further by (3.3) we obtain

$$\alpha_n = c_d(d+1)^{n-2} + (d+1)^{n-2} + \alpha_{n-1}$$

which is solved under the initial condition $\alpha_1 = d$ just as (3.4) by

$$\alpha_n = \beta_n + d.$$

$\square$

The following proposition is the analogon of Theorem I.16 for the adjacency spectrum. Theorem I.16 will then follow immediately.

Proposition I.36. *Let $d > 1$ and $\mathscr{A}_i$ and $\mathscr{B}_i$ be the sequences recursively obtained as*

$$\mathscr{A}_i := g^{-i}(0), \qquad \mathscr{B}_i := g^{-i}(-2)$$

for the polynomial

$$g(\zeta) = \zeta^2 - (d-1)\zeta - (d+1).$$

Then $\{\mathscr{A}_i, \mathscr{B}_i \mid i \in \mathbb{N}\}$ are mutually disjoint and the sequence of shifted spectral quantile functions

$$\Lambda(A(G_n)) = \Lambda(\Xi_n)$$

converges to the unique increasing step function Λ on $[0,1]$ attaining values in

$$\bigcup_{i=0}^{\infty} \mathscr{A}_i \cup \bigcup_{j=0}^{\infty} \mathscr{B}_j$$

such that for $x \in \mathscr{A}_i \cup \mathscr{B}_i$ the value $(d+1) - x$ is attained on an interval of length

$$\frac{d-1}{2(d+1)^{i+1}}$$

in $L^1([0,1])$.

Proof. The claims follow immediately from the factorization provided for $D_n(\mu, \lambda)$ in Proposition I.34 - note that the eigenvalues of Ξ_n are given by the roots of $D_n(\mu, 1)$. Under the assumption $\lambda = 1$ we obtain the following relations

$$D_0(\mu, 1) = \mu - (d+1),$$

$$A_1(\mu, 1) = \mu,$$

$$B_1(\mu, 1) = \mu + 2,$$

$$\Phi(\mu, 1) = g(\mu),$$

$$\Phi_\theta(\mu, 1) = g(\mu) - \theta.$$

Giving us the full description of the spectral distribution. The multiplicities are given by the exponents with which the factors appear.

Thus by $\mathscr{A}_i = g^{-i}(0)$ and $\mathscr{B}_i = g^{-i}(-2)$ the spectrum of Ξ_n as a set decomposes as

$$\bigcup_{i=0}^{n-1} \mathscr{A}_i \cup \bigcup_{i=0}^{n-2} \mathscr{B}_i.$$

We will show that this is in fact a partition (i.e. $\{\mathscr{A}_i, \mathscr{B}_i \mid i \in \mathbb{N}\}$ are mutually disjoint). Furthermore by the above equations the eigenvalues in $\mathscr{A}_i$ and $\mathscr{B}_i$ are precisely the roots of the factors A_{i+1} and B_{i+2} which occur with exponent α_{n-i} and β_{n-i} in D_n, respectively.

In what follows we will also see that the factors A_i and B_i do not have multiple roots so that the multiplicities of eigenvalues of Ξ_n in $\mathscr{A}_i$ and $\mathscr{B}_i$ are α_{n-i} and β_{n-i}, respectively.

The mutual disjointness of $\mathscr{A}_i$ with any $\mathscr{B}_j$ can be seen in the following way: First note that $g(c \cdot (d+1)) > c \cdot (d+1)$ for every $c > 1$;

$$\begin{aligned} g(c \cdot (d+1)) &= c^2(d+1)^2 - c(d-1)(d+1) - (d+1) \\ &= (d+1)(c^2(d+1) - c(d-1) - 1) \\ &> (d+1)(c(d+1) - c(d-1) - 1) \\ &= (d+1)(2c-1) \\ &> c(d+1) \end{aligned}$$

Thus if for given x the sequence $g^n(x)$ surpasses the value $(d+1)$ it must be strictly increasing.

Assume now that there is $x \in \mathscr{A}_i$ such that there is $j \in \mathscr{B}_j$ with $x \in \mathscr{B}_j$. In this case we have

$$g^i(x) = 0$$

and

$$g^j(x) = -2.$$

Thus in case $i < j$ we have found that

$$g^{j-i}(0) = -2$$

and in case $i > j$ we obtain

$$g^{i-j}(-2) = 0.$$

We will exclude both cases by observing the first few elements of the sequence before it is forced to be strictly increasing by the above observation. Note that

$$g(0) = -(d+1) \neq -2$$

since $d > 1$. Furthermore

$$g^2(0) = f(-(d+1)) = (d+1)(d+1+d-1-1) = (d+1)\underbrace{(2d-1)}_{>1}$$

thus not attaining the value -2.

For the sequence with $x = -2$ we have

$$g(-2) = 4 + 2(d-1) - (d+1) = d+1$$

and

$$g(d+1) = (d+1)(d+1-(d-1)-1) = (d+1)$$

thus stabilizing at $d+1$ not attaining 0.

This way we have seen that $\mathscr{A}_i$ and $\mathscr{B}_j$ must be disjoint. To see the mutual disjointness of the $\mathscr{A}$'s assume there is x such that $x \in \mathscr{A}_i \cap \mathscr{A}_j$ and assume further that $i < j$. Then obviously we have

$$g^{j-i}(0) = 0$$

which we have shown to be false.

For the $\mathscr{B}$'s assume analogously that there is $x \in \mathscr{B}_i \cap \mathscr{B}_j$ such that $i < j$. Then again

$$g^{j-i}(-2) = -2.$$

But this is false since the sequence stabilizes at $d+1$ immediately. Thus the mutual disjointness of $\{\mathscr{A}_i, \mathscr{B}_i \mid i \in \mathbb{N}\}$ follows.

That A_i and B_i do not have multiple roots can be obtained as follows: Obviously A_1 and B_2 do not have multiple roots. The roots of A_{i+1} and B_{i+1} are obtained from the roots of A_i and B_i, respectively, by taking g^{-1}, i.e. if λ is a root of A_i it induces two roots of A_{i+1}, namely

$$\left\{\frac{d-1}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 + (d+1) + \lambda}\right\}.$$

In particular $\mathscr{A}_{i+1}$ does split in two sets

$$\mathscr{A}_{i+1} = \mathscr{A}_{i+1}^{+} \cup \mathscr{A}_{i+1}^{-}$$

each in bijection to $\mathscr{A}_i$ over the maps $\frac{d-1}{2} + r$, $\frac{d-1}{2} - r$ for

$$r(x) = \sqrt{\left(\frac{d-1}{2}\right)^2 + (d+1) + x}.$$

Note that r is non-negative and $r(x) = 0$ iff $x = -(1 + ((d+1)/2)^2) < -2$ since $d > 1$. Thus the sets $\mathscr{A}_i^{+}$ and $\mathscr{A}_i^{-}$ have to be disjoint and thus no roots of A_i can be multiple. Analogously we obtain the same for $\mathscr{B}_i$.

Now we will show the convergence of the spectral cdf of G_n towards the function Λ claimed to be the limit.

The convergence of $\Lambda_n = \Lambda(A(G_n)) = \Lambda(\Xi_n)$ towards the increasing step function Λ with step values in $\bigcup_{i=0}^{\infty} \mathscr{A}_i \cup \bigcup_{i=1}^{\infty} \mathscr{B}_i$ and step length

$$\frac{d-1}{2(d+1)^{i+1}}$$

for values in $\mathscr{A}_i \cup \mathscr{B}_i$ follows from the fact that the step length of eigenvalues $\lambda \in \mathscr{A}_i$ or $\lambda \in \mathscr{B}_i$ in Λ_n is given by

$$\frac{\alpha_{n-i}}{(d+1)^n} = \frac{d-1}{2}\frac{(d+1)^{n-i-1}}{(d+1)^n} + \frac{d+1}{2(d+1)^n} \xrightarrow{n\to\infty} \frac{d-1}{2(d+1)^{i+1}}$$

or

$$\frac{\beta_{n-i}}{(d+1)^n} = \frac{d-1}{2}\frac{(d+1)^{n-i-1}}{(d+1)^n} - \frac{d-1}{2(d+1)^n} \xrightarrow{n\to\infty} \frac{d-1}{2(d+1)^{i+1}},$$

respectively.

Thus the step lengths of the steps in Λ_n converge uniformly towards their respective step length in Λ and the values not attained by Λ_n are

$$\bigcup_{i=n}^{\infty} \mathscr{A}_i \cup \bigcup_{i=n-1}^{\infty} \mathscr{B}_i.$$

They are attained by Λ on a joint volume of

$$\begin{aligned}
\sum_{i=n}^{\infty} 2^i \frac{d-1}{2(d+1)^{i+1}} + \sum_{i=n-1}^{\infty} 2^i \frac{d-1}{2(d+1)^{i+1}} &= 2^{n-1}\frac{d-1}{2(d+1)^n} + \sum_{i=n}^{\infty} 2^i \frac{d-1}{(d+1)^{i+1}} \\
&\overset{d\neq 1}{=} 2^{n-2}\frac{d-1}{(d+1)^n} + \frac{2^n}{(d+1)^n} \\
&= \frac{2^{n-2}(d+3)}{(d+1)^n}.
\end{aligned}$$

Obviously since $d+1>2$ this volume vanishes asymptotically.

Thus for $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for every $n\geq n_0$ it holds that

$$\frac{2^{n-2}(d+3)}{(d+1)^n}<\varepsilon$$

so that also

$$\delta:=\frac{d+1}{2(d+1)^n}<\frac{d+1}{2^{n-1}(d+3)}\varepsilon.$$

Let such an n be fixed for now; then by the above considerations we might modify Λ by setting all steps with values in

$$\bigcup_{i=n}^{\infty}\mathscr{A}_i\cup\bigcup_{i=n-1}^{\infty}\mathscr{B}_i$$

to zero, i.e.

$$\tilde{\Lambda}(x):=\Lambda(x)\cdot\mathbf{1}_{\Lambda(x)\in F}$$

for

$$F=\bigcup_{i=0}^{n-1}\mathscr{A}_i\cup\bigcup_{i=0}^{n-2}\mathscr{B}_i.$$

Obviously since the absolute values of elements in $\bigcup_{i=0}^{\infty}\mathscr{A}_i\cup\bigcup_{i=0}^{\infty}\mathscr{B}_i$ are bounded by $d+1$ from the above we obtain

$$||\Lambda-\tilde{\Lambda}||_{L^1([0,1])}<\varepsilon\cdot(d+1).$$

Note that $\Lambda-\tilde{\Lambda}$ is supported on a set of measure less than ε.

Subsequently we do the same modification for Λ_n, i.e. $\tilde{\Lambda}_n(x):=\Lambda_n(x)\cdot\mathbf{1}_{\Lambda(x)\in F}$ so that both $\tilde{\Lambda}$ and $\tilde{\Lambda}_n$ are zero on $\Lambda^{-1}(F^c)$. We obviously have by the same reasoning as for Λ that

$$||\tilde{\Lambda}_n-\Lambda_n||_{L^1([0,1])}<\varepsilon(d+1).$$

Thus on $\Lambda^{-1}(F^c)$ $\tilde{\Lambda}$ aswell as $\tilde{\Lambda}_n$ are zero so that we might consider both functions to be step function on $[0,1-\varepsilon)$ (this can be done because we modified Λ only on a countable union of intervals from $[0,1]$). We will denote those step functions by $\tilde{\Lambda}$ and $\tilde{\Lambda}_n$ aswell as their L^1-distance stay the same under this transition.

Now order the eigenvalues in F as

$$\lambda_1<...<\lambda_k$$

and note that

$$k=\sum_{i=0}^{n-1}2^i+\sum_{i=0}^{n-2}2^i=2^n+2^{n-1}-2.$$

Furthermore denote by ℓ_i and ℓ_i' the length of the step with value λ_i in $\tilde{\Lambda}$ and $\tilde{\Lambda}_n$, respectively. Note that ℓ_i' might be 0 if the entire step of λ_i in Λ_n was contained in $\Lambda^{-1}(F^c)$.

As both $\tilde{\Lambda}$ and $\tilde{\Lambda}_n$ have the same finite image set F we can bound the L^1 distance in terms of their jumps. Note that the discrepancy $\delta_i:=|\ell_i-\ell_i'|$ induces a shift in the subsequent steps by δ_i where the shift is accounted for in L^1-distance by the integration of every subsequent jump over an interval of length δ_i. We decompose δ_i in two parts;

$$\delta_i\leq\delta+f_i,$$

where δ is the discrepancy introduced by the differing length of steps coming from Λ and Λ_n themselves while f_i is introduced by deleting parts of the steps in the process of going over to $\tilde{\Lambda}_n$ from Λ_n. Since the range on which we delete is of measure ε we obtain

$$\sum_{i=1}^{k} f_i = \varepsilon.$$

Thus letting the jump height of the i-th jump be $h_i = \lambda_{i+1} - \lambda_i$ we obtain

$$\begin{aligned} ||\tilde{\Lambda} - \tilde{\Lambda}_n||_{L^1([0,1-\varepsilon))} &\leq \sum_{i=1}^{k} \delta_i \sum_{j=i}^{k-1} h_j \\ &\leq \sum_{i=1}^{k} \delta \sum_{j=i}^{k-1} h_j + \sum_{i=1}^{k} f_i \sum_{j=i}^{k-1} h_j. \end{aligned}$$

Obviously $\sum_{j=i}^{k-1} h_j \leq d+3$ since $F \subset [-2, d+1]$. Thus using $\sum_{i=1}^{k} f_i = \varepsilon$

$$||\tilde{\Lambda} - \tilde{\Lambda}_n||_{L^1([0,1-\varepsilon))} \leq (d+3)(k\delta + \varepsilon).$$

This can be bounded using the equations for δ and k by

$$k\delta < (2^n + 2^{n-1} - 2)\frac{d+1}{2^{n-1}(d+3)}\varepsilon < 3\frac{d+1}{d+3}\varepsilon.$$

Thus

$$||\tilde{\Lambda} - \tilde{\Lambda}_n||_{L^1([0,1-\varepsilon))} < (3(d+1) + d + 3)\varepsilon = (4d+6)\varepsilon;$$

showing the claim. □

Proof of Theorem I.16. Theorem I.16 immediately follows from Proposition I.36 by the observation

$$\Delta_n = \Delta(\mathrm{cd}^n \Delta_d) = (d+1) \cdot I - A(G_n) = (d+1) \cdot I - \Xi_n.$$

Thus the spectral cdfs satisfy the following condition:

$$\Lambda(\Delta_n)(x) = (d+1) - \Lambda(\Xi_n)(1-x).$$

So that by definition both limits are equal if for the shifting function $\sigma(x) := (d+1) - x$ we have

$$\mathscr{P}_i = \sigma(\mathscr{A}_i)$$
$$\mathscr{Q}_i = \sigma(\mathscr{B}_i)$$

for every i.

For $i = 0$ this is obviously true since $\sigma(0) = d+1$ and $\sigma(-2) = d+3$ and for $i > 0$ we can see this by the following inductive consideration:

Let $\lambda \in \mathscr{A}_i$ then $g^{-1}(\lambda) \subseteq \mathscr{A}_{i+1}$. Let $\lambda' \in \mathscr{A}_{i+1}$ be given so that

$$g(\lambda') = \lambda.$$

For $\mu = \sigma(\lambda) \in \mathscr{P}_i$ we obtain $\mu' = \sigma(\lambda')$ from μ over f as follows:

$$\begin{aligned} d+1-\mu=\lambda &= g(\lambda') \\ &= (d+1-\mu')^2-(d-1)(d+1-\mu')-(d+1) \\ &= d+1-(d+3)\mu'+(\mu')^2 \\ &= d+1-f(\mu'). \end{aligned}$$

Thus $\mu = f(\mu')$. The same calculation gives us the equality of $\mathscr{P}_i$ and $\sigma(\mathscr{A}_i)$. Thus the induction holds.

Analogously we can see $\mathscr{Q}_i = \sigma(\mathscr{B}_i)$ and so the limit of $\Lambda(\Delta_n)$ is the claimed step function. $\square$

Example: Barycentric Subdivision for $d = 2$

Barycentric subdivision is an inclusion-uniform subdivision (in all dimensions) as one readily checks. As mentioned in the introduction for $d = 1$ the universal limit is determined (see Section 3 on the cone division as the two procedures coincide for $d = 1$).

Unfortunately already for $d = 2$ the determination of the universal limit was not accessible by the framework presented here; the reason being that as soon as the input dimension $\dim K$ is greater than or equal 2 the associated fractal is not finitely ramified anymore (see Section 4). Note that the spectral analysis of non-finitely ramified fractals is generally considered a hard task in fractal analysis. For some fractals symmetry arguments can augment the computation and make a universal limit accessible (see Section 3).

The role finite ramification plays in the calculation of e.g. the universal limit of the cone division will become clear from the Schreier graph approximation of the dual graph associated to barycentric subdivision. We proceed similar to the Schreier graph approximation of the previous example.

Assume the input complex K has dimension $d = 2$. As every facet of K gets subdivided into 6 triangles we will construct the dual graph $\Gamma_n := \Gamma^{(2)}(\mathrm{sd}^n \Delta_d)$ as the Schreier graph of a subgroup G acting on a rooted senary (i.e. 6-ary) tree of infinite depth. Again let $\mathscr{T}$ denote this tree and let X be a set of 6 elements so that the vertex set of $\mathscr{T}$ can be associated to X^*, e.g. $X = [6]$. We will further denote $\mathscr{T}$ itself by X^*.

Note again that given a self-similar subgroup $G \leq \mathrm{Aut}(X^*)$ and a set of generators $S \subseteq G$ we denote the n-th level Schreier graph (with respect to S) by

$$G_n := (X^n, E_n),$$

where

$$E_n := \{(w, s \cdot w) \mid w \in X^n, s \in S\}.$$

In order to obtain an undirected graph we assume $\{s^{-1} \mid s \in S\} = S$. Furthermore we want S to act on X^n so that for every $w \in X^n$ and $s_1, s_2 \in S$ from $s_1 \cdot w = s_2 \cdot w$ it follows that $s_1 = s_2$. This way the adjacency matrix of G_n can be expressed in terms of representations of the elements s as

$$A(G_n) = \sum_{s \in S} \rho(s),$$

where $\rho(s)$ is the permutation matrix representation of the action of s on the elements of X^n.

We will now determine a set of three elements generating the subgroup G of $\mathrm{Aut}(X^*)$ which induces as a Schreier graph sequence the sequence of dual graphs of barycentric subdivisions of Δ_2.

The first two elements generate the dual graph in the interior of every subdivided triangle, i.e. a hexagon for each elementary cell. We will order the facets of $\mathrm{sd}\sigma$ of every $\sigma \in F_2(\mathrm{sd}^{n-1}\Delta_2)$ cyclically as shown in 3.4.

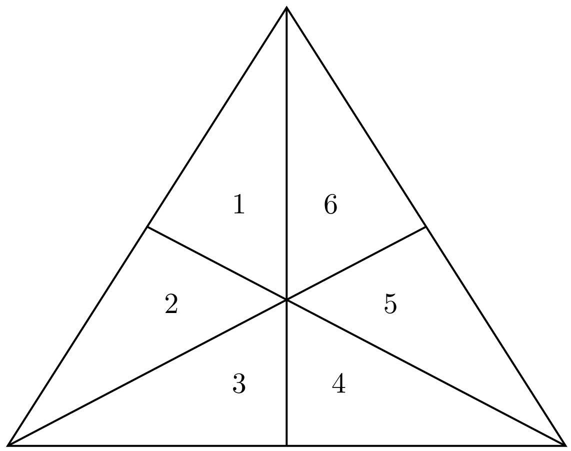


Figure 3.4: Internal numbering scheme of the faces of $\mathrm{sd}\sigma$.

Let τ_* and t_* denote the involutions

$$\tau_*(wx) := w\tau(x),$$

$$t_*(wx) := wt(x)$$

for the permutations $\tau, t \in S_6$ given by

$$\tau = (12)(34)(56)$$

$$t = (16)(23)(45).$$

Similar to the group generated by a in last section t and τ will generate a fixed graph in every elementary cell, i.e. $wX := \{wx \mid x \in X\}$, and so the n-th level Schreier graph of C with respect to $S = \{\tau, t\}$ is a graph of hexagons - one for each cell $wX := \{wi \mid i \in X\}$. See τ and t in Figure 3.5 to see how they generate the hexagons in each cell.

Now we define b and then show that this indeed models the adjacencies of barycentric subdivision. Let

$$b(wx) := \begin{cases} \tau_*(w)x & , \ x \in \{1,2\} \\ t_*(w)x & , \ x \in \{3,4\} \\ b(w)x & , \ x \in \{5,6\} \end{cases}.$$

and $b(x) = x$ for $x \in [6]$.

Remark 1. Note that the last condition is what prevents us from applying the same procedure as in the previous section; while we can handle the appearance of multiple copies of "fixed" matrices like t_* and τ_* in $A(G_n)$ it is not convenient to apply Schur renormalization when a recurrent matrix like b appears in multiple blocks. This is due to the fact that in this case we

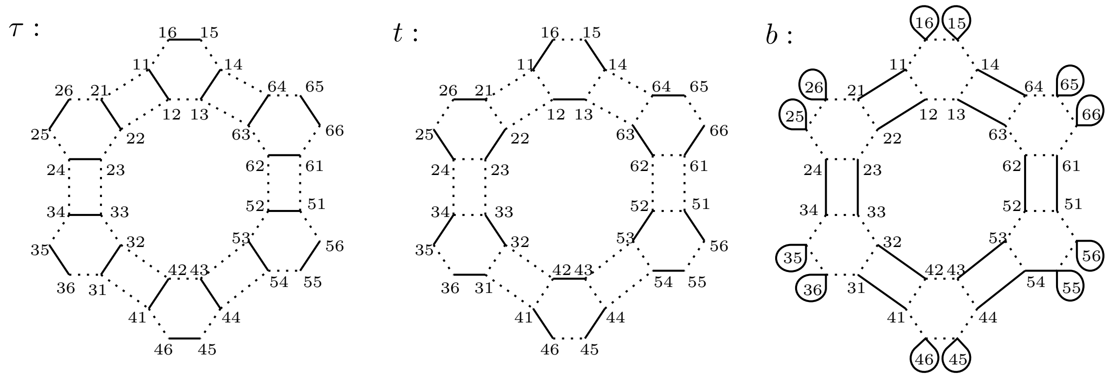


Figure 3.5: Decomposition of the dual graph of $\mathrm{sd}^2\Delta_2$ in Schreier graphs of the generator elements τ, t, b. The dual graph of $\mathrm{sd}^2\Delta_2$ is illustrated via dotted lines.

need to invert sums $\alpha t_* + \beta\tau_* + \gamma b + \eta \cdot 1$ for reals $\alpha, \beta, \gamma, \eta$ which (in this case) introduces terms of the form $t_* b, \tau_* b$ and so in order to obtain a full recursion we would need to introduce another auxiliary parameter (such as λ in the previous section); this has the effect that it fills the matrix with λ-scaled copies of b in even more blocks so that computation is not feasible. Thus we see that finite ramification is indeed an important feature as finitely ramified subdivisions only have one recurrent block in the matrix representation.

Theorem I.37. *G_n as defined above approximates $\Gamma^{(2)}(sd^n\Delta_2)$.*

Proof. We will show this result by introducing an addressing scheme of the facets of $\mathrm{sd}^n\Delta_2$.

We will define this scheme inductively. Let $n_\sigma(\nu)$ for $\sigma \in \mathrm{sd}^{n-1}\Delta_2$ and $\nu \in \mathrm{sd}\sigma \subseteq F_2(\mathrm{sd}^n\Delta_2)$ denote the number of ν in σ then ν has an address $n(\nu) \in X^n$ given by

$$n(\nu) = n_{\Delta_2}(\sigma_1)n_{\sigma_1}(\sigma_2)...n_{\sigma_{n-1}}(\nu),$$

where $\sigma_i \in F_2(\mathrm{sd}^i\Delta_2)$ is a sequence such that $\sigma_{i+1} \in F_2(\mathrm{sd}\sigma_i)$ and $\nu \in F_2(\sigma_{n-1})$. In case $n = 1$ we set $n(\nu) = n_{\Delta_2}(\nu)$.

Note that in case $n_\sigma(\nu) \neq n_\sigma(\nu')$ for $\nu \neq \nu'$ this addressing scheme is a bijection between the vertex set of the n-th layer of X^*, i.e. X^n, and $F_2(\mathrm{sd}^n\Delta_2)$.

Now we will define this addressing scheme in a way that it has further favorable properties. First for every edge $e \in F_1(\mathrm{sd}^{n-1}\Delta_2)$ and every coface $e \leq \sigma \in F_2(\mathrm{sd}^{n-1}\Delta_2)$ we introduce a number $\iota_\sigma(e) \in [3]$. If $n = 1$ we will enumerate the three edges arbitrarily so that each edge has a unique number. For $n > 1$ there are three cases which we will differentiate. The first case is that e is "outwards pointing" from σ, which means that there is a facet $\sigma' \in F_2(\mathrm{sd}^{n-2}\Delta_2)$ such that $\sigma \in \mathrm{sd}\sigma'$ and e is contained in the boundary of σ'. In this case we set $\iota_\sigma(e) = 3$.

If e is an internal edge, i.e. there is a face $\sigma' \in F_2(\mathrm{sd}^{n-2}\Delta_2)$ such that $\sigma \in \mathrm{sd}\sigma'$ and e is not contained in the boundary of σ', we differentiate the case if e contains a vertex of ν nor not. If so $\iota_\sigma(e) = 2$ and if not $\iota_\sigma(e) = 1$. See Figure 3.6 for the ordering of the edges.

Note that indeed the number $\iota_\sigma(e)$ is indepednent of σ. However we will keep this notation as a different ordering might be chosen leading to different group generators inducing this as a Schreier graph.

Now that every facet $\sigma \in F_2(\mathrm{sd}^{n-1}\Delta_2)$ has an edge ordering we can order its vertices dual to it, i.e. if τ_1, τ_2, τ_3 are the edges of σ so that $\iota_\sigma(\tau_i) = i$ we can order the vertices as

$$\sigma = (\sigma \setminus \tau_1, \sigma \setminus \tau_2, \sigma \setminus \tau_3).$$

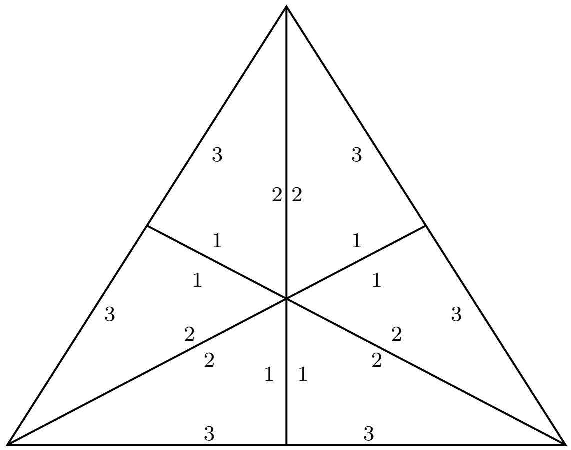


Figure 3.6: Edge numbering scheme $\iota_\sigma(e)$.

It is immediate to see (e.g. by a symmetry argument) that the orientations induced by $e < \sigma, \sigma'$ on an edge $e \in F_1(\mathrm{sd}^{n-1}\Delta_2)$ agree.

Furthermore this edge ordering induces a canonical ordering of the facets ν of $\mathrm{sd}\sigma$. For every edge $e \in F_1(\sigma)$ let e_1, e_2 denote the edges into which e gets subdivided. We choose the numbering e_1, e_2 to be consistent with the orientation of e induced by the orientation of σ, i.e. if $e = (v, w)$ for vertices $v, w \in F_0(\sigma)$ we let e_1 be the edge incident to v and e_2 the edge incident to w. Then let

$$n_\sigma(e_i * v_\sigma) := 2(\iota_\sigma(e) - 1) + i.$$

Since neighboring simplices $\sigma, \sigma' \in F_2(\mathrm{sd}^{n-1}\Delta_2)$ induce the same orientation on $e = \sigma \cap \sigma'$ and $\iota_\sigma(e) = \iota_{\sigma'}(e)$ we have that for every $\nu \in F_1(\mathrm{sd}\sigma)$

$$n_\sigma(\nu * v_\sigma) = n_{\sigma'}(\nu * v_{\sigma'}).$$

Note that for every $\nu * v_\sigma$ in $\Gamma^{(2)}(\mathrm{sd}^n\Delta_2)$ there are two edges for adjacencies internal to σ and possibly one outwards pointing edge. By definition of n_σ if $n_\sigma(\nu * v_\sigma) = i$ the internal adjacencies are with $i - 1$ (or 6 in case $i = 1$) and $i + 1$ (or 1 in case $i = 6$). Note that if i is odd then $\tau(i) = i + 1$ so that the adjacency to the above is given by τ_* on X^n, while $t(i) = i - 1$ for $i \neq 1$ and $t(1) = 6$ so that the adjacency to the lower vertices is given by t_*. One can argue analogously for i even.

In case ν is a boundary edge we do not have another edge in $\Gamma^{(2)}(\mathrm{sd}^n\Delta_2)$ but note that in this case the address has to consist purely of the numbers 5 and 6 by definition (since every edge containing ν in the sequence has to have number 3) so that $b(n(\nu * v_\sigma)) = n(\nu * v_\sigma)$ and in G_n we have a loop instead. If $\nu * v_\sigma$ is incident to another face $\nu * v_{\sigma'}$ σ and σ' have to be adjacent to each other. Note that by $n_\sigma(\nu * v_\sigma) = n_{\sigma'}(\nu * v_{\sigma'})$ and by definition of b we thus only need to check that

$$p_n(b(n(\nu * v_\sigma))) = p_n(b(n(\nu * v_{\sigma'}))),$$

where $p_n : X^n \to X^{n-1}$ is the map neglecting the last number, i.e. $p_n(wx) = w$. In case $\sigma, \sigma' \in \mathrm{sd}\sigma''$ for $\sigma'' \in F_2(\mathrm{sd}^{n-2}\Delta_2)$ then by the above we have $\tau_*(n(\sigma)) = n(\sigma')$ or $t(n(\sigma)) = n(\sigma')$ (in the respective cases); by definition of $n_{\sigma''}$ it is readily checked that these cases are consistent with the definition of b.

If σ and σ' are not contained in a facet $\sigma'' \in F_2(\mathrm{sd}^{n-2}\Delta_2)$ then there are $\gamma, \gamma' \in F_2(\mathrm{sd}^{n-2}\Delta_2)$, $\gamma \neq \gamma'$, $\sigma \in \mathrm{sd}\gamma$ and $\sigma' \in \mathrm{sd}\gamma'$ such that the adjacency of σ and σ' is outwards pointing and so $n_\gamma(\sigma) = n_{\gamma'}(\sigma')$. Note that in this case $b(n(\sigma)) = b(n(\gamma))n_\gamma(\sigma)$ and $b(n(\sigma')) = b(n(\gamma'))n_{\gamma'}(\sigma')$. It thus is to check that $b(n(\gamma)) = b(n(\gamma'))$, where γ, γ' now take the role of σ and σ' and σ, σ' take the role of $\nu * v_\sigma$, $\nu * v_{\sigma'}$. The claim thus can be shown inductively. Note also that this induction ends in an internal adjacency since boundary edges only have one coface so that ν can't be boundary.

In conclusion we obtain $b(n(\nu * v_\sigma)) = b(n(\nu * v_{\sigma'}))$.

The only thing left to show is thus that the number of loops of G_n, $\ell(G_n)$, suffices $\ell(G_n) \ll v(G_n)$, where $v(G_n)$ is the number of vertices. To this end note that a boundary face induces precisely two boundary faces in the subdivision while every facet induces 6 facets in the subdivision, i.e.

$$\ell(G_n) = 3 \cdot 2^n \ll 6^n = v(G_n).$$

□

Corollary I.38. $\Lambda_2^{(sd)}$ *is point symmetric about* 3*.*

Proof. It is immediate to see that $\tilde{G}_n$ (i.e. G_n with loops removed) is bipartite by the bipartition defined inductively by the following function: Let $\varphi_1(x) = x \bmod 2$ which is easily seen to define a bipartition of the hexagon G_1.

The function $\varphi_n : V(G_n) \to \{0, 1\}$ is then inductively defined as

$$\varphi_n(wx) = |\varphi_1(x) - \varphi_{n-1}(w)|,$$

i.e. if w is assigned 1 we invert the bipartition of the cell wX.

This is a bipartition on each cell because on every cell wX $\varphi_{n-1}(w)$ is constant and φ_1 defines a bipartition. On the other hand since by definition adjacent vertices in different cells have to be of the form wx, $w'x$ for $w \neq w'$ adjacent in G_{n-1} (since b leaves the last letter invariant), we have that $\varphi_1(x)$ is the same and $\varphi_{n-1}(w) \neq \varphi_{n-1}(w')$ because φ_{n-1} defines a bipartition on G_{n-1}.

The claim then follows by [Big74, Proposition 8.2]. □

Corollary I.39. *The spectrum of* $\Gamma^{(2)}(sd^n\Delta_2)$ *can be determined by the following matrix recursion (up to an asymptotically negligible* L^1*-error):*

Let $\Delta_0 := t_0 + \tau_0$*,* $W_0 = diag(1, 1, 0, 0, 0, 0)$*,* $V_0 = diag(0, 0, 1, 1, 0, 0)$*,* $K = diag(0, 0, 0, 0, 1, 1)$ *and set*

$$\Delta_n := \begin{pmatrix} \Delta_{n-1} & W_{n-1} & 0 & 0 & 0 & V_{n-1} \\ W_{n-1} & \Delta_{n-1} & V_{n-1} & 0 & 0 & 0 \\ 0 & V_{n-1} & \Delta_{n-1} & W_{n-1} & 0 & 0 \\ 0 & 0 & W_{n-1} & \Delta_{n-1} & V_{n-1} & 0 \\ 0 & 0 & 0 & V_{n-1} & \Delta_{n-1} & W_{n-1} \\ V_{n-1} & 0 & 0 & 0 & W_{n-1} & \Delta_{n-1} \end{pmatrix}$$

for $n > 0$ *and* $W_n = W_{n-1} \otimes K$*,* $V_n = V_{n-1} \otimes K$*.*

Proof. First observe by the above that we have the following representation of the adjacency matrix of $\Gamma^{(2)}(\mathrm{sd}^n\Delta_2)$ (up to loops);

$$A(G_n) = t_n + \tau_n + b_n = \begin{pmatrix} \tau_{n-1} & 1 & 0 & 0 & 0 & 1 \\ 1 & \tau_{n-1} & 1 & 0 & 0 & 0 \\ 0 & 1 & t_{n-1} & 1 & 0 & 0 \\ 0 & 0 & 1 & t_{n-1} & 1 & 0 \\ 0 & 0 & 0 & 1 & b_{n-1} & 1 \\ 1 & 0 & 0 & 0 & 1 & b_{n-1} \end{pmatrix},$$

where $\tau_{n-1} = \tau_0 \otimes 1_{6^{n-2}}$ and $t_{n-1} = t_0 \otimes 1_{6^{n-2}}$ with τ_0 and t_0 being the permutation matrix representations of τ and t. Furthermore b_{n-1} is given by the block diagonal matrix

$$b_{n-1} = \mathrm{diag}(\tau_{n-2}, \tau_{n-2}, t_{n-2}, t_{n-2}, b_{n-2}, b_{n-2})$$

with $b_0 = 1_6$.

Note that since when writing $A(G_n)$ as a block matrix of 6×6 blocks the number of b_0 blocks vanishes compared to the total matrix (see proof of Theorem I.37 for example) the choice of the initial condition b_0 does not matter for the asymptotic spectrum in L^1 (see proof of universal limit theorem). We denote the matrix sequence obtained in the same way as $A(G_n)$ but with $b_0 = 0$ by A_n and note that the spectral limit of these sequences agree.

In terms of the Kronecker product this matrix sequence can be written as

$$A_n = W_0 \otimes \tau_{n-1} + V_0 \otimes t_{n-1} + K \otimes b_{n-1} + \tau_0 \otimes 1_{6^{n-1}} + t_0 \otimes 1_{6^{n-1}}.$$

Note that b_{n-1} can be written as

$$b_{n-1} = W_0 \otimes \tau_{n-2} + V_0 \otimes t_{n-2} + K \otimes b_{n-2} = W_0 \otimes \tau_0 \otimes 1_{6^{n-3}} + V_0 \otimes t_0 \otimes 1_{6^{n-3}} + K \otimes b_{n-2}$$

and thus by an argument over induction (with $b_0 = 0$) we obtain

$$b_{n-1} = \sum_{i=0}^{n-3} K^{\otimes i} \otimes (W_0 \otimes \tau_0 + V_0 \otimes t_0) \otimes 1_{6^{n-i-3}}.$$

For A_n this means

$$A_n = \sum_{i=0}^{n-2} K^{\otimes i} \otimes (W_0 \otimes \tau_0 + V_0 \otimes t_0) \otimes 1_{6^{n-i-2}} + \tau_0 \otimes 1_{6^{n-1}} + t_0 \otimes 1_{6^{n-1}}.$$

In particular if T_n denotes the map exchanging the first $n-1$ scales of tensor products with the last scale, i.e.

$$T_n(v_1 \otimes \ldots \otimes v_n) = v_n \otimes v_1 \otimes \ldots \otimes v_{n-1},$$

we have

$$\begin{aligned} T_n A_n T_n^{-1} &= 1_6 \otimes \Big(\tau_0 \otimes 1_{6^{n-2}} + t_0 \otimes 1_{6^{n-2}} + \sum_{i=0}^{n-3} K^{\otimes i} \otimes (W_0 \otimes \tau_0 + V_0 \otimes t_0) \otimes 1_{6^{n-i-3}} \Big) \\ &\quad + \tau_0 \otimes K^{\otimes i} \otimes W_0 + t_0 \otimes K^{\otimes i} \otimes V_0 \\ &= 1_6 \otimes A_{n-1} + \tau_0 \otimes K^{\otimes i} \otimes W_0 + t_0 \otimes K^{\otimes i} \otimes V_0. \end{aligned}$$

Applying the same transforms $T_{n-1}, ..., T_2$ now on the last $n-1, ..., 2$ scales and letting S_n denote this operator we obtain for the sequence of matrices $\Delta_n := S_n A_n S_n^{-1}$:

$$\Delta_n = 1_6 \otimes \Delta_{n-1} + \tau_0 \otimes W_0 \otimes K^{\otimes i} + t_0 \otimes V_0 \otimes K^{\otimes i}$$

which is precisely the claimed matrix sequence. That their spectra agree is immediate from the similarity of the matrices via S.

□

Remark 2. We can argue the same way for an ordering $\iota_\sigma(e)$ in the proof of Theorem I.37 which induces opposite orientations on each edge. In this case the group is generated by a cyclic group element a of order 6 generating the hexagon with set of generators $\{a, a^t\}$, i.e. $a(wx) := w(x+1)$ for $x < 6$ and $a(w6) = w1$ and the element b defined as

$$b(wx) = \begin{cases} a(w)4 & , \ x = 1 \\ a(w)3 & , \ x = 2 \\ a^t(w)2 & , \ x = 3 \\ a^t(w)1 & , \ x = 4 \\ b(w)6 & , \ x = 5 \\ b(w)5 & , \ x = 6 \end{cases}.$$

This group will give another matrix recursion in Corollary I.39 of the form

$$\begin{pmatrix} \Delta_{n-1} & W_{n-1} & 0 & 0 & 0 & W_{n-1}^t \\ W_{n-1}^t & \Delta_{n-1} & W_{n-1} & 0 & 0 & 0 \\ 0 & W_{n-1}^t & \Delta_{n-1} & W_{n-1} & 0 & 0 \\ 0 & 0 & W_{n-1}^t & \Delta_{n-1} & W_{n-1} & 0 \\ 0 & 0 & 0 & W_{n-1}^t & \Delta_{n-1} & W_{n-1} \\ W_{n-1} & 0 & 0 & 0 & W_{n-1}^t & \Delta_{n-1} \end{pmatrix}$$

for $W_{n-1} = W_{n-1} \otimes K$ with

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 3. The Schreier graph approximation for higher dimensional barycentric subdivision should be feasible in a similar way; the intuition of why the approximation via the group generated by τ and t is the more natural choice rather than the group generated by a and b is that τ is the group element which generates the "internal" edges of the subdivision of the boundary edge to which the two facets $e_1 * v_\sigma$ and $e_2 * v_\sigma$ are incident. In a similar way by how barycentric subdivision is constructed inductively for higher dimensional input complexes we subdivide the boundary $\partial\sigma$ of σ and take the cone over it. The internal incidences are thus just given by the incidences of $d+1$ subdivided $(d-1)$-simplices joined together pairwise via the outwards pointing group element b of one dimension lower. The incidence of σ and σ' is then given by "bridging" these copies of the dual graphs of subdivided $(d-1)$-simplices (which is what b does in $d = 2$).

Example: Edgewise Subdivision for $d = 2$

We have now introduced one finitely ramified example where the determination of limiting distribution is cumbersome but still possible and have demonstrated a non-finitely ramified example where the process fails. However there are highly symmetrical non-finitely ramified subdivisions of which we still can determine the limiting distribution. The edgewise subdivision is one such case. The method we use in this section to calculate the universal limit of 2-dimensional edgewise subdivision is parallel to the method of determining the integrated density of states of archimedean tilings of $\mathbb{R}^2$ in [PT21]. We can apply this procedure to our needs since the honeycomb tiling (6^3) (see [PT21]) is the dual of a tiling of $\mathbb{R}^2$ by triangles which is invariant under edgewise subdivision in the sense that subdividing corresponds to a scaling of the tiling (see Experiments section for an illustration).

Note however that we have to adjust the procedure since we only work on finite complexes and so this invariant tiling is not feasible as it is an infinite simplicial complex. Note also that it is not immediate that the spectrum of a sequence of complexes growing towards an infinite complex converges to the integrated density of states of the infinite complex. We will show however that in this particular constellation it does.

We will now construct a triangulation T_1 of the torus T^2 which has a $C_r \times C_r$ action on it and so that $T_n := \mathrm{esd}_r^{n-1} T_1$ has a $C_{r^n} \times C_{r^n}$ action on it. The following is the Floquet theory of Section 2 in [PT21] in our (finite) setting.

Assume we have a finite graph $G = (V, E)$ and C_k^d acts simply transitive on it. Let F be a fundamental domain of the action. We will always denote by C_k its additive representative and by Ξ_k we denote the set of k-th roots of unity. Denote for $\eta \in \Xi_k^d$

$$\ell^2(V)_\eta := \{f : V \to \mathbb{C} \mid \forall_{v \in V} : \ f(\gamma v) = \prod_{i=1}^{d} \eta_i^{\gamma_i} f(v)\}$$

with inner product

$$\langle f, g \rangle_\eta := \sum_{v \in F} f(v) \overline{g(v)}.$$

For convenience we will denote

$$\eta^\gamma := \prod_{i=1}^{d} \eta_i^{\gamma_i}.$$

We further define on $\ell^2(V)_\eta$ the Laplacian Δ^η the same way as on G but with domain restricted to $\ell^2(V)_\eta$.

Since $f \in \ell^2(V)_\eta$ only has $\#F$ degrees of freedom this operator can be viewed as a $\#F \times \#F$-matrix. Note now that by the conditions on the action we can decompose

$$\ell^2(V) = \bigoplus_{v \in F} \ell^2(C_k^d)$$

where the tuple $(f_v)_{v \in F}$ corresponds to the function $f(\gamma v) := f_v(\gamma)$ (note that γ is unique by the action being free and transitive).

Note in particular that this map is an isometry (since the inner product in $\ell^2(V)$ simply sums over the inner products of each component). We now apply the fourier transform to each component to obtain for $f = (f_v)_{v \in F}$ a function

$$\hat{f} \in \bigoplus_{v \in F} \ell^2(\Xi_k^d),$$

with $\hat{f} = (\hat{f}_v)_{v \in F}$ and

$$\hat{f}_v(\eta) := \sum_{\gamma \in C_k^d} \eta^{-\gamma} f_v(\gamma).$$

Note that the irreducible complex representations of the cyclic group are the roots of unity. In praticular $f \mapsto \hat{f}$ is an isometry with norms

$$||f||_{\ell^2(V)} = \sum_{v \in V} |f(v)|^2 = \sum_{v \in F} \sum_{\gamma \in C_k^d} |f(\gamma v)|^2$$

and

$$||\hat{f}||_{\bigoplus_{v \in F} \ell^2(\Xi_k^d)} := \sum_{v \in F} ||\hat{f}_v||^2_{\ell^2(\Xi_k^d)}$$

for

$$||g||_{\ell^2(\Xi_k^d)} = \frac{1}{k^d} \sum_{\eta \in \Xi_k^d} |g(\eta)|^2.$$

We switch parameters and thus write

$$\tilde{f}_\eta(v) := \hat{f}_v(\eta)$$

and extend this function to a function in $\ell^2(V)_\eta$ via

$$\tilde{f}_\eta(\gamma v) = \eta^\gamma \tilde{f}_\eta(v)$$

for $v \in F$. Thus we have

$$\ell^2(V) \cong \bigoplus_{\eta \in \Xi_k^d} \ell^2(V)_\eta.$$

Now observe that for $v = \gamma v_0$, $v_0 \in F$, we have

$$\begin{aligned}
\tilde{f}_\eta(v) &= \tilde{f}_\eta(\gamma v_0) \\
&= \eta^\gamma \tilde{f}_\eta(v_0) \\
&= \eta^\gamma \hat{f_{v_0}}(\eta) \\
&= \sum_{\alpha \in C_k^d} \eta^{\gamma - \alpha} f(\alpha v_0) \\
&= \sum_{\alpha \in C_k^d} \eta^{-(\alpha - \gamma)} f((\alpha - \gamma)\gamma v_0) \\
&= \sum_{\beta \in C_k^d} \eta^\beta f(\beta v).
\end{aligned}$$

Remember that the Laplacian operator Δ of G is defined as

$$\Delta f(v) = \sum_{w \sim v} (f(v) - f(w)).$$

Thus it holds that $\Delta f = g$ if and only if

$$\begin{aligned}
\tilde{g}_\eta(v) &= \sum_{\gamma \in C_k^d} \eta^\gamma g(\gamma v) \\
&= \sum_{\gamma \in C_k^d} \eta^\gamma \Big(\deg(\gamma v) f(\gamma v) - \sum_{w \sim \gamma v} f(w) \Big) \\
&= \deg v \tilde{f}_\eta(v) - \sum_{\gamma \in C_k^d} \eta^\gamma \sum_{w \sim v} f(\gamma w) \\
&= \deg v \tilde{f}_\eta(v) - \sum_{w \sim v} \tilde{f}_\eta(w) \\
&= \Delta^\eta \tilde{f}_\eta(v),
\end{aligned}$$

so that

$$\Delta = \bigoplus_{\eta \in \Xi_r^d} \Delta^\eta$$

under the identification $\ell^2(V) \cong \bigoplus_{\eta \in \Xi_k^d} \ell^2(V)_\eta$. In particular for the normalized eigenvalue counting function

$$F_\Delta(x) = \frac{1}{v(G)} \#\{i \in [v(G)] \mid \lambda_i(\Delta) \leq x\}$$

it thus holds that

$$F_\Delta(x) = \frac{1}{v(G)} \sum_{\eta \in \Xi_k^d} \#\{i \in [v(F)] \mid \lambda_i(\Delta^\eta) \leq x\}.$$

By $v(G) = \#\Xi_k^d \cdot v(F)$ we have concluded the following theorem which is a discrete version of [PT21, Theorem 2.1].

Theorem I.40.

$$F_\Delta(x) = \frac{1}{k^d} \sum_{\eta \in \Xi_k^d} F_{\Delta^\eta}(x).$$

Now that we have Theorem I.40 at hand we can determine the universal limit of the edgewise subdivision for $d = 2$. Let $r > 1$ be fixed and let $T_1(r)$ denote the triangulation of the torus T^2 as in Figure 3.7. Note that it is not really a triangulation of the torus in the simplicial sense when $r = 2$. In this case we just set r to be 4 which gives us the same limit so that we can assume without loss of generality that $r > 2$.

It is immediate that C_r^2 acts simply transitively on this subdivision by rotating rows and columns. In particular since after subdivision

$$T_2(r) := \mathrm{esd}_r T_1(r) = T_1(r^2)$$

we have a $C_{r^2}^2$ action on $T_2(r)$. Inductively we obtain a sequence of triangulations of the torus

$$T_{n+1}(r) := \mathrm{esd}_r^n T_1(r) = T_1(r^n)$$

each with a $C_{r^n}^2$ action. Note that this action induces a simply transitive action on the dual graph $G_n = (V_n, E_n)$. Furthermore by definition of the action its fundamental domain looks

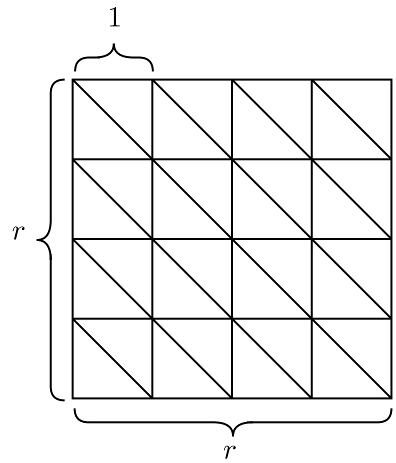


Figure 3.7: The triangulation $T_1(r)$ of T^2.

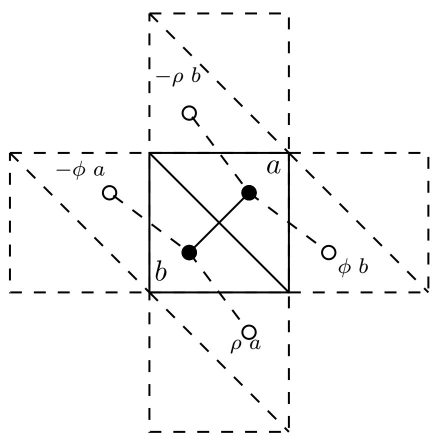


Figure 3.8: Fundamental domain of G_n. The adjacent vertices are indicated with their respective transitive group element.

like Figure 3.8. Let ρ_n and ϕ_n denote generators of C_{r^n} such that the neighborhoods of a and b in F are $\{b, \phi b, \overline{\rho} b\}$ and $\{a, \rho a, \overline{\phi} a\}$.

We determine the spectrum of $\Delta^\eta(G_n)$ now. Let $f \in \ell^2(V_n)_\eta$, i.e. f is defined by the choices $f(a) = \alpha$, $f(b) = \beta$ via

$$f(\gamma a) = \eta^\gamma \alpha$$

and

$$f(\gamma b) = \eta^\gamma \beta$$

for $\gamma \in C^2_{r^n}$. In particular Δ^η has a matrix representation of the following form:

$$\Delta^\eta = \begin{pmatrix} 3 & -(1 + \overline{\eta_1} + \eta_2) \\ -(1 + \eta_1 + \overline{\eta_2}) & 3 \end{pmatrix}.$$

Which (unsurprisingly) coincides with the matrix representation of Δ^θ for the (6^3)-tiling of [PT21] for $\eta_1 = e^{-i\theta_1}$ and $\eta_2 = e^{i\theta_2}$ (after scaling by $1/3$, i.e. normalizing the Laplacian). We will denote this pair by $\eta(\theta)$ for now. Note also that the eigenvalues of this matrix are determined in [PT21]; since they are continuous functions in the parameter $\theta \in T^2$ the functions $F_{\Delta^{\eta(\theta)}}(x)$ are Riemann integrable in θ for fixed x.

Note that the points $\eta \in \Xi^2_{r^n}$ form a mesh on $T^2 = S^1 \times S^1$, where $f:\ \theta \mapsto (e^{-i\theta_1}, e^{i\theta_2})$ is a parameterization of T^2 on $[0, 2\pi) \times [0, 2\pi)$. We have

$$X^2_{r^n} := f^{-1}(\Xi^2_{r^n}) = \{0, 2\pi/r^n, ..., 2\pi(r^n-1)/r^n\}^2 \subseteq \mathbb{R}^2 \Big/ (2\pi\mathbb{Z})^2$$

so that Theorem I.40 can be written as

$$F_\Delta(x) = \frac{1}{(2\pi)^2}\frac{(2\pi)^2}{r^{2n}} \sum_{\theta \in X^2_{r^n}} F_{\Delta^{f(\theta)}}(x)$$

which obviously as a Riemann sum over a Riemann integrable function

$$\mathbb{R}^2 \Big/ (2\pi\mathbb{Z}) \ni \theta \mapsto F_{\Delta^{f(\theta)}}(x)$$

converges point-wise to

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2 \big/ (2\pi\mathbb{Z})^2} F_{\Delta^{f(\theta)}}(x)\,\mathrm{d}\theta.$$

Which (after a a parameter scaling) is determined in [PT21] as

$$F_\Delta(x) \to F(x) := \begin{cases} 0 & , \text{ if } E < 0 \\ \int_{1-\frac{1}{2}E}^{1} g(t)\,\mathrm{d}t & , \text{ if } 0 \le E < 2 \\ \frac{1}{2} - \int_{-2+\frac{3}{2}E}^{1-\frac{3}{2}E} g(t)\,\mathrm{d}t & , \text{ if } 2 \le E < 3 \\ \frac{1}{2} + \int_{1-\frac{3}{2}E}^{-2+\frac{3}{2}E} g(t)\,\mathrm{d}t & , \text{ if } 3 \le E < 4 \\ 1 - \int_{1-\frac{3}{2}E}^{1} g(t)\,\mathrm{d}t & , \text{ if } 4 \le E < 6 \\ 1 & , \text{ if } 6 \le E \end{cases}$$

for $g(t) = \frac{1}{\pi^2}\frac{\arccos\left(\frac{(3-E)^2-1}{4t} - t\right)}{\sqrt{1-t^2}}$.

Note also that this function is strictly increasing so that its quantile function Q_Δ is continuous. In particular the point-wise convergence $F_\Delta(x) \to F(x)$ thus implies the point-wise convergence $Q_\Delta(u) \to Q(u)$, where Q is the quantile function of F. We have thus shown:

Theorem I.41. *The universal limit of edgewise subdivision for $d = 2$ is Q, i.e. the quantile function of F.*

Compare e.g. [PT21, Figure 5] and Figure 3.9. Note further that the universal limit does not depend on the subdivision parameter r.

Remark 4. Note that the same procedure can be applied to a $(C_{r^n})^d$ action on the triangulation $T^d(r^n)$ of the d-dimensional torus T^d given by a $r \times ... \times r$ grid of copies of triangulations of the cube $Q^d = [0,1]^d$ with faces of the $[0,r]^d$-cube identified according the identification rule of T^d. The triangulation of the cube Q^d can be chosen so that $\mathrm{esd}_r T^d(r^n) = T^d(r^{n+1})$ and a limit can thus be calculated by an integral of the eigenvalue counting functions of the triangulations of this base cube (so that the universal limit is solely calculable by the spectrum

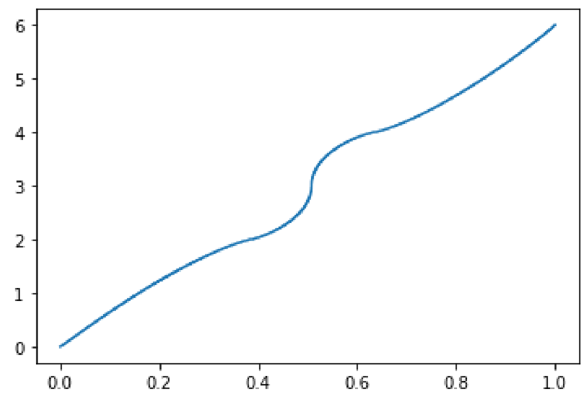


Figure 3.9: $\Lambda(\mathrm{esd}^5 \Delta_2)$

of Q^d and the action of T^d on T^d as above). In particular the limiting distribution again is independent of r.

Note however that the higher-dimensional edgewise subdivision does not really fit our framework as it is not inclusion-uniform; for every $\sigma \in K$ its subdivision depends on a vertex ordering of σ. Thus $\mathrm{esd}_r(K)$ is really only well-defined for vertex-ordered complexes, $\mathrm{esd}_r(K, \leq)$. Note however that $\mathrm{esd}_r(K, \leq)$ can be ordered in such a way that

$$\mathrm{esd}_s \mathrm{esd}_r(K, \leq) = \mathrm{esd}_{s+r}(K, \leq)$$

and so a well-defined limiting distribution of the sequence

$$\mathrm{esd}_r^n(K, \leq) = \mathrm{esd}_{r^n}(K, \leq)$$

exist by the argument as used to prove Theorem I.1.

This limit however is not (necessarily) the same for different orderings $\leq, \leq'$ of $F_0(K)$ so that no universal limit exists in this case (in the sense that for every two complexes the limit only depends on the dimension). Note that for $d \leq 2$ we have that esd_r does not depend on an ordering $\leq$ of the vertices and so for these cases universality holds.

Relations to Fractal Theory

Given a inclusion-uniform subdivision div we associate to it the sequence $(\Gamma_n)_{n\in\mathbb{N}} = (\Gamma^{(d)}(\mathrm{div}^n\Delta_d))_{n\in\mathbb{N}}$ of simple graphs. This sequence can be considered self-similar when relaxing known constructions of graph-directed self-similar sets. This relaxation has to be made to common definitions since fractals associated to inclusion-uniform subdivisions are not finitely-ramified in general and thus the orientation matters when joining graphs. In the following construction we will compensate for the ambiguity introduced by dependence on orientation. Note however that all considered examples can be oriented in a more convenient way - thus also giving rise to a Schreier graph approximation of the sequence.

The construction which dualizes iterated subdivision by a inclusion-uniform operation as a graph-sequence approximation of a self-similar set approximation is the following:

The data: Let $d, n, N \in \mathbb{N}$. We start with an initial graph Γ_0 on vertex set $\{1, ..., N\}$ with degrees bounded by $d+1$ and a dedicated S_{d+1}-action on it. We will formally let S_{d+1} act on $\{0, ..., d\}$ for the purpose of this construction and denote action of $\sigma \in S_{d+1}$ on $i \in V(\Gamma_0) = \{1, ..., N\}$ by σi opposed to the evaluation $\sigma(i)$ at $i \in \{0, ..., d\}$. $\sigma_* : E(\Gamma_0) \to E(\Gamma_0)$ denotes the push forward action on the edge set, i.e.

$$\sigma_*\{v, w\} := \{\sigma v, \sigma w\} \in E(\Gamma_0)$$

for $\{v, w\} \in E(\Gamma_0)$.

Further let $\partial_i\Gamma_0$, $i = 0, ..., d$, denote $(d+1)$-many dedicated n-element boundary sets of Γ_0 such that $\partial\Gamma_0 := \bigcup_{i=0}^{d} \partial_i\Gamma_0$ is the set of vertices with degree $< d+1$ in Γ_0. Furthermore for every vertex $v \in \partial\Gamma_0$ we require that

$$\deg v + \underbrace{\#\{i \in \{0, ..., d\} \mid v \in \partial_i\Gamma_0\}}_{=:b_v} = d+1.$$

We further want the above sets $\partial_i\Gamma_0$ to be compatible with the group action in the sense that

$$\partial_i\Gamma_0 = \tau_{ij}\partial_j\Gamma_0 \tag{$*$}$$

for the transposition $\tau_{ij} = (i\ j)$ and the set $\partial_i\Gamma_0$ has to be invariant under $S_d^i = \{\sigma \in S_{d+1} \mid \sigma(i) = i\}$, $i = 0, ..., d$.

In order to make Γ_0 $(d+1)$-regular we add b_v many loops to the vertices $v \in \partial\Gamma_0$ and denote the loop added to $v \in \partial_i\Gamma_0$ for the i-th boundary by $\ell_v(i)$. The action of S_{d+1} extends to this graph in a natural way by

$$\sigma_*\ell_v(i) := \ell_{\sigma v}(\sigma(i)).$$

We will thus denote the graph obtained by this addition as Γ_0 from now on.

For every loop $e = \ell_v(i) \in E(\Gamma_0)$ let $\kappa_v(e) := i$ and for every edge $e = \{v, w\} \in E(\Gamma_0)$ choose $\kappa_v(e), \kappa_w(e) \in \{0, ..., d\}$ such that

$$\{\kappa_v(e) \mid v \in e \in E(\Gamma_0)\} = \{0, ..., d\}.$$

Further let $\rho_{ij} : \{0, ..., d\} \to \{0, ..., d\}$ be a bijection such that

$$\rho_{ij}(\kappa_i(e)) = \kappa_j(e)$$

and $\rho_{ji} := \rho_{ij}^{-1}$ aswell as the compatibility with the S_{d+1}-action as $\rho_{\sigma i\ \sigma j} = \nu_{\sigma,j}\rho_{ij}\nu_{\sigma,i}^{-1}$, where $\nu_\sigma \in S_{d+1}$ is given as

$$\nu_{\sigma,i}(\kappa_i(e)) := \kappa_{\sigma i}(\sigma_* e)$$

The construction: Having this information fixed we construct a self-similar sequence $(\Gamma_k)_{k\in\mathbb{N}}$. Assume Γ_{k-1} and $\partial_i\Gamma_{k-1}$ with a S_{d+1}-action sufficing the same conditions as the action on Γ_0 are constructed. Then we construct Γ_k as a graph on vertex set $V(\Gamma_k) = V(\Gamma_0) \times V(\Gamma_{k-1}) = [N]^{k+1}$ and denote a vertex by a pair (i, v) for $i \in V(\Gamma_0)$ and $v \in V(\Gamma_{k-1})$. We include the edge $\{(i, v), (j, w)\} \in E(\Gamma_k)$ if one of the following two conditions is met:

- Either $i = j$ and $\{v, w\} \in E(\Gamma_{k-1})$ or
- $i \neq j$, $e = \{i, j\} \in E(\Gamma_0)$, $v \in \partial_{\kappa_i(e)}\Gamma_{k-1}$, $w \in \partial_{\kappa_j(e)}\Gamma_{k-1}$ and

$$\rho_{ij}v = w.$$

Furthermore we include the loops at vertices (i, v) for every loop at v in Γ_{k-1}.

We will now show that Γ_k again admits an S_{d+1}-action and construct sets $\partial_i\Gamma_k$. The sets $\partial_i\Gamma_k$ are given by

$$\partial_i\Gamma_k := \partial_i\Gamma_{k-1} \times \partial_i\Gamma_0 = (\partial_i\Gamma_0)^{k+1}.$$

The graph Γ_k can be assigned an S_{d+1}-action as follows:

$$\sigma \cdot (i, v) := (\sigma i, \nu_{\sigma,i}v),$$

where $\nu_{\sigma,i}$ as above is the permutation in S_{d+1} given by $\nu_{\sigma,i} : \kappa_i(e) \mapsto \kappa_{\sigma i}(\sigma_* e)$ for all $e \in E(\Gamma_0)$ with $i \in e$. We will now show that this action is compatible with the multiplication of S_{d+1}. To this end note that $\nu_{\mathrm{id},i}(\kappa_i(e)) = \kappa_i(e)$ and thus $\nu_{\mathrm{id},i} = \mathrm{id}$. Let $\sigma, \tau \in S_{d+1}$ then

$$\nu_{\sigma\tau,i}(\kappa_i(e)) = \kappa_{(\sigma\tau)i}((\sigma\tau)_* e) = \kappa_{\sigma(\tau i)}(\sigma_*(\tau_* e)) = \nu_{\sigma,\tau i}(\kappa_{\tau i}(\tau_* e)) = \nu_{\sigma,\tau i}(\nu_{\tau,i}(\kappa_i(e)))$$

such that $\nu_{\sigma\tau,i} = \nu_{\sigma,\tau i}\nu_{\tau,i}$.

In particular we have

$$(\sigma\tau)(i, v) = ((\sigma\tau)i, \nu_{\sigma\tau,i}v) = (\sigma(\tau i), \nu_{\sigma,\tau i}(\nu_{\tau,i}v)) = \sigma(\tau i, \nu_{\tau,i}v) = \sigma(\tau(i, v)).$$

Thus we obtain a well-defined action of S_{d+1} on $V(\Gamma_k)$. We will show that it preserves edge relations, which is immediate for edges of the first type.

Assume we have an edge of the second type, i.e. $e = \{(i, v), (j, w)\}$ and $i \neq j$, $e' = \{i, j\} \in E(\Gamma_0)$, $v \in \partial_{\kappa_i(e')}\Gamma_{k-1}$, $w \in \partial_{\kappa_j(e')}\Gamma_{k-1}$ and

$$\rho_{ij}v = w.$$

Let $\sigma \in S_{d+1}$ be given. Obviously $\sigma i \neq \sigma j$ and $\sigma_* e' \in E(\Gamma_0)$. Since for $\ell, \ell' \in \{0, ..., d\}$ $\partial_\ell \Gamma_{k-1}$ are invariant under S_d^ℓ and $\partial_\ell \Gamma_{k-1} = \tau_{\ell\ell'} \partial'_\ell \Gamma_{k-1}$ we have

$$\sigma \partial_\ell \Gamma_{k-1} = ((\ell\ \sigma(\ell)) \circ \tau) \partial_\ell \Gamma_{k-1} = \partial_{\sigma(\ell)} \Gamma_{k-1}$$

for $\tau \in S_d^\ell$ given as $\tau(p) = \sigma(p)$ if $p \notin \{\ell, \sigma^{-1}(\ell)\}$ and $\tau(\ell) = \ell$, $\tau(\sigma^{-1}(\ell)) = \sigma(\ell)$. Denote now $\ell = \kappa_i(e')$, $\ell' = \kappa_j(e')$ so that $\nu_{\sigma,i} v \in \partial_{\nu_{\sigma,i}(\ell)} \Gamma_{k-1}$ and $\nu_{\sigma,j} w \in \partial_{\nu_{\sigma,j}(\ell')} \Gamma_{k-1}$. Thus the last condition to show is that

$$\rho_{\sigma i\ \sigma j}(\nu_{\sigma,i}(v)) = \nu_{\sigma,j}(w).$$

This is by definition equivalent to

$$(\nu_{\sigma,j}^{-1} \rho_{\sigma i\ \sigma j} \nu_{\sigma,i}) v = w = \rho_{ij} v.$$

But by the condition on ρ_{ij} we have

$$\rho_{\sigma i\ \sigma j} = \nu_{\sigma,j} \rho_{ij} \nu_{\sigma,i}^{-1}$$

showing the claim.

So that all conditions for Γ_k are met in order to iteratively define the next graph in the sequence.

Duality to iterated subdivision: Let div be a inclusion-uniform subdivision acting on d-dimensional complexes. We apply the above construction with the given d, $n = f_{d-1}(\mathrm{div})$ and $N = f_d(\mathrm{div})$. We can equip the d-dual graph of the subdivision of Δ_d,

$$\Gamma_0 := \Gamma^{(d)}(\mathrm{div}\Delta_d),$$

with an S_{d+1}-action by the condition of being inclusion-uniform; let $\sigma \in S_{d+1}$ and $K = \Delta_d$ in Definition I.10; then σ canonically defines a bijective vertex-identification

$$\sigma : F_0(\Delta_d) \to F_0(\Delta_d)$$

and thus extends to a (geometric) simplicial isomorphism of

$$\tilde{\sigma} : \mathrm{div}\Delta_d \to \mathrm{div}\Delta_d$$

sufficing $\tilde{\sigma}(\{i\}) = \{\sigma(i)\}$. Note that $\tilde{\sigma}$ defines an action on Γ_0 by

$$\sigma\tau := \tilde{\sigma}(\tau)$$

for $\tau \in V(\Gamma_0) = F_d(\mathrm{div}\Delta_d)$. Obviously this action preserves the edge-relation since $\tilde{\sigma}$ is a simplicial isomorphism.

We enumerate the facets $F_d(\mathrm{div}\Delta_d) = \{\tau_1, ..., \tau_N\}$ and assume τ_i corresponds to the vertex $i \in V(\Gamma_0)$.

In order to define the boundary sets let $\sigma_i = (0, ..., \hat{i}, ..., d) \in F_{d-1}(\Delta_d)$ and let

$$\Sigma_i := F_{d-1}(\mathrm{div}_{\Delta_d} \sigma_i),$$

where $s : \mathrm{div}\Delta_d \to \Delta_d$ denotes the subdivision map. Then let

$$\partial_i \Gamma_0 := \{\tau \in F_d(\mathrm{div}\Delta_d) \mid F_{d-1}(\tau) \cap \Sigma_i \neq \emptyset\}.$$

Obviously since the isomorphism is geometric it has to restrict to an isomorphism on the boundaries; thus $\tilde{\tau}_{ij}$ is mapping Σ_i to Σ_j and so it maps the respective unique facets of the $(d-1)$-faces in Σ_i to their counterparts in Σ_j, i.e.

$$\partial_i \Gamma_0 = \tau_{ij} \partial_j \Gamma_0.$$

Furthermore assuming $\sigma \in S_d^i$ we have that it leaves σ_i invariant as a set of vertices so that $\sigma \Sigma_i = \Sigma_i$ and analogously

$$\sigma \partial_i \Gamma_0 = \partial_i \Gamma_0.$$

The labels $\kappa_i(e)$, $i \in [N]$, can be chosen at will respecting the condition that $\kappa_i(\ell_i(j)) = j$ if $i \in \partial_j \Gamma_0$.

We now identify an edge $\{i, j\} \in E(\Gamma_0)$ with the face $\tau_i \cap \tau_j \in F_{d-1}(\mathrm{div}\Delta_d)$ for their respective facets $\tau_i, \tau_j \in F_d(\mathrm{div}\Delta_d)$. The bijections ρ_{ij}, $\{i, j\} \in E(\Gamma_0)$, are then given by the maps

$$\rho_{ij}(\kappa_i(\nu)) := \kappa_j(\nu')$$

for $\nu \in F_{d-1}(\tau_i)$ fixed and ν' the unique face in $F_{d-1}(\tau_j)$ such that $\nu \cap \nu' \in F_{d-2}(\tau_i \cap \tau_j)$.

The equation for ρ_{ij} making it compatible with the S_{d+1}-action can be seen as follows; assume $\{i, j\} \in E(\Gamma_0)$ and let $\nu \in F_{d-1}(\tau_i)$ be given and $\nu' \in F_{d-1}(\tau_j)$ be the unique face such that $\nu \cap \nu' \in F_{d-2}(\tau_i \cap \tau_j)$. By definition of the action $\sigma \in S_{d+1}$ acts on Γ_0 as the isomorphism $\tilde{\sigma}$ acts on the facets. Since $\tilde{\sigma}$ is a simplicial isomorphism we have that

$$\tilde{\sigma}(\nu) \cap \tilde{\sigma}(\nu') \in F_{d-2}(\tilde{\sigma}(\tau_i) \cap \tilde{\sigma}(\tau_j)).$$

Furthermore we know that if the edges $e, e' \in E(\Gamma_0)$ correspond to ν, ν', respectively, then the edges corresponding to $\tilde{\sigma}(\nu)$ and $\tilde{\sigma}(\nu')$ are given by $\sigma_* e$ and $\sigma_* e'$ by definition, respectively.

Thus

$$\rho_{\sigma i\ \sigma j}(\nu_{\sigma,i}(\kappa_i(e))) = \rho_{\sigma i\ \sigma j}(\kappa_{\sigma i}(\sigma_* e)) = \kappa_{\sigma j}(\sigma_* e') = \nu_{\sigma,j}(\kappa_j(e')) = \nu_{\sigma,j}(\rho_{ij}(\kappa_i(e))),$$

showing the compatibility with the S_{d+1}-action.

Theorem I.42. *In the above setting it holds that $\tilde{\Gamma}_k \cong \Gamma^{(d)}(\mathrm{div}^{k+1}\Delta_d)$ for all $k \geq 0$, where $\tilde{\Gamma}_k$ results from Γ_k by removing loops.*

In case div *acts non-trivial on d-faces the graph Γ_k contains $(d+1)f_{d-1}(\mathrm{div})^{k+1}$ loops. In particular Γ_k approximates $\Gamma^{(d)}(\mathrm{div}^{k+1}\Delta_d)$.*

Proof. The claim is trivially true for $k = 0$. Thus let $k > 0$ and assume the claim is satisfied for $k - 1$.

Let $i \in [N]$ be fixed for now. Note that the choice of $\kappa_i(e)$ for $e \in E(\Gamma_0)$, $i \in e$, corresponds to an ordering of the vertices of τ_i as follows:

For $e \in E(\Gamma_0)$ with $i \in e$ let $\nu_e \in F_{d-1}(\tau_i)$ denote the face generating e in Γ_0, i.e. if $e = \{i, j\}$ for $j \in [N]$ then $\nu_e = \tau_i \cap \tau_j$ and if $e = \ell_i(j)$ for some $j \in \{0, ..., d\}$ we let ν_e denote the unique face in $\Sigma_j \cap F_{d-1}(\tau_i)$. Now we order the vertices of τ_i in a way compatible with how we ordered the boundary sets - i.e. we let $v \in F_0(\tau_i)$ be at position

$$\kappa_i(e)$$

where $e \in E(\Gamma_0)$, $i \in e$, is the unique edge such that $\nu_e = \tau_i \setminus \{v\}$. This vertex will be denoted $v^i_{\kappa_i(e)}$ from now on such that

$$\tau_i = (v^i_0, ..., v^i_d)$$

is now an ordered simplex.

Now note that

$$\mathrm{div}^{k+1}\Delta_d = \mathrm{div}^k \mathrm{div}\Delta_d.$$

Furthermore note that the subdivision procedure div^k is itself inclusion-uniform and thus for every face $\tau_i = \{v_0, ..., v_d\} \in \mathrm{div}\Delta_d$ and bijection

$$f : \{v_0, ..., v_d\} \to \{0, ..., d\}$$

there exists a unique isomorphism

$$f^{(k)} : \mathrm{div}^k_{\mathrm{div}\Delta_d}\tau_i \to \mathrm{div}^k\Delta_d,$$

such that $f^{(k)}(v_j) = f(v_j)$, $j = 0, ..., d$. In particular the dual graph of $\mathrm{div}^k_{\mathrm{div}\Delta_d}\tau_i$ is isomorphic to Γ_{k-1} by induction hypothesis. Note that the dual graph of $\mathrm{div}^k_{\mathrm{div}\Delta_d}\tau_i$ is the restriction of $\Gamma^{(d)}(\mathrm{div}^{k+1}\Delta_d)$ to the set of facets added in the interior of τ_i.

Thus we take N copies of Γ_{k-1} - one for each facet τ_i. In order to obtain an isomorphism $\tilde{\Gamma}_k \cong \Gamma^{(d)}(\mathrm{div}^{k+1}\Delta_d)$ we only need to show that the edges of second kind in the construction are indeed the edges obtained by gluing the copies of $\mathrm{div}^k\Delta_d$ along their boundaries.

To this end assume two faces $\tau_i, \tau_j \in F_d(\mathrm{div}\Delta_d)$ are given such that $e = \{i, j\} \in E(\Gamma_0)$. By definition they meet in the common face

$$\tau_i \setminus \{v^i_{\kappa_i(e)}\} = \tau_j \setminus \{v^j_{\kappa_j(e)}\}.$$

Note that the action of ρ_{ij} induced on τ_i maps $v^i_{\kappa_i(e)}$ to $v^j_{\kappa_j(e)}$. ρ_{ij} delivers even more; by definition the action of ρ_{ij} on the vertices maps $v^i_{\kappa_i(e')}$ to $v^j_{\kappa_j(e'')}$ whenever the vertices opposed to e' and e'' in i and j, respectively, are geometrically identical (i.e. when they are identified in the gluing process). This can be seen by the $(d-2)$-adjacency of the edges e' and e'' (or rather their generating faces) in the boundary of $\tau_i \cap \tau_j$. In particular if we consider $\mathrm{div}\Delta_d$ to be obtained by a gluing $\mathscr{G}_*(\sigma_1, ..., \sigma_N)$ for N copies of the standard simplex Δ_d with σ_i corresponding to τ_i in the glued complex $\mathscr{G}_*(\sigma_1, ..., \sigma_N) \cong \mathrm{div}\Delta_d$. We will identify σ_i with τ_i by the ordering fixed above; i.e. the canonical inclusion of the i-th standard simplex is given by

$$\iota_i : \ell \mapsto v^i_\ell.$$

Under this identification the restriction of ρ_{ij} to $\{\kappa_i(e') \mid e' \in E(\Gamma_0),\ i \in e'\} \setminus \{\kappa_i(e)\}$ is thus precisely the map

$$\iota_j^{-1} \circ \iota_i$$

and so by definition of the S_{d+1}-action acts as its extended isomorphism

$$\widetilde{\iota_j^{-1} \circ \iota_i}$$

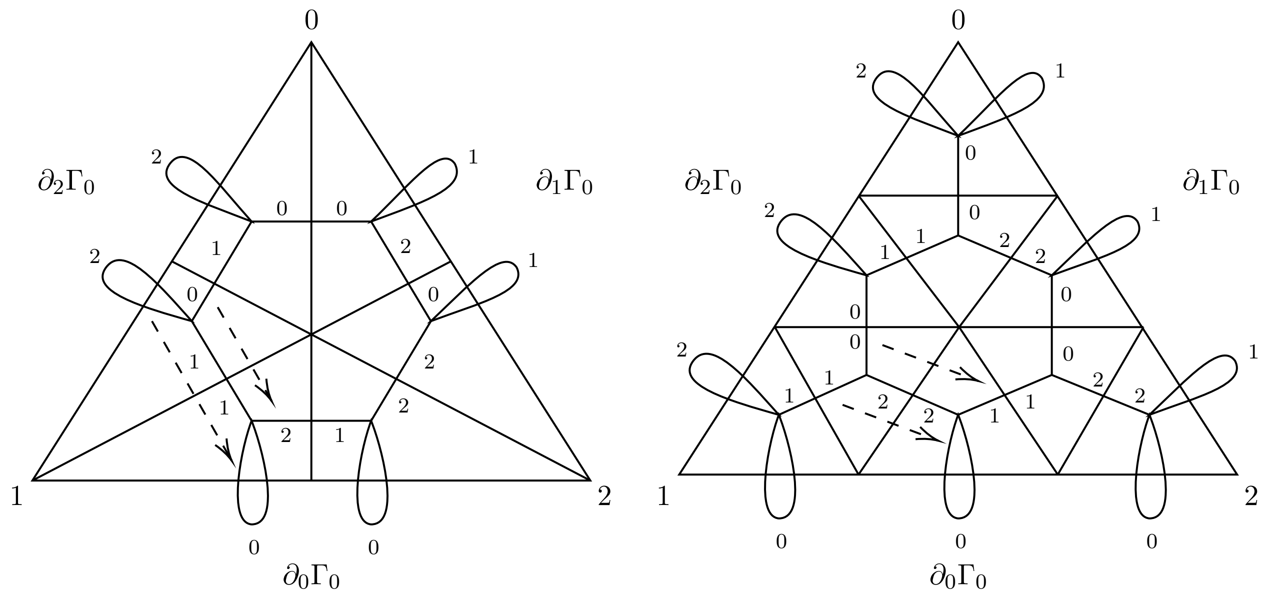
which by equation (2.1) from Section 2 gives exactly the vertex bijection of the relation $\mathscr{G}'$ for obtaining the subdivision as the glued complex

$$\mathscr{G}'_*(\mathrm{div}^k\sigma_1, ..., \mathrm{div}^k\sigma_N)$$

which is isomorphic to the graph $\tilde{\Gamma}_k$ by how ρ_{ij} identifies the boundaries. $\square$

Note that this fractal process can be applied not only to Γ_0 being the dual graph of divΔ_d. Having constructed this sequence of fractals for $\Gamma_0 = \Gamma^{(d)}(\mathrm{div}\Delta_d)$. We can also define fractal sequences on any given pseudo-manifold K in the same fashion by fixing the same maps κ_i and ρ_{ij} (which amounts to chosing an ordering for every simplex in K) and perform gluing along the boundary by utilizing the S_{d+1}-action on the already constructed sequence of subdivided standard simplices. The fractal sequence arising from this is the sequence of dual graphs of the iterated subdivisions of K.

In Figure 4.1 we have illustrated the input data for Γ_0, κ and ρ in order to generate the barycentric or edgewise subdivision (with parameter 3) of a 2-simplex respectively.



(a) Barycentric subdivision div = sd and $d = 2$.

(b) Edgewise subdivision for parameter $r = 3$ and $d = 2$.

Figure 4.1: Two subdivision procedure of infinite ramification and their respective Γ_0 with added loops. The dashed arrow indicates the map ρ_{ij} for a particular edge $\{i, j\}$. The numbers along the edge indicate the values of κ_i associated to the vertex of the edge closer to the label.

In case of finite ramification: The generic case of finite ramification is the case where $n = 1$. This is due to the fact that we call a self-similar set construction of the above type *finitely ramified* if every copy of Γ_{k-1} in Γ_k can be isolated by the removal of a bounded number of edges (independent of k). However the boundaries to be joined have n^{k+1} elements which is only bounded by a constant if $n = 1$.

In this case the above construction reduces to a construction related to a graph sequence approximating a self-similar set in the sense of Sabot, [Sab03]. We assume Γ_0 to be equipped with the enumeration κ_i of edges at every vertex $i \in V(\Gamma_0) = [N]$ and view Γ_0 as generated by a relation $\mathscr{R}$ on the set $[d+1] \times [N]$, i.e. $\mathscr{R}$ is generated by the set of relations

$$(\kappa_i(\{i,j\}), i)\mathscr{R}(\kappa_j(\{i,j\}), j)$$

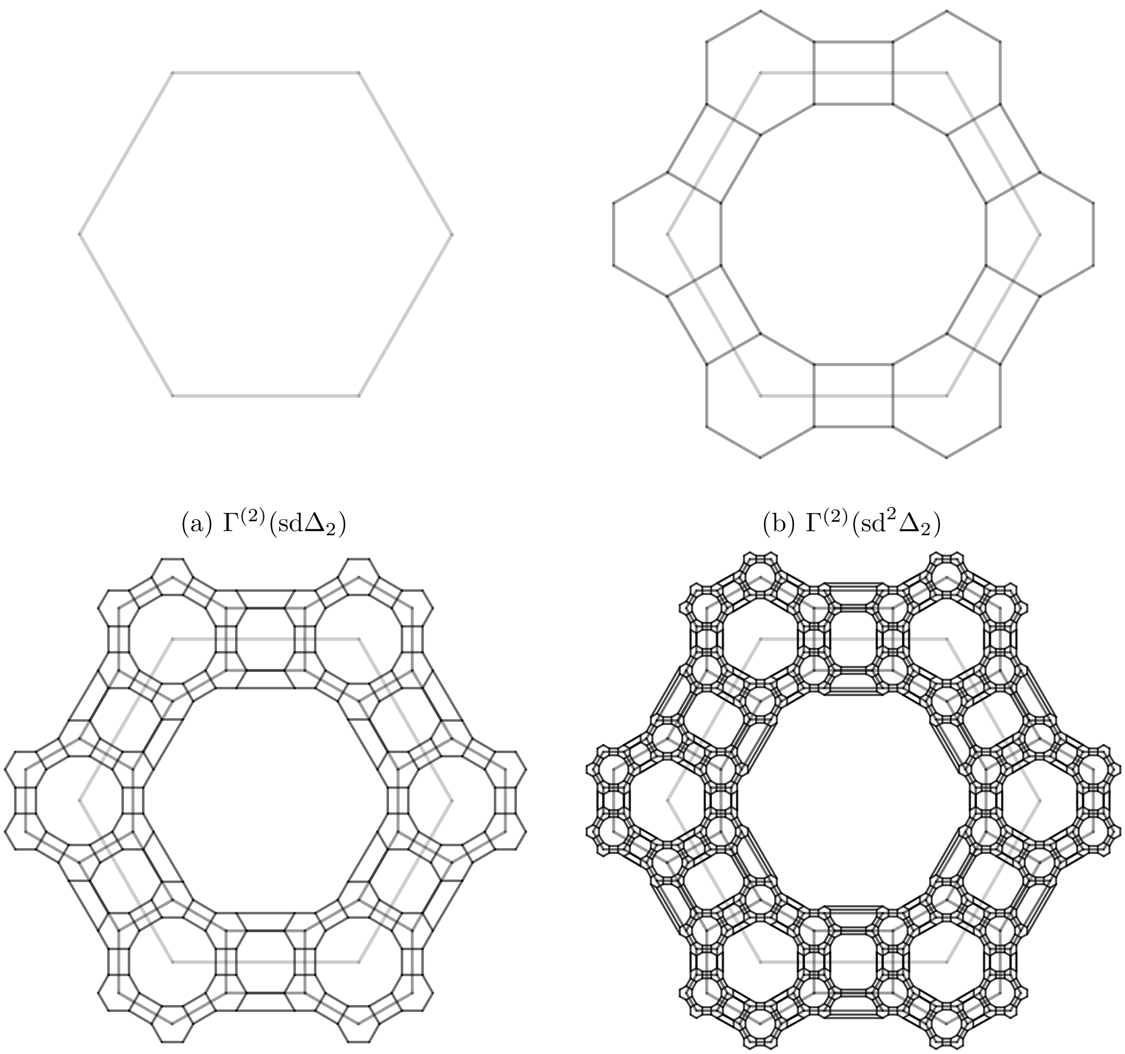
for every $\{i, j\} \in E(\Gamma_0)$.

Since $n = 1$ we can identify $\partial_i \Gamma_{k-1}$ with $\partial_i \Gamma_0$ and thus push the equivalence relation $\mathscr{R}$ from Γ_0 to the k-th level. Let Γ^i_{k-1} denote the i-th copy of Γ_{k-1} in Γ_k. We then join the vertex in $\partial_r \Gamma^i_{k-1}$ with the vertex in $\partial_\ell \Gamma^j_{k-1}$ iff

$$(r, i)\mathscr{R}(\ell, j),$$

i.e. iff $r = \kappa_i(\{i, j\})$, $\ell = \kappa_j(\{i, j\})$ and $\{i, j\} \in E(\Gamma_0)$. Note that ρ_{ij} does not play a role here since the restriction of the isomorphism induced by ρ_{ij} over the S_{d+1}-action on $\partial_{\kappa_i(\{i,j\})}\Gamma_{k-1}$ then just maps this singleton onto the singleton $\partial_{\kappa_j(\{i,j\})}\Gamma_{k-1}$.

The above construction can then be transferred to the setting of Sabot by taking the line graph and adjusting the elementary cell Γ_0 accordingly. By [BKPS18] the spectral effects of taking the line graph is known in case the graph is regular.



(a) $\Gamma^{(2)}(\mathrm{sd}\Delta_2)$

(b) $\Gamma^{(2)}(\mathrm{sd}^2\Delta_2)$

(c) $\Gamma^{(2)}(\mathrm{sd}^3\Delta_2)$

(d) $\Gamma^{(2)}(\mathrm{sd}^4\Delta_2)$

Figure 4.2: Sequence of dual graphs of the 2-dimensional barycentric subdivision. The tree hierarchy of the fractal is indicated by the previous graphs grayed out behind the current one.

Strong Universal Limit Theorem

We can prove an even stronger version of the Theorem as in Chapter 3 by a result from [Ele08a]. We can apply Proposition 3.2 in [Ele08a] to fractals generated in the sense of last chapter directly in order to find that their spectral distribution functions $F_{\Delta(G_n)}$ uniformly converge. This is due to the fact that it is immediate for our fractal construction that the sequences resulting from it are strongly converging. We will thus apply one result from [Ele08b] in order to show the following result.

Theorem I.43 (Strong Universal Limit Theorem)**.** *Let $d \geq 1$ be an integer and let* div *be an inclusion-uniform subdivision acting non-trivially on d-dimensional complexes.*

Then there exists a function $F_d^{(\mathrm{div})} \in L^\infty(\mathbb{R})$ such that for every d-dimensional complex K it holds

$$F(\mathscr{L}(\mathrm{div}^n K)) \xrightarrow{n\to\infty}_{L^\infty} F_d^{(\mathrm{div})},$$

i.e. essentially uniformly.

Note that every result from the above can be shown for this strong version by taking the probabilistic inverse. In particular for the edgewise subdivision we don't need to do the unnatural inversion we had to perform to get to $\Lambda_d^{(\mathrm{div})}$.

We will now reproduce the core concepts used for the proof of this result from [Ele08b]. Let G and H be graphs on the same finite vertex set $V(G) = V(H) = V$. We define a distance

$$\delta(G,H) := \mu_V(x \in V \mid \mathrm{St}_x(G) \neq \mathrm{St}_x(H))$$

for $\mathrm{St}_x(G)$ being the star of G at $x \in V$, i.e. the vertex-induced subgraph of G on $\{x\} \cup N_G(x)$, where $N_G(x) = \{y \in V \mid \{x,y\} \in E(G)\}$. μ_V denotes the uniform measure on the vertex set. In particular $\delta(G,H) = 0$ iff $G = H$.

Note that δ defines a metric on the set of graphs on vertex set V (which can be identified with $2^{\binom{V}{2}}$). In order to δ into a metric on isomorphism classes we will allow a symmetric reordering of H, i.e.

$$\delta_S(G,H) := \inf_{\sigma \in S_V} \delta(G, H^\sigma),$$

where H^σ is the graph on V with edge set

$$E(H^\sigma) := \{\{\sigma x, \sigma y\} \mid \{x,y\} \in E(H)\}.$$

This defines a metric on isomorphism classes of graphs on vertex set V. In particular δ_S is defined for isomorphism classes of graphs on sets other than V with only the same cardinality.

Given two graphs G and H on finite vertex sets which not need to be equal we can define the *geometric distance* of G and H via

$$\delta_\rho(G,H) := \inf_{\{q,r \mid qv(G) = rv(H)\}} \delta_S(qG, rH),$$

where qG denotes the disjoint union of q copies of G.

Note that by [Ele08b, Proposition 2.1] δ_ρ defines a metric on isomorphism classes of graphs on finite vertex sets.

Definition I.44. A sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs is said to be strongly convergent if $(G_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence with respect to the δ_ρ metric.

Now we can formulate [Ele08b, Lemma 3.5].

Lemma I.45 ([Ele08b])**.** *Let $C : \mathbb{R}^n \to \mathbb{R}^n$, $D : \mathbb{R}^n \to \mathbb{R}^n$ denote self-adjoint linear transformations such that*

$$rk(C - D) \leq \varepsilon n.$$

Then it holds

$$||F_C - F_D||_\infty \leq \varepsilon.$$

Which will take the place of Corollary I.5 in this stronger version of Theorem I.1.

Proposition I.46 ([Ele08b])**.** *Let $(G_n)_{n\in\mathbb{N}}$ be strongly convergent of bounded vertex degree d. Then the sequence $(F_{\Delta(G_n)})_{n\in\mathbb{N}}$ converges uniformly.*

Lemma I.47. *Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of graphs as defined in Section 4 such that G_0 is not contained in one $\partial_i G_0$. Then $(G_n)_{n\in\mathbb{N}}$ is strongly convergent and has vertex degree bound $d+1$.*

Proof. We show that the corresponding sequence is Cauchy. Let the data be given by $d, n, N \in \mathbb{N}$ and some initial graph G_0 on vertex set $[N]$ with degrees bounded by $d+1$ and a dedicated S_{d+1}-action on it. Let $\partial_i G_0$ denote the dedicated boundary sets, i.e. a partition of the set of vertices of degree $< d+1$ compatible with the S_{d+1}-action.

Now let the sequence $(G_n)_{n\in\mathbb{N}}$ be generated according to this input. Note that the boundary $\partial_i G_n = (\partial_i G_0)^{n+1}$ so that the boundary ∂G_n has

$$(d+1)(\#\partial_0 G_0)^{n+1}$$

elements. Let the number of boundary elements in one group of the boundary be denoted by b, then we have

$$(d+1)b^{n+1}$$

boundary elements in G_n.

Now let $n \leq m$ be given. We will calculate an upper bound of $\delta_\rho(G_n, G_m)$ by calculating $\delta(qG_n, (rG_m)^\sigma)$ in the particular case of $\sigma = \mathrm{id}_{V(G_m)}$ and $q = N^{m-n}$, $r = 1$.

To this end note that by our construction G_m is made up of N^{m-n} copies of G_n where we add edges between some boundary vertices of these copies. In particular the number of vertices for which the neighborhood is altered is bound from above by

$$N^{m-n}(d+1)b^{n+1}.$$

In particular for all other vertices (i.e. in the interior of the copies of G_n) the stars agree. We obtain

$$\mu_{V(G_m)}(x \in V(G_m) \mid \mathrm{St}_x(qG_n) \neq \mathrm{St}_x(G_m)) = \frac{N^{m-n}(d+1)b^{n+1}}{N^{m+1}} = (d+1)\Big(\frac{b}{N}\Big)^{n+1}.$$

We further note the fact that

$$\frac{b}{N} < 1$$

by the assumption on the fractal data.

In particular this sequence is Cauchy. $\square$

Corollary I.48. *Let* div *be an inclusion-uniform subdivision procedure acting non-trivially on complexes of dimension* $d \geq 1$. *Then the sequence* $\left(\Gamma^{(d)}(\mathrm{div}^n \Delta_d)\right)_{n\in\mathbb{N}}$ *strongly converges.*

Proof. This is immediate by Theorem I.42 and the consideration that

$$b = f_{d-1}(\mathrm{div}\Delta_{d-1}) < f_d(\mathrm{div}\Delta_d) = N$$

in the notation of the previous Lemma I.47. $\square$

Proposition I.49 (Convergence of local spectra)**.** *Let* div *be an inclusion-uniform subdivision procedure acting non-trivially on complexes of dimension* $d \geq 1$. *Then there is a function* $F_d^{(\mathrm{div})} \in L^\infty(\mathbb{R})$ *such that*

$$F_{\mathscr{L}(\mathrm{div}^n\Delta_d)} \xrightarrow{n\to\infty}_{L^\infty} F_d^{(\mathrm{div})}.$$

Proof. Follows from Corollary I.48 and Proposition I.46 aswell as the fact that F^∞ is complete. $\square$

The universality can be shown the same way as in Section 3; we only have to show that the perturbation matrix is of rank $o(n)$.

Proposition I.50 (Dominance of local spectra)**.** *Let* div *denote an inclusion-uniform subdivision procedure acting non-trivially on d-dimensional complexes and* K *denote an arbitrary (finite) simplicial complex of dimension* d. *It holds that*

$$||F_{\mathscr{L}(\mathrm{div}^n K)} - F_{\mathscr{L}(\mathrm{div}^n\Delta_d)}||_\infty \xrightarrow{n\to\infty} 0.$$

In particular the sequence $\mathscr{L}(\mathrm{div}^n K)$ *has a uniformly converging sequence of spactral cdf's and the uniform limit is the same as for* $\mathscr{L}(\mathrm{div}^n\Delta_d)$.

Proof. Analogously to Section 3 let $K = \mathscr{G}(\Delta_d, ..., \Delta_d)$ and so

$$\mathrm{div}^n K = \mathscr{G}^{(n)}(\mathrm{div}^n\Delta_d, ..., \mathrm{div}^n\Delta_d).$$

Again let $r_{d-1}(\mathscr{G}^{(n)}) = (f_{d-1}(\mathrm{div}))^n r_{d-1}(\mathscr{G})$ denote the number of gluing operations needed in the n-th step. As we have already seen the difference matrix

$$E := \mathscr{L}(\mathrm{div}^n K) - \bigoplus_{\sigma\in F_d(K)} \mathscr{L}(\mathrm{div}^n\Delta_d)$$

then is of the form

$$\mathscr{L}_n = \begin{pmatrix} D_1 & G_{12} & G_{13} & \dots & G_{1N} \\ G_{12}^t & D_2 & G_{23} & \dots & G_{2N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & G_{(N-1)N} \\ G_{1N}^t & \dots & G_{(N-2)N}^t & G_{(N-1)N}^t & D_N \end{pmatrix},$$

where D_k corrects the degrees on the diagonal along the boundary of the k-th copy of $\mathrm{div}^n \Delta_d$, i.e. every D_k has at most $D f_{d-1}(\mathrm{div}\Delta_{d-1})^{n+1}$ non-zero entries, where D denotes the maximal down-degree of a facet of K.

Furthermore only $r_{d-1}(\mathscr{G})$-many of the G_{ij}'s are non-zero and the non-zero G_{ij}'s have at most $(f_{d-1}(\mathrm{div}))^n$ non-zero entries so that from the G-matrices we obtain another

$$r_{d-1}(\mathscr{G})(f_{d-1}(\mathrm{div}))^n$$

non-zero entries.

We obtain a total number of

$$(f_d(K)D + 2r_{d-1}(\mathscr{G}))f_{d-1}(\mathrm{div})^n$$

non-zero entries. In particular this is an upper bound on the rank of the perturbation E, thus

$$\mathrm{rk}(E) = c_n n,$$

where

$$c_n = \frac{(f_d(K)D + 2r_{d-1}(\mathscr{G}))f_{d-1}(\mathrm{div})^n}{f_d(K)f_d(\mathrm{div})^n},$$

which we have already shown in Section 3 to asymptotically vanish. In particular for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the rank of the perturbation E is

$$\mathrm{rk}(E) \leq c_n n \leq \varepsilon n$$

and so by Lemma I.45 the result follows. □

Experiments

In this section we will present python code in order to give empirical evidence for the validity of Proposition I.18 and Theorem I.16 in the cone division case and Theorem I.37 for the barycentric subdivision.

Regarding Cone Division

In this section we will present python code in the context of Theorem I.16. The codes will generate the spectrum by explicitly calculating it from subdivision and plot it against the spectrum calculated in the way of Theorem I.16.

Code Block 6.1: Python code to calculate and plot the spectrum described by polynomial recursion in Theorem I.16 and the actual Laplacian matrix. The euclidean distance of the spectra is printed.

```
import numpy as np
from matplotlib import pyplot as plot
from sympy.solvers import solve
import sympy

def plot_spec(spec, label='step'):
    spec = np.insert(spec, 0, spec[0])
    plot.step(np.linspace(0, 1, spec.shape[0]), spec, label=label)

# introduce variable for getting roots of the recursion polynomial via sympy
x = sympy.Symbol('x')
d = 3 # dimension of input complex
pol = x * (d + 3 - x) # polynomial of Theorem I.16

# function to determine f^{-1}(e) for a set e
def recurse(e):
    new_e = np.array([])
    for r in e:
        f = pol - r
        new_e = np.concatenate((new_e, solve(f, x))).astype(float)
    return np.unique(new_e)

# start with A_1, A_2 and B_2, i.e. the spectrum of sd^2 Δ_d
```

```python
e_0 = [np.array([d+1])]
e_2 = [np.array([d+3])]
e_0 = e_0 + [recurse(e_0[0])]

# functions returning multiplicities
def beta(n):
    return int((d-1) / 2 * ((d+1)**(n-1) - 1))
def alpha(n):
    return beta(n) + d

# calculate the spectrum by polynomial recursion as in Theorem I.16
e_pol = np.array([])
for j in range(len(e_0)):
    e_pol = np.concatenate((e_pol, np.tile(e_0[j], alpha(2-j))))
for j in range(len(e_2)):
    e_pol = np.concatenate((e_pol, np.tile(e_2[j], beta(2-j))))
e_pol = np.sort(e_pol)
e_pol = np.insert(e_pol, 0, 0)
plot_spec(e_pol)
plot.show()

# calculate spectrum numerically from adjacency matrix of
#   Schreier graph sequence by Proposition I.18
# n = 1
pattern_a = np.zeros((d+1,d+1))
for i in range(d+1):
    pattern_a[i, (i+1) % (d+1)] = 1
J = np.ones((d+1,d+1)) - np.identity(d+1)
b = np.identity(d+1)

# Adjacency matrix of sd^2Δ_d
i = 2

M = np.zeros((d+1,d+1))
M[d,d] = 1
b = np.kron(M, b)
for j in range(1,d+1):
    M = np.zeros((d+1,d+1))
    M[j-1,d-j] = 1
    b = b + np.kron(M,
                    np.kron(
                        np.linalg.matrix_power(pattern_a, j),
                        np.identity((d+1)**(i-2))
                    )
            )

delta = (d+1) * np.identity((d+1)**i) - (np.kron(J, np.identity((d+1)**(i-1))) + b)
```

```python
e_exact, v = np.linalg.eigh(delta)
plot_spec(e_exact)
plot.show()

# Print euclidean distance of the spectra
print(np.linalg.norm(e_pol - e_exact))

# iterate for $sd^3\Delta_d, sd^4\Delta_d, sd^5\Delta_d$
for i in range(3, 6):
    e_0 = e_0 + [recurse(e_0[-1])]
    e_2 = e_2 + [recurse(e_2[-1])]

    e_pol = np.array([])
    for j in range(len(e_0)):
        e_pol = np.concatenate((e_pol, np.tile(e_0[j], alpha(i-j))))
    for j in range(len(e_2)):
        e_pol = np.concatenate((e_pol, np.tile(e_2[j], beta(i-j))))
    e_pol = np.sort(e_pol)
    e_pol = np.insert(e_pol, 0, 0)
    plot_spec(e_pol)
    plot.show()

    M = np.zeros((d+1,d+1))
    M[d,d] = 1
    b = np.kron(M, b)
    for j in range(1,d+1):
        M = np.zeros((d+1,d+1))
        M[j-1,d-j] = 1
        b = b + np.kron(M,
                        np.kron(
                            np.linalg.matrix_power(pattern_a, j),
                            np.identity((d+1)**(i-2))
                        )
                )

    delta = (d+1) * np.identity((d+1)**i) - (np.kron(J, np.identity((d+1)**(i-1))) + b)
    e_exact, v = np.linalg.eigh(delta)
    plot_spec(e_exact)
    plot.show()

    print(np.linalg.norm(e_pol - e_exact))
```

Code Block 6.2: Output of Code Block 6.2, i.e. euclidean distance of the spectra which are shown to be equal by Theorem I.16

```
7.840360590302494e-15
2.0515268049222782e-14
5.0682272876712425e-14
1.1265875013606188e-13
```

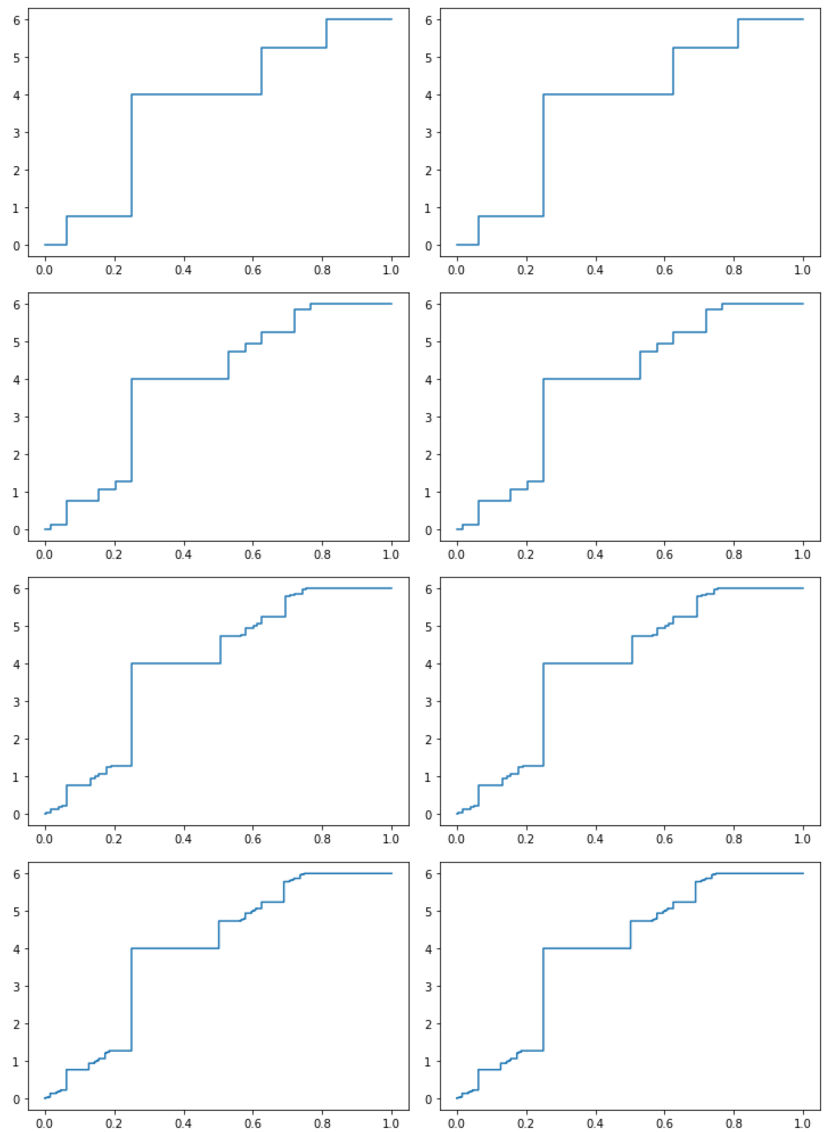


Figure 6.1: Output of Code Block 6.2. Left are the spectra calculated by polynomial recursion and the actual spectra are on the right.

Regarding Barycentric Subdivision

This section is devoted to experiments on the 2-dimensional barycentric subdivision. We will present two python codes; Code Block 6.3 and 6.4. Both will calculate a perturbed version of the adjacency spectrum $\Lambda(A(\Gamma^{(2)}(\mathrm{sd}^n\Delta_2)))$ for $n = 1, ..., 4$. The perturbation is induced by adding loops to the boundary as described in Chapter 4. Note that as described in Section 3 the perturbed spectra deviate from the actual spectrum of $A(\Gamma^{(2)}(\mathrm{sd}^n\Delta_2))$ by a L^1-error which converges to 0 as n becomes large.

The first Code Block, 6.3, uses the C++ library ANUBiS, [Mä], in order to calculate barycentric subdivisions of the standard 2 simplex Δ_2 calculates the perturbed adjacency matrix from the Laplacian generated by the library. Thus this code shows the spectrum calculated in the "naive" way and serves us a base case. Code Block 6.4 calculates the same spectrum via the matrix recurrence obtained from Schreier graph approximations as in Theorem I.37. As seen by their outputs displayed in Figures 6.2 and 6.3 the spectra are the same which follows from Theorem I.37.

The matrices returned by ANUBiS are, unlike the matrices generated by Code Block 6.4, not given with respect to the lexicographic ordering of the vertex set. This ordering exhibits a nice structure of the adjacency matrix, i.e. the adjacency matrix can be decomposed in a sum of a symmetric (block) circulant matrix $t_n + \tau_n = t_0 \otimes I_{6^{n-1}} + \tau_0 \otimes I_{6^{n-1}}$ and a block-diagonal matrix $b_n = \mathrm{diag}(\tau_{n-1}, \tau_{n-1}, t_{n-1}, t_{n-1}, b_{n-1}, b_{n-1})$. However since b_n contains two blocks equal to the matrix b_{n-1} we can't apply Schur complement as in Section 3 in order to deduce a closed recursive formula.

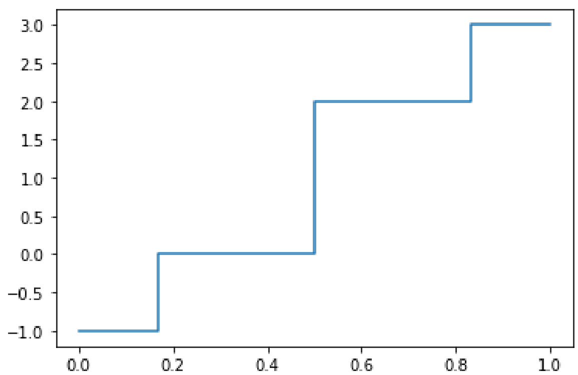
Note that there are useful results for determinants of such matrices, e.g. [Mol08], but at one point in the calculation we always have to invert a matrix of the form $\alpha t_{n-1} + \beta\tau_{n-1} + \gamma b_{n-1}$ for $\alpha, \beta, \gamma \in \mathbb{R}$ which itself involves factors of the kind $t_{n-1}\tau_{n-1}$ or $t_{n-1}b_{n-1}$ (similar to how we had to add powers of a to the adjacency matrix of the cone division). In order to obtain a closed iteration we thus would need to adjust the determinant by a summand of this kind which iteratively fills in the nice structure of the adjacency matrix making it more dense (and thus the results on block tridiagonal matrices not applicable anymore).

Notably however as indicated by the experiments (see Figure 6.2) there is a large spectral gap around a prescribed (wave-like) function (in the limit) of eigenvalues between -1 and 1, while the eigenvalues not in this range - which we will call the "boundary part" of the spectrum - seem to exhibit a self-similar nature; see Code Block 6.5 and Figure 6.4. It is immediate to see that the large spectral gaps seem to reproduce in the "boundary part" (again with a prescribed function in the range which corresponded to $[-1, 1]$ before transformation). But even more notably is the fact that when comparing the "boundary part" of the spectrum in the n-th step with the (complete) previous spectrum (after appropriate rescaling) the "boundary part" seems to be a distorted version of the previous spectrum; see Figure 6.4. Note how prominent features of the spectral CDFs (like gaps and dips) are paralleled exactly from one CDF to the other. This prompts the conjecture that also barycentric subdivision obeys a (possibly more complicated) spectral decimation rule on the "boundary part" with a fixed spectrum introduced between -1 and 1.

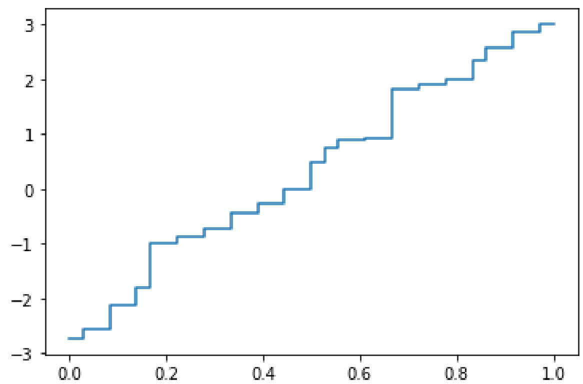
In order to plot higher-dimensional spectra of barycentric subdivisions the code from Code Block 6.3 is not feasible anymore since matrix sizes grow exponentially with the number of subdivisions and the base is $(d+1)!$ for d being the input dimension. ANUBiS offers a functionality to determine all eigenvalues of a matrix - not preserving multiplicities though. The outputs of this algorithm up to dimension 4 are shown in Figure 6.7. Note already that there is distortion of the two-dimensional spectrum compared to Figure 6.2 because multiplicities are not preserved.

Code Block 6.3: Python code for calculating spectra of iterated barycentric subdivision of 2-dimensional simplex by explicitly subdividing the complex using ANUBiS. Note that this code calculates adjacency spectrum of $\Gamma^2(2)(\mathrm{sd}^i\Delta_2)$ (up to loops on the boundary) in order to maintain comparability to the output of Code Block 6.4.

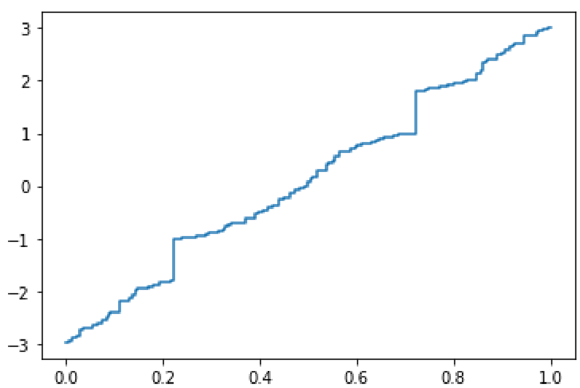
```python
import anubis
from matplotlib import pyplot as plt
import numpy as np

# function to plot the spectral CDF
def plot_spec(spec):
    # this insertion only has rendering purposes
    spec = np.insert(spec, 0, spec[0])
    plt.step(np.linspace(0, 1, spec.shape[0]), spec)

# load a standard 2 simplex $\Delta_2$ into S_TREE memory format of anubis
# standard_2d contains "Delta2 := [[0,1,2]]"
c = anubis.complex.from_file(anubis.S_TREE, "../standard_2d")

for i in range(1,5):
    # c_next = sd(c)
    c_next = c.barycentric()
    # free C++ memory occupied by the old complex c
    del c
    # c = $sd^i\Delta_2$
    c = c_next

    # get top-Laplacian $\mathscr{L}(c)$
    delta = np.matrix(c.laplacian_down(2), dtype=np.float64)
    # determine its spectrum using numpy
    e, v = np.linalg.eigh(delta)

    # plot $\Lambda(3 - \mathscr{L}(c))$
    plot_spec(np.sort(3 - e))
    plt.show()
```



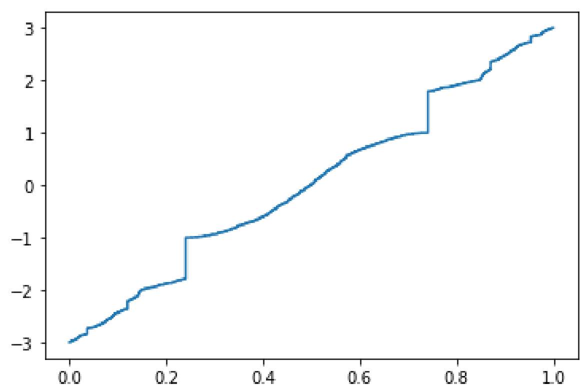
(a) $\Lambda(A(G_1))$



(b) $\Lambda(A(G_2))$



(c) $\Lambda(A(G_3))$



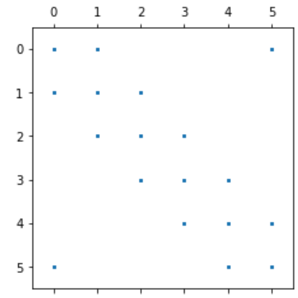
(d) $\Lambda(A(G_4))$

Figure 6.2: Output of Code Block 6.3; i.e. the spectral CDFs for G_n the sequence of graphs $\Gamma^2(2)(\mathrm{sd}^i\Delta_2)$ with loops added to the boundary. This sequence is equal to $\Lambda(A(\Gamma^{(2)}(\mathrm{sd}^i\Delta_2)))$ up to asymptotically negligible L^1-error introduced by loops on the boundary.

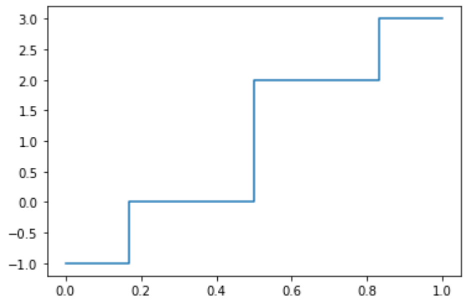
Code Block 6.4: Python code for calculating spectra of iterated barycentric subdivision of 2-dimensional simplex by matrix recursion from Theorem I.37.

```
import numpy as np
from matplotlib import pyplot as plot

def plot_spec(spec, label='step'):
    spec = np.insert(spec, 0, spec[0])
    plot.step(np.linspace(0, 1, spec.shape[0]), spec, label=label)

# initializing patterns of taking the Kronecker product in recursion
pattern_tau = np.matrix([   [0,1,0,0,0,0],
                            [1,0,0,0,0,0],
                            [0,0,0,1,0,0],
                            [0,0,1,0,0,0],
                            [0,0,0,0,0,1],
                            [0,0,0,0,1,0]   ])
pattern_t = np.matrix([     [0,0,0,0,0,1],
                            [0,0,1,0,0,0],
                            [0,1,0,0,0,0],
                            [0,0,0,0,1,0],
                            [0,0,0,1,0,0],
                            [1,0,0,0,0,0]   ])

pattern_btau = np.matrix([  [1,0,0,0,0,0],
                            [0,1,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,0,0]   ])
pattern_bt = np.matrix([    [0,0,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,1,0,0,0],
                            [0,0,0,1,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,0,0]   ])
pattern_bb = np.matrix([    [0,0,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,0,0],
                            [0,0,0,0,1,0],
                            [0,0,0,0,0,1]   ])

# initialize the matrix representations on layer 1
tau = pattern_tau.copy()
```

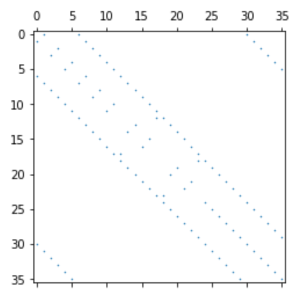
```
t = pattern_t.copy()
b = np.identity(6)

# L(sdΔ2) is the adjacency matrix of the hexagon
delta = tau + t + b
plot.spy(delta, markersize=2) # spy used to plot sparse matrices
plot.show()

# calculate and plot spectrum of L(Δ2)
e, v = np.linalg.eigh(delta)
plot_spec(e)
plot.show()

for i in range(2, 5):
    # representation of b on the next layer is a recurrent Kronecker product of t,τ and b
    #   representations of the previous layer (self-similarity of group generators)
    b = np.kron(pattern_btau, tau) + np.kron(pattern_bt, t) + np.kron(pattern_bb, b)
    # τ and t only act non-trivial on the last letter of each word so that their
    #   representations are obtained as Kronecker products with identity
    tau = np.kron(tau, np.identity(6))
    t = np.kron(t, np.identity(6))
    # L(sd^i Δ2) is the sum of the representations of generators
    delta = tau + t + b
    plot.spy(delta, markersize=0.5) # plot sparse matrix
    plot.show()
    # calculate spectrum of L(sd^i Δ2)
    e, v = np.linalg.eigh(delta)
    # plot Λ(L(sd^i Δ2))
    plot_spec(e)
    plot.show()
```



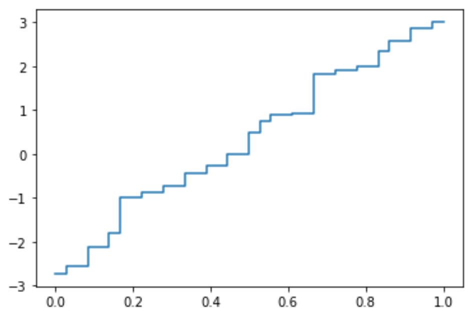
(a) $A(G_1)$



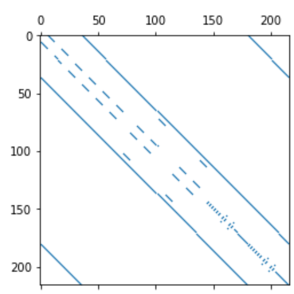
(b) $\Lambda(A(G_1))$



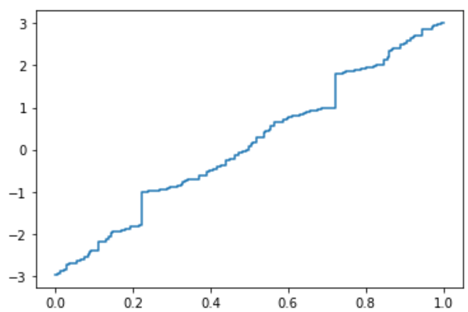
(c) $A(G_2)$



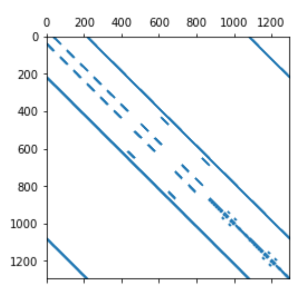
(d) $\Lambda(A(G_2))$



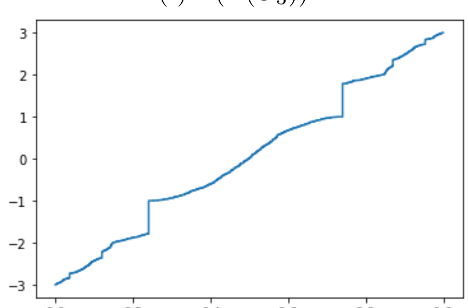
(e) $A(G_3)$



(f) $\Lambda(A(G_3))$



(g) $A(G_4)$

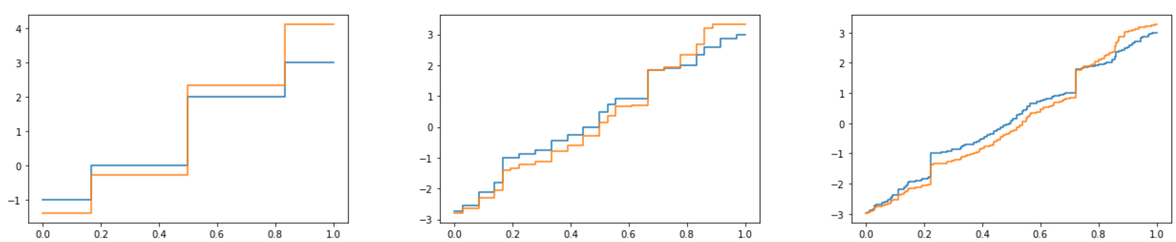


(h) $\Lambda(A(G_4))$

Figure 6.3: Output of Code Block 6.4

Code Block 6.5: Python code plotting the left boundary part of the spectrum against the previous complete spectrum.

```
# /**
# insert the initialization, i.e. lines 1-55 from Code Block 6.4
# **/

# save the previous spectrum in the e_old field
e_old = e

for i in range(2, 5):
    b = np.kron(pattern_btau, tau) + np.kron(pattern_bt, t) + np.kron(pattern_bb, b)
    tau = np.kron(tau, np.identity(6))
    t = np.kron(t, np.identity(6))
    delta = tau + t + b
    plot.spy(delta, markersize=0.5) # plot sparse matrix
    plot.show()
    e, v = np.linalg.eigh(delta)

    # plot e_old next to the first len(e_old)-many eigenvalues in e (after rescaling)
    plot_spec(e_old)
    plot_spec(6 * (e[:len(e_old)] + 3) - 3)
    plot.show()

    e_old = e

```



(a) Boundary part of $\mathrm{sd}^2\Delta_2$ against spectrum of $\mathrm{sd}\Delta_2$.

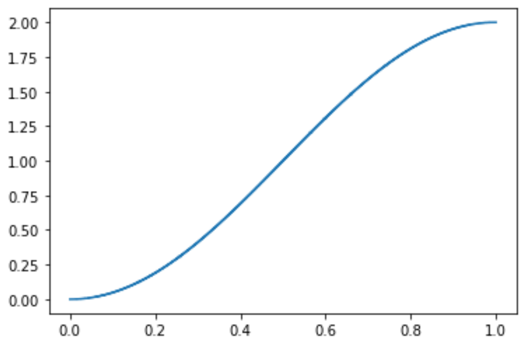
(b) Boundary part of $\mathrm{sd}^3\Delta_2$ against spectrum of $\mathrm{sd}^2\Delta_2$.

(c) Boundary part of $\mathrm{sd}^4\Delta_2$ against spectrum of $\mathrm{sd}^3\Delta_2$

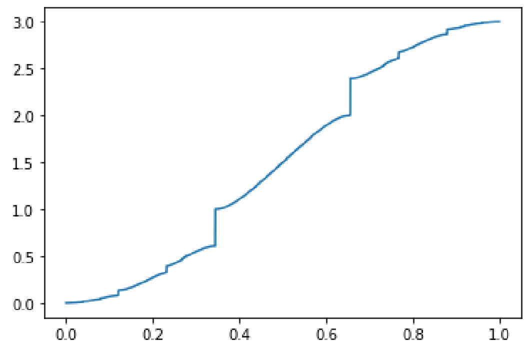
Figure 6.4: Output of Code Block 6.5. The orange curve is the boundary part of the current spectrum and the blue curve is the previous spectrum.

Code Block 6.6: Python code plotting the left boundary part of the spectrum against the previous complete spectrum.

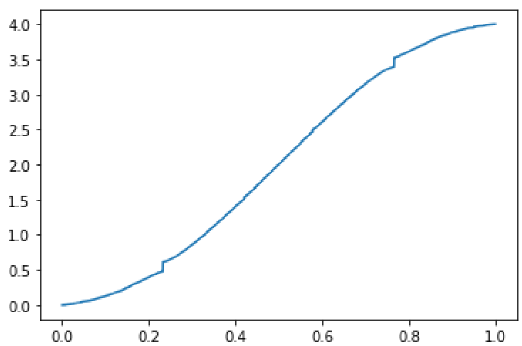
```python
import anubis
from matplotlib import pyplot as plt
import numpy as np

def plot_spec(spec):
    spec = np.insert(spec, 0, spec[0])
    plt.step(np.linspace(0, 1, spec.shape[0]), spec)

d = # put dimension here

c = anubis.complex.from_file(anubis.S_TREE, "../standard_{d}d".format(d=d))

for i in range(1,3):
    c_next = c.barycentric()
    del c
    c = c_next

e = np.array(c.laplacian_spectrum(d-1))
plot_spec(np.sort(e))
plt.show()

```



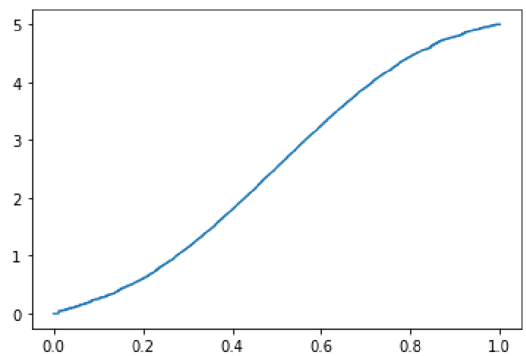
(a) $\mathrm{sd}^9\Delta_1$



(b) $\mathrm{sd}^5\Delta_2$



(c) $\mathrm{sd}^3\Delta_3$



(d) $\mathrm{sd}^2\Delta_4$

Figure 6.5: Output of Code Block 6.6. Eigenvalues of higher dimensions. Note that multiplicities are not correct.

Regarding Edgewise Subdivision

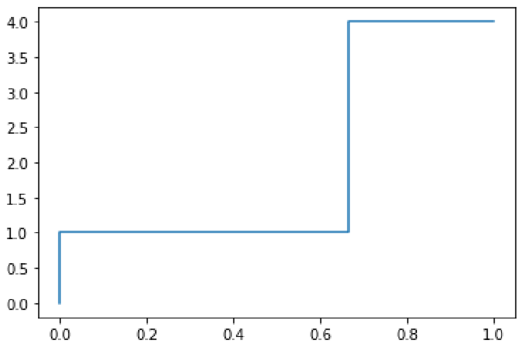
In this section we will present some python code in order to plot the actual spectra of edgewise subdivision. Note that we calculated the universal limit of edgewise subdivision using the same method as in [PT21]; however in our setting we are interested in the quantile function of the distribution which is why we only calculated $\Lambda_2^{(\mathrm{esd})}$ implicitly. Compare the plots produced by the following code to Figure 5 of [PT21] to see that it actually is the quantile function (up to a scaling of the parameter by 3).

Code Block 6.7: Python code plotting the first iterations of spectral quantile functions and dual graphs of edgewise subdivision.

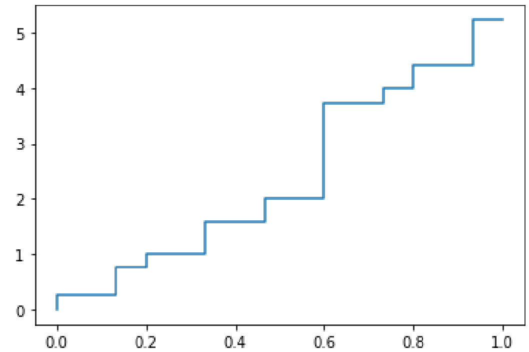
```python
import networkx as nx
import numpy as np
import itertools
from matplotlib import pyplot as plt

def plot(spec):
    np.insert(spec, 0, spec[0])
    plt.step(np.linspace(0,1,len(spec)), spec, where="pre")
    plt.show()

d = 2 # dimension
r = d # subdiv param

def partitionfunc(n,k,l=0):
    '''n is the integer to partition, k is the length of partitions,
    l is the min partition element size'''
    if k < 1:
        raise StopIteration
    if k == 1:
        if n >= l:
            yield (n,)
        return
    for i in range(l,n+1):
        for result in partitionfunc(n-i,k-1,i):
            yield (i,)+result

def iota(t):
    '''Discrete integration, i.e. r[j] = t[j] + ... + t[0] for all j.'''
    r = [0] * (d+1)
    r[0] = t[0]
    for j in range(1, d+1):
        r[j] = t[j] + r[j-1]
    return r

```

```python
def shift_v(vert, n, facet):
    '''For array "vert" and a mask "facet" of the same size only containing integers < n
    return an array of size n which has vert[j] at the facet[j]-th place.'''
    l_vert = [0] * n
    j = 0
    for i in facet:
        l_vert[i] = vert[j]
        j = j + 1
    return tuple(l_vert)

def dual(G):
    '''Returns graph dual to G.'''
    H = nx.Graph()
    cliques = nx.enumerate_all_cliques(G)
    facets = set()
    for c in cliques:
        if len(c) == d+1:
            facets.add(tuple(c))
    for facet in facets:
        H.add_node(facet)
    for pair in itertools.product(facets, repeat=2):
        if len(set(pair[0]).intersection(set(pair[1]))) == d:
            H.add_edge(pair[0], pair[1])

    return H

def edgewise(G):
    '''Given the graph of a clique-2-complex generate its edgewise subdivision.'''
    cliques = nx.enumerate_all_cliques(G)
    facets = set()
    for c in cliques:
        if (len(c) == d+1):
            facets.add(tuple(c))

    n = G.number_of_nodes()

    verts = []
    edges = set()
    for facet in facets:
        parts = partitionfunc(r, d+1)
        v = set()

        for t in parts:
            v.update(list(itertools.permutations(t)))

        v = list(v)
        indices = [-1] * len(v)
```

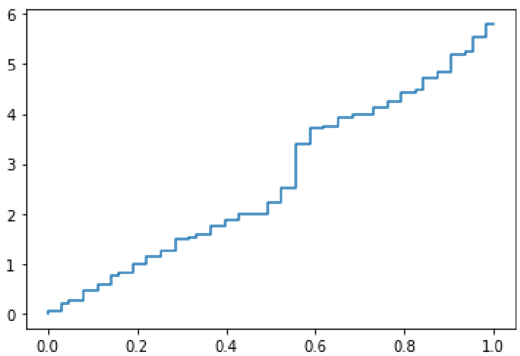
```
        for i in range(len(v)):
            el = shift_v(v[i], n, facet)
            try:
                indices[i] = verts.index(el)
            except ValueError:
                indices[i] = len(verts)
                verts.append(el)

        for pair in itertools.product(range(len(v)), repeat=2):
            x = iota(v[pair[0]])
            y = iota(v[pair[1]])
            diff = [x[i] - y[i] for i in range(d+1)]
            pos = True
            neg = True
            for i in range(d+1):
                if (diff[i] not in [0,1]):
                    pos = False;
                if (diff[i] not in [0,-1]):
                    neg = False;
            if (pos or neg):
                edges.add((indices[pair[0]], indices[pair[1]]))
    verts = list(verts)
    H = nx.Graph()
    for i in range(len(verts)):
        H.add_node(i)
    for e in edges:
        H.add_edge(e[0], e[1])

    return H, verts

G = nx.complete_graph(d+1) # start with standard simplex

it = 6

for i in range(it):
    G, v = edgewise(G) # get edgewise subdivision

    H = dual(G) # get dual graph
    nx.draw(H, nx.spring_layout(H, iterations=5000), node_size = 0.5) # plot dual graphs
    plt.show()
    plot(nx.linalg.laplacian_spectrum(H)) # plot spectrum
    plt.show()
```



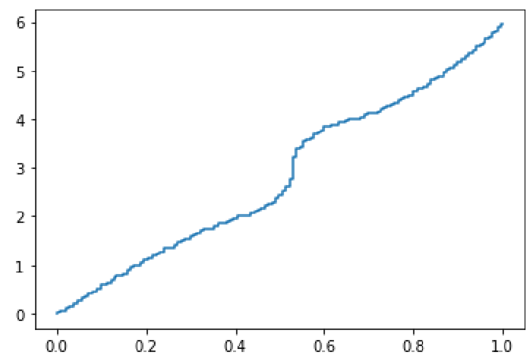
(a) $\Lambda(\mathrm{esd}_2^1\Delta_2)$



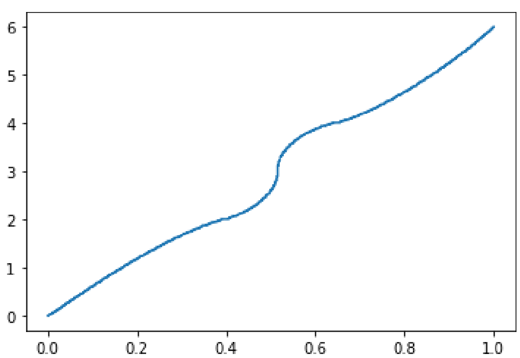
(b) $\Lambda(\mathrm{esd}_2^2\Delta_2)$



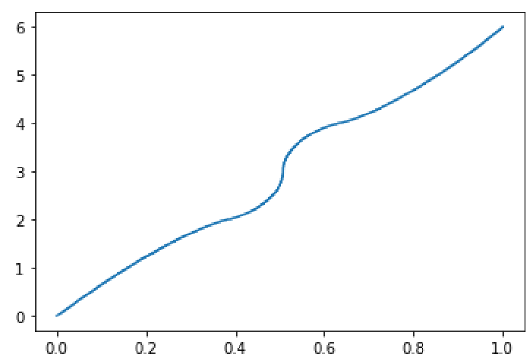
(c) $\Lambda(\mathrm{esd}_2^3\Delta_2)$



(d) $\Lambda(\mathrm{esd}_2^4\Delta_2)$

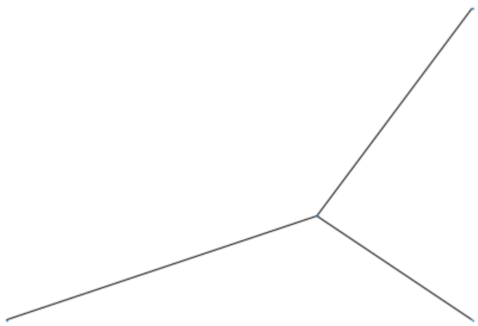


(e) $\Lambda(\mathrm{esd}_2^5\Delta_2)$

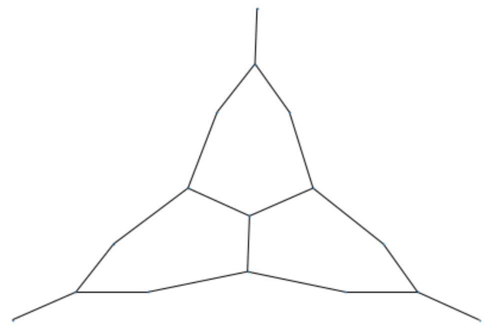


(f) $\Lambda(\mathrm{esd}_2^6\Delta_2)$

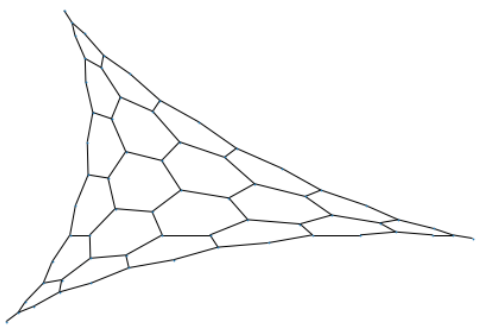
Figure 6.6: Spectral output of Code Block 6.7.



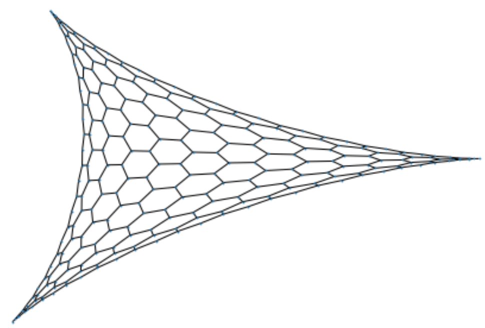
(a) $\Gamma^{(2)}(\mathrm{esd}_2^1\Delta_2)$



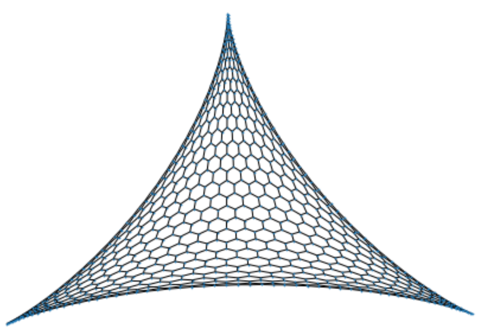
(b) $\Gamma^{(2)}(\mathrm{esd}_2^2\Delta_2)$



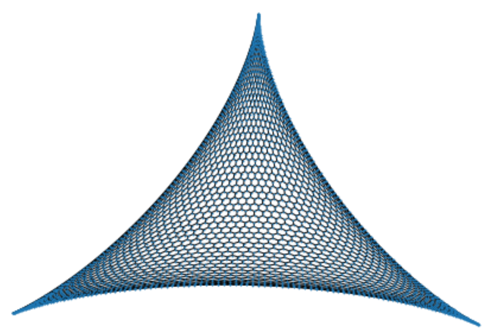
(c) $\Gamma^{(2)}(\mathrm{esd}_2^3\Delta_2)$



(d) $\Gamma^{(2)}(\mathrm{esd}_2^4\Delta_2)$



(e) $\Gamma^{(2)}(\mathrm{esd}_2^5\Delta_2)$



(f) $\Gamma^{(2)}(\mathrm{esd}_2^6\Delta_2)$

Figure 6.7: Graph output of Code Block 6.7. Generated by networkx, [HSS08].

Open Questions

The following questions were considered but not solved in the course of this work.

The most interesting open question on the above is the one for limiting objects of the fractal procedure described in Section 4. One object of particular interest is the case of the barycentric subdivision as it is the most natural example for which finite ramification does not hold.

As seen in Section 5 the sequence of barycentric subdivisions are "strongly convergent" in the sense of [Ele08b]. However this only means that the sequence of graphs is a Cauchy sequence with respect to the geometric distance (see Section 5 or [Ele08b] for its definition) and this metric space need not be complete. Note that the strong convergence in this case also implies Benjamini-Schramm convergence; let G_n denote the dual graph of the n-th barycentric subdivision of $\Delta^{(2)}$ and let μ be the uniform probability measure on the set of words $[6]^\infty$ of infinite length, i.e. the measure generated by $\mu(w \in [6]^\infty \mid w_n = \ell) = 1/6$ for $n \in \mathbb{N}$, $\ell \in [6]$. Note that the vertex set of G_n is $[6]^n$ and thus the vertex set of disjoint union

$$\boldsymbol{G}_n := \bigsqcup_{w \in [6]^\infty} G_n$$

also is $[6]^\infty$ by the identification $(w, x_1 ... x_n) \mapsto x_n ... x_1 w$. Note that by the section on the barycentric subdivision of dimension 2 we can describe G_n as a Schreier graph via three group generators τ, t and b which act right-to-left in the word sequence $x_1 ... x_n$. Let τ_*, t_* and b_* denote the corresponding operators acting from left-to-right on the flipped word $x_n ... x_1$. Intuitively the corresponding limit of the graph whose edge relations are induced by left-to-right application of τ_*, t_*, b_* would be the graph G_∞ on vertex set $[6]^\infty$ with edge relations given by application of τ_*, t_*, b_* on the infinite sequence. Note hereby that either the calculation ends after finitely many steps (i.e. at the first 5 or 6) or the word consists of only 5's and 6's and so b doesn't act on the element at all. G_∞ can be thought of as a hexagon from which can be zoomed out in the sense of the sequence $(G_n)_{n \in \mathbb{N}}$ infinitely wide.

Now note that the stars of the graphs G_∞ and $\boldsymbol{G}_n$ agree on all vertices except for those on the boundary of $\boldsymbol{G}_n$. These elements are precisely of the form $x_n ... x_1 w$ for $x_1 ... x_n$ on the boundary of G_n. Thus the event

$$\{w \in [6]^\infty \mid \mathrm{St}_w(\boldsymbol{G}_n) \neq \mathrm{St}_w(G_\infty)\} \subseteq \partial G_n \times [6]^\infty$$

completely defined by the first n letters. In particular since $\# \partial G_n = 6 \cdot 2^{n-1}$ we obtain in measure that

$$\mu(w \in [6]^\infty \mid \mathrm{St}_w(\boldsymbol{G}_n) \neq \mathrm{St}_w(G_\infty)) = \frac{1}{3^{n-1}}.$$

In particular under some modifications (which make the geometric distance only into a pseudo-metric) this meant convergence of G_n to G_∞ in the strong sense (see Section 5 or [Ele08b]). This implies Benjamini-Schramm convergence in particular; let (H, o) be an arbitrary rooted graph of radius $k \in \mathbb{N}$. First of all note again, that the probability

$$\mu(w \in [6]^\infty \mid (H, o) \hookrightarrow (\boldsymbol{G}_n, w)) = \mu_n(x \in [6]^n \mid (H, o) \hookrightarrow (G_n, x)),$$

where μ_n denotes the (finite) uniform distribution on $V(G_n) = [6]^n$. Since all stars of $\boldsymbol{G}_n$ and G_∞ coincide in the interior of $\boldsymbol{G}_n$ for every word $w \in [6]^\infty$ the embeddability of a k-radius rooted graph into $(\boldsymbol{G}_n, w)$ and (G_∞, w) coincides as long as w has distance $\geq k$ from $\partial \boldsymbol{G}_n$. Let $D_{H,n}$ denote the set of vertices $w \in [6]^\infty$ such that (H, o) embeds in $(\boldsymbol{G}_n, w)$ but not in (G_∞, w) or vice versa. Thus by the above we have $H \subseteq \partial \boldsymbol{G}_n$. Now since G_n has degrees ≤ 3 for every boundary vertex $x \in \partial G_n$ the k-ball contains at most $2^{k+1} - 1$ vertices. Thus the number of vertices of distance $\leq k$ from ∂G_n is

$$6 \cdot 2^{n-1}(2^{k+1} - 1) \leq 6 \cdot 2^{n+k}.$$

In particular the measure of such points in $\boldsymbol{G}_n$ is

$$\mu(D_{H,n}) \leq \frac{2^{k+1}}{3^{n-1}}.$$

By definition it holds

$$\left|\mu(w \in [6]^\infty \mid (H, o) \hookrightarrow (\boldsymbol{G}_n, w)) - \mu(w \in [6]^\infty \mid (H, o) \hookrightarrow (G_\infty, w))\right| \leq \mu(D_{H,n}) \to 0.$$

for every given (H, o).

By all the above the graph G_∞ seems to be a reasonable choice. Note however that this graph G_∞ behaves weirdly as it is not connected for example. One could modify the strong convergence to obtain as a limit one of the countable connected components of the above, which as a vertex-induced subgraph has G_n on the set $\Sigma^n \cdot \overline{0}$, where $\overline{0}$ is the constant 0-word of infinite length.

Question 1. Is there a reasonable limiting object of the barycentric subdivision fractal process preserving spectral convergence? In particular; does there exist an object from which the universal limit gets induced in a natural way?

Furthermore more generally we could ask ourselves the following:

Question 2. Under what conditions does such a limiting object exist for a general fractal sequence coming from a subdivision procedure as described in Section 4.

Part II

Differential Theory

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Introduction

Wave propagation in graph-like media is a topic of increasing interest in the last decades due to its applications in chemistry, mesoscopic physics and photonic crystals. We refer the reader to [Kuc02] and [BCFK06] and the references therein for an overview of systems modelled by thin structures associated to graphs.

Certain aspects a system of graph-like nature exhibits are approximated by the graph-model. As shown in [EP05] one such aspect is the spectral dynamics of a thin medium, i.e. for sufficiently thin systems the spectra of their Laplacians are approximated by those of the quantum graph which models the system. Note that the transition to a quantum graph can simplify the complexity drastically as quantum graphs are combinatorial graphs with (curvi-)linear differential geometry assigned to each edge.

In this work we will generalize the notion of a quantum graph to quantum simplicial complexes so that the modelling of systems with an arbitrarily large number of dimensions not confined by a small diameter is permissible. The notion of quantum simplicial complexes is introduced in Section 2. The concept of thin systems modelled by such a complex will be made precise in Section 3. Section 4 presents some inequalities generalizing the ones used in the proof of the main theorem of [EP05]. Our main contribution is the deduction of a higher-dimensional Kirchhoff condition allowing for the definition of a Laplacian on a quantum simplicial complex which has discrete spectrum. Thus associating to a geometry on a complex a spectral theory. We present conjectures on the spectral behaviour of this operator; in particular we conjecture that for an equilateral 2-dimensional quantum complex the spectrum is completely determined by the spectrum of the combinatorial 1-up Laplacian of the complex (as is the case for quantum graphs and their graph Laplacians). Furthermore we conjecture that the Riemannian Laplacian of thin structures modelled around a quantum complex converges to the spectrum of the quantum complex itself. More speculatively we also ask whether or not a phase transition law holds for spectral convergence with respect to the dimension of the modelling complex (as is the case for graphs).

Metric simplicial Complexes and their Laplacians

We firstly fix some conventions. All *simplicial complexes* considered in this work are finite abstract simplicial complexes, i.e. a finite collection of finite non-empty sets such that for $\tau \in K$ and non-empty $\tau' \subset \tau$ it holds that $\tau' \in K$. The sets contained in K are called *faces* of K and by $F_i(K)$ we denote the collection of i-faces of K, i.e. the collection of sets of cardinality $i+1$ in K. For sets $\tau \in F_i(K)$ we set their *dimension* to $\dim \tau = i$. The *dimension* of K is the maximal dimension of one of its elements. We denote the inclusion of sets in K by $\leq$ and call the maximal elements of K with respect to $\leq$ its *facets*. Let $\tau \leq \tau' \in K$; we say τ is a *face* of τ' and τ' is a *coface* of τ, depending on the frame of reference. We say $\dim \tau' - \dim \tau$ is the *codimension* of the pair τ', τ. Furthermore all complexes we work with will be *pure* which means that all facets have the same dimension. For an in-depth introduction to simplicial topology we refer the reader to [Mun18].

We will distinguish faces with certain properties by notation in order to facilitate readability. σ will be reserved for a facet, τ and ν are lower-dimensional faces where ν has dimension less than τ. Most of the times ν will be a codimension-1-face of τ, i.e. $\nu \leq \tau$ and $\dim \tau - \dim \nu = 1$.

In order to assign to every facet of K a geometry we need the notion of a geometric simplex. We say d points $v_0, ..., v_d \in \mathbb{R}^n$ are affinely independent if they are not contained in an affine subspace of $\mathbb{R}^n$ of dimension $< d$. In this case we say the convex hull

$$\Sigma = \operatorname{conv}(v_0, ..., v_d)$$

is a d-dimensional geometric simplex. We remark that

$$\Sigma = \Big\{ \sum_{i=0}^{d} \lambda_i v_i \mid \sum_{i=0}^{d} \lambda_i = 1,\ \lambda_i \geq 0 \Big\}.$$

Note that every selection $\{i_0, ..., i_k\} \subseteq \{0, ..., d\}$ gives a unique k-simplex $\operatorname{conv}(v_{i_0}, ..., v_{i_k})$ contained in the boundary of Σ. Those k-simplices are called the k-faces of Σ and the set of all k-faces of Σ is denoted by $F_k(\Sigma)$. In particular we have $F_0(\Sigma) = \{v_0, ..., v_d\}$. Assuming for Σ, Σ' two d-simplices we have a bijective map $f : F_0(\Sigma) \to F_0(\Sigma')$ we immediately obtain a geometric extension of f, denoted $\tilde{f}$, which is actually a bijective map

$$\tilde{f} :\ \Sigma \to \Sigma'; \quad \sum_{i=0}^{d} \lambda_i v_i \mapsto \sum_{i=0}^{d} \lambda_i f(v_i).$$

The following definition is similar to the one in [BH99] or the definition of a affine realization in [Chr07].

Definition II.1. A metric simplicial complex is a pair (K, Σ) of a finite pure d-dimensional abstract simplicial complex K and Σ assigns to every $\sigma \in F_d(K)$ a d-dimensional geometric simplex $\Sigma_\sigma \subset \mathbb{R}^d$ together with a bijective map

$$\varphi_\sigma : \ \sigma \to F_0(\Sigma_\sigma)$$

such that for $\tau \leq \sigma, \sigma' \in F_d(K)$ the map

$$\widetilde{\psi_{\sigma'} \circ \psi_\sigma^{-1}}$$

is a geometric isometry for $\psi_\sigma = \varphi_{\sigma|_\tau}$ and $\psi_{\sigma'} = \varphi_{\sigma'|_\tau}$. We further fix one isometric copy of the span of $\varphi_\sigma(\tau)$ in $\mathbb{R}^{\dim \tau}$ and denote it by $\Sigma_\tau \subset \mathbb{R}^{\dim \tau}$.

By a slight abuse of notation we will denote the face $\Sigma_{\sigma|_\tau}$ spanned by $\varphi_\sigma(\tau)$ by Σ_τ aswell even though they are technically only isometric to each other. However for our analysis this won't make any difference as their L^2-spaces are isometric aswell via transformation rule.

Note that every geometric simplicial complex is a metric simplicial complex; having an embedding in some $\mathbb{R}^N$ fixed we fix isometric copies of every facet in $\mathbb{R}^d$ and the faces obviously obey the condition we imposed on the maps φ. However not every metric simplicial complex has an embedding in some euclidean space.

We will now introduce a Hamiltonian operator on a metric simplicial complex which is a reasonable candidate to parallel Riemannian geometry. First we need to introduce function spaces associated to a metric simplicial complex $M = (K, \Sigma)$. We will employ the following function spaces defined for a compact domain $\Sigma \subset \mathbb{R}^d$: $L^2(\Sigma) = L^2(\Sigma, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_d)$ denotes the common L^2-space with the inner product given as

$$\langle u, v \rangle_\Sigma = \int_\Sigma uv \, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_d$$

and the corresponding norm denoted $|| \cdot ||_\Sigma$; $C(\Sigma)$ denotes the space of continuous functions and $H^k(\Sigma)$ the Sobolev space of degree k, i.e. the space of functions on Σ k-times weakly differentiable in $L^2(\Sigma)$. $H^k(\Sigma)$ will be seen mainly as a subspace of $L^2(\Sigma)$ for the purposes of this work, however we will make usage of the Sobolev norm which is denoted as

$$||u||_{k,\Sigma}^2 := \sum_{\alpha \in \mathbb{N}^d : \ |\alpha| \leq k} ||\partial^{|\alpha|} u / \partial x^\alpha||_\Sigma^2,$$

where $|\alpha| = \alpha_1 + ... + \alpha_d$ and $\partial^{|\alpha|} u / \partial x^\alpha := \Big(\partial_{x_1}\Big)^{\alpha_1} ... \Big(\partial_{x_d}\Big)^{\alpha_d} u$.

Using these function spaces we can finally define the corresponding ones for the metric simplicial complex M. Let

$$L^2(M) := \bigoplus_{\sigma \in F_d(K)} L^2(\Sigma_\sigma, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_d)$$

be the space of L^2-integrable functions,

$$C(M) := \{u \in \bigoplus_{\sigma \in F_d(K)} C(\Sigma_\sigma) \mid \forall_{\sigma, \sigma' > \tau \in K} : \ (u_\sigma)\big|_{\Sigma_{\sigma|_\tau}} \equiv (u_{\sigma'})\big|_{\Sigma_{\sigma'|_\tau}} \} \subseteq L^2(M)$$

be the space of continuous functions and

$$H^k(M) := \bigoplus_{\sigma \in F_d(K)} H^k(\Sigma_\sigma) \subset L^2(K)$$

be the space of k-times weakly differentiable functions on M, where $x_1, ..., x_d$ are the cartesian coordinates of $\mathbb{R}^d$. $L^2(M)$ becomes a Hilbert space via the inner product

$$\langle u, v \rangle_M := \sum_{\sigma \in F_d(K)} \langle u, v \rangle_{\Sigma_\sigma}$$

and the induced norm is thus

$$||u||_M^2 := \sum_{\sigma \in F_d(K)} ||u||_{\Sigma_\sigma}^2.$$

We now denote the quadratic form

$$q_M(u) := ||\nabla u||_M^2$$

for $u \in H^1(M) \cap C(M)$, where ∇u denotes the weak gradient of u, i.e.

$$(\nabla u)_\sigma := \Big(\frac{\partial u_\sigma}{\partial x_1}, ..., \frac{\partial u_\sigma}{\partial x_d}\Big),$$

for $\sigma \in F_d(K)$.

We have two distinct norms with respect to which we can view $H^k(M)$ - namely the common L^2-norm (and thus H^k as subspace of L^2) or the Sobolev norm denoted

$$||u||_{k,M}^2 := \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} ||\partial^\alpha u / \partial x^\alpha||_M^2,$$

as above.

We now associate to M a cotangential bundle T^*M as

$$T^*M := \bigoplus_{\sigma \in F_d(M)} T^*\Sigma_\sigma$$

and on it the L^2-space

$$L^2(T^*M) := \bigoplus L^2(T^*\Sigma_\sigma)$$

for

$$L^2(T^*\Sigma_\sigma) := \Big\{\alpha \in T^*\Sigma_\sigma \mid \alpha = \alpha_1 \, \mathrm{d}x_1 + ... + \alpha_d \, \mathrm{d}x_d, \; \alpha_1, ..., \alpha_d \in L^2(\Sigma_\sigma)\Big\}$$

with the L^2-norm

$$\Big\langle \sum_{i=1}^d \alpha_i \, \mathrm{d}x_i, \sum_{i=1}^d \beta_i \, \mathrm{d}x_i \Big\rangle_{\Sigma_\sigma} := \sum_{i=1}^d \langle \alpha_i, \beta_i \rangle_{\Sigma_\sigma}.$$

The cotangential bundle on M then gets assigned a function space in the same fashion as M itself, i.e.

$$L^2(T^*M) := \bigoplus_{\sigma \in F_d(K)} L^2(T^*\Sigma_\sigma)$$

with the product inner norm

$$\langle \alpha, \beta \rangle_M := \sum_{\sigma \in F_d(K)} \sum_{i=1}^{d} \langle \alpha_i^\sigma, \beta_i^\sigma \rangle_{\Sigma_\sigma},$$

where α_i^σ and β_i^σ are the coefficient functions for $\mathrm{d}x_i$ in α_σ and β_σ, respectively. Analogously we can define $H^k(T^*M)$ and $C(T^*M)$ as subspaces of $L^2(T^*M)$ by imposing the condition suggested by notation on every coefficient function.

In order to obtain a correspondence to the Laplacian of classical Riemannian geometry we will now define the exterior derivative $\mathrm{d} = \mathrm{d}_M$ as an operator from $H^1(M) \cap C(M)$ into $L^2(T^*M)$.

The exterior derivative is then given as

$$\mathrm{d}: \; H^1(M) \cap C(M) \to L^2(\Lambda^1(M))$$

via

$$(\mathrm{d}u)_\sigma = \frac{\partial u_\sigma}{\partial x_1} \,\mathrm{d}x_1 + \ldots + \frac{\partial u_\sigma}{\partial x_d} \,\mathrm{d}x_d.$$

In order to define our Hamiltonian we now need the following lemma.

Lemma II.2. *The operator* $\mathrm{d} : H^1(M) \cap C(M) \to L^2(T^*M)$ *has the formal adjoint operator* $\mathrm{d}^* : \operatorname{dom} \mathrm{d}^* \to L^2(M)$ *given by*

$$(\mathrm{d}^*\alpha)_\sigma := -\Big(\frac{\partial \alpha_1^\sigma}{\partial x_1} + \ldots + \frac{\partial \alpha_d^\sigma}{\partial x_d} \Big)$$

on domain

$$\operatorname{dom} \mathrm{d}^* = \{ \alpha \in H^1(T^*M) \mid \forall_{\tau \in F_{d-1}(K),\; x \in \Sigma_\tau} : \sum_{\sigma > \tau} \alpha_\sigma(\nu_\sigma)_{|x} = 0 \}$$

Proof. Let $\alpha \in \operatorname{dom} \mathrm{d}^*$ and $f \in \operatorname{dom} \mathrm{d}$ be an arbitrary test-function. The necessary condition for d^* being adjoint to d is

$$\langle \mathrm{d}f, \alpha \rangle_M = \langle f, \mathrm{d}^*\alpha \rangle_M$$

which we will use as defining equation for d^*. Integration by parts gives:

$$\begin{aligned}
\langle \mathrm{d}f, \alpha \rangle_M &= \sum_{\sigma \in F_d(K)} \int_{\Sigma_\sigma} \langle df_\sigma, \alpha_\sigma \rangle_{\Sigma_\sigma} \,\mathrm{d}\Sigma_\sigma \\
&= \sum_{\sigma \in F_d(K)} \int_{\Sigma_\sigma} \Big(\frac{\partial f_\sigma}{\partial x_1} \alpha_1^\sigma + \ldots + \frac{\partial f_\sigma}{\partial x_d} \alpha_d^\sigma \Big) \,\mathrm{d}\Sigma_\sigma \\
&= \sum_{\sigma \in F_d(K)} \int_{\Sigma_\sigma} \nabla f_\sigma \bullet \alpha^\sigma \,\mathrm{d}\Sigma_\sigma \\
&= \sum_{\sigma \in F_d(K)} \Big(\int_{\partial\Sigma_\sigma} f_\sigma \cdot \alpha^\sigma \bullet \nu_\sigma \,\mathrm{d}\partial\Sigma_\sigma - \int_{\Sigma_\sigma} f_\sigma \nabla \bullet \alpha^\sigma \,\mathrm{d}\Sigma_\sigma \Big) \\
&= \sum_{\sigma \in F_d(K)} \Big(\sum_{\tau <_1 \sigma} \int_{\Sigma_\tau} f_\sigma \cdot \alpha^\sigma \bullet \nu_\sigma \,\mathrm{d}\Sigma_\tau - \int_{\Sigma_\sigma} f_\sigma \nabla \bullet \alpha^\sigma \,\mathrm{d}\Sigma_\sigma \Big) \\
&= \sum_{\tau \in F_{d-1}(K)} \int_{\Sigma_\tau} f_\tau \cdot \sum_{\sigma > \tau} \underbrace{\alpha^\sigma \bullet \nu_\sigma}_{= \alpha_\sigma(\nu_\sigma)} \,\mathrm{d}\Sigma_\tau - \langle f, d^*\alpha \rangle_M.
\end{aligned}$$

Since f is an arbitrary test function from dom$d$ the (almost everywhere) vanishing of the function

$$\Sigma_\tau \to \mathbb{R};\ x \mapsto \sum_{\sigma > \tau} \alpha_\sigma(\nu_\sigma)_{|x}$$

is sufficient for the claim to hold. □

This lemma immediately gives rise to a well-defined Laplacian operator

$$\Delta := \mathrm{d}^* \, \mathrm{d}$$

densely defined on the domain

$$\{u \in \mathrm{dom} d \mid \ \mathrm{d}u \in \mathrm{dom} d^*\}$$

which when regarding all the above can be written as

$$H^2_K(M) := \{u \in H^2(M) \cap C(M) \mid \forall_{\tau \in F_{d-1}(K),\ x \in \Sigma_\tau} :\ \sum_{\sigma > \tau} \partial_{\nu_\sigma} u_\sigma(x) = 0\} \subset L^2(M),$$

where the subscript K should imply that for the functions $u \in H^2_K(M)$ the Kirchhoff boundary condition for higher-dimensional domains holds.

Definition II.3. A quantum simplicial complex is a triple $M = (K, \Sigma, \Delta)$ for a metric simplicial complex (K, Σ) with its Laplacian operator

$$\Delta := \mathrm{d}^* \, \mathrm{d}.$$

We also write $\Delta(M) := \Delta$.

Thin Manifolds with model Complex

Now that we have defined the concept of a quantum simplicial complex $M = (K, \Sigma, \Delta)$ we can define what a thin manifold with model complex M should be.

Definition II.4. Let $M_0 = (K, \Sigma, \Delta)$ be a quantum simplicial complex. A Riemannian manifold (M, g) is said to have *model complex* M_0 if M has a decomposition

$$M = \bigcup_{\tau \in K} X_\tau$$

into closed sets X_τ sufficing the following properties:

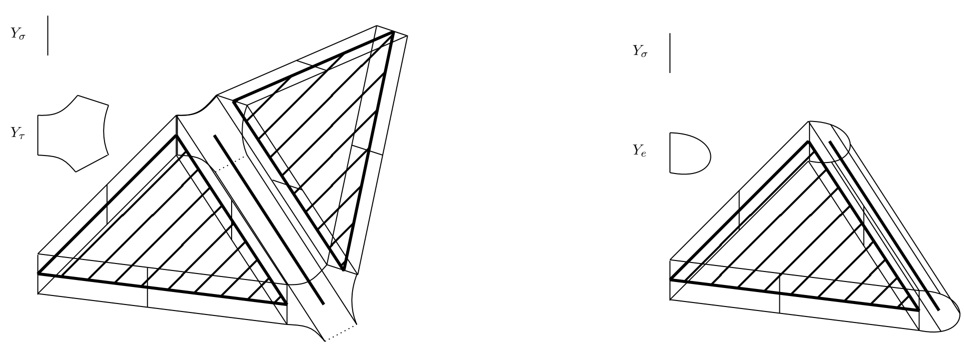
- $\{\mathring{X}_\tau\}_{\tau \in K}$ are mutually disjoint.
- $X_\tau \cap X_{\tau'} = \emptyset$ if neither $\tau \leq \tau'$ nor $\tau' \leq \tau$.
- For $\tau \in F_k(K)$ there exists a connected, compact Riemannian manifold (Y_τ, h_τ) such that X_τ is as a Riemannian manifold with the metric induced from g isomorphic to

$$(\Sigma_\tau \times Y_\tau, g_\tau).$$

- For $\tau \leq \tau'$ it holds that

$$X_\tau \cap X'_\tau \cong \Sigma_\tau \times Y_{\tau'}.$$

Besides the case when $\dim K \leq 1$ we can only illustrate cases when $\dim Y_\sigma = 1$ and $\dim K = 2$. So for example consider $Y_\sigma = [-1/2, 1/2]$ for $\dim K = 2$. The fibers along edges $e \in F_1(K)$ with more than one coface need to be junctions of thickened simplices as in Figure 3.1a. Finally open edges, i.e. edges with only one coface, are capped off as in Figure 3.1b. The vertices fill in and smooth out the remaining voids.



(a) Example of a three-junction. The lower simplex is not drawn to maintain clarity.

(b) A capped edge.

Figure 3.1: Examples of the decomposition $M = \bigcup_{\tau \in K} X_\tau$ for $\dim K = 2$ and $Y_\sigma = [-1/2, 1/2]$. The linear directions are indicated by thickened lines and hatched areas.

In order to define when a sequence of such manifolds behaves similar to the standard product metric on every X_τ we will introduce the notion of asymptotic equivalence of a metric to the product metric. Let $(M_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of Riemannian manifolds with model complex M_0 such that $M_\varepsilon = (M, g_\varepsilon)$ (i.e. M_ε are all defined on the same differential structure with varying Riemannian manifolds). We say that g_ε is asymptotically equivalent to the product metric on X_τ if

$$(G_\varepsilon)_\tau - \begin{pmatrix} 1 & 0 \\ 0 & (H_\varepsilon)_\tau \end{pmatrix} = \begin{pmatrix} o(1) & 0 \\ 0 & o(\varepsilon^2) \end{pmatrix},$$

where $(G_\varepsilon)_\tau$ denotes the matrix of the metric $(g_\varepsilon)_\tau$ and $(H_\varepsilon)_\tau$ denotes the metric of $(h_\varepsilon)_\tau$ (i.e. the metric on Y_τ in M_ε) in some chart of $\Sigma_\tau \times Y_\tau$ (where we take the standard chart in the linear direction Σ_τ). Here $o(a_n)$ denotes a sequence of matrices where every entry is $o(a_n)$.

Let $\dim \tau = i$, we then write

$$(g_\varepsilon)_\tau \sim \mathrm{d}x_1^2 + ... + \mathrm{d}x_i^2 + (h_\varepsilon)_\tau.$$

The idea behind this ease of condition on the metric is that we allow a wider range of manifolds smoothing out between the different parts X_τ of the manifold.

The object we define next formalizes the process of interpolating between a modelled manifold and its underlying complex via scaling the fiber by a ε-homothety.

Definition II.5. Let $M_0 = (K, \Sigma, \Delta)$ be a quantum simplicial complex. A one-parameter family $(M_\varepsilon)_{0<\varepsilon\leq 1}$ is called a *thin manifold with model complex* M_0 if for every $0 < \varepsilon \leq 1$ M_ε is a Riemannian manifold

$$M_\varepsilon = (M, g_\varepsilon),$$

where the differential structure is a fixed manifold M, M_1 has model complex M_0 and g_ε is asymptotically equivalent to the product metric of the standard metric on Σ_τ and the ε-homothety of h_τ on every X_τ, i.e.

$$(g_\varepsilon)_\tau \sim \mathrm{d}x_1^2 + ... + \mathrm{d}x_i^2 + \varepsilon^2 h_\tau$$

for $i = \dim \tau$. Here h_τ denotes some fixed Riemannian metric of Y_τ.

We denote for a thin manifold M_ε by

$$\operatorname{ess\,dim} M_\varepsilon = \dim K$$

the essential dimension of $\{M_\varepsilon\}_{0<\varepsilon\leq 1}$. Note that we will say M_ε is a thin manifold though this notion really only makes sense for an entire family of manifolds.

First note that in the beginning we fixed K to be a finite complex and by definition every fiber manifold Y_τ is compact. Thus for our purposes M_ε is always a compact manifold.

It is immediate that this is a generalization of compact graph-like thin manifolds in case $\operatorname{ess\,dim} M_\varepsilon = 1$. In what follows we will denote by $\theta_\tau = \dim Y_\tau$ the fiber dimension over Σ_τ and by $\theta = \theta_\sigma$ for some $\sigma \in F_d(K)$ the principal transversal dimension.

Manifolds and their Laplacian

We assign to M_ε a Laplacian $\Delta(M_\varepsilon)$ now. Assume (M, g) is a compact Riemannian manifold with piecewise smooth boundary and denote by $L^2(M) = L^2(M, \mathrm{d}M)$ the space of square integrable functions on M with respect to the volume form $\mathrm{d}M$. We denote the norm of this space by $||\cdot||_M$. For a function $u \in C^\infty(M)$ we set

$$q_M(u) := ||\,\mathrm{d}u||_M^2 := \int_M |\,\mathrm{d}u|^2 \,\mathrm{d}M,$$

where $|\,\mathrm{d}u|^2$ is the norm on 1-forms defined via Hodge-$*$-operator as

$$|\,\mathrm{d}u|^2 \,\mathrm{d}M = \mathrm{d}u \wedge *\,\mathrm{d}u,$$

i.e. in a chart it holds that

$$|\,\mathrm{d}u|^2 = \partial u G^{-1} (\partial \overline{u})^t,$$

where G is the matrix representing g in the chart and ∂u is the tangent vector of partial derivatives of u with respect to that chart (in order to distinguish it from the gradient ∇u). q_M is defined on the domain $H^1(M) := \{u \in L^2(M) \mid \mathrm{d}u \in L^2(T^*M)\}$, i.e. the L^2 function with weak outer derivative in L^2. The Laplacian of M, $\Delta(M)$, is then given as the unique self-adjoint and non-negative operator associated with the form q_M.

Remark 5. It is well known by the theory of elliptic partial differential equations that $\Delta(M)$ has purely discrete spectrum if M is compact. The same is true for the common Neumann Laplacian on open bounded subsets of $\mathbb{R}^d$ with piecewise smooth boundary. The same line of reasoning as for the Neumann Laplacian can be applied in order to see that for M a quantum complex the operator $\Delta(M)$ also must have discrete spectrum. Thus henceforth for a manifold or quantum complex M we will denote by $\lambda_k(M)$ the eigenvalues listed in ascending order with multiplicities.

Some elementary results

In this section we deduce partial results similar to those used in the proof of the main result of [EP05].

Lemma II.6 (Poincaré-type estimate, [EP05])**.** *Let X be a connected, compact manifold with smooth boundary ∂X. For $u \in H^1(X)$ we let u_0 denote its average*

$$u_0 := \frac{1}{volX} \int_X u(x)\,\mathrm{d}x.$$

The following estimates hold:

$$||u_0||_X^2 \leq ||u||_X^2$$
$$||u - u_0||_X^2 \leq \frac{1}{\lambda_2^N(X)} ||\,\mathrm{d}u||_X^2$$
$$||u||_X^2 - ||u_0||_X^2 \leq \frac{1}{\delta \lambda_2^N(X)} ||\,\mathrm{d}u||_X + \delta ||u||_X^2$$

for any $\delta > 0$. Here $\lambda_2^N(X)$ denotes the first non-zero eigenvalue of the Neumann-Laplacian of X.

We will also make use of the following two variations of the trace theorem.

Lemma II.7 (Trace Theorem for manifolds, Proposition 4.5 in [Tay10])**.** *Let X be a smooth, compact manifold with smooth boundary ∂X. There exists a constant $c > 0$ such that for $u \in H^1(X)$ it holds that*

$$||\gamma u||_{\partial X}^2 \leq c(||u||_X^2 + ||\,\mathrm{d}u||_X^2),$$

where $\gamma : H^1(X) \to L^2(\partial X)$ denotes the trace operator.

Lemma II.8. *[Trace Theorem for Lipschitz domains, [Din96]] Let Ω be a Lipschitz domain. Then it exists a constant $C > 0$ such that*

$$||\gamma u||_{L^2(\partial\Omega)} \leq C||u||_{1,\Omega} = C(||u||_{L^2(\Omega)} + ||\nabla u||_{L^2(\Omega)}).$$

for all $u \in H^1(\Omega)$, where $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ denotes the trace operator.

Spectral Asymptotics

We will show a relaxation of the key lemma in [EP05]. This would allow for us to only work on C^∞ functions on the manifold and complex (which replaces some assumptions that can be made for the graph case, e.g. Sobolev functions being continuous). It is a consequence of the min-max principle for eigenvalues of operators with purely discrete spectrum.

We thus assume Q and Q' are densely defined, self-adjoint, non-negative operators on seperable Hilbert spaces $(\mathscr{H}, ||\cdot||)$ and $(\mathscr{H}', ||\cdot||')$ associated to the quadratic forms q and q' defined on $\mathscr{D} \subset \mathscr{H}$ and $\mathscr{D}' \subset \mathscr{H}'$ with purely discrete spectra $(\lambda_i)_{i\in\mathbb{N}}$ and $(\lambda'_i)_{i\in\mathbb{N}}$. Further for a multi-index $\boldsymbol{n} \in \mathbb{N}^k$ we define a norm by

$$||u||^2_{Q,\boldsymbol{n}} := ||u||^2 + ||Q^{n_1/2}u||^2 + ... + ||Q^{n_k/2}u||^2.$$

The version of the min-max principle we use can be found in [Sch12, Section 12.1] and it allows us to compute the k-th eigenvalue of Q in the above setting as

$$\lambda_k(Q) = \inf_{L_k} \sup_{u\in L_k\setminus\{0\}} \frac{||q(u)||^2}{||u||^2},$$

where L_k runs over all k-dimensional subspaces of $\mathscr{D}$, the domain of definition of the form q associated to Q.

Then the key lemma of [EP05] can be generalized to the following result.

Lemma II.9. *Let $\phi : \mathscr{D} \to \mathscr{D}'$ be a linear map between the domains of definition of q and q'. Assume that all eigenvectors of Q are contained in the subspace $\mathscr{E} \leq \mathscr{H}$. If there exist multi-indices $\boldsymbol{n} \in \mathbb{N}^i, \boldsymbol{m} \in \mathbb{N}^\ell$ and $\delta_1, \delta_2 \geq 0$ such that*

$$\mathscr{E} \subseteq dom(Q^{\max(\max \boldsymbol{n}, \max \boldsymbol{m})/2})$$

and

$$||u||^2 \leq ||\phi u||'^2 + \delta_1 ||u||^2_{Q,\boldsymbol{n}}$$
$$q(u) \geq q'(\phi u) - \delta_2 ||u||^2_{Q,\boldsymbol{m}}$$

for all $u \in \mathscr{E}$, then for every k there exists a positive function $\eta_k(\lambda_k, \delta_1, \delta_2) \to 0$ for $\delta_1, \delta_2 \to 0$ such that

$$\lambda_k \geq \lambda'_k - \eta_k.$$

Proof. As in [EP05] let $\phi_1, ..., \phi_k$ be an orthonormal system of eigenvectors corresponding to the eigenvalues $\lambda_1, ..., \lambda_k$ of Q. For $u \in E_k$ the linear span of $\phi_1, ..., \phi_k$ we obviously have $u \in \mathscr{E}$. Furthermore for $\boldsymbol{n} \in \mathbb{N}^i$ it holds that

$$\begin{aligned} ||u||^2_{Q,\boldsymbol{n}} &= ||u||^2 + ||Q^{n_1/2}u||^2 + ... + ||Q^{n_i/2}u||^2 \\ &\leq ||u||^2 + \lambda_k^{n_1}||u||^2 + ... + \lambda_k^{n_i}||u||^2 \\ &= \underbrace{(1 + \lambda_k^{n_1} + ... + \lambda_k^{n_i})}_{=:\Lambda_{k,\boldsymbol{n}}} ||u||^2. \end{aligned}$$

By the presumed inequalities on $\mathscr{E}$ we obtain

$$\begin{aligned} \frac{q(\phi u)}{||\phi u||'^2} - \frac{q(u)}{||u||^2} &\leq \frac{1}{||\phi u||'^2}\Big(q(u) + \delta_2||u||^2_{Q,\boldsymbol{m}} - \frac{q(u)}{||u||^2}(||u||^2 - \delta_1||u||^2_{Q,\boldsymbol{n}})\Big) \\ &= \frac{1}{||\phi u||'^2}\Big(\delta_1||u||^2_{Q,\boldsymbol{n}}\frac{q(u)}{||u||^2} + \delta_2||u||^2_{Q,\boldsymbol{m}}\Big) \\ &\leq \frac{||u||^2}{||\phi u||'^2}\Big(\delta_1\lambda_k\Lambda_{k,\boldsymbol{n}} + \delta_2\Lambda_{k,\boldsymbol{m}}\Big). \end{aligned}$$

We rearrange the first of the presumed inequalities on $\mathscr{E}$ to obtain

$$||\phi u||'^2 \geq ||u||^2 - \delta_1 ||u||^2_{Q,\boldsymbol{n}} \geq (1 - \delta_1 \Lambda_{k,\boldsymbol{n}})||u||^2. \tag{4.1}$$

Thus combining the above we obtain

$$\frac{q(\phi u)}{||\phi u||'^2} - \frac{q(u)}{||u||^2} \leq \frac{\delta_1 \lambda_k \Lambda_{k,\boldsymbol{n}} + \delta_2 \Lambda_{k,\boldsymbol{m}}}{1 - \delta_1 \Lambda_{k,\boldsymbol{n}}} =: \eta_k(\lambda_k, \delta_1, \delta_2) =: \eta_k$$

for $\delta_1 < 1/\Lambda_{k,\boldsymbol{n}}$. Further for such values of δ_1 equation (4.1) implies the injectivity of ϕ so that $\phi(E_k)$ is k-dimensional. Thus applying the min-max principle to q' we obtain

$$\begin{aligned} \lambda'_k = \inf_{L_k} \sup_{u \in L_k \setminus \{0\}} \frac{q'(u)}{||u||'^2} &\leq \sup_{u \in \phi(E_k) \setminus \{0\}} \frac{q'(u)}{||u||'^2} \\ &= \sup_{u \in E_k \setminus \{0\}} \frac{q'(\phi u)}{||\phi u||'^2} \\ &\leq \sup_{u \in E_k \setminus \{0\}} \frac{q(u)}{||u||^2} + \eta_k \\ &= \lambda_k + \eta_k. \end{aligned}$$

□

Transversal Averaging and useful inequalities

For the rest of the section let $(M_\varepsilon)_{0<\varepsilon\leq 1}$ denote a thin manifold with model complex M_0.

We will show that we can immediately work with the simplified assumption that

$$(g_\varepsilon)_\tau = \mathrm{d}x_1^2 + ... + \mathrm{d}x_i^2 + \varepsilon^2 h_\tau$$

as the error terms introduced in all quantities handled by the following results only changes by a $(1+o(1))$ coefficient (in the linear direction) or a $o(\varepsilon)$ coefficient in the fiber direction.

Lemma II.10 ([EP05])**.** *Let $(g_\varepsilon)_\tau$ be the Riemannian metric of $(M_\varepsilon)_{0<\varepsilon\leq 1}$ and let $(\tilde{g}_\varepsilon)_\tau = \mathrm{d}x_1^2 + ... + \mathrm{d}x_i^2 + \varepsilon^2 h_\tau$ denote the product metric ε-homothetic to h_τ in the fiber. The following asympototics hold:*

$$\begin{aligned} (\det(G_\varepsilon)_\tau)^{1/2} &= (1+o(1))(\det(\tilde{G}_\varepsilon)_\tau)^{1/2} \\ ((G_\varepsilon)_\tau)^{-1} &= 1 + o(1) \\ |\,\mathrm{d}_{\Sigma_\tau} u|^2 &\leq (1+o(1))|\,\mathrm{d}u|^2_{(g_\varepsilon)_\tau} \\ |\,\mathrm{d}_{Y_\tau} u|^2_{h_\tau} &= o(\varepsilon)|\,\mathrm{d}u|^2_{g_\varepsilon}. \end{aligned}$$

Proof. The first equation follows as in [EP05] from

$$\det((G_\varepsilon)_\tau(\tilde{G}_\varepsilon)_\tau^{-1}) = \det\begin{pmatrix} 1+o(1) & 0 \\ 0 & 1+o(1) \end{pmatrix} = 1 + o(1)$$

by the continuity of the determinant.

The second equation follows from the continuity of Inversion around 1.

The third inequality can be shown the same way as in [EP05] via

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq (1 + o(1))(G_\varepsilon)_\tau^{-1}$$

in the sense of quadratic forms which follows if we have

$$\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \leq (1 + o(1))(G_\varepsilon)_\tau^{-1}$$

or equivalently

$$(G_\varepsilon)_\tau \leq (1 + o(1)) \begin{pmatrix} 1 & 0 \\ 0 & \delta^{-1} \end{pmatrix}.$$

Note by definition that

$$(G_\varepsilon)_\tau = \begin{pmatrix} 1 + o(1) & 0 \\ 0 & O(\varepsilon^2) \end{pmatrix} \leq (1 + o(1)) \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

as $O(\varepsilon)$ is $o(1)$. So that the inequality involving δ holds whenever $\delta < c^{-1}$. The fourth inequality is proven similarly. $\square$

The previous Lemma allows us to work only with $(\tilde{g}_\varepsilon)_\tau$ from now on; we thus assume that the family $(M_\varepsilon)_{0<\varepsilon\leq 1}$ is locally the product Riemannian manifold on every X_τ without mentioning that we can transition to this case because of Lemma II.10. The reason that the coefficients introduced by transitioning to the product metric are negligable lays in Lemma II.9 and the following estimates; note that the coefficients are always compatible with the coefficients we'll obtain. All $o(1)$ and $o(\varepsilon)$ coefficients could be collected in δ_1, δ_2 in Lemma II.9 if the required estimate holds for the product metric.

First we analyze the behaviour of the function norm with respect to the homothetic parameter ε.

Lemma II.11. *The following L^2-relations hold:*

- *Let $f \in L^2(M_\varepsilon)$, then*
$$||f||_{X_{\tau,\varepsilon}} = \varepsilon^{\theta_\tau/2} ||f||_{X_\tau}.$$
- *Let $f \in H^1(Y_\tau)$, then*
$$||\,\mathrm{d}_{Y_\tau} f||_{Y_{\tau,\varepsilon}} = \varepsilon^{\theta_\tau/2-1} ||\,\mathrm{d}_{Y_\tau} f||_{Y_\tau}.$$

Proof. Note that the Riemannian metric of M_ε is locally (on X_τ) given by the matrix

$$\begin{pmatrix} \mathrm{Id}_{\dim \tau} & 0 \\ 0 & \varepsilon^2 H_\tau \end{pmatrix},$$

where H_τ is the matrix corresponding to the Riemannian metric fixed for Y_τ.

In particular

$$\int_{X_{\tau,\varepsilon}} |f(\zeta)|^2 \,\mathrm{d}\zeta = \int_{\Sigma_\tau} \int_{Y_{\tau,\varepsilon}} |f(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x.$$

Since the integral over Y_τ is given as a sum of localized integrals it suffices to carry on the analysis on a local chart $\varphi : U \to V$ for open subsets $U \subseteq \mathbb{R}^{\theta_\tau}$, $V \subseteq Y_\tau$. The localized integral then has the form

$$\int_{\Sigma_\tau} \int_U |f(x, \varphi(y))|^2 \sqrt{\det(\varepsilon^2 H_\tau)}\, \mathrm{d}y\, \mathrm{d}x = \varepsilon^{\theta_\tau} \int_{\Sigma_\tau} \int_U |f(x, \varphi(y))|^2 \sqrt{\det H_\tau}\, \mathrm{d}y\, \mathrm{d}x,$$

where the integral on the right-hand side is the integral of $|f|^2$ over $X_{\tau,1}$ localized on the chart (φ, U). Thus

$$||f||^2_{X_{\tau,\varepsilon}} = \varepsilon^{\theta_\tau} ||f||^2_{X_\tau}.$$

For the second part observe that

$$||\, \mathrm{d}_{Y_\tau} f||^2_{Y_{\tau,\varepsilon}} = \int_{Y_{\tau,\varepsilon}} |\, \mathrm{d}f|^2_{\varepsilon^2 h},$$

where

$$|\, \mathrm{d}f|^2_{\varepsilon^2 h} = (\nabla_Y f)^t (\varepsilon^2 H^{-1})(\nabla_Y f),$$

where H is the local matrix representation of the Riemannian metric h.

Thus

$$||\, \mathrm{d}_{Y_\tau} f||^2_{Y_{\tau,\varepsilon}} = \frac{1}{\varepsilon^2} \int_{Y_{\tau,\varepsilon}} |\, \mathrm{d}f|^2_h$$

and with the same argument as in the first part we obtain

$$||\, \mathrm{d}_{Y_\tau} f||^2_{Y_{\tau,\varepsilon}} = \varepsilon^{\theta_\tau - 2} ||\, \mathrm{d}_{Y_\tau} f||^2_{Y_\tau}.$$

□

Let $u \in H^2(M_\varepsilon)$ for a thin manifold M_ε. Then we denote for $\tau \in M_0$ by $\overline{u_\tau}$ the transversal averaging function

$$\overline{u_\tau} : \ \Sigma_\tau \to \mathbb{R};\ x \mapsto \frac{1}{\mathrm{vol} Y_\tau} \int_{Y_\tau} u(x,y) dy$$

where $u(x, y)$ is the function u on $X_\tau \cong \Sigma_\tau \times Y_\tau$ under the obvious identifications. This operation will be useful in order to pull back H^2-functions from M_ε to M_0. The following result motivates the construction of the L^2-space of a quantum simplicial complex M_0 as the one integrating only the top-dimensional simplices.

Lemma II.12. *Let M_ε be a thin manifold and let $u \in L^2(M_\varepsilon)$ such that u is constant in transversal direction, i.e.*

$$u_{|X_\tau}(x, y) = u_\tau(x)$$

for $u_\tau \in L^2(\Sigma_\tau)$. Then

$$||u||^2_{X_{\tau,\varepsilon}} = O(\varepsilon^{\theta_\tau}) ||u_\tau||^2_{\Sigma_\tau}.$$

In particular it holds that

$$||u||^2_{M_\varepsilon} = O(\varepsilon^\theta) \sum_{i=0}^{d} \sum_{\tau \in F_i(K)} \varepsilon^{d-i} ||u_\tau||_{\Sigma_\tau}.$$

Proof. The second part is immediate if the first part holds.

For the first part observe that by Lemma II.11

$$\begin{aligned}
||u||^2_{X_{\tau,\varepsilon}} &= \int_{Y_{\tau,\varepsilon}} \int_{\Sigma_\tau} |u(x,y)|^2 \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{Y_{\tau,\varepsilon}} \int_{\Sigma_\tau} |u_\tau(x)|^2 \,\mathrm{d}y \,\mathrm{d}x \\
&= \varepsilon^{\theta_\tau} \mathrm{vol}(Y_\tau) \int_{\Sigma_\tau} |u_\tau(x)|^2 \,\mathrm{d}x \\
&= O(\varepsilon^{\theta_\tau}) ||u_\tau||^2_{\Sigma_\tau}.
\end{aligned}$$

□

Lemma II.13. *Let $u \in L^2(M_\varepsilon)$ and $\tau \in K$. Then it holds that*

$$||\overline{u_\tau}||^2_{X_{\tau,\varepsilon}} \leq ||u||^2_{X_{\tau,\varepsilon}}.$$

Proof. In view of Lemma II.12 and the Cauchy-Schwarz inequality it holds that

$$\begin{aligned}
||\overline{u_\tau}||^2_{X_{\tau,\varepsilon}} &= \mathrm{vol}(Y_\tau)\varepsilon^{\theta_\tau} \int_{\Sigma_\tau} \Big| \frac{1}{\mathrm{vol} Y_\tau} \int_{Y_\tau} u(x,y) \,\mathrm{d}y \Big|^2 \mathrm{d}x \\
&\leq \varepsilon^{\theta_\tau} \int_{\Sigma_\tau} \int_{Y_\tau} |u(x,y)|^2 \,\mathrm{d}y \,\mathrm{d}x \\
&= ||u||^2_{X_{\tau,\varepsilon}}.
\end{aligned}$$

□

Next we show a Poincaré-type theorem for thin manifolds.

Lemma II.14. *Let $u \in H^1(M_\varepsilon)$ and $\nu \in M_0$. Then it holds that*

$$||u - \overline{u_\nu}||_{X_{\nu,\varepsilon}} \leq \varepsilon || \,\mathrm{d}u||_{X_{\nu,\varepsilon}}$$

and

$$||u||^2_{X_{\nu,\varepsilon}} - ||\overline{u_\nu}||^2_{X_{\nu,\varepsilon}} \leq O(\varepsilon^{1/2})(||u||^2_{X_{\nu,\varepsilon}} + || \,\mathrm{d}u||^2_{X_{\nu,\varepsilon}}).$$

Proof. By using Lemma II.11 we obtain

$$||u - \overline{u_\nu}||^2_{X_{\nu,\varepsilon}} \leq \varepsilon^{\theta_\nu} ||u - \overline{u_\nu}||^2_{X_\nu}$$

and can apply the Poincaré-type estimate, Lemma II.6, now to obtain

$$||u - \overline{u_\nu}||^2_{X_\nu} = \int_{\Sigma_\nu} ||u(x,\cdot) - u_0(x)||^2_{Y_\nu} \,\mathrm{d}x \leq \frac{1}{\lambda_2^N(Y_\nu)} \int_{\Sigma_\nu} || \,\mathrm{d}_{Y_\nu} u(x,\cdot)||^2_{Y_\nu} \,\mathrm{d}x = C \int_{\Sigma_\nu} \varepsilon^{2-\theta_\nu} || \,\mathrm{d}_{Y_\nu} u(x,\cdot)||^2_{Y_{\nu,\varepsilon}} dx,$$

where we applied the second part of Lemma II.11 in the last equality. The claim then follows by the obvious inequality

$$|| \,\mathrm{d}_{Y_\nu} u||_{X_{\nu,\varepsilon}} \leq || \,\mathrm{d}Y_\nu u||_{X_\nu}.$$

For the second inequality we again use Lemma II.6 to obtain

$$||u(x,\cdot)||^2_{Y_\nu} - |\overline{u_\nu}(x)|^2 \leq \frac{1}{\delta \lambda_2^N(Y_\nu)} || \,\mathrm{d}_Y u(x,\cdot)||^2_{Y_\nu} + \delta ||u(x,\cdot)||^2_{Y_\nu}$$

□

Lemma II.15. *Let $u \in H^1(M_\varepsilon)$ and $\nu \in M_0$ be a non-facet, i.e.* $\dim \nu < d$, *aswell as $\tau > \nu$ be a codimension-1-coface. It holds that*

$$||\overline{u_\nu} - \overline{u_\tau}_{|X_\nu}||_{X_{\nu,\varepsilon}} \leq O(\varepsilon)||\,\mathrm{d}u||_{X_{\nu,\varepsilon}}.$$

Proof.

$$\begin{aligned} ||\overline{u_\nu} - \overline{u_\tau}_{|X_\nu}||^2_{X_{\nu,\varepsilon}} &= \varepsilon^{\theta_\nu}\mathrm{vol}Y_\nu \int_{\Sigma_\nu} |\overline{u_\nu} - \overline{u_\tau}_{|X_\nu}|^2 \\ &= C\varepsilon^{\theta_\nu}\int_{\Sigma_\nu} \Big|\frac{1}{\mathrm{vol}Y_\tau}\int_{Y_\tau} \overline{u_\nu}(x)\,\mathrm{d}y - \frac{1}{\mathrm{vol}Y_\tau}\int_{Y_\tau} u(x,y)\,\mathrm{d}y\Big|^2\,\mathrm{d}x \\ &= \frac{C\varepsilon^{\theta_\nu}}{(\mathrm{vol}Y_\tau)^2}\int_{\Sigma_\nu}\Big|\int_{Y_\tau}(\overline{u_\nu}(x) - u(x,y))\,\mathrm{d}y\Big|^2\,\mathrm{d}x \\ &\leq \frac{C\varepsilon^{\theta_\nu}}{\mathrm{vol}Y_\tau}\int_{\Sigma_\nu}\int_{Y_\tau}|u(x,y) - \overline{u_\nu}(x)|^2\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

We now analyze the interior integral using Lemma II.7; to this end observe that

$$\int_{Y_\tau}|u(x,y) - \overline{u_\nu}(x)|^2\,\mathrm{d}y \leq \int_{\partial Y_\nu}|u(x,y) - \overline{u_\nu}(x)|^2\,\mathrm{d}y \leq c(||u(x,\cdot) - \overline{u_\nu}(x)||^2_{Y_\nu} + ||\,\mathrm{d}_{Y_\nu}u(x,\cdot)||^2_{Y_\nu}).$$

By Lemma II.11 this can be bound above by

$$c\varepsilon^{-\theta_\nu}(||u(x,\cdot) - \overline{u_\nu}||^2_{Y_{\nu,\varepsilon}} + \varepsilon^2||\,\mathrm{d}_{Y_\nu}u(x,\cdot)||^2_{Y_{\nu,\varepsilon}}).$$

In summary we obtain

$$||\overline{u_\tau} - \overline{u_\nu}_{|X_\nu}||^2_{X_{\nu,\varepsilon}} \leq O(1)(||u - \overline{u_\nu}||^2_{X_{\nu,\varepsilon}} + \varepsilon^2||\,\mathrm{d}u||^2_{X_{\nu,\varepsilon}})$$

and thus by the previous Lemma II.14

$$||\overline{u_\tau} - \overline{u_\nu}_{|X_\nu}||^2_{X_{\nu,\varepsilon}} \leq O(\varepsilon^2)||\,\mathrm{d}u||^2_{X_{\nu,\varepsilon}}.$$

□

Lemma II.16. *Let $u \in H^1(M_\varepsilon)$ and $\nu \in M_0$ be a non-facet, i.e.* $\dim \nu < d$, *aswell as $\tau > \nu$ be a codimension-1-coface. It holds that*

$$||\overline{u_\tau}_{|X_\nu}||^2_{X_{\nu,\varepsilon}} \leq O(\varepsilon)(||u||^2_{X_{\tau,\varepsilon}} + ||\,\mathrm{d}u||^2_{X_{\tau,\varepsilon}}).$$

Proof. Obviously we have

$$\begin{aligned} ||\overline{u_\tau}_{|X_\nu}||^2_{X_{\nu,\varepsilon}} &= \varepsilon^{\theta_\nu}||\overline{u_\tau}_{|X_\nu}||^2_{\Sigma_\nu} \\ &= \varepsilon^{\theta_\nu}\int_{\Sigma_\nu}\Big|\int_{Y_\tau} u(x,y)\,\mathrm{d}y\Big|^2\,\mathrm{d}x \\ &\leq \mathrm{vol}Y_\tau\varepsilon^{\theta_\nu}\int_{\Sigma_\nu}\int_{Y_\tau}|u(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \mathrm{vol}Y_\tau\varepsilon^{\theta_\nu}\int_{\partial\Sigma_\tau}\int_{Y_\tau}|u(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x \\ &= O(\varepsilon^{\theta_\nu})\int_{Y_\tau}||u_{|\partial\Sigma_\tau}(\cdot,y)||^2_{\partial\Sigma_\tau}\,\mathrm{d}y. \end{aligned}$$

We will now employ Lemma II.8 in order to estimate

$$||u_{|\partial\Sigma_\tau}(\cdot,y)||^2_{\partial\Sigma_\tau} \leq O(1)(||u(\cdot,y)||^2_{\Sigma_\tau} + ||d_\tau u(\cdot,y)||^2_{\Sigma_\tau}).$$

Thus together with the above we yield

$$\begin{aligned} ||\overline{u_\tau}_{|X_\nu}||^2_{X_{\nu,\varepsilon}} &\leq O(\varepsilon^{\theta_\nu})(||u||^2_{X_\tau} + ||d_\tau u||^2_{X_\tau}) \\ &= O(\varepsilon^{\theta_\nu-\theta_\tau})(||u||^2_{X_{\tau,\varepsilon}} + ||\,\mathrm{d}_\tau u||^2_{X_{\tau,\varepsilon}}) \\ &\leq O(\varepsilon)(||u||^2_{X_{\tau,\varepsilon}} + ||\,\mathrm{d}u||^2_{X_{\tau,\varepsilon}}). \end{aligned}$$

□

The following lemma will be crucial in our analysis since it allows us to estimate the parts of the L^2-norm induced by domains in the manifold corresponding to lower-dimensional simplices in M_0.

Lemma II.17. *Let $u \in H^1(M_\varepsilon)$ and $\nu \in M_0$ be a non-facet, i.e.* $\dim \nu < d$*. Then for any given codimension-*1*-coface $\tau > \nu$ it holds that*

$$||u||^2_{X_{\nu,\varepsilon}} \leq O(\varepsilon)(||u||^2_{X_{\nu,\varepsilon}\cup X_{\tau,\varepsilon}} + ||\,\mathrm{d}u||^2_{X_{\nu,\varepsilon}\cup X_{\tau,\varepsilon}}).$$

Proof. We have

$$||u||_{X_{\nu,\varepsilon}} \leq ||u - \overline{u_\nu}||_{X_{\nu,\varepsilon}} + ||\overline{u_\nu} - \overline{u_\tau}_{|X_\nu}||_{X_{\nu,\varepsilon}} + ||\overline{u_\tau}_{|X_\nu}||_{X_{\nu,\varepsilon}}.$$

The first norm can be estimated by Lemma II.14, the second one by Lemma II.15 and the third by Lemma II.16. Note that need to apply the square root to the third estimate first. Thus we obtain the upper bound

$$||u||_{X_{\nu,\varepsilon}} \leq O(\varepsilon^{1/2})\sqrt{||u||^2_{X_{\nu,\varepsilon}\cup X_{\tau,\varepsilon}} + ||\,\mathrm{d}u||^2_{X_{\nu,\varepsilon}\cup X_{\tau,\varepsilon}}}.$$

□

Corollary II.18. *Let $u \in H^1(M_\varepsilon)$. Then it holds that*

$$||u||^2_{M_\varepsilon} \leq \sum_{\sigma\in F_d(M_0)} ||u||_{X_{\sigma,\varepsilon}} + O(\varepsilon)(||u||_{M_\varepsilon} + ||\,\mathrm{d}u||_{M_\varepsilon}).$$

Proof. This result immediately follows from the previous lemma.

$$||u||^2_{M_\varepsilon} = \sum_{i=0}^{d} \sum_{\tau\in F_i(M_0)} ||u||^2_{X_{\tau,\varepsilon}}.$$

Now in order to bound the summands for $i < d$ for every $\nu \in F_i(M_0)$ we choose an arbitrary codimension-1-coface τ and thus obtain

$$||u||^2_{X_{\nu,\varepsilon}} \leq O(\varepsilon)(||u||_{X_{\nu,\varepsilon}} + ||\,\mathrm{d}u||_{X_{\nu,\varepsilon}} + ||u||_{X_{\tau,\varepsilon}} + ||\,\mathrm{d}u||_{X_{\tau,\varepsilon}}).$$

In particular summing over the norms for ν we obtain an upper bound of $O(\varepsilon)(||u||_{M_\varepsilon} + ||\,\mathrm{d}u||_{M_\varepsilon})$, while for the τ-terms every τ can have a maximum number of $d+1$ faces so that we obtain an upper bound of

$$(d+1)O(\varepsilon)(||u||_{X_{\tau,\varepsilon}} + ||\,\mathrm{d}u||_{X_{\tau,\varepsilon}}).$$

This shows the claim. □

Open Questions

The following questions were considered but not solved in the course of this work.

As we generalized the results from [EP05] it is reasonable to ask whether they can be used to generalize the main theorem therein.

Question 3. Let $(M_\varepsilon)_{0<\varepsilon\leq 1}$ be a thin manifold with model complex $M_0 = (K, \Sigma, \Delta)$. Does the spectral asymptotic

$$\lambda_k(M_\varepsilon) \to \lambda_k(M_0);\ \varepsilon \to 0$$

hold?

We can similarly define p-forms on M and give a Kirchhoff condition for the formal adjointness of d and d* (i.e. an analogue of Lemma II.2) in this higher-order case. This has been done for graphs in [EP17]; in the same paper it was also shown that there holds a phase transition property for convergence of eigenvalues of the Hodge-p-Laplacian in case all transversal manifolds X_e have trivial $(p-1)$-th cohomology. We thus ask if this phase transition property holds in higher dimensions aswell.

However we won't introduce this operator particularly; it acts as the Hodge-p-Laplacian on every Σ_σ and along the boundaries a Kirchhoff condition must hold for forms in its domain. We denote by $\lambda_k^{(p)}(M)$ its k-th largest eigenvalue. Analogously for a Riemannian manifold (N, g) we denote by $\lambda_k^{(p)}(N)$ the k-th largest eigenvalue of its Riemannian Hodge-p-Laplacian.

Question 4. Let $(M_\varepsilon)_{0<\varepsilon\leq 1}$ be a thin manifold with model complex $M_0 = (K, \Sigma, \Delta)$ of dimension d. Does for $p < d$ hold that

$$\lambda_k^{(p)}(M_\varepsilon) \to \lambda_k^{(p)}(M_0);\ \varepsilon \to 0?$$

Is there a condition for the transversal manifolds X_τ, $\tau \in K$, under which all eigenvalues for $d+1 \leq p \leq n-1$ diverge; i.e.

$$\lambda_1^{(p)}(M_\varepsilon) \to \infty?$$

I.e. does a phase transition law in the asymptotic behavior of spectra hold at $\operatorname{ess\,dim} M_0$?

By [Cat97] it is known that in case a quantum graph is equilateral, i.e. every edge has the same length $\ell \in (0, \infty)$ associated to it, the spectrum of the continuous Laplacian can completely be determined by the (finite) spectrum of the graphs combinatorial Laplacian. The main ingredient of the proof is the exact form of a solution to the Dirichlet boundary value problem

$$\begin{cases} \Delta u(t) = \lambda u(t) & ,\ t \in [0, \ell] \\ u(0) = a,\ u(\ell) = b \end{cases}$$

on every edge (i.e. $[0, \ell]$) with given values $a, b \in \mathbb{R}$. Note that for the containment of the spectrum of the continuous Laplacian in the right-hand side (which is completely defined by the combinatorial Laplacian) it is enough to use the Kirchhoff condition given at every vertex together with a form of the normal derivative of u at the boundary 0 and ℓ in every edge (depending only on a, b and λ) deduced from the exact description of the Laplacian eigenfunctions on the edge. The normal derivative of u solving this problem is called the Neumann data of the problem; thus we seek to describe the Neumann data by the Dirichlet data on every edge in order to show the first containment.

Now; in case we look at a quantum simplicial complex $M = (K, \Sigma, \Delta)$ of dimension $d = 2$, a reasonable generalization of equilaterality of the graph is the condition of every facet $\sigma \in F_2(K)$ to be embedded in euclidean space by an equilateral triangle $\Sigma_\sigma = \Sigma_0$ of edge length ℓ, where ℓ is the same for every σ. By [DF05] we can then obtain the Neumann data on every facet solely by λ and the Dirichlet data on this face; the Dirichlet data in this case is the value of u on the $(d-1)$-faces. After then substituting this Neumann data into the Kirchhoff condition of $\Delta(M)$ we might rearrange the equation to obtain a similar result to [Cat97]. By the form of the Neumann data we immediately see that this obtained equation only depends on the values of u on $(d-1)$-faces. Thus under suitable assignment of a function $f \in \mathbb{R}^{f_{d-1}(K)}$ to every function $u \in H^2(M)$ we might obtain after rearranging an equation of the form

$$\mathscr{L}_1^{\mathrm{up}}(K) f = p_2(\lambda) f$$

for a suitable function p_1. The author did not find any work on Dirichlet-to-Neumann data transfer for the regular tetrahedron or the standard-d-simplex in general. However, it seems reasonable that the data transfer only depends on the boundary values of u and this boundary behavior can be split up via restriction to each $(d-1)$-face. So that naturally the following question arises.

Question 5. Let $M = (K, \Sigma, \Delta)$ be a quantum simplicial complex of dimension d. Is there a function p_d such that for every eigenvalue λ of $\Delta(M)$ $p_d(\lambda)$ is an eigenvalue of $\mathscr{L}_{d-1}^{\mathrm{up}}$? If not, for what values of d does this hold?
More speculatively; if $p_d(\lambda)$ is an eigenvalue of $\mathscr{L}_{d-1}^{\mathrm{up}}$ is λ and eigenvalue of $\Delta(M)$?

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