Explicit Solution of a Linearly Constrained Infinite Quadratic Problem

Gilberto Calvillo-Vives[1*], Jesús López-Estrada[2]
David Romero[1]
[1] Instituto de Matemáticas, UNAM
62210 Cuernavaca, Mor., Mexico.
[2] Facultad de Ciencias, UNAM
04510 Ciudad Universitaria, Mexico City, Mexico.
[*] e-mail: gilberto.calvillo@im.unam.mx

Abstract

A norm minimization problem posed in K-dimensional finite arrays without non-negativity constraints was efficiently solved by Romero Romero [11]. Going beyond we provide here an explicit, exact solution in case the arrays are replaced by L_2-Hilbert spaces. Furthermore, we propose a polynomial procedure yielding an approximate optimal solution when non-negativity constraints must be taken into account for $K = 2$.

Key words: Input-output integral equations; Quadratic optimization; Continuous transportation constraints; Norm minimization; Radon transform.

Introduction

Closely related to optimization problems that arise when updating input-output matrices in economic studies [6], inferring migration patterns in demography [10], and reconstructing images in tomography Kak and Slaney [5], or among other applications in the realm of statistical sampling [3], Romero [11] considered

Problem 1.1. *Given real vectors*

$$\underset{\sim}{u} = (u_1, \ldots, u_m) \text{ and } \underset{\sim}{v} = (v_1, \ldots, v_n)$$

satisfying $\sum_{i=1}^{m} u_i = \sum_{j=1}^{n} v_j$*, and an m-by-n real matrix* $A = (A_{ij})$*, find* $X \in \mathbb{R}^{m\times n}$ *that minimizes* $\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - A_{ij})^2$ *subject to linear constraints* $\sum_{j=1}^{n} X_{ij} = u_i$*, for* $i = 1, \ldots, m$*, and* $\sum_{i=1}^{m} X_{ij} = v_j$*, for* $j = 1, \ldots, n$*.*

and proved that its optimum $X^* = (X^*_{ij})$ can be found by means of the explicit, compact formula:

$$X^*_{ij} = A_{ij} + \frac{u_i - u'^{i}}{n} + \frac{v_j - v'^{j}}{m} - \frac{\varphi}{mn},$$

$$i = 1, \ldots, m; \; j = 1, \ldots, n,$$

where $u'_i = \sum_{\ell=1}^{n} A_{i\ell}$, $v'_j = \sum_{k=1}^{m} A_{kj}$, and $\varphi = \sum_{i=1}^{m}(u_i - u'_i)$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, thus yielding a solution algorithm of linear computational complexity in the number mn of variables.

Calvillo and Romero [2] furnished a polynomial algorithm to solve Problem 1.1 when non-negativity constraints are added, namely,

Problem 1.2. *Given real vectors* $(u_1,\dots,u_m)$ *and* $(v_1,\dots,v_n)$ *satisfying* $\sum_{i=1}^{m} u_i = \sum_{j=1}^{n} v_j$*, and an* m*-by-*n *real matrix* $A = (A_{ij})$*, find* $X \in \mathbb{R}^{m\times n}$ *minimizing* $\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - A_{ij})^2$ *subject to* $\sum_{j=1}^{n} X_{ij} = u_i$*,* $\sum_{i=1}^{m} X_{ij} = v_j$*, and* $X_{ij} \geq 0$*, for* $i = 1,\dots,m$*, and* $j = 1,\dots,n$*.*

On the other hand, in [11] a generalization of Problem 1.1 to K-dimensional arrays ($K \geq 2$) was described and explicitly solved as follows.

For given positive integers $K, d_1, d_2, \dots, d_K$ ($K \geq 2$), let $\mathcal{K}$ be the set of the first K positive integers and let G be the grid

$$I^1 \times I^2 \times \dots \times I^K\,,$$

where I^k ($k \in \mathcal{K}$) denotes the set of the first d_k positive integers. Further, let $A : G \to \mathbb{R}$ be a K-dimensional array of real numbers over G, and for every $k \in \mathcal{K}$ let

$$\hat{b}_k = (\hat{b}_{k1}, \hat{b}_{k2}, \dots, \hat{b}_{kd_k}) : I^k \to \mathbb{R}$$

be a one-dimensional array of real numbers. Finally, letting α denote an element $(i_1,\dots,i_K)$ of G, $W := \{\,(k,\ell) \mid k \in \mathcal{K},\ \ell \in I^k\,\}$ and $S_{k\ell} := \{\,\alpha \in G \mid i_k = \ell\,\}$, consider

Problem 1.3.

$$P_K \begin{cases} \textit{minimize} \ \ \displaystyle\sum_{\alpha\in G}(X_\alpha - A_\alpha)^2 \\ \textit{subject to} \ \ \displaystyle\sum_{\alpha\in S_{k\ell}} X_\alpha = \ \hat{b}_{k\ell}\,,\ (k,\ell)\in W. \end{cases}$$

Note that adding non-negativity constraints to the norm minimization problem P_K yields an axial transportation polytope as defined in Yemelichev et al. [13], which has been object of numerous studies (see for example [9], and references therein). Assuming feasibility, namely,

$$\sum_{\ell\in I^j} \hat{b}_{j\ell} = \sum_{\ell\in I^k} \hat{b}_{k\ell} =: \hat{\varphi}\,,$$

for every $j,k \in \mathcal{K}$, the following result arises.

Theorem 1.1. *[11]Romero The optimal solution* $X^* = (X^*_{i_1,\dots,i_K})$ *to* P_K *is given by*

$$X^*_{i_1,\dots,i_K} = A_{i_1,\dots,i_K} + \frac{1}{\Omega}\left[\sum_{j=1}^{K} d_j b_{ji_j} - (K-1)\varphi\right],$$

$$(i_1,\dots,i_K) \in G,$$

where $\varphi = \hat{\varphi} - \sum_{\alpha\in G} A_\alpha$*,* $\Omega = \prod_{k=1}^{K} d_k$*, and* $b_{k\ell} = \hat{b}_{k\ell} - \sum_{\alpha\in S_{k\ell}} A_\alpha$ *for* $(k,\ell) \in W$*.*

The aim of this paper is twofold: first to generalize Theorem 1.1, then to tackle Problem 1.2 by algorithmic means, both considering the minimization of the quadratic functional that arises when the arrays are replaced by L_2-Hilbert spaces as is explained now.

Let $B \in \mathbb{R}^K$ be the K-box $B \equiv [0,d_1]\times\cdots\times[0,d_K]$, and $B^{(j)}_{x_j} \in \mathbb{R}^K$, for $x_j \in [0,d_j]$, be the slice

$$B^{(j)}_{x_j} \equiv [0,d_1]\times\cdots\times[0,d_{j-1}]\times\{x_j\}\times[0,d_{j+1}]\times \cdots\times[0,d_K].$$

Also, we denote by $vol(B) \equiv \prod_{j=1}^{K} d_j$ the K-dimensional content of box B, and by

$$vol(B^{(j)}_{x_j}) \equiv \prod_{k=1,k\neq j}^{K} d_k$$

the $(K-1)$-dimensional content of slice $B^{(j)}_{x_j}$. As the content of $vol(B^{(j)}_{x_j})$ is independent of x_j we write $vol(B^{(j)})$ instead of $vol(B^{(j)}_{x_j})$. Now, consider the space of *observable* real functions

$$\mathbb{Y} \equiv L_2[0,d_1]\times\cdots\times L_2[0,d_K], \tag{1}$$

where $L_2[0,d_j]$ is the classical space of quadratic integrable real functions in the Lebesgue sense over $[0,d_j]$, equipped with the inner product $\langle y,z\rangle_{\mathbb{Y}} \equiv \sum_{j=1}^{K}\int_0^{d_j} z(x_j)\,y(x_j)\,dx_j$. Clearly, $\mathbb{Y}$ is a Hilbert space. We take as *reconstruction* space the classical space $L_2(B)$ of the quadratic integrable real functions in the Lebesque sense over B, equipped with the usual inner product $\langle X,Y\rangle_{L_2(B)} \equiv \int_B Y(\underset{\sim}{x})X(\underset{\sim}{x})d\underset{\sim}{x}$.

Problem 1.4. *Given the observable functions $f_j \in L_2[0,d_j]$ of X, for $j=1,\dots,K$, find $X \in L_2(B)$ minimizing $\Psi(X) \equiv \frac{1}{2}\|X\|^2_{L_2(B)}$, subject to linear integral constraints*

$$\int_{B^{(j)}} X(\underset{\sim}{x})\,d\underset{\sim}{x}^{[j]} = f_j(x_j), \tag{2}$$

$$x_j \in [0,d_j], \quad j=1,\dots,K,$$

where $d\underset{\sim}{x}^{[j]} \equiv dx_1\dots dx_{j-1}dx_{j+1}\dots dx_K$.

Our aim in this paper is twofold. First, we study the Problem 1.4 and then we discuss the natural generalization of Problem 1.2 in an infinite dimensional context. In Section 3, we provide an operator formulation of Problem 1.4, which is used in Section 3 to furnish an explicit formula for its solution. In Section 4, a natural generalization to Problem 1.2 is given, and it is solved by a polynomial algorithm relaying on a discretization of the problem. Finally, Section 5 is devoted to our conclusions.

Operator formulation

We start by reformulating Problem 1.4 in terms of continuous linear operator

$$\begin{aligned} &R_j[X](p) \equiv R[X](\hat{e}_j,p) \\ &= \textstyle\int_{B_p^{(j)}} X(x_1,..,x_{j-1},p,x_{j+1},..,x_K)d\underset{\sim}{x}^{[j]} \\ &= \textstyle\int_{B_p^{(j)}} X(\underset{\sim}{x})d\underset{\sim}{x}^{[j]}. \end{aligned}$$

close related with the Radon transform

$$R[X](\underset{\sim}{\omega},p) = \int_{\underset{\sim}{\omega}\cdot\underset{\sim}{x}=p} X(\underset{\sim}{x})d\underset{\sim}{x}$$

where $\underset{\sim}{\omega} \in S^{K-1}$, namely, the unit sphere in $\mathbb{R}^K$, and p is a real number[1].

Proposition 2.1. *The linear transforms*

$$R_j : L_2(B) \to L_2[0,d_j]\,,$$

for $j=1,\dots,K$, are continuous.

Proof. For $X \in L_2(B)$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} &|R_j[X](p)|^2 = \left|\textstyle\int_{B_p^{(j)}} 1\cdot X(\underset{\sim}{x})d\underset{\sim}{x}^{[j]}\right|^2 \\ &\le \textstyle\int_{B_p^{(j)}} |1\cdot X(\underset{\sim}{x})|^2 d\underset{\sim}{x}^{[j]} \\ &\le \left|\left(\textstyle\int_{B_p^{(j)}} 1\,d\underset{\sim}{x}^{[j]}\right)^{\frac{1}{2}}\left(\textstyle\int_{B_p^{(j)}} |X(\underset{\sim}{x})|^2\,d\underset{\sim}{x}^{[j]}\right)^{\frac{1}{2}}\right|^2 \\ &= vol(B^{(j)})\textstyle\int_{B^{(j)}} |X(\underset{\sim}{x})|^2\,d\underset{\sim}{x}^{[j]} \end{aligned}$$

Consequently,

$$\| R_j[X] \|^2_{L_2[0,d_j]} \le \int_0^{d_j}\left(vol(B^{(j)})\int_{B_p^{(j)}} |X(\underset{\sim}{x})|^2\,d\underset{\sim}{x}\right.$$

where the right-hand side becomes

$$vol(B^{(j)})\int_0^{d_j}\left(\int_{B_p^{(j)}} |X(\underset{\sim}{x})|^2\,d\underset{\sim}{x}^{[j]}\right)dp$$

$$= vol(B^{(j)})\int_B |X(\underset{\sim}{x})|^2\,d\underset{\sim}{x}.$$

So, we get that $R_j[X] \in L_2[0,d_j]$, and

$$\| R_j[X] \|_{L_2[0,d_j]} \le (vol(B^{[j]})^{1/2}\,\|X\|_{L_2(B)}\,,$$

and the proposition follows. □

Now, in terms of the *observational* operator $A : L_2(B) \to \mathbb{Y}$ given by

$$A[X] \equiv \begin{bmatrix} R_1[X] \\ \vdots \\ R_K[X] \end{bmatrix}, \tag{3}$$

we see that Problem 1.4 can be reformulated as

[1]See Natterer Natterer [8] and Helgason Helgason [4] for applications of the Radon transform in Computarized Tomography, and in Integral Geometry and Partial Differential Equations, respectively.

Problem 2.1. *Given $f \in \mathbb{Y}$, find $X \in L_2(B)$ minimizing $\phi(X) \equiv \frac{1}{2}\|X\|^2_{L_2(B)}$ and subject to $A[X] = f$.*

Proposition 2.2. *A sufficient and necessary condition for Problem 2.1 to be feasible is*

$$\varphi = d_1\,\bar{f}_1 = d_2\,\bar{f}_2 = \cdots = d_K\,\bar{f}_K, \qquad (4)$$

where $\bar{f}_j \equiv \dfrac{1}{d_j}\int_0^{d_j} f_j(x_j)\,dx_j$ is the mean value of f_j on $[0, d_j]$, for $j = 1, \ldots, K$.

Proof. Let $X \in L_2(B)$ be a feasible solution of Problem 2.1. Then, by Fubini's theorem,

$$\begin{aligned}\int_B X\,d\underset{\sim}{x} &= \int_0^{d_j}\left(\int_{B^{(j)}} X\,d\underset{\sim}{x}^{[j]}\right)dx_j\\ &= \int_0^{d_j} f(x_j)\,dx_j = d_j\bar{f}_j\,,\end{aligned}$$

for $j = 1, \ldots, K$.

Thus, we get (4).

To prove the sufficiency, let $f \in \mathbb{Y}$ satisfy the feasibility condition (4), take

$$X(\underset{\sim}{x}) = \varphi^{-(K-1)}\prod_{j=1}^{K} f_j(x_j)\,,$$

and apply the Fubini theorem to obtain

$$\begin{aligned}R_j[X](x_j) &= \int_{B^{(j)}} X\,d\underset{\sim}{x}^{[j]}\\ &= \int_{B^{(j)}} \varphi^{-(K-1)} f_j(x_j)\prod_{i\neq j}^{K} f_i(x_i)\,d\underset{\sim}{x}^{[j]}\\ &= \varphi^{-(K-1)} f_j(x_j)\int_{B^{(j)}}\prod_{i\neq j}^{K} f_i(x_i)\,d\underset{\sim}{x}^{[j]}\\ &= \varphi^{-(K-1)} f_j(x_j)\prod_{i\neq j}^{K}\int_{L_2[0,d_i]} f_i(x_i)\,dx_i\\ &= \varphi^{-(K-1)} f_j(x_j)\left(\prod_{i\neq j}^{K} d_i\bar{f}_i\right)\end{aligned}$$

so that $R_j[X](x_j) = f_j(x_j)$, for $j = 1, \ldots, K$. Hence, the set of feasible solutions is not empty. □

Corollary 2.1. *If (4) holds for $f \in \mathbb{Y}$ given, then the set of feasible solutions of Problem 2.1 is an affine subspace of $L_2(B)$.*

Proof. It follows directly from propositions 2.1 and 2.2. □

Proposition 2.3. *Problem 2.1 has a unique optimal solution $\hat{X} \in \mathcal{L}_2(B)$. Further, if the operator AA^* is invertible, where A^* is the adjoint of operator A, then*

$$\hat{X} = A^*(AA^*)^{-1}[f]. \qquad (5)$$

In other words, if P is the orthogonal projection operator over $N(A)^\perp$ then $P = A^*(AA^*)^{-1}$.

Proof. See Luenberger [7], section 6.10. □

Problem solution

In this section we obtain an explicit solution to Problem 2.1 (or, equivalently, to Problem 1.4).

Lemma 3.1. *The adjoint operator $A^* : \mathbb{Y} \to L_2(B)$ of A is linear, continuous, and given by*

$$A^*[f](\underset{\sim}{x}) \equiv \sum_{j=1}^{K} f_j(x_j). \qquad (6)$$

Proof. From the general theory of linear continuous operators in Hilbert spaces (Luenberger [7], page 151), the adjoint A^* of A given by (3) is also a linear continuous operator, and

$$\|A^*\|_{L(\mathbb{Y},L_2(B))} = \|A\|_{L(L_2(B),\mathbb{Y})}.$$

Now, observing that

$$\begin{aligned}\langle A[X], f\rangle_{\mathbb{Y}} &= \textstyle\sum_{j=1}^{K}\int_0^{d_j} f_j(x_j)R_j[X](x_j)\,dx_j\\ &= \textstyle\sum_{j=1}^{K}\int_0^{d_j} f_j(x_j)\left(\int_{B^{(j)}} X(\underset{\sim}{x})\,d\underset{\sim}{x}^{[j]}\right)dx_j\\ &= \textstyle\int_0^{d_j}\int_{B^{(j)}} X(\underset{\sim}{x})\left(\sum_{j=1}^{K} f_j(x_j)\right)d\underset{\sim}{x}^{[j]}\,dx_j\\ &= \textstyle\int_B X(\underset{\sim}{x})\left(\sum_{j=1}^{K} f_j(x_j)\right)d\underset{\sim}{x}\\ &= \langle X, A^*[f]\rangle_{L_2(B)}\end{aligned}$$

we get $A^*[f](\underset{\sim}{x}) \equiv \sum_{j=1}^{K} f_j(x_j)$. □

Corollary 3.1. $ker(A)^\perp = \{X \in L_2(B) \mid X(\underset{\sim}{x}) = \sum_{j=1}^{K} f_j(x_j),\ f \in \mathbb{Y}\}$.

Proof. It follows directly from the known fact $ker(A)^{\perp} = im(A^*)$ (Luenberger [7], page 156), and Lemma 3.1. $\square$

Lemma 3.2. *The operator* $AA^* : \mathbb{Y} \to \mathbb{Y}$ *is linear, continuous, and given by*

$$AA^*[f](\underset{\sim}{x}) \equiv \begin{bmatrix} v(B^{(1)})(f_1(x_1) + \sum_{j\neq 1}^{K} \bar{f}_j) \\ v(B^{(2)})(f_2(x_2) + \sum_{j\neq 2}^{K} \bar{f}_j) \\ \vdots \\ v(B^{(K)})(f_K(x_K) + \sum_{j\neq K}^{K} \bar{f}_j) \end{bmatrix} \tag{7}$$

Proof. Since AA^* is the composition of linear continuous operators, it is a linear continuous operator itself. Now, we have

$$\begin{aligned} (AA^*)[f](\underset{\sim}{x}) &= A(A^*[f])(\underset{\sim}{x}) \\ &= A[\,\textstyle\sum_{j=1}^{K} f_j(x_j)\,] \\ &= \begin{bmatrix} R_1[\,\sum_{j=1}^{K} f_j(x_j)\,] \\ \vdots \\ R_K[\,\sum_{j=1}^{K} f_j(x_j)\,] \end{bmatrix} \end{aligned} \tag{8}$$

and

$$\begin{aligned} & R_k\left[\textstyle\sum_{j=1}^{K} f_j(x_j)\right] = \int_{B^{(k)}_{x_k}} \left(\sum_{j=1}^{K} f_j(x_j)\right) d\underset{\sim}{x}^{[k]} \\ & = \textstyle\sum_{j=1}^{K} \int_{B^{(k)}_{x_k}} f_j(x_j)\, d\underset{\sim}{x}^{[k]} \\ & = vol(B^{(k)})\, f_k(x_k) \\ & + \textstyle\sum_{\substack{j=1 \\ j\neq k}}^{K} \int_{B^{(k)}_{x_k}} f_j(x_j)\, d\underset{\sim}{x}^{[k]}. \end{aligned} \tag{9}$$

Besides, for $j \neq k$:

$$\begin{aligned} & \int_{B^{(k)}_{x_k}} f_j(x_j)\, d\underset{\sim}{x}^{[k]} \\ & = \int_{B^{(j,k)}} \left(\int_0^{d_k} f_j(x_j)\, dx_j\right) d\underset{\sim}{x}^{[j,k]} \\ & = \int_{B^{(j,k)}} \left(d_j \bar{f}_j\right) d\underset{\sim}{x}^{[j,k]} \\ & = vol(B^{(k)})\, \bar{f}_j, \end{aligned} \tag{10}$$

where[2] $B^{(j,k)} = \prod_{\ell\neq j,k}[0, d_\ell]$, and in $d\underset{\sim}{x}^{[j,k]}$ all differentials dx_ℓ appear whenever $\ell \neq j, k$. Thus, (7) follows from (8), (9), and (10). $\square$

[2] For $j < k$ we write $B^{(j,k)} = \prod_{\ell\neq j,k}[0, d_\ell]$ instead of $B^{(j,k)}_{x_j,x_k} = [0, d_1] \times \cdots \times [0, d_{j-1}] \times \{x_j\} \times [0, d_{j+1}] \times [0, d_{k-1}] \times \{x_k\} \times [0, d_{k+1}] \times \cdots \times [0, d_K]$.
A similar consideration is applied for $k < j$.

Lemma 3.3. $\mathbb{W} = \{f \in \mathbb{Y} \,|\, (4)$ *is satisfied* $\}$ *is a closed linear subspace of* $\mathbb{Y}$ *and it is invariant under* AA^*.

Proof. If $f, g \in \mathbb{W}$ then $\varphi = d_1\bar{f}_1 = \cdots = d_K\bar{f}_K$ and $\psi = d_1\bar{g}_1 = \cdots = d_K\bar{g}_K$, obtaining $\varphi + \psi = d_1\overline{f_1 + g_1} = \cdots = d_K\overline{f_K + g_K}$, and $f + g \in \mathbb{W}$. Similarly, for $\alpha \in \mathbb{R}$ we have $\alpha\, f \in \mathbb{W}$. Hence, $\mathbb{W}$ is a linear subspace of $\mathbb{Y}$. Further, the set $\mathbb{W}_{j,k} = \{f \in \mathbb{Y} \,|\, \int_0^{d_j} f_j dx_j = \int_0^{d_k} f_k dx_k\}$, $j \neq k$, is clearly closed in $\mathbb{Y}$, and since $\mathbb{W}$ is a finite intersection of the closed sets $\mathbb{W}_{j,k}$, $j \neq k$, it follows that $\mathbb{W}$ is also closed. To prove that $\mathbb{W}$ is invariant under AA^*, from Lemma 3.2 it is sufficient to show that, for $f \in \mathbb{W}$,

$$\int_0^{d_k} vol(B^k)\left(f(x_k) + \sum_{j\neq k}^{k} \bar{f}_j\right) dx_k$$

has a constant value, independent of k. In fact, we have

$$\begin{aligned} & \int_0^{d_k} vol(B^k)\left(f(x_k) + \textstyle\sum_{j\neq k}^{K} \bar{f}_j\right) dx_k \\ & = vol(B^k)\left(\varphi + d_k \textstyle\sum_{j\neq k}^{K} \varphi/d_j\right) \\ & = \varphi\, vol(B^k)\left(1 + d_k \textstyle\sum_{j\neq k}^{K} 1/d_j\right) \\ & = \varphi\, vol(B^k)\left(1 + \frac{\sum_{j\neq k}^{K} vol(B^j)}{vol(B^k)}\right). \end{aligned}$$

This is to say

$$\int_0^{d_k} vol(B^k)\left(f(x_k) + \sum_{j\neq k}^{K} \bar{f}_j\right) dx_k$$

$$= \left(\sum_{j=1}^{K} vol(B^j)\right)\, \varphi\,,$$

and the proof is completed. $\square$

Proposition 3.1. *The linear operator* $AA^* : \mathbb{Y} \rightarrow \mathbb{Y}$ *restricted to observable functions* $f \in \mathbb{W}$ *is invertible and given by*

$$(AA^*)^{-1}[f](\underset{\sim}{x}) \equiv \begin{bmatrix} \frac{f_1(x_1)}{vol(B^{(1)})} + \frac{\sum_{k\neq 1}^{K} \bar{f}_k}{\sum_{k=1}^{K} vol(B^{(k)})} \\ \vdots \\ \frac{f_K(x_K)}{vol(B^{(K)})} + \frac{\sum_{k\neq K}^{K} \bar{f}_k}{\sum_{k=1}^{K} vol(B^{(k)})} \end{bmatrix}. \tag{11}$$

Proof. By Lemma 3.2, $(AA^*)[f] = 0$ implies, for $j = 1, \ldots, K$,

$$f_j(x_j) + \sum_{k=1,k\neq j}^{K} \bar{f}_k = 0, \tag{12}$$

which leads to

$$f_j(x_j) = \bar{f}_j. \tag{13}$$

and $d_j \bar{f}_j + d_j \sum_{k=1,k\neq j}^{K} \bar{f}_k = 0$. So, using the solubility conditions (4), from (12) and (13) we get

$$\begin{aligned} 0 &= \bar{f}_j + \sum_{\substack{k=1 \\ k\neq j}}^{K} \bar{f}_k \\ &= \bar{f}_j + \sum_{\substack{k=1 \\ k\neq j}}^{K} \frac{d_j}{d_k} \bar{f}_j \\ &= \bar{f}_j + d_j \bar{f}_j \sum_{\substack{k=1 \\ k\neq j}}^{K} \frac{1}{d_k} \\ &= \bar{f}_j + \bar{f}_j \frac{\sum_{\substack{k=1 \\ k\neq j}}^{K} vol(B^{(k)})}{vol(B^{(j)})}, \end{aligned}$$

which comes to $0 = \left(\sum_{k=1}^{K} vol(B^{(k)})\right) \bar{f}_j$, and we have $f_j(x_j) \equiv \bar{f}_j = 0$, for $j = 1, \ldots, K$. Therefore, $f \in N(AA^*)$ leads to $f = 0$ and so AA^* is invertible.

Now, taking $f(\underset{\sim}{x}) = (AA^*)[z](\underset{\sim}{x})$ with $z \in \mathbb{Y}$ satisfying the solubility conditions (4), and applying Lemma 3.2 leads to

$$f_j(x_j) = vol(B^{(j)}) \left(z_j(x_j) + \sum_{\substack{k=1 \\ k\neq j}}^{K} \bar{z}_k \right), \tag{14}$$

for $j = 1, \ldots, K$.
Hence,

$$z_j(x_j) = \frac{f_j(x_j)}{vol(B^{(j)})} - \sum_{\substack{k=1 \\ k\neq j}}^{K} \bar{z}_k. \tag{15}$$

From (14) we have

$$\begin{aligned} d_j \bar{f}_j &= \int_0^{d_j} f(x_j)\, dx_j \\ &= vol(B) \left(\bar{z}_j + \sum_{k=1,k\neq j}^{K} \bar{z}_k \right), \end{aligned}$$

thus

$$\bar{f}_j = vol(B^{(j)}) \left(\bar{z}_j + \sum_{k=1,k\neq j}^{K} \bar{z}_k \right).$$

Now, using the solubility conditions $d_k \bar{z}_k = d_j \bar{z}_j$, for all k, j, we have

$$\begin{aligned} \bar{f}_j &= vol(B^{(j)}) \bar{z}_j + \left(\sum_{k=1,k\neq j}^{K} \frac{d_j vol(B^{(j)})}{d_k} \right) \bar{z}_j \\ &= \left(\sum_{k=1}^{K} vol(B^{(k)}) \right) \bar{z}_j. \end{aligned}$$

Thus, $\bar{z}_j = \frac{\bar{f}_j}{\sum_{k=1}^{K} vol(B^{(k)})}$ and

$$\sum_{k=1,k\neq j}^{K} \bar{z}_k = \frac{\sum_{k=1,k\neq j}^{K} \bar{f}_k}{\sum_{k=1}^{K} vol(B^{(k)})}. \tag{16}$$

Therefore, (11) follows from (15) and (16), and the proof is complete. □

Theorem 3.1. *If $f \in \mathbb{W}$ then the unique solution of Problem 2.1 is*

$$\hat{X}(\underset{\sim}{x}) = \frac{\sum_{j=1}^{K} d_j\, f_j(x_j) - (K-1)\varphi}{vol(B)}. \tag{17}$$

Proof. From Propositions 2.3 and 3.1, $\hat{X} = A^*(AA^*)^{-1}[f]$ is the solution of Problem 2.1. Thus applying Lemma 3.1 and Proposition 3.1 we get

$$\begin{aligned} \hat{X}(\underset{\sim}{x}) &= A^*(AA^*)^{-1}[f](\underset{\sim}{x}) \\ &= A^* \begin{bmatrix} \frac{f_1(x_1)}{vol(B^{(1)})} + \frac{\sum_{k=1,k\neq K}^{K} \bar{f}_k}{\sum_{k=1}^{K} vol(B^{(k)})} \\ \vdots \\ \frac{f_K(x_K)}{vol(B^{(K)})} + \frac{\sum_{k=1,k\neq K}^{K} \bar{f}_k}{\sum_{k=1}^{K} vol(B^{(k)})} \end{bmatrix}. \end{aligned}$$

Therefore, an explicit solution of the quadratic optimization problem with linear constraints (2) is given by

$$\hat{X}(\underset{\sim}{x}) = \sum_{j=1}^{K} \frac{f_j(x_j)}{vol(B^{(j)})} - (K-1) \frac{\sum_{j=1}^{K} \bar{f}_j}{\sum_{j=1}^{K} vol(B^{(j)})}$$

or by

$$\begin{aligned}\hat{X}(\underset{\sim}{x}) &= \frac{1}{vol(B)}\sum_{j=1}^{K} d_j f_j(x_j) \\ &- (K-1)\frac{\sum_{j=1}^{K}\bar{f}_j}{\sum_{j=1}^{K} vol(B^{(j)})}.\end{aligned}$$

Now, substituting $\bar{f}_j = \varphi/d_j$, for all j, we obtain (17) after a trite simplification. □

Theorem 3.1 generalizes the solution given by Romero [11] for this problem in the finite dimension case (see Theorem 1.1), which is continuous if the observable functions are continuous. Our next result establishes that Problem 2.1 is well-posed in the Hadamard sense.

Proposition 3.2. *If $f \in \mathbb{W}$ then*

$$\|\hat{X}\|_{L_2(B)} \leq \|f\|_{\mathbb{Y}}.$$

Proof. It follows directly from Proposition 2.3, and from the fact that every orthogonal projection operator P has norm equal to the unity. □

Non-Negativity constraints case

In this section we still consider Problem 2.1, this time incorporating non-negative constraints over $X \in L_2(B)$. More specifically, we discuss

Problem 4.1. *Given $f \in \mathbb{Y}$, find $X \in L_2(B)$ minimizing $\phi(X) = \frac{1}{2}\|X\|^2_{L_2(B)}$, subject to $A[X] = f$, and $X \geq 0$ almost everywhere in B.*

Subsection 4.1 is devoted to prove that this problem is well-posed in the Hadamard's sense. Then, in Subsection 4.2 we propose a polynomial procedure to approach the optimal solution of the two-dimensional case of Problem 4.1.

Feasibility

Clearly, $C_0 = \{X \in L_2(B) \mid X \geq 0$ almost everywhere in $B\}$ is a non-empty closed cone, and under feasibility conditions (4) $C_f = \{X \in L_2(B) \mid A[X] = f\}$ is a non-empty, closed linear manifold.

Lemma 4.1. $N(A) \subsetneq \{X \in L_2(B) \mid \overline{X} = 0\}$.

Proof. Let $X \in N(A)$ be given. Then, as - $R_j[X](x_j) \equiv 0$ implies $\overline{R_j[X]} = 0$, for $j = 1, \ldots, K$, by the Fubini theorem we get

$$\begin{aligned} 0 &= \int_0^{d_j} R_j[X](x_j)\,dx_j \\ &= \int_0^{d_j}\left(\int_{B^j} X(\underset{\sim}{x}^{[j]}, x_j)\,d\underset{\sim}{x}^{[j]}\right)dx_j \\ &= \int_B X(\underset{\sim}{x})\,d\underset{\sim}{x},\end{aligned}$$

where for convenience we have written

$$R_j[X](x_j) = \int_{B^j} X(\underset{\sim}{x}^{[j]}, x_j)\,d\underset{\sim}{x}^{[j]}.$$

Hence, $\overline{X} = 0$, and $N(A) \subset \{X \in L_2(B) \mid \overline{X} = 0\}$.

Now, let $X \in L_2(B)$ be such that $\overline{X} = 0$. Applying the Fubini theorem again, we have

$$\begin{aligned} 0 &= \int_B X(\underset{\sim}{x})\,d\underset{\sim}{x} \\ &= \int_0^{d_j}\left(\int_{B^j} X(\underset{\sim}{x}^{[j]}, x_j)\,d\underset{\sim}{x}^{[j]}\right)dx_j \\ &= \int_0^{d_j} R_j[X](x_j)\,dx_j,\end{aligned}$$

thus $\overline{R_j[X]} = 0$ and $R_j(x_j) \not\equiv 0$, for $j = 1, \ldots, K$. In conclusion, we come to $N(A) \subsetneq \{X \in L_2(B) \mid \bar{X} = 0\}$, and the proof is completed. □

Proposition 4.1. *The closed subspace $N(A)$ is a support subspace at the origin of the positive cone set C_0 in $L_2(B)$.*

Proof. Clearly, the intersection of C_0 with the set $\{X \in L_2(B) \mid \bar{X} = 0\}$ is $X \equiv 0$. So, the proof of this proposition follows from Lemma 4.1. □

Given $f \in \mathbb{Y}$, by $f \geq 0$ we mean $f_j(x_j) \geq 0$ almost everywhere in $[0, d_j]$, for $j = 1, \ldots, K$.

Corollary 4.1. *If $f \in \mathbb{Y}$ satisfies the feasibility conditions (4) and $f \geq 0$ then $C_{(f)} = C_0 \cap C_f$ is non-empty set.*

Now, under the hypothesis of Collorary 4.1, $C_{(f)} = C_0 \cap C_f$ is a non-empty, closed linear subset. Namely, we can see Problem 4.1 as Problem

2.1 over the closed and convex subset C_0, and Problem 4.1 thus consists in finding $X \in C_{(f)}$ that is closest to the origin $O \in \mathcal{L}_2(B)$. Therefore, Problem 4.1 can be seen as a special case ($F = 0$) of

Problem 4.2. *Let H be a Hilbert space, and $C \in H$ a closed and convex subset. Given $F \in H$, find $\hat{X} \in C$ such that*

$$\|F - \hat{X}\|_{L_2(B)} \leq \| \ F - X\|_{L_2(B)}, \quad \textit{for all } X \in C.$$

The characterization of the solution to Problem 4.2 is given by the following result (for a proof, see Brézis [1], Siddiqi [12]).

Theorem 4.1. (*Convex Projection*) *Let H be a Hilbert space. If $C \subset H$ is a closed convex set then, given $F \in H$, Problem 4.2 has a unique solution $\hat{X} \in C$, characterized by the variational inequality*

$$\langle F - \hat{X}, X - \hat{X}\rangle_H \leq 0, \quad \textit{for all } X \in C. \quad (18)$$

Furthermore, the mapping $P_C : H \to C$ given by $F \mapsto \hat{X}$ is a nonlinear projection operator satisfying

$$\|P_C\, G - P_C\, F\|_H \leq \|G - F\|_H. \quad (19)$$

Observe that the orthogonal projection is a particular case of the convex projection when the closed convex C is a closed subspace. In general, it appears very difficult to furnish an explicit solution to Problem 4.2, since to give an explicit expression of the convex projector P_C does not seem easy.

Coming back to Problem 4.1, from Theorem 4.1 it has an unique solution $\hat{X}^c \in \mathcal{L}_2(B)$ characterized by the variational inequality

$$\langle \hat{X}^c, \ X - \hat{X}^c\rangle_{L_2(B)} \geq 0, \ \text{ for all } X \in C_{(f)}, \ (20)$$

which says that $\hat{X}^c$ is orthogonal to $C_{(f)}$, and from which it is not obvious that $\hat{X}^c$ depends continuously on $f \in \mathbb{Y}$.

To prove that $\hat{X}^c$ depends continuously with respect to $f \in \mathbb{Y}$, it is convenient to first consider some aspects of the problem data. We denote by $\mathbb{W}_0 \subset \mathbb{Y}$ the subset of all $f \in \mathbb{W}$ (see Lemma 3.3) with non-negative f_j components, namely, $f_j \geq 0$ almost everywhere in $[0, d_j]$, for $j = 1, \ldots, K$.

Lemma 4.2. *$im(A^*|_{\mathbb{W}_0}) \subset C_0$.*

Proof. It is a direct consequence of Lemma 3.1. □

Lemma 4.3. *$im((AA^*)^{-1}|_{\mathbb{W}_0}) \subset C_0$.*

Proof. It is a direct consequence of Proposition 3.1. □

Theorem 4.2. *If $\hat{X}^c_f$ and $\hat{X}^c_g$ are the solutions to Problem 2.1 for f and $g \in \mathbb{W}_0$, respectively, then $\|\hat{X}^c_g - \hat{X}^c_f\|_{L_2(B)} \leq \|g - f\|_{\mathbb{Y}}$.*

Proof. If $h \in \mathbb{W}_0$, then $\hat{X}^c_h = \hat{X}_h$ follows from Lemmas 4.2 and 4.3. That is, when $h \in \mathbb{W}_0$ the convex and orthogonal projections of h coincide. So, applying Proposition 3.2 the proof is completed. □

In other words, Problem 4.1 is well-posed in the Hadamard's sense.

Approximate solution for the 2-D case

Let us consider Problem 4.1 in the two-dimensional case; more specifically

Problem 4.3. *Given positive reals d_1, d_2, and non-negative real functions $f(x)$, $g(y)$, satisfying $\int_0^{d_1} f(x)\,dx = \int_0^{d_2} g(y)\,dy$, find a non-negative real function $h(x,y)$ that minimizes*

$$\int_0^{d_1} \int_0^{d_2} [h(x,y)]^2 \,dx\,dy,$$

subject to

$$\int_0^{d_1} h(x,y)\,dx = g(y), \ \textit{for } y \in [0, d_2],$$

and

$$\int_0^{d_2} h(x,y)\,dy = f(x), \ \textit{for } x \in [0, d_1].$$

We were not able to find an explicit solution to Problem 4.3. However, the polynomial procedure **A** below —strongly relying on the exact method proposed in [2] to solve Problem 1.2 for any m-by-n matrix— can be applied in case an approximate optimal solution suffices.

Lines (1)–(6) are aimed to first determine suitable discretization intervals Δ_1 and Δ_2, where the parameter $\epsilon > 0$ is meant to control the desired accuracy level, and $\nu(w, h, d)$ denotes the integral of a function h in the interval $[0, d]$, approximated with the trapezoid rule on w points. The dimensions m and n of the working matrix are computed in line (7). Then, in lines (8)–(12) an approximation to the optimal solution $h^*(x, y)$ of Problem 4.3 is produced as $h'(x, y)$ for discretized values x, y. Note that vectors $(u_1, \dots, u_m)$ and $(v_1, \dots, v_n)$ are non-negative, and satisfy $\sum_{i=1}^m u_i = \sum_{j=1}^n v_j$. They are taken as input to determine in line (11) the optimal solution X^* of Problem 1.2 with the polynomial method proposed in [2].

PROCEDURE **A**

(1) $k_1 \leftarrow 2$; **While** $|\nu(k_1+1, f, d_1) - \nu(2k_1+1, f, d_1)| > \epsilon$ **do** $k_1 \leftarrow 2k_1$;

(2) $k_2 \leftarrow 2$; **While** $|\nu(k_2+1, g, d_2) - \nu(2k_2+1, g, d_2)| > \epsilon$ **do** $k_2 \leftarrow 2k_2$;

(3) $\Delta \leftarrow \min\{\, d_1/k_1,\ d_2/k_2 \,\}$;
$\Delta_1' \leftarrow d_1/\lceil \frac{d_1}{\Delta} \rceil$;
$\Delta_1'' \leftarrow d_1/\lfloor \frac{d_1}{\Delta} \rfloor$;

(4) **If** $|\Delta - \Delta_1'| \le |\Delta - \Delta_1''|$ **then** $\Delta_1 \leftarrow \Delta_1'$ **else** $\Delta_1 \leftarrow \Delta_1''$;

(5) $\Delta_2' \leftarrow d_2/\lceil \frac{d_2}{\Delta} \rceil$; $\Delta_2'' \leftarrow d_2/\lfloor \frac{d_2}{\Delta} \rfloor$;

(6) **If** $|\Delta - \Delta_2'| \le |\Delta - \Delta_2''|$ **then** $\Delta_2 \leftarrow \Delta_2'$ **else** $\Delta_2 \leftarrow \Delta_2''$;

(7) $m \leftarrow d_1/\Delta_1$; $n \leftarrow d_2/\Delta_2$;

(8) **For** $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ **do**

$$x_i \leftarrow i\Delta_1; \quad y_j \leftarrow j\Delta_2;$$
$$\psi_{ij} \leftarrow \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} h(x, y)\, dy\, dx;$$

(9) **For** $i \in \{1, \dots, m\}$ **do** $u_i \leftarrow \sum_{j=1}^n \psi_{ij}$;

(10) **For** $j \in \{1, \dots, n\}$ **do** $v_j \leftarrow \sum_{i=1}^m \psi_{ij}$;

(11) Find the optimal solution $X^* = (X_{ij}^*)$ to Problem 1.2;

(12) **For** $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ **do**

$$h'(i - \tfrac{\Delta_1}{2}, j - \tfrac{\Delta_2}{2}) \leftarrow \frac{X_{ij}^*}{\Delta_1 \Delta_2}.$$

Example. Let $f(x) = 496\,x^2 - 1\,984\,x + 1\,984$, $g(y) = 35\,y^3 - 455\,y^2 + 1\,470\,y + 70$, $d_1 = 5$, $d_2 = 8$. Since $\int_0^5 f(x) = \int_0^8 g(y) = 5\,786 + 2/3$, feasibility is verified. Fixing $\epsilon = 0.1$, lines (1)-(7) yield $k_1 = 2^7$, $k_2 = 2^8$, $\Delta = \min\{5/2^7, 8/2^8\} = 0.03125$, $\Delta_1 = 0.03125$, $\Delta_2 = 0.03125$, $m = 160$, and $n = 256$. Then, in line (8) we get vectors $(u_1, \dots, u_{160})$ and $(v_1, \dots, v_{256})$, which in turn are used as the input data for the optimization process in line (9). Finally, line (10) obtains an approximate optimal solution, as depicted in Figure 1.

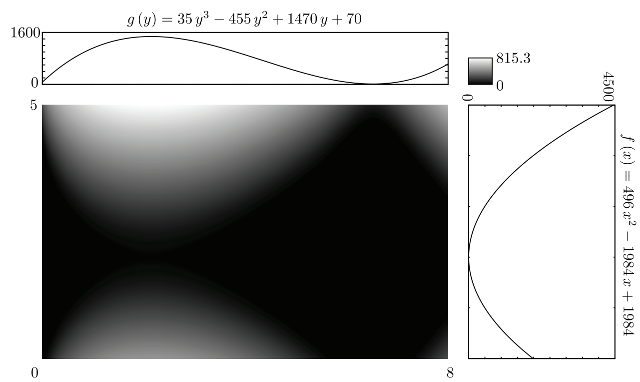

Figure 1: Approximate solution to Problem 4.3 for $f(x) = 496\,x^2 - 1\,984\,x + 1\,984$, $g(y) = 35\,y^3 - 455\,y^2 + 1\,470\,y + 70$, $d_1 = 5$, and $d_2 = 8$.

Conclusions

In this paper the problem of finding the closest point to the origin in transportation polytopes is studied in a Hilbert space framework. We show that this point can be found by the explicit formula (17). In the finite dimensional context, an analogous explicit formula was proposed by Romero [11].

We are not aware of an explicit solution to Problem 2.1 in case non-negative constraints are added. However, we establish its well-posedness as a direct consequence of the convex projection theorem.

Finally, a two-dimensional instance of Problem 2.1 with non-negative constraints is numerically solved by means of a discretization procedure based on the trapezoid rule, together with the exact, polynomial method proposed in [2].

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