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ESTIMATION OF DISTRIBUTION ALGORITHM FOR CONSTRAINED OPTIMIZATION IN STRUCTURAL DESIGN

Marco Martino Rosso[1]*, Jonathan Melchiorre[1], Raffaele Cucuzza[1], Amedeo Manuello[1], and Giuseppe Carlo Marano[1]

[1] Department of Structural, Geotechnical and Building Engineering Politecnico di Torino, Corso Duca degli Abruzzi, 24 - 10129. Torino, Italy
*Corresponding author E-mail: marco.rosso@polito.it

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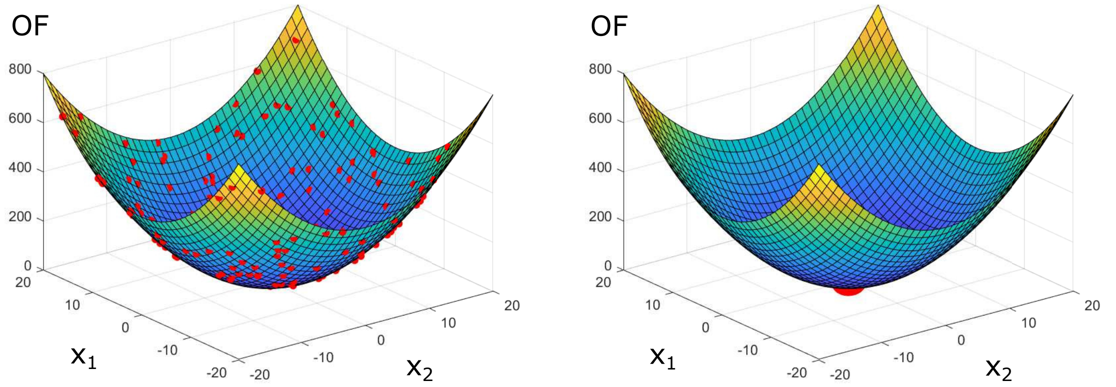
Abstract. *Nowadays, the need to deal with limited resources together with the newly discovered awareness of the human over-exploitation of the environment, has made the optimization a cutting edge topic both in scientific research and in the different professional fields. In this paper, a particular evolutionary optimization algorithm is presented: The Estimation of Distribution Algorithm (EDA). This type of algorithm has been developed to be used in search-based constrained optimization problems which are difficult and time-consuming to be solved by other general algorithms. Being an evolutionary algorithm, the main idea is to generate a population of solutions and evaluate the objective function of each one of them. Then, using the information obtained from the previous generations, the algorithm step-by-step will generate new populations that will tend to the best value of the objective function. In EDA, the population of solutions defines a probability density function (PDF) and, by integration, a cumulative density function (CDF), which is used for the generation of the next generation. In structural design optimization problems, it is very common that the best solution is very close to the constraint function. The main advantage of applying the EDA for constrained optimization problems is that each generation of solutions is obtained starting from a PDF that is defined on the whole domain. This means that, for each generation, the solutions have a probability to be on the unfeasible domain space, maintaining the information about the objective function in the evolution process. In the present research, an original EDA and related self-made code are presented together with a specific application to structural optimization problems, in order to show the effectiveness of the obtained results and to make a comparison with other evolutionary optimization algorithms.*

1 INTRODUCTION

In the last two decades, computational intelligence methods have been successfully applied in many engineering applications and structural optimization problems, showing their potentiality solving hard real-world constrained optimization problems [1–13]. The versatility of the evolutionary algorithms (EAs) is due to not requiring any gradient information, promoting their simple implementation and therefore their rapid widespread and accessibility even for practitioners. These methods are often based on a population-based approach. As depicted in Figure 1, a population-based strategy relies on a multi-start approach, i.e., considering simultaneously a number of candidate solutions in the design space and attempting to learn the most promising direction in which evolve the population towards the global optima of the objective function (OF). The genetic algorithm (GA) by J. Holland [14] is one of the most fa-



(a) **(b)**

Figure 1: Example of visualization of the population-based optimization technique applied on an unconstrained convex problem, i.e. a paraboloid objective function (global minimum at $x* = [0,0]^T$). (a): Initial population.(b) Final population.

mous population-based tools, which mimics Darwinian's Theory of biological evolutionary processes of species. Inspired by bird flockings, school fishing, or swarming of insects behavioural models, Kennedy and Eberhart [15] proposed another well-acknowledged EA called Particle Swarm Optimization (PSO). Among of the others, relative attractive tool is nowadays represented by estimation of distribution algorithms (EDAs) [16–19]. These algorithms are able to solve hard optimization problems by adopting an initial probabilistic model over the design variables of the optimization problem and iteratively providing a series of incremental updates until the probabilistic model is able to sample only in the nearby of the global optima [20]. In population-based EDAs, starting with an initial random population over the design space, each candidate solution is evaluated and the most promising ones in terms of Objective Function (OF) evaluation will be selected to adjust the probabilistic model during each iteration, in the spirit of EAs. From this model, offspring are sampled by the model encoded from the best solutions until the current iteration. Then, the evaluation and selection cycle starts again, until achieving some stopping conditions. The EDA is a quite attractive tool because it incorporates the learning problem into an optimization framework with a simple implementation based on the generate-update logic, able to learn a probabilistic model from the current population of candidate solutions. Population-based EDAs [21] rely on a quite large population size and building each probabilistic model from scratch [20]. In the current study, an implementation of the EDAs is presented based on an iterative update of a gaussian mixture model (GMM) with controlling the covariance matrix to boost the EDA performances. In section 2, the current implementation is explained in detail, then in section 3, two numerical benchmark problems are tested to evaluate the current EDA performances compared with other EAs, i.e. GA and PSO. Finally, a structural benchmark case study related to engineering structural optimization truss design is further explored at the end of section 3.

2 METHODOLOGY

In the current study a novel implementation of the EDA is presented. The starting population of size m is generated by using the Latin Hyper-cube Sampling method (LHS) [22]. When dealing with constrained optimization problems with equality $h_r(x)$ or inequality constraints $g_q(x)$, such as

$$\begin{aligned} & \min_{x \in \Omega} \{f(x)\} \\ \text{s.t.} \quad & g_q(x) \leq 0 \quad \forall q = 1, \dots, n_q \\ & h_r(x) = 0 \quad \forall r = 1, \dots, n_r \end{aligned} \tag{1}$$

if the entire population of candidate solutions is unfeasible, the optimization goal is redirected to minimize the constraints violation in the attempt to find the feasible region. When the feasible region is found, the actual EDA can start. Even for unconstrained problems (the same statement of equation 1 without any constraints), typically the n_{dim} design variables collected in x are usually limited in a range of interest between a lower x_j^{lb} and an upper bound x_j^{ub} of each $j = 1, ..., n_{dim}$ design variable. The design space is thus the Cartesian product $\Omega = [x_1^{lb}; x_1^{ub}] \times ... \times [x_{n_{dim}}^{lb}; x_{n_{dim}}^{ub}]$. The probabilistic model which is iteratively generated by the EDA on the population of candidate solutions has been chosen by the Authors as the gaussian mixture model (GMM). The mixture models provide a semi-parametric tool able to model quite complex unknown distributional shapes, in which parametric approach dreadfully fail [23, 24]. The GMM can be obtained by a linear superposition of n gaussian PDFs with non-negative mixing proportions or weights π_i, which sum to unity, applied to each PDF component densities p_i of the mixture [16, 24]:

$$p(x) = \sum_{i=1}^{n} \pi_i \cdot p_i = \sum_{i=1}^{n} \pi_i \cdot N(x|\mu_i, \Sigma_i) \qquad \text{with} \quad \sum_{i=1}^{n} \pi_i = 1, \quad 0 \leq \pi_i \leq 1 \tag{2}$$

In general, each component density has its own mean and covariance matrix. To implement GMM, the Matlab command from "*Statistics and Machine Learning Toolbox*" [25] has been adopted. Setting the mean μ coincident with the candidate solutions x in the design space, and a starting diagonal covariance matrix Σ of size $n_{dim} \times n_{dim}$ for each element of the population, the implemented GMM consist of a multivariate normal distribution with arbitrary standard deviations for each variable (diagonal covariance matrix) [20]. In this case, the current GMM actually performs similarly to a kernel density estimation (KDE) non-parametric algorithm [24]. In this study, to improve the optimization process toward the optima, the mixing weights π_i of GMM are related to the OF of each candidate solution in the design space. Denoting k_{max} as the maximum number of iterations, in order to promote exploration in the early beginning of the iterative optimization process, and promoting exploitation towards the end of the iterative process, the current EDA adopts a dynamic covariance adaptation approach. Denoting with σ_j the variance diagonal term of the covariance matrix Σ referred to the design variable j, the σ_j has been bounded between an admissible range

$$\sigma_{j,max} = \alpha \frac{\left| x_j^{ub} - x_j^{lb} \right|}{m} ; \qquad \sigma_{j,min} = \alpha \frac{\left| x_j^{ub} - x_j^{lb} \right|}{m \cdot k_{max}} \tag{3}$$

where the parameter $\alpha > 0$ is a static user-defined hyperparameter which heuristically controls the exploration and exploitation: increasing α the exploration is promoted, on the contrary, the exploitation is enhanced. σ_{max} and σ_{min} represent the extremes of the dynamic variation of σ_j during iterations, as

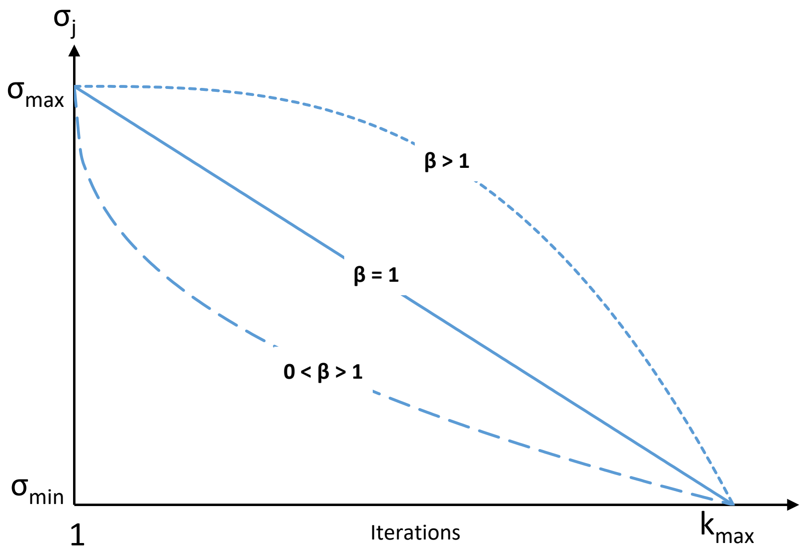


Figure 2: Dynamic variation of σ as function of the parameter β

depicted in Figure 2. Tu further control these aspects, a continuous dynamic decreasing law of variation for σ_j during iterations $0 \leq k \leq k_{max}$ has been defined as:

$$\sigma_j(k) = \sigma_{j,max} - \frac{\sigma_{j,max} - \sigma_{j,min}}{k_{max}^{\beta}} k^{\beta} \tag{4}$$

where the parameter $\beta > 0$ is another static user-defined hyperparameter which heuristically governs the decreasing law trend during iteration, having three possible outcomes as depicted in Figure 2: • Sub-linear behaviour with $0 < \beta < 1$; • Linear behaviour with $\beta = 1$; • Super-linear behaviour with $\beta > 1$. After defining the GMM distribution, a number of offsprings have been sampled from this distribution by peaking randomly the PDF components of the GMM and sampling from that normal. Since the GMM is weighted according to the OF, even the sampled offspring will have more chance to be sampled from normal components related to the current best individuals of the population. The entire population considering parents and offspring compete for survival and only the best individuals survive to the next iteration. The unfeasible points are handled with a sort of adaptive penalty-based approach which relies on the constraints violation degree of the candidate solutions, on the number of constraints violated, and also on the euclidean distance with the nearest feasible point. At this point, the evolution cycle can start again with the generation of a new GMM based on new mean and covariance matrices, sampling new offsprings which will compete with the parents for survival. The stopping criteria have been set as a predefined maximum number of iterations, a stagnation criterion with no improvements of the OF within a certain tolerance, and a maximum allowable computational time criterion.

3 RESULTS AND DISCUSSION

In the present section, the proposed EDA implementation with GMM using a dynamic covariance matrix adaptation has been tested on some numerical and structural optimization benchmark problems.

Table 1: Numerical benchmark results comparisons: unconstrained optimization problem, Rastrigin function.

Obj. Fun.	Real Optimum	PSO-ES [26]	GA [25]	EDA $\alpha = 1$ $\beta = 0.5$	EDA $\alpha = 1$ $\beta = 1$	EDA $\alpha = 1$ $\beta = 1.5$	EDA $\alpha = 1$ $\beta = 2$	EDA $\alpha = 1$ $\beta = 2.5$
best	0	0	3.73E-10	6.17E-07	8.13E-07	4.77E-07	5.55E-07	1.26E-06
worse	-	0	3.98	3.46E-05	4.29E-05	0.99	9.93E-05	1.47E-04
mean	-	0	0.46	7.24E-06	1.41E-05	0.02	2.90E-05	3.09E-05
std. dev.	-	0	0.76	5.93E-06	1.21E-05	0.14	2.50E-05	3.21E-05

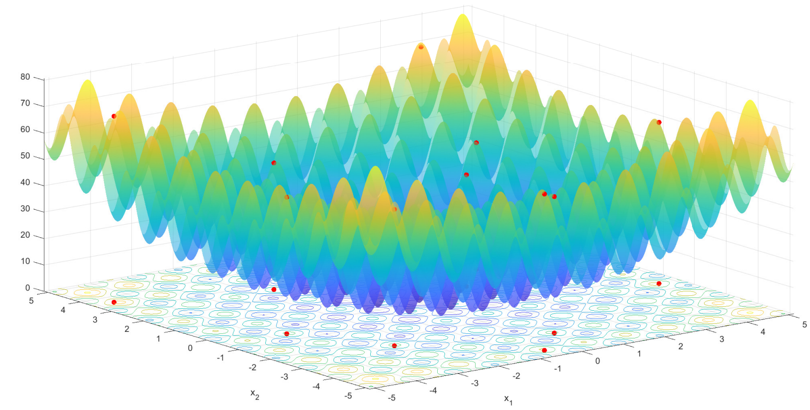
Since the proposed approach permits the user to vary the hyperparameters α and β to govern the dynamic covariance matrix adaptation during the optimization process, the benchmark problems have been tested with different values of β leave α fixed to 1 in all the cases, exploring the effects of a covariance matrix with a sub-linear, linear or super-linear trends with iterations k. In all the analyzed problems, the population size is set to 50 individuals and the maximum number of iterations is $k_{max} = 500$. For the sake of comparisons, the EDA performances have been compared with other different EAs to explore the reliability and the effectiveness of the optimization algorithm. Specifically, the GA from standard Matlab implementation [25] and the PSO, in the variant of the enhanced multi-strategy PSO (PSO-MS) [26], have been selected as alternative completely independent implementation EAs for an objective evaluation of the robustness, reliability, and performance of the current EDA.

3.1 Unconstrained optimization problem: Rastrigin function

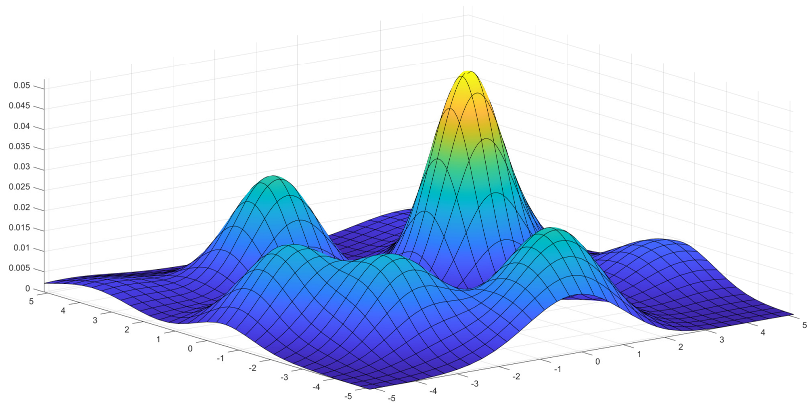
The first numerical benchmark problem adopted to test the current EDA implementation performances is the unconstrained optimization problem better acknowledged as Rastrigin function. The mathematical statement is

$$\min_{x_i \in [-5.12;5.12]} f(x) = A \cdot n + \sum_{i=1}^{n} \left[x_i^2 - A\cos(2\pi x_i)\right] \qquad \text{for } i = 1,2 \tag{5}$$

where A is a constant equal to 10. This problem is a non-convex problem with a global minimum at $x^* = 0$, where the OF is equal to $f(x) = 0$. Achieving the global minimum is quite difficult due to Rastrigin large search space and its large number of local minima. The OF on the design space domain of interest is depicted in Figure 3 (a). In Figure 3 (b) is also depicted an example of the PDF generated by the GMM model, which rely on the initial population in this case. The next generation of offsprings will be sampled from this multimodal non-parametric bivariate PDF. Table 1 presents the results obtained from the EDA for 50 independent runs of the algorithm, with $\alpha = 1$ and β varying from 0.5 to 2.5 with 0.5 step size. The EDA results appear very promising, actually close to the global optimum and with very low variability with respect to the GA. The PSO-MS appears very robust since it provides, for all the 50 independent runs, the global optima solution without any variation. The effect of β on the optimal results appears evident, even if it is very low in this case, providing in all the cases, but $\beta = 1.5$, noticeable results, especially compared with standard GA.



(a)



(b)

Figure 3: Example of optimization of the Rastrigin's Objective Function with $n_{pop} = 10$. (a): Initial population; (b): generated probability density function provided by the weighted Gaussian Mixture Model based on the initial population.

3.2 Constrained optimization problem: Sickle function (banana function)

The second numerical benchmark problem adopted to test the current EDA implementation performances is a constrained optimization problem, better acknowledged as Sickle or Banana function [26]. The mathematical statement is

$$\begin{aligned} \min_{x \in \Omega} \quad & f(x) = (x_1 - 10)^3 + (x_2 - 20)^3 \\ \text{s.t.} \quad & g_1(x) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0 \\ & g_2(x) = (x_1 - 6)^2 - (x_2 - 5)^2 - 82.81 \leq 0 \end{aligned} \tag{6}$$

Table 2: Numerical benchmark results comparisons: constrained optimization problem, Sickle function (banana function) .

Obj. Fun.	Real Optimum	PSO-ES [26]	GA [25]	EDA $\alpha=1$ $\beta=0.5$	EDA $\alpha=1$ $\beta=1$	EDA $\alpha=1$ $\beta=1.5$	EDA $\alpha=1$ $\beta=2$	EDA $\alpha=1$ $\beta=2.5$
best	-6961.81388	-6961.8	-9000.0	-6958.69	-6961.46	-6953.94	-6959.11	-6959.71
worse	-	-6958.4	-1611.7	-6944.94	-6925.59	-6937.86	-6928.62	-6924.13
mean	-	-6960.7	-7070.8	-6954.39	-6947.41	-6946.49	-6946.68	-6949.61
std. dev.	-	0.9752	1553.8	3.54006	8.66695	5.31385	8.54892	8.236062

where Ω is the domain of the design vector, described by the Cartesian product among the admissible intervals of the two design variables, i.e. $x_1 \in [13;100]$ and $x_2 \in [0;100]$. $g_1(x)$ and $g_2(x)$ are the two inequality constraints which create a very narrow feasible region with a shape which resembles a sickle or a banana, hence the name. The optimum is located at $x^* = [14.095; 0.84296]$, where $f(x^*) = -6961.81388$. Table 2 illustrates the results obtained from the EDA for 50 independent runs of the algorithm, with $\alpha = 1$ and β varying from 0.5 to 2.5 with 0.5 step size. The EDA results appear very promising even with a constrained optimization, achieving optimum always appreciably close to the global optimum and in a reliable way. The GA does not perform sufficiently well with this problem, because the results are generally far from the optimal ones. On the other hand, the PSO-MS appears very robust since it provide for all the 50 independent runs the global optima solution with a fairly small standard deviation. The effect of β on the optimal results is sufficiently appreciable. All the EDA results are in the same order of magnitude with the global optimum and they are consistent with PSO-MS ones. In this case, a sub-linear trend, $\beta = 0.5$, appear to be the best hyperparameter choice to get optimal results with a mildly low standard deviation with the same computational cost (population size and maximum number of iterations).

3.3 Constrained structural optimization problem: twenty-five bars truss design problem

The current EDA implementation has been tested structural benchmark case study. A three-dimensional truss tower structure composed of 25 bar members [27] represents the constrained structural optimization design problem which is depicted in Figure 4. The truss tower has a squared footprint of side 200 in, and it is tapered with elevation, reaching a side of 75 in at an elevation of 100 in. The truss tower is 200 in high. The truss design constrained optimization problem can be mathematically stated as:

$$\begin{aligned} \min_{x\in\Omega} \quad & f(x) = \sum_{i=1}^{n} \rho_i \ell_i A_i \\ \text{s.t.} \quad & A_i^{\text{lb}} \le A_i \le A_i^{\text{ub}} \\ & \sigma_i \le \sigma_{\max} \\ & \delta \le \delta_{\max} \end{aligned} \tag{7}$$

where n refers to the number of members in the truss design domain and ℓ_i represents the length of every element. The structural steel of the truss members has a Young's modulus of 10^7 psi and unit weight of $\rho_i = \rho = 0.1$ lb/in^3. The design vector x considers the cross section areas of each truss element , gathered into eight groups, as depicted in Figure 4 (b). They are considered as continuous variables in the range determined by a lower A_i^{lb} and an upper A_i^{ub} bound, i.e. $[0.01, 3.40]$ in^2. The usual structural constraints

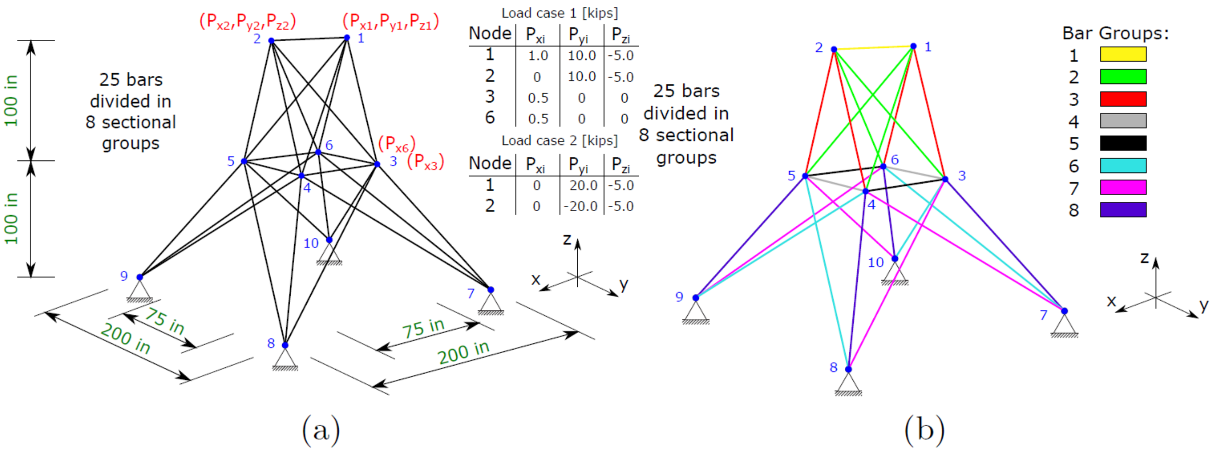


Figure 4: Structural design constrained optimization benchmark illustration. (a): geometry and load conditions of the twenty-five bars truss problem. (b): eight bar groups representation.

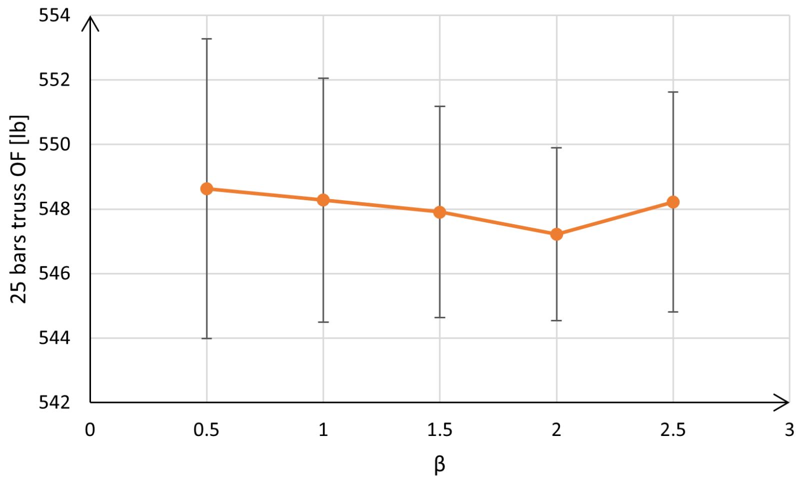


Figure 5: Effects of different values of β on the optimal results for 100 independent run of the structural optimization problem twenty-five bars truss design.

are referred to respect safety requirements of the structure expressed in terms of maximum allowable displacement, i.e. $\delta \leq \delta_{max} = \pm 0.35$ in in every direction, and maximum axial stress of each member, e.g. $\sigma \leq \sigma_{max} = \pm 40$ ksi which is limited to take into account the buckling phenomena that represent the most severe verifications for this type of structure [28–31]. During the optimization process, two different load cases have been taken into consideration, as depicted in Figure 4 (a).

Table 3 illustrates the results obtained from the EDA for 50 independent runs of the algorithm, with $\alpha = 1$ and β varying from 0.5 to 2.5 with 0.5 step size. The EDA results appear remarkable dealing with a benchmark structural constrained optimization problem, achieving sometimes optimal results even better than the reference solution illustrated in [27]. EDA appears robust and reliable with a variance with the

Table 3: Structural optimization twenty-five bars truss design problem results comparisons.

	Cross Section [in^2]							
Bar groups	Ref. Sol. [27]	PSO-ES [26]	GA [25]	EDA $\alpha = 1$ $\beta = 0.5$	EDA $\alpha = 1$ $\beta = 1$	EDA $\alpha = 1$ $\beta = 1.5$	EDA $\alpha = 1$ $\beta = 2$	EDA $\alpha = 1$ $\beta = 2.5$
1	0.100	0.011	0.010	0.010	0.034	0.015	0.012	0.01
2	1.800	1.976	2.023	1.967	2.015	1.978	1.937	1.956
3	2.300	2.989	2.941	3.028	2.965	2.99	3.066	3.04
4	0.200	0.010	0.010	0.010	0.012	0.013	0.017	0.011
5	0.100	0.011	0.010	0.010	0.014	0.018	0.021	0.011
6	0.800	0.690	0.671	0.678	0.688	0.688	0.678	0.681
7	1.800	1.689	1.673	1.680	1.667	1.689	1.691	1.683
8	3.000	2.654	2.694	2.660	2.666	2.655	2.646	2.652
OF best [lb]	546.010	545.249	545.236	545.265	548.277	545.533	545.662	545.246
OF worse [lb]	-	552.378	557.755	562.650	560.608	557.586	560.698	561.039
OF mean [lb]	-	546.003	547.828	548.631	548.277	547.906	547.221	548.479
OF std. dev. [lb]	-	0.7879	2.0743	4.6380	3.7812	3.2743	2.6815	4.3482

same order of magnitude as the other EAs tested. However, the PSO-MS appears again remarkably robust since it provides, for all the 50 independent runs, the global optima solution with a noticeably small standard deviation. The effect of β on the optimal results is fairly appreciable, and for the sake of clarity, it has been depicted in Figure 5. This figure resembles a statistical graph related to the analysis of variance (ANOVA) for one factor β. Observing this Figure, it is possible to notice that the variations of the optimal results over the 50 independent runs appear to be small and, from a statistical point of view, it would virtually possible to test the hypothesis to check if the influence of β is relevant on the optimal results. From a visual inspection of the graph, it is possible to affirm that the best results are obtained with $\beta = 2$ with a quadratic dynamic covariance matrix adaptation law during iterations.

3.4 Discussion and summary

In this section, some numerical benchmark was adopted to test the performances of the current EDA implementation with the proposed dynamic covariance matrix adaptation during the optimization iterative process. A first numerical unconstrained benchmark problem, the Rastrigin function, has been successfully tested. Table 1 evidenced the EDA capabilities to deal with unconstrained optimization problems in a robust and reliable way, even compared with other independent EAs implementations (GA and PSO-MS). Exploring the capability of the current EDA to deal with constrained optimization problems, the Sickle (or banana) function has been considered as numerical constrained benchmark problem. Table 2 evidences once again the performance of the EDA, both its reliability and robustness even considering constraints. In this problem, considering the early exploration phase with minimization of the constraint violation, the EDA is able to locate fairly quick the very narrow feasible region, starting then the exploitation phase, enhanced also by the dynamic covariance adaptation of the GMM. Finally, the current EDA has been successfully tested on a benchmark structural optimization problem related to the twenty-five bars truss design benchmark. The EDA performed great, as well as the other EAs, and in the same order of magnitude as the optimal reference solution. Table 3 and Figure 5 illustrate the effect of varying β parameter, which is fairly evident but substantially does not influence remarkably the optimal results in terms of standard deviation around the mean value of the 50 independent runs.

4 CONCLUSIONS

In the current study, the EAs acknowledged as EDA has been implemented to solve structural optimization problems with a GMM as the probabilistic framework. A dynamic covariance matrix adaptation has been stated, with the advantage to permit the user to tune the diagonal covariance matrix terms with respect to the design space amplitude (hyperparameter α) and governing its decreasing dynamic trend during the optimization process iterations (hyperparameter β). Two numerical benchmark problems have been tested, the Rastrigin function as an unconstrained optimization problem and the Sickle (or banana) function as a constrained one. The EDA performances and its reliability have been statistically compared using the results of 50 independent runs compared with two other EAs (the GA and the PSO). In these examples the effects of tuning β have been further explored, providing the optimal settings for the hyperparameter to enhance at the best the EDA's performances. Eventually, a benchmark structural optimization problem has been tested demonstrating the EDA's success and its noticeable capability to deal with hard structural optimization problems in a reliable way. Furthermore, from the statistical analysis of the results from 50 independent runs, it was observed not so remarkable changes with different values of the hyperparameter β, underlining its robustness with structural optimization problems.

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REFERENCES

[1] Marano, G., Trentadue, F., & Greco, R. (2007). Stochastic optimum design criterion of added viscous dampers for buildings seismic protection. *Structural Engineering and Mechanics*, *25*(1), 21–37. https://doi.org/10.12989/sem.2007.25.1.021

[2] Cucuzza, R., Rosso, M. M., Aloisio, A., Melchiorre, J., Giudice, M. L., & Marano, G. C. (2022). Size and shape optimization of a guyed mast structure under wind, ice and seismic loading. *Applied Sciences*, *12*(10). https://doi.org/10.3390/app12104875

[3] Sardone, L., Rosso, M. M., Cucuzza, R., Greco, R., & Marano, G. C. (2021). COMPUTATIONAL DESIGN OF COMPARATIVE MODELS AND GEOMETRICALLY CONSTRAINED OPTIMIZATION OF a MULTI DOMAIN VARIABLE SECTION BEAM BASED ON TIMOSHENKO MODEL. *14th International Conference on Evolutionary and Deterministic Methods for Design, Optimization and Control*. https://doi.org/10.7712/140121.7961.18535

[4] Cucuzza, R., Rosso, M. M., & Marano, G. (2021). Optimal preliminary design of variable section beams criterion. *SN Applied Sciences*, *3*. https://doi.org/10.1007/s42452-021-04702-5

[5] Cucuzza, R., Costi, C., Rosso, M. M., Domaneschi, M., Marano, G. C., & Masera, D. (0). Optimal strengthening by steel truss arches in prestressed girder bridges. *Proceedings of the Institution of Civil Engineers - Bridge Engineering*, *0*(0), 1–21. https://doi.org/10.1680/jbren.21.00056

[6] Fiore, A., Marano, G., Greco, R., & Mastromarino, E. (2016). Structural optimization of hollow-section steel trusses by differential evolution algorithm. *International Journal of Steel Structures*, *16*(2), 411–423. https://doi.org/10.1007/s13296-016-6013-1

[7] Aloisio, A., Pasca, D. P., Battista, L., Rosso, M. M., Cucuzza, R., Marano, G., & Alaggio, R. (2022). Indirect assessment of concrete resistance from fe model updating and young's modulus estimation of a multi-span psc viaduct: Experimental tests and validation. *Elsevier Structures*, *37*, 686–697. https://doi.org/10.1016/j.istruc.2022.01.045

[8] Marano, G., Trentadue, F., & Petrone, F. (2014). Optimal arch shape solution under static vertical loads. *Acta Mechanica*, *225*(3), 679–686. https://doi.org/10.1007/s00707-013-0985-0

[9] Melchiorre, J., Bertetto, A. M., & Marano, G. C. (2021). Application of a machine learning algorithm for the structural optimization of circular arches with different cross-sections. *Journal of Applied Mathematics and Physics*, *9*(5), 1159–1170.

[10] Marano, G. C., Trentadue, F., & Petrone, F. (2014). Optimal arch shape solution under static vertical loads. *Acta Mechanica*, *225*(3), 679–686.

[11] Manuello, A. (2020a). Multi-body rope approach for grid shells: Form-finding and imperfection sensitivity. *Engineering Structures*, *221*, 111029.

[12] Fiore, A., Quaranta, G., Marano, G. C., & Monti, G. (2016). Evolutionary polynomial regression–based statistical determination of the shear capacity equation for reinforced concrete beams without stirrups. *Journal of Computing in Civil Engineering*, *30*(1), 04014111.

[13] Marano, G. C., Trentadue, F., & Greco, R. (2006). Optimum design criteria for elastic structures subject to random dynamic loads. *Engineering Optimization*, *38*(7), 853–871.

[14] Martí, R., Pardalos, P. M., & Resende, M. G. C. (2018). *Handbook of heuristics* (1st). Springer Publishing Company, Incorporated. https://doi.org/10.1007/978-3-319-07124-4

[15] Kennedy, J., & Eberhart, R. (1995). Particle swarm optimization. *Proceedings of ICNN'95 - International Conference on Neural Networks*, *4*, 1942–1948 vol.4. https://doi.org/https://www.doi.org/10.1109/ICNN.1995.488968

[16] Bishop, C. M., & Nasrabadi, N. M. (2006). *Pattern recognition and machine learning* (Vol. 4). Springer.

[17] Paul, T. K., & Iba, H. (2002). Linear and combinatorial optimizations by estimation of distribution algorithms. *9th MPS symposium on Evolutionary Computation, IPSJ, Japan*, *12*.

[18] Hauschild, M., & Pelikan, M. (2011). An introduction and survey of estimation of distribution algorithms. *Swarm and evolutionary computation*, *1*(3), 111–128.

[19] Gao, S., & de Silva, C. W. (2018). Estimation distribution algorithms on constrained optimization problems. *Applied Mathematics and Computation*, *339*, 323–345.

[20] Pelikan, M., Hauschild, M. W., & Lobo, F. G. (2015). Estimation of distribution algorithms, 899–928. https://doi.org/10.1007/978-3-662-43505-2_45

[21] Kashani, A. R., Camp, C. V., Rostamian, M., Azizi, K., & Gandomi, A. H. (2021). Population-based optimization in structural engineering: A review. *Artificial Intelligence Review*, 1–108.

[22] Monti, G., Quaranta, G., & Marano, G. (2010). Genetic-algorithm-based strategies for dynamic identification of nonlinear systems with noise-corrupted response. *Journal of Computing in Civil Engineering - J COMPUT CIVIL ENG*, *24*. https://doi.org/10.1061/(ASCE)CP.1943-5487.0000024

[23] McLachlan, G. J., & Basford, K. E. (1988). *Mixture models: Inference and applications to clustering* (Vol. 38). M. Dekker New York.

[24] McLachlan, G., & Peel, D. (2000). *Finite mixture models*. New York: Wiley.
[25] *MATLAB version 9.10.0.1649659 (R2021a) Update 1*. (2021).
[26] Rosso, M. M., Cucuzza, R., Aloisio, A., & Marano, G. C. (2022). Enhanced multi-strategy particle swarm optimization for constrained problems with an evolutionary-strategies-based unfeasible local search operator. *Applied Sciences*, *12*(5). https://doi.org/10.3390/app12052285
[27] Camp, C., & Farshchin, M. (2014). Design of space trusses using modified teaching–learning based optimization. *Engineering Structures*, *62-63*, 87–97. https://doi.org/https://doi.org/10.1016/j.engstruct.2014.01.020
[28] Piana, G., Lofrano, E., Manuello, A., Ruta, G., & Carpinteri, A. (2017). Compressive buckling for symmetric twb with non-zero warping stiffness. *Engineering Structures*, *135*, 246–258.
[29] Bazzucchi, F., Manuello, A., & Carpinteri, A. (2017). Instability load evaluation of shallow imperfection-sensitive structures by form and interaction parameters. *European Journal of Mechanics-A/Solids*, *66*, 201–211.
[30] Manuello, A. (2020b). Semi-rigid connection in timber structure: Stiffness reduction and instability interaction. *International Journal of Structural Stability and Dynamics*, *20*(07), 2050072.
[31] Piana, G., Lofrano, E., Manuello, A., & Ruta, G. (2017). Natural frequencies and buckling of compressed non-symmetric thin-walled beams. *Thin-Walled Structures*, *111*, 189–196.