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AN EQUAL-ORDER FINITE ELEMENT FRAMEWORK FOR INCOMPRESSIBLE NON-NEWTONIAN FLOW PROBLEMS

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Abstract. Various materials of engineering and biomedical interest can be modelled as generalised Newtonian fluids, i.e., via an apparent viscosity depending locally on the flow field. In spite of the particular features of those models, they are often handled in practice by classical numerical techniques originally conceived for Newtonian fluids. Methods designed specifically for the generalised case are rather scarce in the literature, as well as their use in practical applications. As it turns out, tackling non-Newtonian problems with standard finite element formulations can have undesired consequences such as the induction of spurious pressure boundary layers and the emergence of natural boundary conditions not suitable for realistic flow scenarios. In this context, we introduce a novel framework that deals with those issues while maintaining simplicity and low computational cost. The new stabilised formulation is based on a modified system combining the continuity equation with a Poisson equation for the pressure and consistent pressure boundary conditions. A weak enforcement of the rheological law is employed to enable full consistency even for first-order finite element pairs. Simple numerical examples are provided to demonstrate the potential of our method in yielding accurate solutions for relevant problems.

1 INTRODUCTION

Although blood can normally be considered as a fluid, it is not a liquid, but an organic tissue consisting of living and non-living components immersed in a liquid matrix. Therefore, modelling its behaviour mathematically is by no means a trivial task. In most arteries of healthy individuals, blood flows basically as a Newtonian fluid with uniform viscosity and density. In low-shear regions such as certain veins, small arteries or aneurysms, non-Newtonian effects like shear thinning and viscoelasticity can play a major role in hemodynamics [1]. The non-Newtonian behaviour most frequently observed in blood flows is shear thinning, characterised by decreasing apparent viscosity in the presence of increasing shear rates. This is mostly due to the three-dimensional structures formed by the aggregation of red blood cells [1]. Since these structures take several seconds to form and cannot withstand high shear rates, this non-Newtonian response is seen mainly in flow regions experiencing low shear rates stably throughout time [2].

In particular, our focus is placed on generalised Newtonian models. This popular approach allows for local variations of blood's effective viscosity depending on shear rate levels. It is capable of accounting for non-Newtonian phenomena such as shear thinning and plug flow, while maintaining a similar mathematical and computational framework as used for Newtonian fluids. In fact, several simple models provide excellent fits to experimental data [1, 3]. Such generalised models, sometimes called quasi-Newtonian, are used not only in hemodynamics but also in polymeric flow simulations of industrial interest. Due to the apparent similarity between Newtonian and quasi-Newtonian equations, one is easily tempted to try solving the generalised problem using classical numerical methods. Although it is often possible to do so by applying certain modifications, that is neither physically nor mathematically ideal. Yet, before commenting on the limitations of the existing methods, we shall briefly outline some typical numerical issues encountered in incompressible flow problems.

When using finite element methods for the approximation of incompressible flows, one must carefully choose velocity and pressure basis functions. Using the same polynomial order for both quantities, for instance, violates the famous Ladyzhenskaya-Babuška-Brezzi (LBB) condition, which leads to unstable methods. However, since the findings of Hughes and co-workers four decades ago [4, 5], it has become standard to employ stabilisation methods such as PSPG or GLS to circumvent the LBB condition and allow equal-order shape functions for all unknowns. Nonetheless, most stabilisation methods are based on perturbed incompressibility equations devised under classical Newtonian assumptions. Probably for that reason, the use of LBB-compatible finite elements is somewhat more frequent in the literature for quasi-Newtonian fluids [6–16]. Although such compatible spaces offer an ideal setting from a theoretical standpoint, they are not always viable options. There is thus great practical appeal for first-order elements – especially in biomedical applications, where higher-order meshes are rarely available.

There are a handful of stabilisation methods which are not residual-based and can therefore be applied to generalised Newtonian problems rather straightforwardly. The most popular of them is probably the penalty method relaxing incompressibility to decouple velocity and pressure. Despite being used quite often for quasi-Newtonian fluids [17–20], this approach was shown by Sobhani et al. [17] to attain poor pressure approximations. The pressure Poisson stabilisation of Brezzi and Pitkäranta [21] was reported by Knauf et al. [22] to also perform poorly in practice. The pressure projection method by Dohrmann and Bochev [23] was used successfully by John et al. [24] and seems to be a good alternative. Time-related methods which can be applied straightforwardly to the quasi-Newtonian case are artificial compressibility [25, 26] and split-step methods [27, 28]. Although residual-based stabilisations such as PSPG and VMS can also be used [29–32], their loss of consistency for linear elements can yield inaccurate results, as discussed in Subsection 3.1.

To the best of our knowledge, there are very few works in the literature dedicated to designing stabilisation techniques for generalised Newtonian problems. Some of them require introducing the viscous stress as an additional tensor-valued unknown [33, 34], leading to prohibitive computational costs in practical applications. The variational muti-scale (VMS) approach proposed by Masud and Kwack [30] offers the efficiency and simplicity of residual-based methods, but can induce spurious pressure boundary layers for low-order discretisations. With these challenges in mind, this work presents, as its main contribution, a new residual-based framework for equal-order finite element approximations of quasi-Newtonian problems. The most important feature of our new method is overcoming the loss of consistency of standard PSPG-like stabilisation terms in the lowest-order case, by augmenting the continuity equation with a pressure Poisson equation [35, 36] with consistent boundary conditions. In doing so, we achieve considerable improvements in robustness and accuracy with respect to state-of-the-art residual-based stabilisations.

2 PRELIMINARIES

The balance of linear momentum and mass for a stationary incompressible flow of a generalised Newtonian fluid can be stated as, respectively,

$$(\rho\nabla\mathbf{u})\,\mathbf{u}-\nabla\cdot(2\mu\nabla^s\mathbf{u})+\nabla p=\mathbf{0}, \tag{1}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2}$$

where $\mathbf{u}$ is the flow velocity, p is the pressure, ρ is the fluid's density, μ is the (apparent) dynamic viscosity and $\nabla^s\mathbf{u}$ is the symmetric part of the velocity gradient, namely,

$$\nabla^s\mathbf{u}:=\frac{1}{2}\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\right].$$

In hemodynamic and polymeric flows, the viscosity is normally modelled through a nonlinear dependence on the shear rate $\dot\gamma:=\sqrt{2\nabla^s\mathbf{u}:\nabla^s\mathbf{u}}$, that is,

$$\mu=\mu(\nabla^s\mathbf{u})=\eta\left(\dot\gamma(\nabla^s\mathbf{u})\right).$$

Before introducing the variational formulations, let us consider a spatial domain $\Omega\subset\mathbb{R}^d$, $d=2,3$, with a Lipschitz boundary $\Gamma:=\partial\Omega$ decomposed into two non-overlapping regions Γ_D and Γ_N. In Γ_D the velocities are prescribed: $\mathbf{u}|_{\Gamma_D}=\mathbf{g}$. The kind of data prescribed on Γ_N depends on the weak form used, as will be discussed later on.

3 STABILISED FORMULATIONS

We now present the most classical residual-based stabilisation for equal-order methods, followed by a new stabilisation approach.

3.1 The pressure-stabilised Petrov-Galerkin method

The pressure-stabilised Petrov-Galerkin (PSPG) method is the most popular stabilisation approach for equal-order Navier-Stokes solvers. It consists of relaxing the incompressibility constraint with a (strong) weighted residual of the momentum equation. The discrete variational formulation is to find $(\mathbf{u}_h,p_h)\in[X_h]^d\times X_h$, with $\mathbf{u}_h|_{\Gamma_D}=\mathbf{g}_h$, such that for all $(\mathbf{w}_h,q_h)\in[X_h]^d\times X_h$, with $\mathbf{w}|_{\Gamma_D}=\mathbf{0}$,

$$\langle\mathbf{w}_h,(\rho\nabla\mathbf{u}_h)\mathbf{u}_h\rangle_\Omega+\langle\nabla^s\mathbf{w}_h,2\mu(\nabla^s\mathbf{u}_h)\nabla^s\mathbf{u}_h\rangle_\Omega-\langle\nabla\cdot\mathbf{w}_h,p_h\rangle_\Omega=\langle\mathbf{w}_h,\mathbf{t}\rangle_{\Gamma_N}, \tag{3}$$

$$\langle q_h,\nabla\cdot\mathbf{u}_h\rangle_\Omega+\sum_{e=1}^{N_e}\left\langle\frac{\alpha {h_e}^2}{\mu(\nabla^s\mathbf{u}_h)}\nabla q_h,\nabla p_h-\nabla\cdot[2\mu(\nabla^s\mathbf{u})\nabla^s\mathbf{u}]+(\rho\nabla\mathbf{u}_h)\mathbf{u}_h\right\rangle_{\Omega_e}=0, \tag{4}$$

where X_h is a continuous finite element space, $\mathbf{t}:=[2\mu(\nabla^s\mathbf{u})\nabla^s\mathbf{u}]\mathbf{n}-p\mathbf{n}$, N_e is the number of elements Ω_e, α is a positive parameter and $\mathbf{n}$ is the outward unit normal vector on Γ. Let us also define $h=\max\{h_e\}$. Using the stabilisation parameter as $O({h_e}^2)$, as done above, is the optimal choice for diffusion-dominated flows [37]. For α sufficiently large, the system is stable for equal-order pairs due to the bilinear form $\langle\nabla q_h,\nabla p_h\rangle_\Omega$, which breaks the saddle-point structure of the problem and the LBB requirement. The idea behind including the remaining terms of the residual is to render the perturbation to the continuity equation smaller where the solution is already accurate enough. This is very important for the accuracy and quality of the approximations [38, 39], and here arises the issue of using PSPG with linear elements. In that case, and when using simplicial elements, the velocity gradient is piecewise constant and, consequently, so is the (discrete) viscous stress tensor $\mathbb{S}_h$. Hence:

$$\sum_{e=1}^{N_e} \left\langle \frac{\alpha {h_e}^2}{\mu(\nabla^s \mathbf{u}_h)} \nabla q_h, \nabla \cdot \mathbb{S}_h \right\rangle_{\Omega_e} = \sum_{e=1}^{N_e} \left\langle \frac{\alpha {h_e}^2}{\mu(\nabla^s \mathbf{u}_h)} \nabla q_h, \mathbf{0} \right\rangle_{\Omega_e} = 0,$$

that is, we completely lose the viscous part of the residual. This means that, no matter how fine the mesh is, the added residual will never vanish. An incomplete residual can spoil coarse grid accuracy and restrict the choice of the stabilisation parameter. As a matter of fact, it can be shown that an "inviscid" residual induces so-called spurious pressure boundary layers [40]. Although remedies exist in the Newtonian case [38, 39, 41], they cannot be directly applied to fluids with shear-dependent viscosity.

Another shortcoming of the standard formulation (3) is its natural boundary conditions (BCs) given in terms of tractions $\mathbf{t}$. Although very often done [24], setting $\mathbf{t} = \mathbf{0}$ on open boundaries leads to spurious velocity and pressure oscillations [42, 43]. For constant viscosity, there is a simple remedy: using the Laplacian form of the stress-divergence, that is,

$$\begin{aligned} \nabla \cdot \mathbb{S} &= \mu \nabla \cdot (2\nabla^s \mathbf{u}) \\ &= \mu [\Delta \mathbf{u} + \nabla (\nabla \cdot \mathbf{u})] \\ &= \mu \Delta \mathbf{u}. \end{aligned}$$

Then, integration by parts will result in normal pseudo-tractions $\tilde{\mathbf{t}} := [\mu(\nabla^s \mathbf{u}) \nabla \mathbf{u}]\mathbf{n} - p\mathbf{n}$ as natural BCs. The outflow boundary condition $\tilde{\mathbf{t}} = \mathbf{0}$ is not only satisfied by Poiseuille and Womersley flows, but also allows vortices to leave the computational domain with minimal upstream disturbance [42]. Therefore, it is the preferred BC for truncated outlets. When the viscosity is no longer constant, we can formulate the problem in a generalised Laplacian form by writing

$$\begin{aligned} \nabla \cdot \mathbb{S} &= \nabla \cdot (2\mu \nabla^s \mathbf{u}) \\ &= \mu [\Delta \mathbf{u} + \nabla (\nabla \cdot \mathbf{u})] + 2\nabla^s \mathbf{u} \nabla \mu \\ &= \mu \Delta \mathbf{u} + 2\nabla^s \mathbf{u} \nabla \mu, \end{aligned} \tag{5}$$

and using integration by parts only on the Laplacian term $\mu \Delta \mathbf{u}$. The weak form of the momentum equation then becomes [44]

$$\langle \mathbf{w}_h, (\rho \nabla \mathbf{u}_h) \mathbf{u}_h \rangle_\Omega + \langle \nabla \mathbf{w}_h, \mu_h \nabla \mathbf{u}_h \rangle_\Omega - \left\langle \mathbf{w}_h, (\nabla \mathbf{u}_h)^\top \nabla \mu_h \right\rangle_\Omega - \langle \nabla \cdot \mathbf{w}_h, p_h \rangle_\Omega = \langle \mathbf{w}_h, \tilde{\mathbf{t}} \rangle_{\Gamma_N}, \tag{6}$$

where μ_h is a continuous projection of the viscosity field, as discussed later on.

3.2 The generalised boundary vorticity stabilisation method

We will next present a generalisation of the boundary vorticity stabilisation (BVS) by Pacheco et al. [45] to quasi-Newtonian problems. The BVS is a residual-based formulation containing a first-order boundary term proportional to the vorticity $\nabla \times \mathbf{u}$, which guarantees consistency even for linear elements. Details on the method for the Newtonian case can be found in our previous article [45].

3.2.1 Strong form

We start by constructing an equivalent PDE system to replace the classical momentum-mass system (1)–(2). The proposed boundary value problem (BVP) reads:

$$(\rho\nabla\mathbf{u})\,\mathbf{u} - \mu\Delta\mathbf{u} - 2\nabla^s\mathbf{u}\nabla\mu + \nabla p = \mathbf{0}, \tag{7}$$

$$-\Delta p = \nabla\cdot\left[(\rho\nabla\mathbf{u})\mathbf{u} - 2\nabla^s\mathbf{u}\nabla\mu\right] + \left[\nabla\times(\nabla\times\mathbf{u})\right]\cdot\nabla\mu - \beta\nabla\cdot\mathbf{u}, \tag{8}$$

$$\left.\frac{\partial p}{\partial n}\right|_\Gamma = \mathbf{n}\cdot\left[2\nabla^s\mathbf{u}\nabla\mu - (\rho\nabla\mathbf{u})\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u})\right], \tag{9}$$

where β is some given positive function to be defined later. The velocity BCs were omitted because they play no role in the following discussion. We wish to show that, for sufficiently regular p and $\mathbf{u}$, the new system is equivalent to the classical one ((1)–(2)). The first step is to apply the divergence operator to Eq. (7), leading to

$$\begin{aligned}\Delta p &= \nabla\cdot\left[\mu\Delta\mathbf{u} + 2\nabla^s\mathbf{u}\nabla\mu - (\rho\nabla\mathbf{u})\mathbf{u}\right]\\ &= \nabla\cdot\left[2\nabla^s\mathbf{u}\nabla\mu - (\rho\nabla\mathbf{u})\mathbf{u}\right] + \nabla\mu\cdot\Delta\mathbf{u} + \mu\nabla\cdot(\Delta\mathbf{u}),\end{aligned} \tag{10}$$

which when added to Eq. (8) gives

$$\begin{aligned}0 &= \left[\nabla\times(\nabla\times\mathbf{u})\right]\cdot\nabla\mu - \beta\nabla\cdot\mathbf{u} + \nabla\mu\cdot\Delta\mathbf{u} + \mu\nabla\cdot(\Delta\mathbf{u})\\ &= \mu\nabla\cdot(\Delta\mathbf{u}) + \nabla\mu\cdot\left[\Delta\mathbf{u} + \nabla\times(\nabla\times\mathbf{u})\right] - \beta\nabla\cdot\mathbf{u},\end{aligned} \tag{11}$$

but $\Delta\mathbf{u} + \nabla\times(\nabla\times\mathbf{u}) \equiv \nabla(\nabla\cdot\mathbf{u})$ and $\nabla\cdot(\Delta\mathbf{u}) \equiv \Delta(\nabla\cdot\mathbf{u})$. Therefore, introducing $\phi := \nabla\cdot\mathbf{u}$, we get the diffusion-reaction equation

$$-\nabla\cdot(\mu\nabla\phi) + \beta\phi = 0, \tag{12}$$

whose Neumann BCs are attained by dotting Eq. (7) with $\mathbf{n}$ and substracting from (9), which leads to

$$\begin{aligned}0 &= \mathbf{n}\cdot\left[\Delta\mathbf{u} + (\nabla\times\nabla\times\mathbf{u})\right]\\ &= \mathbf{n}\cdot\left[\nabla(\nabla\cdot\mathbf{u})\right]\\ &= \frac{\partial\phi}{\partial n}.\end{aligned} \tag{13}$$

The solution of Eq. (12) is thus $\phi \equiv 0$, that is, $\nabla\cdot\mathbf{u} = 0$ in Ω, as we wanted. Now that we have recovered Eq. (2), the equivalence between Eqs. (7) and (1) is straightforward. Proving the other direction, i.e., that the standard Navier-Stokes system implies the modified BVP is similar but simpler, so we omit this part.

3.2.2 Weak formulation

With the new modified system we can start deriving our stabilised formulation. Integration by parts gives

$$\langle\nabla q, \nabla p\rangle_\Omega - \left\langle q, \frac{\partial p}{\partial n}\right\rangle_\Gamma = \left\langle q, \nabla\cdot\left[(\rho\nabla\mathbf{u})\mathbf{u} - 2\nabla^s\mathbf{u}\nabla\mu\right]\right\rangle_\Omega + \left\langle q, \left[\nabla\times(\nabla\times\mathbf{u})\right]\cdot\nabla\mu - \beta\nabla\cdot\mathbf{u}\right\rangle_\Omega.$$

Integrating the first term on the right-hand side by parts and enforcing the Neumann BC (9) yields

$$\begin{aligned} &\langle q, \beta \nabla \cdot \mathbf{u} \rangle_\Omega + \langle \nabla q, \nabla p + (\rho \nabla \mathbf{u})\mathbf{u} \rangle_\Omega = \\ &\left\langle \nabla q, \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right] \nabla \mu \right\rangle_\Omega + \langle q, [\nabla \times (\nabla \times \mathbf{u})] \cdot \nabla \mu \rangle_\Omega - \langle q\mathbf{n}, \mu \nabla \times (\nabla \times \mathbf{u}) \rangle_\Gamma , \end{aligned} \tag{14}$$

but

$$\begin{aligned} \langle q\mathbf{n}, \mu \nabla \times (\nabla \times \mathbf{u}) \rangle_\Gamma &= \langle q, \mu \overbrace{\nabla \cdot [\nabla \times}^{\equiv 0} (\nabla \times \mathbf{u})] \rangle_\Omega + \langle \nabla (q\mu), \nabla \times (\nabla \times \mathbf{u}) \rangle_\Omega \\ &= \langle \nabla q, \mu \nabla \times (\nabla \times \mathbf{u}) \rangle_\Omega + \langle q, \nabla \mu \cdot [\nabla \times (\nabla \times \mathbf{u})] \rangle_\Omega . \end{aligned}$$

Therefore, the right-hand side of Eq .(14) becomes

$$\left\langle \nabla q, \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right] \nabla \mu \right\rangle_\Omega - \langle \nabla q, \mu \nabla \times (\nabla \times \mathbf{u}) \rangle_\Omega .$$

As in the PSPG formulation, there is a second-order term which cannot be approximated by first-order finite element spaces. To preserve the viscous residual we must rewrite this second-order term with first-order derivatives only. Using integration by parts once again gives

$$\langle \mu \nabla q, \nabla \times (\nabla \times \mathbf{u}) \rangle_\Omega = \langle \nabla q \times \mathbf{n}, \mu \nabla \times \mathbf{u} \rangle_\Gamma + \langle \nabla \times (\mu \nabla q), \nabla \times \mathbf{u} \rangle_\Omega ,$$

but

$$\begin{aligned} \langle \nabla \times (\mu \nabla q), \nabla \times \mathbf{u} \rangle_\Omega &\equiv \langle \mu \overbrace{\nabla \times (\nabla q)}^{\equiv \mathbf{0}} + \nabla \mu \times \nabla q, \nabla \times \mathbf{u} \rangle_\Omega \\ &= \langle \nabla q, (\nabla \times \mathbf{u}) \times \nabla \mu \rangle_\Omega \\ &\equiv \left\langle \nabla q, \left[\nabla \mathbf{u} - (\nabla \mathbf{u})^\top \right] \nabla \mu \right\rangle_\Omega . \end{aligned}$$

Thus, the weak form of our pressure Poisson problem finally simplifies to

$$\langle q, \beta \nabla \cdot \mathbf{u} \rangle_\Omega + \left\langle \nabla q, \nabla p + (\rho \nabla \mathbf{u})\mathbf{u} - 2 (\nabla \mathbf{u})^\top \nabla \mu \right\rangle_\Omega + \langle \nabla q \times \mathbf{n}, \mu \nabla \times \mathbf{u} \rangle_\Gamma = 0.$$

The last issue to be addressed is the presence of the term $\nabla \mu$ in both momentum and pressure Poisson equations. Since the viscosity in general depends on $\nabla^s \mathbf{u}$, computing $\nabla \mu$ in the variational formulation would increase the regularity requirements on the velocity space and therefore preclude the use of standard Lagrangian finite elements. We can avoid this by projecting the viscosity onto a continuous space, or, in other words, enforcing the rheological law weakly. Our discrete variational formulation then reads: Find $(\mathbf{u}_h, p_h, \mu_h) \in [X_h]^d \times X_h \times X_h$, with $\mathbf{u}_h|_{\Gamma_D} = \mathbf{g}_h$, such that for all $(\mathbf{w}_h, q_h, \upsilon_h) \in [X_h]^d \times X_h \times X_h$, with $\mathbf{w}_h|_{\Gamma_D} = \mathbf{0}$,

$$\langle \mathbf{w}_h, (\rho \nabla \mathbf{u}_h)\mathbf{u}_h \rangle_\Omega + \langle \nabla \mathbf{w}_h, \mu_h \nabla \mathbf{u}_h \rangle_\Omega - \left\langle \mathbf{w}_h, (\nabla \mathbf{u}_h)^\top \nabla \mu_h \right\rangle_\Omega - \langle \nabla \cdot \mathbf{w}_h, p_h \rangle_\Omega = \langle \mathbf{w}_h, \tilde{\mathbf{t}} \rangle_{\Gamma_N}, \tag{15}$$

$$\langle q_h, \beta \nabla \cdot \mathbf{u}_h \rangle_\Omega + \left\langle \nabla q_h, \nabla p_h + (\rho \nabla \mathbf{u}_h)\mathbf{u}_h - 2 (\nabla \mathbf{u}_h)^\top \nabla \mu_h \right\rangle_\Omega + \langle \nabla q_h \times \mathbf{n}, \mu_h \nabla \times \mathbf{u}_h \rangle_\Gamma = 0, \tag{16}$$

$$\langle \upsilon_h, \mu_h - \eta (\dot{\gamma}(\nabla^s \mathbf{u}_h)) \rangle_\Omega = 0, \tag{17}$$

with X_h being once again a continuous finite element space. If we have a homogeneous Newtonian fluid, then $\nabla\mu \equiv \mathbf{0}$ and we recover the original boundary vorticity stabilisation [45]. We therefore denote the present method as *generalised boundary vorticity stabilisation* (GBVS). For optimal convergence, the function β must be taken as the inverse of the PSPG stabilisation parameter, which for a diffusion-dominated flow gives $\beta|_{\Omega_e} = (\alpha {h_e}^2/\mu_h)^{-1}$, resulting in

$$\langle q_h, \beta \nabla \cdot \mathbf{u}_h \rangle_\Omega = \alpha^{-1} \sum_{e=1}^{N_e} {h_e}^{-2} \langle q_h, \mu_h \nabla \cdot \mathbf{u}_h \rangle_{\Omega_e} . \tag{18}$$

A more detailed discussion on the choice of β is presented by Pacheco et al. [45].

Finally, we have attained a stabilised formulation retaining full consistency even for linear elements, since all second-order derivatives have been eliminated and no part of the residual is lost. This, along with the fact that our stabilisation term is constructed from a Poisson equation with consistent pressure boundary conditions, results in a formulation that is free from spurious pressure boundary layers [45]. This will be illustrated with a numerical example.

4 NUMERICAL EXAMPLES

We now consider two simple benchmark tests to showcase the performance of our new stabilised formulation. Bilinear quadrilateral elements are used in both examples, with the popular Carreau-Yasuda rheological model

$$\eta(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + |\lambda\dot{\gamma}|^a\right)^{\frac{n-1}{a}} , \tag{19}$$

with $a > 0$, $n < 1$ and $\mu_0 > \mu_\infty > 0$ for shear-thinning fluids such as blood. We begin with a simple two-dimensional Poiseuille flow benchmark. The domain is the straight channel $\Omega = (0, L) \times \left(-\frac{H}{2}, \frac{H}{2}\right)$, with $L = 3H = 3$ mm, and the rheological parameters are those of blood for a representative composition [3]: $\rho = 1050$ kg/m^3, $\mu_\infty = 3.45$ mPa·s, $\mu_0 = 56$ mPa·s, $n = 0.3568$, $a = 2$ and $\lambda = 3.313$ s. We impose no slip on the walls, i.e., $\mathbf{u}|_{y=\pm H/2} = \mathbf{0}$, and enforce a 9 Pa pressure drop by setting $\tilde{\mathbf{t}}|_{x=0} = \{p_{\text{in}}, 0\}^\top$ and $\tilde{\mathbf{t}}|_{x=L} = \mathbf{0}$, with $p_{\text{in}} = 9$ Pa. The exact solution has perfectly vertical, uniformly spaced pressure isolines, which gives an ideal scenario to investigate the issue of spurious pressure boundary layers. In order to do that, we consider a mesh of 96×32 identical square elements and a stabilisation parameter $\alpha = 20$. This is rather higher than used in practice, which allows us to really highlight the spurious effects that arise when the stabilisation term is not fully consistent. The isobars produced by the PSPG and GBVS methods are depicted in Figure 1. One immediately spots the numerical boundary layer attained by the classical method, which does not happen with the present approach. Of course, for finer meshes and smaller stabilisation parameters the difference tends to become less visible, but the spurious boundary behaviour will always be there (the thickness of the boundary layer in PSPG is actually proportional to $\sqrt{\alpha h^2}$ [46]). A quantitative assessment of the boundary accuracy can be found in the first study [45] where the Newtonian BVS was originally proposed.

As a second example, we choose the backward-facing step benchmark set up by Masud and Kwack [30], with the parameters $\rho = 1060$ kg/m^3, $\mu_\infty = 3.45$ mPa·s, $\mu_0 = 56$ mPa·s, $n = 0.22$, $\lambda = 1.902$ s and $a = 1.25$. Considering the domain $\Omega = ((0, 12H) \times (0, 2H)) \setminus ([0, 4H] \times [0, H])$ shown in Figure 2, we prescribe a parabolic inflow at $x = 0$, zero pseudo-traction on the outlet $x = 12H$ and no slip on the remainder of the boundary. The Reynolds number for this example can be written as $\text{Re} = \rho Q/\mu_\infty$, with

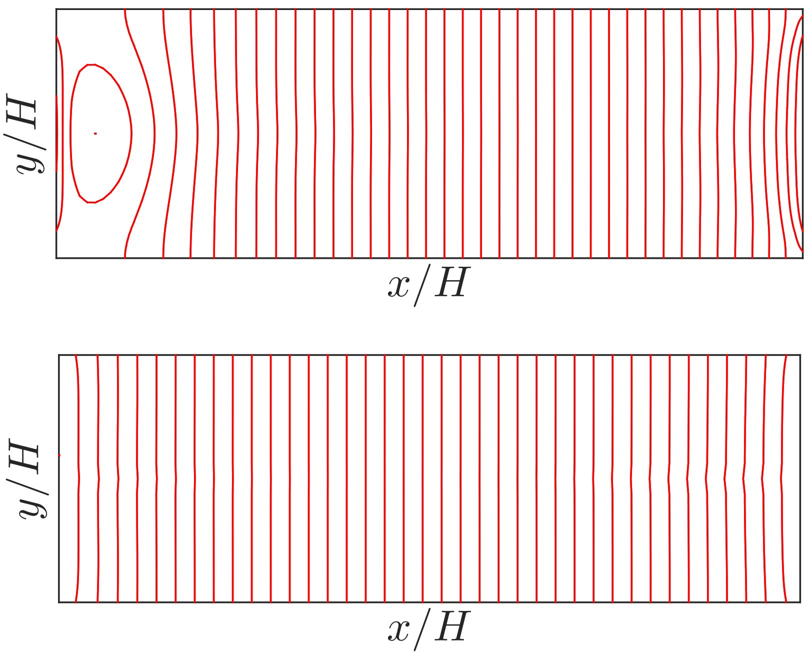

Figure 1: Poiseuille-Carreau flow: pressure isolines yielded by the PSPG (top) and GBVS (bottom) methods.

Q being the inflow rate. Results for Re $= 25$ have been reported by Masud and Kwack [30] for three channel widths: $H = 0.5$, 5 and 50 mm. We compare them with respect to the normalised wall shear stress $\overline{\tau_w}$ downstream of the step, along the line $y = 0$:

$$\overline{\tau_w} := \frac{\left[\eta\left(\dot{\gamma}\right)\frac{\partial u}{\partial y}\right]\Big|_{y=0}}{6\mu_\infty Q/(2H)^2},$$

whose denominator corresponds to the wall shear stress of a developed Newtonian flow. For the numerical tests we set $\alpha = 0.1$ and a uniform mesh with 50,000 identical square elements. Figure 3 shows the comparison between our results and those attained by Masud and Kwack [30] through a variational multiscale method. The comparison reveals good agreement between the solutions, especially for the narrower channels.

5 CONCLUSIONS

In this contribution, we have presented a new residual-based stabilisation method for equal-order finite element approximations of incompressible quasi-Newtonian flow problems. The stabilisation term is formed by combining the divergence-free constraint with a Poisson equation for the pressure with consistent boundary conditions. Furthermore, using appropriate simplifications of the nonlinear viscous term, we have managed to construct a weak formulation relying only on first-order derivatives. As a consequence, the new stabilisation term is fully consistent even for lowest-order pairs, in contrast to

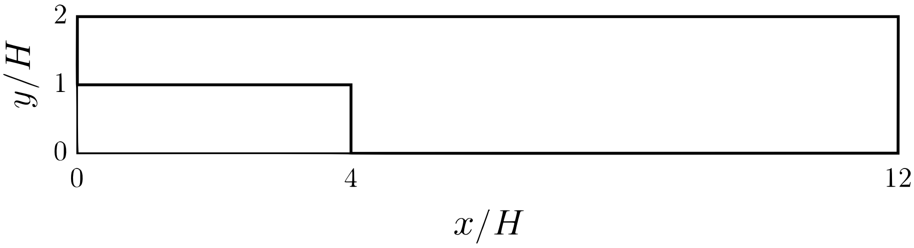

Figure 2: Geometry considered for the backward-facing step problem.

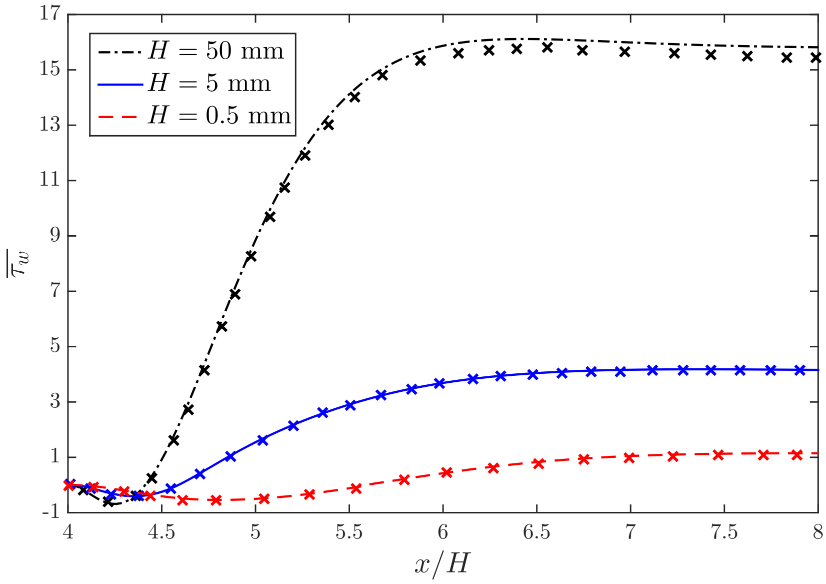

Figure 3: Backward-facing step: wall shear stress at $y = 0$ for the GBVS (lines) and VMS (markers [30]) methods.

most residual-based stabilisation techniques such as PSPG, GLS and VMS. The result is a formulation completely free from spurious pressure boundary layers and improved conservation mass. As a matter of fact, numerical tests for the Newtonian case indicate improved accuracy even for higher-order finite element pairs [45]. Ongoing work includes the development of fast solvers for the time-dependent setting, including split-step and space-time schemes.

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