The 8th European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS Congress 2022
5–-9 June 2022, Oslo, Norway

ON THE EXISTENCE OF COMPRESSION-ONLY DISCRETE FORCE NETWORKS THAT SUPPORT ASSIGNED SETS OF NODAL FORCES

ADA AMENDOLA[1], ORNELLA MATTEI[2] ANTONIO FORTUNATO[1], PIERRE SEPPECHER[3], FERNANDO FRATERNALI[1], GRAEME W. MILTON[4]

[1]Department of Civil Engineering, University of Salerno, Fisciano (SA), Italy.
[2]Department of Mathematics, San Francisco State University, CA, USA.
[3]Institut de Mathematiques de Toulon, Université de Toulon, France.
[4]Department of Mathematics, University of Utah, Salt Lake City, UT, USA.

Key words: Discrete element models, Compression-only response, Limit analysis, Masonry structures

Abstract. We formulate a linear programming approach to the limit analysis of discrete element models of masonry structures. Numerical results show the ability of the given procedure to predict the limit value of the multiplier of variable horizontal forces, which are applied to masonry walls in association with a fixed vertical loading.

1 INTRODUCTION

Discrete element models of masonry structures that generalize the thrust line construction to 3D systems of general shape are frequently encountered in the literature (refer, e.g., to [1, 2, 3, 4, 5, 6] and references therein). The limit analysis of such structures, subjected to a combination of fixed loads and variable forces that grow proportionally to a scalar multiplier λ, is usually performed through suitable generalizations of the well known master safe theorem of Heyman for masonry arches [7, 8, 9] The present work enriches this literature by presenting a linear programming (LP) procedure for the limit analysis of 'strut nets' formed by pairwise connections of the points of application of forces acting on truss models of masonry structures. The given procedure generalizes recent results dealing with cable web models [10]. The limiting values of the load scaling factor, which ensure existence of compression-only supporting truss structures, are identified with the lower bounds of the collapse multipliers of the variable forces [8, 9]. Numerical examples illustrate the application of the given LP algorithm to the prediction of the limit multiplier of horizontal forces that act on benchmark examples of masonry walls, in association with fixed vertical loading.

2 A LINEAR PROGRAMMING APPROACH TO THE LIMIT ANALYSIS OF STRUT NETS

Let us consider a strut net $\mathcal{S}$ consisting of a *complete web* formed by the pairwise connections of a set of N nodes $\mathbf{X} = (\mathbf{x}_1, ..., (\mathbf{x}_N)$, such that it results

$$||\mathbf{x}_i - \mathbf{x}_j|| \ \leq \ \ell \tag{1}$$

ℓ denoting a maximum connection length (figure 1).

We write the equilibrium problem of $\mathcal{S}$ into the following matrix form

$$\mathbf{AP} \ = \ \lambda \, \mathbf{F} \ + \ \mathbf{G} \tag{2}$$

where $\mathbf{A}$ is the equilibrium matrix of the complete web [11]; $\mathbf{P}$ is the vector of the axial forces carried by the members of $\mathcal{S}$; $\mathbf{G}$ is the vector of the fixed nodal forces; and $\mathbf{F}$ is the vector of the nodal forces growing proportionally to the scalar multiplier λ. The following assumes that the equilibrium problem of $\mathcal{S}$ admits solution for $\lambda = 0$.

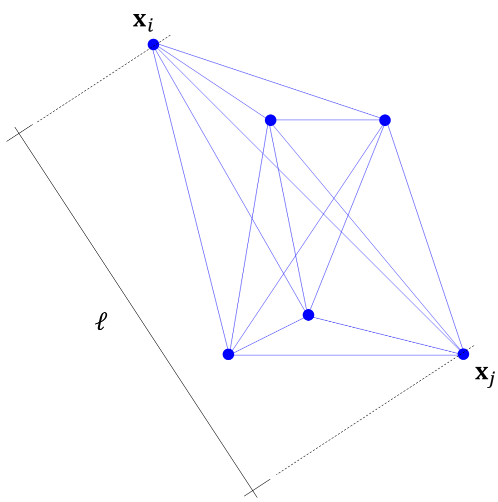

Figure 1: Maximum connection distance ℓ between the nodes of the complete web.

We account for the presence of constraints imposing that selected nodal displacements of $\mathcal{S}$ are zero by dropping the equilibrium equations associated with such degrees of freedom into (2) (see, e.g., [12], page 75). Post-multiplying both members of equation (2) by $\mathbf{F}$ (through a scalar product) and solving for λ, we obtain

$$\lambda \ = \ \frac{1}{F^2}\mathbf{AP} \cdot \mathbf{F} \ - \ \frac{1}{F^2}\mathbf{F} \cdot \mathbf{G} \tag{3}$$

with $F^2 = \mathbf{F} \cdot \mathbf{F}$, $(\cdot)$ denoting the symbol of the scalar product between vectors.

We search for the *limit load multiplier* of the proportional loads $\mathbf{F}$ by solving the following LP problem

$$\begin{aligned} &\underset{\mathbf{P}}{\text{maximize}} \quad \bar{\lambda} \; = \; \bar{\mathbf{C}} \cdot \mathbf{P} \\ &\text{subject to} \quad \begin{cases} \bar{\mathbf{A}}\mathbf{P} = \bar{\mathbf{F}} \\ \mathbf{l_b} \leq \mathbf{P} \leq \mathbf{u_b} \end{cases}, \end{aligned} \tag{4}$$

Here, we have set

$$\bar{\mathbf{C}} \; = \; \frac{1}{F^2}\mathbf{A}^T\mathbf{F}; \quad \bar{\mathbf{A}} \; = \; \mathbf{A} \; - \; \frac{\mathbf{F} \otimes \mathbf{F}}{F^2}\mathbf{A}; \quad \bar{\mathbf{F}} \; = \; \mathbf{G} \; - \; \frac{\mathbf{F} \cdot \mathbf{G}}{F^2}\mathbf{F} \tag{5}$$

where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$, and we have employed the $\otimes$ symbol to denote the tensor product between matrices. In Eqn. (5), $\mathbf{l_b}$ is a vector with all $-\infty$ entries, while $\mathbf{u_b}$ is a vector with all 0 entries. If problem (4) admits an optimal solution, we compute a first limit load multiplier λ^+ of $\mathcal{S}$ through

$$\lambda^+ = \; \bar{\lambda}^+ \; - \; \frac{1}{F^2}\mathbf{F} \cdot \mathbf{G} \tag{6}$$

where $\bar{\lambda}^+$ is the optimal value of the objective function $\bar{\lambda}$. By turning problem (4) into a minimization problem we next obtain a second limit load multiplier λ^-. The vector of axial forces $\mathbf{P}$ that corresponds to prescribing $\lambda = \lambda^+$ (or $\lambda = \lambda^-$) in the complete web (*limit load strut net*) is given by the solution of the LP problem (4).

3 Numerical results

The following sections present applications of the above LP procedure to problems dealing with the equilibrium of 2D and 3D models of masonry walls. It is easy to show that it results $\lambda^+ = -\lambda^- = \lambda_{lim}$ in all the examples that follow.

3.1 Masonry wall under vertical loading and a concentrated horizontal force

Let us consider a rectangular masonry wall with horizontal span L, height h and thickness th, which is subject to a uniformly distributed vertical load q (per unit length L) on the top side of the middle section, and a horizontal force $F = \lambda q L$ applied to a corner of this section (see section 3.2 of [4]). We lump the vertical load q in correspondence to the seven nodes shown in figure $2a$, and discretize the bottom side of the wall by introducing an equal number of nodes vertically aligned with those of the top side. The bottom nodes are fixedly constrained. Figure $2a$ shows the complete web $\mathcal{S}$ assumed for the current example, which is formed by all the pairwise connections of the top and bottom nodes ($\ell = \sqrt{L^2 + h^2}$). The LP problem (4) returns $\lambda_{lim} = 1/3$, which corresponds to the exact solution of the continuum problem [4]. The strut net corresponding to the limit load of $\mathcal{S}$ is shown in figure $2b$. Upon inserting a third row of seven nodes at the mid-height of the wall in the complete web (figure $2c$), we obtain a limit load strut net that more closely approximates the distribution of forces at collapse for the continuous problem [4]. The latter features the reactive part of the wall separated by the non-reactive part through the curve Γ marked with a red dashed line in figure $2d$.

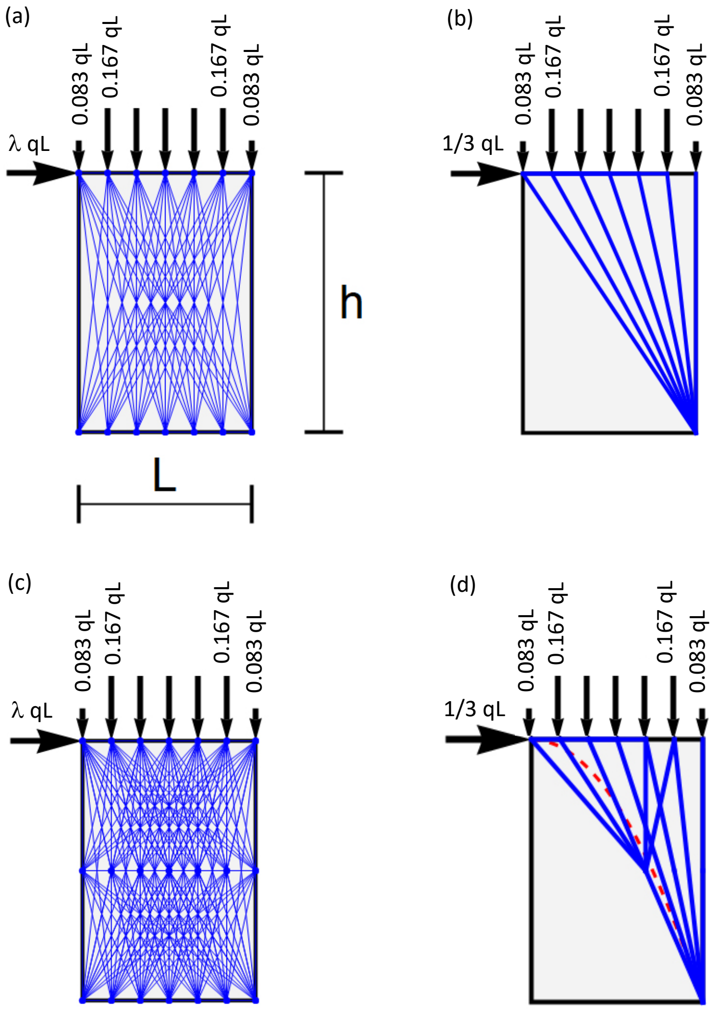

Figure 2: Limit analysis problem of a discrete element model of a masonry wall loaded by a uniform vertical load and a concentrated horizontal force. (a) The complete web is formed by all the pairwise connections of seven nodes placed on the top and bottom sides of the middle section. (b) The limit load multiplier $\lambda_{lim} = \lambda^+ = -\lambda^-$ is equal to 1/3, as in the exact solution of the continuum problem [4]. The distribution of compression forces in correspondence to the limit load is supported by struts placed above a diagonal of the panel. (c) A more refined complete web is obtained by introducing a third row of seven nodes at the mid-height of the wall. (c) The limit load strut net corresponding to such a web more accurately follows the discontinuity line Γ that characterizes the distribution of forces at collapse for the continuum problem, which is marked by a dashed red line [4]. When adopting the refined complete web, one again obtains $\lambda_{lim} = 1/3$.

3.2 C-shaped wall

A second and final example deals with a C-shaped masonry wall loaded out-of-plane by inertial forces induced by the horizontal shaking of the base of the wall [13]. The frontal panel of the examined C-wall has a span of 630 mm, while the lateral panels have 350 mm span. The height of the wall is 345 mm and the thickness of the blocks is 23 mm. We introduce a strut net model formed by all the the pairwise connections of the nodes shown in figure 3a, over one half of the wall (due to symmetry). Constant vertical forces g_i and variable horizontal forces $f_i = \lambda g_i$ are applied to the nodes of such a strut net, which respectively model the action of self weight, and an equivalent static description of the inertia forces generated by the horizontal motion of the wall [13] (here, λ denotes the ratio between the horizontal acceleration applied at the base of the wall and the gravitational acceleration). The limit load strut net depicted in figure 3a corresponds to $\lambda_{lim} = 0.1127$, and has been determined through the LP procedure illustrated in section 2. A shake table test conducted on a physical model composed of wooden masonry [15, 13] showed the occurrence of sliding movements between the blocks for λ greater than 0.125. We are therefore led to observe that the limit load strut net of figure 3a well captures the no-tension response of the wall, before sliding phenomena occur. In particular, one notes the absence of diagonal struts in such a model in correspondence to the top central region of the frontal panel, and the top-corner region of the lateral panel, where the experiment illustrated in [15, 13] highlights out-of-plane collapse of the blocks (compare figures 3a,b with figure 3c). The strut net model is therefore able to predict the regions where the sliding motion of the blocks may occur, due to insufficient compaction forces, even if it does not model such a phenomenon directly. As anticipated, the experimental results presented in [15, 13] highlight that the an appreciable relative sliding motion of the blocks initiates for $\lambda \approx 0.125$. It leads to the out-of-plane collapse of the wall for $\lambda = 0.178$ (the experimental collapse configuration is illustrated in figure 3c).

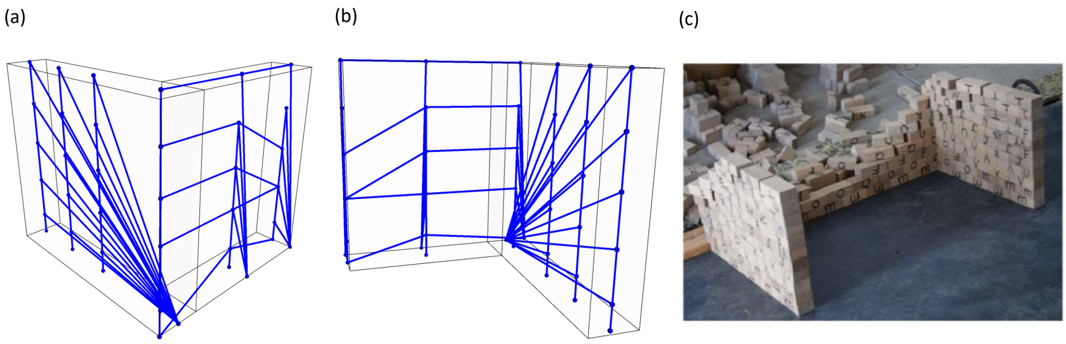

Figure 3: (a,b) Different views of the strut net model of a C-shaped dry-jointed masonry wall under the action of constant vertical forces g_i, acting on a grid of 67 nodes (one-half of the wall is modeled), and horizontal forces $f_i = \lambda g_i$. The displayed strut net is associated with the limit load multiplier $\lambda_{lim} = 0.1127$. (c) Collapse configuration observed on a physical sample tested on a tilting table (reproduced with permission from [13]). (Online version in color.)

4 CONCLUSIONS

The LP procedure presented in this work permits the limit analysis of 2D and 3D discrete element models of masonry structures. We address the generalization of such a procedure to problems dealing with multiply-connected domains, as well as the modeling of elastic and dynamic effects, to future work.

ACKNOWLEDGMENTS

OM and GWM are grateful to the National Science Foundation for support through Research Grants DMS-2008105 and DMS-2107926. AA and FF are grateful to the Italian Ministry of University and Research for support through the PRIN 2017 grant 2017J4EAYB.

REFERENCES

[1] O'Dwyer D. 1999 Funicular analysis of masonry vaults. *Comput. Struct.* **73**, 187–197.

[2] Block P, Ochsendorf J. 2007 Thrust network analysis: A new methodology for three-dimensional equilibrium *J. Int. Assoc. Shell Spat. Struct.* **48**, 167–173.

[3] Fraternali F. 2010 A thrust network approach to the equilibrium problem of unreinforced masonry vaults via polyhedral stress functions. *Mech. Res. Commun.* **37**, 198–204.

[4] Angelillo M., Fortunato A., Montanino A., Lippiello M. 2014. Singular stress fields in masonry structures: Derand was right. *Meccanica* **49**, 1243-1262.

[5] De Faveri S., Freddi L., Paroni R. 2013. No-tension bodies: A reinforcement problem. *European Journal of Mechanics, A/Solids* **39**, 163-169

[6] Marmo F, Rosati L. 2017 Reformulation and extension of the thrust network analysis. *Comput. Struct.* **182**. 104–118.

[7] Heyman J. 1995. *The Stone Skeleton: Structural Engineering of Masonry Architecture* Cambridge: Cambridge University Press.

[8] Del Piero G. 1998 Limit analysis and no-tension materials. *International Journal of Plasticity.* **14**, 259-271

[9] Como M. 2017 *Statics of Historic Masonry Constructions*, 3rd edn. Heidelberg: Springer.

[10] Bouchitté G., Mattei O., Milton G.W., Seppecher P. 2019 On the forces that cable webs under tension can support and how to design cable webs to channel stresses *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.* **475**, 2223

[11] Skelton, R. E., de Oliveira, M. C., 2010c. Tensegrity Systems. Springer.

[12] Gill, P.E., Murray, W., Wright, M.H. , 1981. *Practical Optimization.* San Diego: Academic Press, Inc.

[13] Orduña A. 2017. Non-linear static analysis of rigid block models for structural assessment of ancient masonry constructions, *Int. J. Solids Struct.*, **128**, 23-35.

[14] Lourenҫ PB, Oliveira DV, Roca P, Orduña A. 2005. Dry joint stone masonry walls subjected to in-plane combined loading. *J. Struct. Eng.*, 131(11), 1665-1673.

[15] Valdez R. 2012 *Block Models of Walls Subject to Lateral Loads*, M.Eng. Thesis, Universidad de Colima.