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Research Article

Robust Evaluation for Transportation Network Capacity under Demand Uncertainty

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As more and more cities in worldwide are facing the problems of traffic jam, governments have been concerned about how to design transportation networks with adequate capacity to accommodate travel demands. To evaluate the capacity of a transportation system, the prescribed origin and destination (O-D) matrix for existing travel demand has been noticed to have a significant effect on the results of network capacity models. However, the exact data of the existing O-D demand are usually hard to be obtained in practice. Considering the fluctuation of the real travel demand in transportation networks, the existing travel demand is represented as uncertain parameters which are defined within a bounded set. Thus, a robust reserve network capacity (RRNC) model using min–max optimization is formulated based on the demand uncertainty. An effective heuristic approach utilizing cutting plane method and sensitivity analysis is proposed for the solution of the RRNC problem. Computational experiments and simulations are implemented to demonstrate the validity and performance of the proposed robust model. According to simulation experiments, it is showed that the link flow pattern from the robust solutions to network capacity problems can reveal the probability of high congestion for each link.

1. Introduction

The capacity of transportation network reflects the supply ability of its infrastructure and service to the travel demand which is generated from the zones covered by the transportation system in a specific period. For many years, transportation planners and managers wanted to understand how many trips can be accommodated at the most by the current or designed network in a certain period of time. This need is more necessary in those developing regions which are confronted with rapid growth of private vehicles and increased urban congestion. Meanwhile, the researchers made a long-term effort to model and estimate the maximum throughput of transportation networks. The achievements include max-flow min-cut theorem [1], incremental assignment approach [2], and later bilevel programming models [3–5].

For the network capacity model, the most popular formulation in passenger transportation system is the bilevel model, which maximizes the traffic flows under the equilibrium constraints. Wong and Yang [3] first incorporated the *reserve capacity* concept into a traffic signal control network. The reserve capacity is defined as the largest multiplier applied to a given O-D demand matrix without violating capacity constraints, so the solution is significantly affected by the predetermined O-D matrix. Ziyou and Yifan [6] extended the reserve capacity model by considering O-D specific demand multipliers, and all demand multipliers should be ensured not lower than a predetermined minimum value. In order to avoid assuming that all O-D flows increase in a same rate, another concept of *ultimate capacity* was proposed [5]. But it assumes that the O-D distribution is totally variable, which may produce unrealistic results that cause the trip productions at some origins below their current levels. Furthermore, Yang et al. [4] suggested that the new increased O-D demand pattern should be variable in both level and distribution, while the current travel demand is fixed. Later, Yang's model was also referred to as the *practical capacity* by Kasikitwiwat and Chen [5]. In summary, although unrealistic, the reserve capacity model is more easy-to-use and has been adopted widely in many researches [7–10]. The ultimate capacity and practical capacity model are more practical but have more parameters to be calibrated when applied, and the formulated models are still difficult to solve [11].

While the deterministic network capacity problem has been explored extensively, few studies have investigated the issue of uncertainties in demand data associated with this problem. The ultimate capacity and practical capacity model are only concerned with the uncertainties related to the new increased travel demand by using combined models [5], while the uncertainties in the current (or existing) demand are not considered. In reality, travel demands in transportation system are always fluctuant day by day, even hour by hour. Besides, errors of survey data also affect the accuracy of the existing O-D matrix. As a consequence, the existing travel demands are usually difficult to be obtained in actual transportation projects and then are not easy to be represented using fixed values. As the existing O-D matrix is usually used as the reference matrix in reserve capacity or practical capacity model and its pattern significantly influences the result of the models, we first consider it as an uncertain variable in this study. And thus the network capacity model is extended to be an optimization with parameter uncertainty.

Researches on other areas of transportation network optimization typically adopted two methods to address the uncertain O-D demand [12]: (i) stochastic optimization aims at maximizing the expected profit by assuming that the demand follows a known probability distribution; (ii) robust optimization aims at maximizing the profit with the worst-case scenario of the demand pattern. Considering the exact probability distribution of the O-D demand is still hard to be obtained, the robust optimization is more effective in dealing with this problem. If a limited number of discrete scenarios of O-D demand patterns are detected, the scenario-based robust optimization [13] is conducted, which is a practical approach usually implemented in transportation projects. It is more general to assume the possibility of the travel demand to be a continuous variable within a bounded set, and the set-based robust optimization can be used for decision-making [14]. The uncertainty set is constructed to include most of possible values of the travel demand. The decision-makers' attitudes to risk should be considered as well when deciding the shapes and size of the uncertainty sets. It is important to make a trade-off between the system performance and the level of robustness achieved [13].

In this study, we propose a robust optimization model for the network capacity problem by using the existing O-D travel demands as uncertain parameters. The existing demand between each O-D pair is assumed to be variable between its upper and lower limits. Besides, three typical uncertainty regions are introduced to provide a bounded set for the uncertain demand. A heuristic solution is developed for the solution to the robust network capacity model. In the next section, the concept of network spare capacity is revisited based on the reserve capacity model. Then, the robust model for network capacity estimation is presented, and the three typical uncertainty sets of existing travel demand are defined. After that, the solution algorithm is described. Computational experiments show the validation and justification of the robust model. Conclusions and perspectives for further research are provided in the last section.

2. Network Spare Capacity and Its Flexibility

The reserve capacity was proposed as the largest multiplier μ applied to a given existing O-D demand matrix that can be allocated to a transportation network without violating any individual link capacity [3]. The product of the largest multiplier and the existing O-D demand (represented by vector $\mathbf{q}$) gives the maximum travel demand which can be loaded to the network. For clarity sake we refer to the maximum travel demand as the value of *network capacity* in rest of this paper. For passenger network, it is well known that multiple O-D pairs exist and demands between different O-D pairs are not exchangeable or substitutable. Thus, the *travel demand pattern* or *matrix* reflects both its quantity and spatial distribution. The method of reserve capacity assumes that the existing O-D demand is scaled with a uniform O-D growth. The largest value of μ indicates whether the current network has spare capacity or not. So the *network spare capacity* is generally explained as follows: if $\mu > 1$, then the network can be loaded more travel demand and the additional demand can be accommodated by the network which is $(\mu - 1)\mathbf{q}$; otherwise, that is, $\mu < 1$, the network is overloaded and the existing O-D demand should decrease by $(1 - \mu)\mathbf{q}$ to satisfy the capacity constraints [10]. In some researches, the demand multiplier μ is regarded as the uncertainties in the future O-D demand [9, 15].

The classical model of reserve network capacity (RNC) is defined as follows:

$$\text{RNC: } \max_{\mu} \quad \mu, \tag{1}$$

$$\text{s.t.} \quad v_a(\mu\mathbf{q}) \le C_a, \quad \forall a \in A, \tag{2}$$

where $v_a(\mu\mathbf{q})$ is obtained by solving the following user equilibrium problem:

$$\min_{\mathbf{f}} \quad \sum_a \int_0^{v_a} t_a(x)\,dx, \tag{3}$$

$$\text{s.t.} \quad \sum_{r\in R_{ij}} f_r^{ij} = \mu q_{ij}, \quad \forall i \in I,\ j \in J, \tag{4}$$

$$v_a = \sum_i\sum_j\sum_r f_r^{ij}\cdot\delta_{a,r}^{ij}, \quad \forall a \in A, \tag{5}$$

$$f_r^{ij} \ge 0, \quad \forall i \in I,\ q \in J,\ r \in R_{ij}, \tag{6}$$

where μ is the O-D demand multiplier to all O-D demands; R is the set of all routes in the network; i is the origin index, $i \in I$, and I is the set of all origin nodes; j is the destination index, $j \in J$, and J is the set of all destination nodes; C_a is the capacity of link a; v_a is the flow on link a, $a \in A$; $\mathbf{v}$ is the vector of all link flows; q_{ij} is the existing trip demand between O-D pair ij; $\mathbf{q}$ is the vector of all O-D demand; f_r^{ij} is the flow on route r, $r \in R$, between O-D pair ij associated with q_{ij}; $\mathbf{f}$ is the vector of flows of all route in R; $\delta_{a,r}^{ij}$ is the link-route

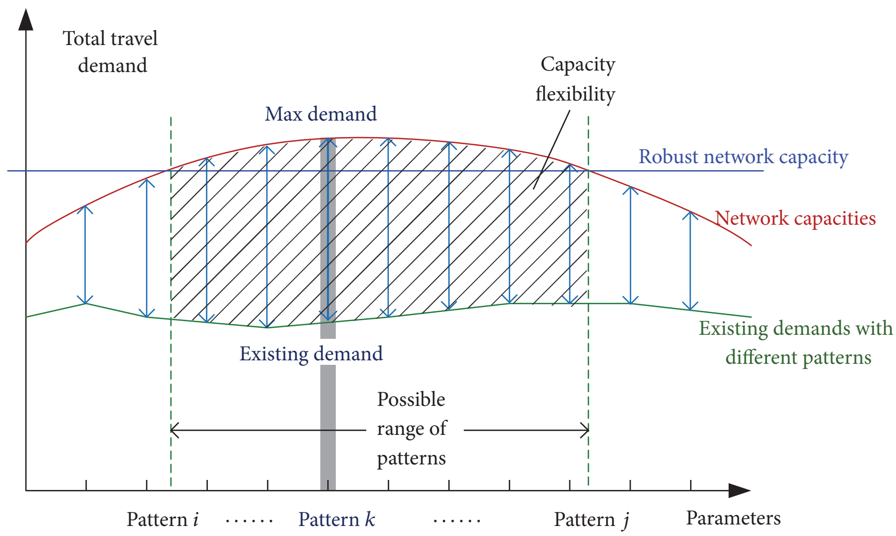


Figure 1: Robust solution of network capacities with different demand pattern.

incidence indicator: 1 if link a is on route r between O-D pair ij and 0 otherwise; $t_a(v_a)$ is the travel cost function for link a.

In the above model, the upper-level model maximizes the O-D matrix multiplier without violating the capacity constraints (2) for every individual link. The parameter $\mathbf{q}$ gives the prescribed O-D travel demand in the network, which can be obtained according to the current trip demand or a predicted demand pattern accordingly. Route choice behavior and congestion effect are considered in the user equilibrium (UE) model as the lower-level model in (3)–(6). Generally, other traffic assignment methods, such as stochastic user equilibrium (SUE) model, can be used in place of the above deterministic UE model as required [10].

The result of the reserve capacity model which is considered may underestimate the capacity of the passenger network, because only the existing O-D demand pattern that is more congruous with the network topology would achieve a higher value of network capacity [16]. Basically, the reserve capacity depends on the initial O-D demand patterns and route choice behavior of the users. Given the lower-level traffic assignment method, the existing O-D demand should be the only determinant to the result of the above model. It means that if the given O-D matrix is not consistent with the network, the reserve capacity model will produce a result having a low level of maximum demand. Otherwise, if the O-D pattern is determined according to the network spatial structure, the travel demand can grow to a very high amount.

Directly applying the result of the reserve capacity may have the following problems. (i) It is hard to decide an exact existing (or predetermined) O-D matrix, because the real travel demand pattern is changing at different hours every day and different days every week. Also, it is still very difficult to obtain the full data of the O-D demands covering many different hours. (ii) In real-world applications, decision-makers tend to be risk averse and may be more concerned with the worst cases. Using only a few situations of the O-D demand pattern may not provide a *robust* answer to the network capacity estimation. Conversely, as long as the system performance reaches an acceptable level, it does not matter how much it changes above that level. Thus, it may be more desirable to have an optimization result that performs better in the worst case.

When estimating the capacity of transportation systems, decision-makers are not only concerned with the extreme results that the total trips can be allocated to a transportation network but also need to evaluate the unknown situations resulted from the fluctuation of the travel demand. Thus, to measure the ability of transportation networks that can deal with the variation of travel demand, Chen and Kasikitwiwat [16] discussed the concept of the *network capacity flexibility* using three typical network capacity models. The network capacity flexibility is defined as the ability of a transport system to accommodate changes in traffic demand while maintaining a satisfactory level of performance [16, 17]. In this study, integrated with the uncertainties from the existing demand in transportation networks, the network capacity flexibility is further illustrated in Figure 1. On the basis of this, the robust estimation of network capacity is defined as *the maximum travel demand can be allocated to a transportation network when satisfying all the possibilities of the uncertain changes in the quantitative and spatial demand pattern.* The robust value of the network capacity is also illustrated in Figure 1.

In this study, we extended the reserve capacity model by considering the existing O-D demands as uncertain parameters within a certain bounded region. Robust solutions to the network capacity can be conducted using the robust optimization. We utilize the classical reserve capacity model to conduct the robust network capacity for two reasons: (i) the reserve capacity is easy to solve, and the O-D travel demand is allowed either increasing or decreasing by applying an O-D matrix multiplier greater than one or less than one; (ii) as the existing O-D matrix is extended to be an uncertain parameter in the reserve capacity model, the O-D distribution is no longer fixed but a variable pattern within some range given by the uncertainty set.

3. Robust Network Capacity Estimation under Demand Uncertainty

In this section, we assume that the prescribed O-D trip demand is unknown but bounded within an uncertainty set Q. Mathematically, the uncertainty set should be closed and convex. In practice, the set of uncertain demand should be derived based on the transportation planners' knowledge on the uncertainty associated with both the current and future O-D travel demand. Nevertheless, it is very difficult to obtain the exact probability distribution of the trip demand between all O-D pairs. If the random demand follows a continuous distribution defined from zero to infinity, for example, normal distribution [12], this would require an infinity capacity to meet all possible demand realizations. Thus, the uncertainty set of travel demand is defined as a bounded region, typically utilizing the highest travel demand, q_{ij}^U, and the lowest demand, q_{ij}^L, for each O-D pair. Using the bounded uncertainty region, the events with low probabilities can be excluded, and then the robust optimization would not provide overly conservative results.

In this study, three typical uncertainty sets were constructed for the existing travel demands.

(1) Interval Constraint [13]. The travel demand between each O-D pair which is assumed varies independently within a given interval of $Q_{ij} = [q_{ij}^L, q_{ij}^U]$. The interval could be the confidence interval of an estimated demand obtained from a survey or by using an O-D estimation model. Without additional restraints, the whole uncertainty set will be a *box* centered at the average travel demand. In this case, it is simple to set $q_{ij} \coloneqq q_{ij}^U$ for all O-D pairs and solve the resulting reserve capacity problem. The capacity value would be estimated in the worst case, which is too conservative. In reality, it is never possible that the demand for all O-D pairs reaches their estimated upper bound at the same time. The travel demand pattern is always fluctuant among the O-D pairs in different directions. Therefore, it is more reasonable to set an upper limit to the summation of the existing travel demand in network, that is, $\sum_{i,j} q_{ij} \le D$. The upper limits D could be estimated by using the maximum trip volume investigated for the existing travel demand.

(2) Ellipsoid [18]. An ellipsoidal set is generally defined as follows:

$$Q = \left\{ \mathbf{q} \mid \sum_{i,j} \left(\frac{q_{ij} - \overline{q}_{ij}}{(1/2)\left(q_{ij}^U - q_{ij}^L\right)} \right)^2 \le \theta^2 \right\}, \tag{7}$$

where $\overline{q}_{ij} = (q_{ij}^U + q_{ij}^L)/2$, the average O-D demand; θ is a parameter that reflects decision-makers' attitudes to risk; and larger θ indicates that it is more adverse to risk. The value of θ is from zero to $\sqrt{|W|}$, where $|W|$ denotes the number of O-D pairs. When $\theta = 1$, the uncertainty region is the largest ellipsoid contained in the box region $Q = \{\mathbf{q} \mid q_{ij}^L \le q_{ij} \le q_{ij}^U, \ \forall i, j\}$.

(3) Polyhedron. The polyhedron is a set of a finite number of linear equalities and inequalities that restrains the travel demand. It is a generalized form of the box uncertainty set. For example, Sun et al. [19] constructed the following polyhedron region for uncertain O-D demand:

$$Q = \left\{ \mathbf{q} \mid \begin{array}{c} q_{ij}^0 - \gamma q_{ij}^0 \le q_{ij} \le q_{ij}^0 + \gamma q_{ij}^0, \ \gamma \in [0,1], \ \forall i, j \\ \sum_i q_{ij} = \sum_i q_{ij}^0, \ \forall j \\ \sum_j q_{ij} = \sum_j q_{ij}^0, \ \forall i \end{array} \right\}, \tag{8}$$

where q_{ij}^0 is the nominal value of the travel demand for O-D pair ij. It may choose the mean value of the interval constraints or an observed result from a survey. This uncertainty set allows the demand pattern varying entirely around its nominal value and involves the implicit possible interactions among O-D demands. Therefore, the overly conservative results may be avoided. The last two sets of constraints require that the uncertain travel demands meet the conservation condition with the nominal demand matrix at the zonal production and attraction.

Note that the shape of uncertainty set affects the efficiency and robustness of network capacity value. Ben-Tal and Nemirovski [14] suggested applying the min–max optimization model. Once the uncertainty set of the travel demand, Q, is determined, the min–max model will find a robust solution that tolerates changes in travel demand up to the given bound. Using any type of the uncertainty sets, Q, the robust reserve network capacity (RRNC) problem can be formulated as follows:

$$\begin{aligned} \text{RRNC: } & \max_{\mu} \min_{\mathbf{q}} \quad \mu, \\ & \text{s.t.} \quad v_a(\mu \mathbf{q}) \le C_a, \\ & \qquad \forall a \in A, \ \text{such that } \mathbf{q} \in Q, \end{aligned} \tag{9}$$

where $v_a(\mu\mathbf{q})$ solves

$$\begin{aligned} \min_{\mathbf{f}} \quad & \sum_a \int_0^{v_a} t_a(x)\,dx, \\ \text{s.t.} \quad & \sum_{r \in R_{ij}} f_r^{ij} = \mu q_{ij}, \quad \forall i \in I, \ j \in J, \ q_{ij} \in Q, \\ & v_a = \sum_i \sum_j \sum_r f_r^{ij} \cdot \delta_{a,r}^{ij}, \quad \forall a \in A, \\ & f_r^{ij} \ge 0, \quad \forall i \in I, \ j \in J, \ r \in R_{ij}. \end{aligned} \tag{10}$$

The above model is referred to as the robust counterpart of the original reserve network capacity problem. The solution of the robust counterpart results in a maximum total travel demand scheme under the corresponding worst-case demand pattern.

4. Solution Algorithm

A heuristic algorithm is proposed to solve the above robust optimization model. It takes a similar framework as the procedure presented in [18], which is referred to as the cutting plane algorithm to robust optimization. The algorithm involves an iterative procedure to solve two inner optimization problems alternately until the convergence criterion is satisfied. The algorithm is presented as follows.

Step 0 (initialization). Give the initial values of the O-D demand $\mathbf{q}^{(0)} \in Q$ (usually $q_{ij}^{(0)} = q_{ij}^{L}, \forall i, j$) and solve the following reserve network capacity (RNC) problem to produce an initial demand multiplier $\mu^{(0)}$:

$$\begin{aligned} \text{RNC: } & \max_{\mu} \quad \mu, \\ & \text{subject to} \quad v_a\left(\mu \mathbf{q}^{(n)}\right) \leq C_a, \quad \forall a \in A, \ (3)\text{–}(6). \end{aligned} \tag{11}$$

Set the iteration counter $n := 0$.

Step 1 (direction finding).

Step 1.1. Solve the following inner (worst-case scenario (WCS)) problem with the determined $\mu^{(n)}$ to obtain the worst-case demand scenario:

$$\begin{aligned} \text{WCS: } & \max_{\mathbf{q}} \quad \sum_{a} \left(\max\left(0, v_a\left(\mu^{(n)}\mathbf{q}\right) - C_a\right)\right) \\ & \text{subject to} \quad \mathbf{q} \in Q, \ (10). \end{aligned} \tag{12}$$

Step 1.2. Formulate a RNC problem with the scheme of existing demand $q_{ij}^{(n)}$ produced from the WCS problem in Step 1.1. Solve this inner problem to find a search direction of the maximum demand multiplier $\widehat{\mu}^{(n)}$.

Step 2 (move). Compute $\mu^{(n)} := \mu^{(n-1)} + \alpha^{(n)}(\widehat{\mu}^{(n)} - \mu^{(n-1)})$, where $\alpha^{(n)}$ is the step length. In this study, the step length is chosen as $\alpha^{(n)} = 1/n$, which is used in the method of successive averages.

Step 3 (convergence check). If the objective value of the WCS problem $y > \varepsilon$ or n reaches the maximum iterations, then stop, where there is a predetermined convergence criterion. Otherwise, denote the solution of the WCS problem as $q_{ij}^{(n+1)}$, and go to Step 1; set $n := n + 1$.

Remark 1. In the above steps, the WCS problem is formulated to find a solution of $\mathbf{q} \in Q$, in which case the traffic flows on all links which exceed their capacities the most. If $\mathbf{q}$ is an optimal solution to the RRNC, the corresponding optimal objective value of the WCS problem must be zero. Otherwise, an improved solution may be obtained by solving the RNC problems in Step 1.2 which is a relaxation of the RRNC problems with a specific demand pattern. In the process of the algorithm, each of these relaxed RRNC problems can approximate the original RRNC better than its predecessors. Although it is still difficult in practice to find a global optimum of the relaxed RRNC and WSC, Yin et al. showed that the cutting plane algorithm is effective in providing a good solution to the robust optimization problem [18]. The relaxed RRNC problems and the WSC problems are solved by the sensitivity analysis based (SAB) algorithm [20].

Remark 2. The second inner problem is a standard RNC model when the existing O-D demand is determined. The RNC can be solved efficiently by applying the SAB algorithm [3]. The SAB algorithm locally approximates the original bilevel problem as a single-level optimization by using first-order Taylor expansion. The derivatives of lower-level decision variables with respect to upper-level ones are utilized for the linear approximation. The derivatives can be conducted from the sensitivity analysis of the lower-level model.

In this study, we used the restriction approach for the sensitivity analysis of the lower-level UE model. The restriction approach was proposed by Tobin and Friesz [21] and then corrected by Yang and Bell [22] for its flaws on selecting the nondegenerate extreme point. One can also refer to Du et al. [23] for the details of this approach. In this section, some necessary results are present without proof.

For the reserve capacity model, the link flows in upper-level, $v_a(\mu\mathbf{q})$, are represented as an implicit function of the O-D matrix multiplier μ as constraint (2) shows. Using the first-order Taylor expansion, it can be approximated as

$$v_a(\mu\mathbf{q}) \approx v_a(\mu^*\mathbf{q}) + \left[\left.\frac{\partial v_a(\mu\mathbf{q})}{\partial\mu}\right|_{\mu=\mu^*}\right](\mu - \mu^*), \quad \forall a \in A, \tag{13}$$

where μ^* is the given solution of the O-D demand multiplier at the current iteration of SAB algorithm.

From the results in Tobin and Friesz [21], the derivatives of the route flows, $\mathbf{f}$, with respect to the O-D demand multiplier μ, are derived as follows:

$$\begin{bmatrix} \nabla_\mu \mathbf{f}^0 \\ \nabla_\mu \boldsymbol{\pi} \end{bmatrix} = \begin{bmatrix} \Delta^{0T}\nabla_{\mathbf{v}}\mathbf{t}(\mathbf{v}^*, 0)\Delta^0 & -\Lambda^{0T} \\ -\Lambda^0 & O \end{bmatrix}^{-1} \begin{bmatrix} O \\ \nabla_\mu(\mu\mathbf{q}) \end{bmatrix}, \tag{14}$$

where the superscript "0" denotes that the variables or matrices are only associated with the restricting subproblem derived by the restriction approach (applying the correction in Yang and Bell [22]) and the superscript "T" represents the transposed matrix. Other notations are defined as follows:

$\Delta = [\delta_{a,r}^{ij}]$ is the link-route incidence matrix;

$\Lambda = [\lambda_r^{ij}]$ is the O-D-route incidence matrix, where λ_r^{ij} equals 1 if O-D pair ij is connected by route r, and 0 otherwise;

$\boldsymbol{\pi}$ is the Lagrangian multiplier associated with constraint (4);

$\mathbf{t}(\mathbf{v}^*, 0)$ is the vector of the travel cost function of all links with the equilibrium link flow $\mathbf{v}^*$ for the perturbation parameters at 0.

Thus, the derivatives of the link flows to the multiplier are obtained by $\nabla_\mu \mathbf{v} = \Delta^0 \cdot \nabla_\mu \mathbf{f}^0$. Based on the above derivations, the SAB algorithm can be used for solving the RNC problem.

Remark 3. The WCS problem is also formulated as a bilevel programming using equilibrium constraints, so the SAB method can also be modified for its solution. The implicit relationship $v_a(\mu\mathbf{q})$ is first-order approximated as

$$v_a(\mu\mathbf{q}) \approx v_a(\mu\mathbf{q}^*) + \sum_{i\in I}\sum_{j\in J}\left[\left.\frac{\partial v_a(\mu\mathbf{q})}{\partial q_{ij}}\right|_{\mathbf{q}=\mathbf{q}^*}\right]\left(q_{ij} - q_{ij}{}^*\right), \quad \forall a \in A, \tag{15}$$

where $\mathbf{q}^*$ is the solution of the existing O-D demand at the current iteration of SAB algorithm.

The derivatives of the route flows, $\mathbf{f}$, with respect to the existing O-D demand $\mathbf{q}$, are derived from the following equation:

$$\begin{bmatrix} \nabla_{\mathbf{q}}\mathbf{f}^0 \\ \nabla_{\mathbf{q}}\boldsymbol{\pi} \end{bmatrix} = \begin{bmatrix} \Delta^{0T}\nabla_{\mathbf{v}}\mathbf{t}(\mathbf{v}^*, 0)\,\Delta^0 & -\Lambda^{0T} \\ -\Lambda^0 & O \end{bmatrix}^{-1} \begin{bmatrix} O \\ \nabla_{\mathbf{q}}(\mu\mathbf{q}) \end{bmatrix}. \tag{16}$$

Because in this inner problem the value of μ is fixed, the derivative $\nabla_{\mathbf{q}}(\mu\mathbf{q})$ will be a diagonal matrix with the value of μ on its diagonal. Then, the derivatives of $\mathbf{v}$ to μ are calculated by $\nabla_{\mathbf{q}}\mathbf{v} = \Delta^0 \cdot \nabla_{\mathbf{q}}\mathbf{f}^0$. Thus, at each iteration of the SAB method, the approximating WCS problem is

$$\begin{aligned} \max_{\mathbf{q}} \quad & \sum_{a\in A}\left(\max\left(0, \nabla_{\mathbf{q}}v_a\mathbf{q} + v_a^* - \nabla_{\mathbf{q}}v_a\mathbf{q}^* - C_a\right)\right) \\ \text{subject to} \quad & \mathbf{q} \in Q. \end{aligned} \tag{17}$$

Note that the above localized approximation problem is a nonlinear problem and a number of optimization tools in commercial software packages could be used for its solution. In this study the approximate WCS problem is solved using MATLAB built-in functions which were converted to a .NET component and used in our solution program in C# language. Similar to [18], we randomly generate 100 vectors of $\mathbf{q}$ from the uncertainty region Q. For each $\mathbf{q}$, the user equilibrium problem with $\mu^{(n)}$ is solved. Let $\widehat{\mathbf{q}}$ be the random $\mathbf{q}$ with the maximum objective value of the WCS problem, and denote the objective value as $\widehat{y}$. If $\widehat{y} > \varepsilon$, then set $\mathbf{q}^{(n+1)} := \widehat{\mathbf{q}}$. Otherwise, use the MATLAB functions to solve the approximate WCS problem with $\widehat{\mathbf{q}}$ as an initial solution. If it gives a solution with an objective larger than ε, the solution is used as $\mathbf{q}^{(n+1)}$ for the next iteration; otherwise, the algorithm is terminated with an optimal solution $\widehat{\mathbf{q}}$.

5. Computational Experiments

5.1. Experiment 1: Nguyen-Dupuis Network. Computational experiments are presented in this section to illustrate the results of the robust network capacity model. The example is based on a road network which is adopted from Nguyen and Dupuis [24] as Figure 2 shows. It consists of 13 nodes, 19 links, and 4 O-D pairs. The nominal value of the existing travel demand is given by the O-D matrix in Figure 2 (denoted by $\mathbf{q}^0$). The characteristics of the links are listed in Table 1. The

Table 1: Link characteristics of the example network.

Link number a	Free-flow time t_a^0	Capacity C_a
1	7.0	800
2	9.0	400
3	9.0	200
4	12.0	800
5	3.0	350
6	9.0	400
7	5.0	800
8	13.0	250
9	5.0	250
10	9.0	300
11	9.0	550
12	10.0	550
13	9.0	600
14	6.0	700
15	9.0	500
16	8.0	300
17	7.0	200
18	14.0	400
19	11.0	600

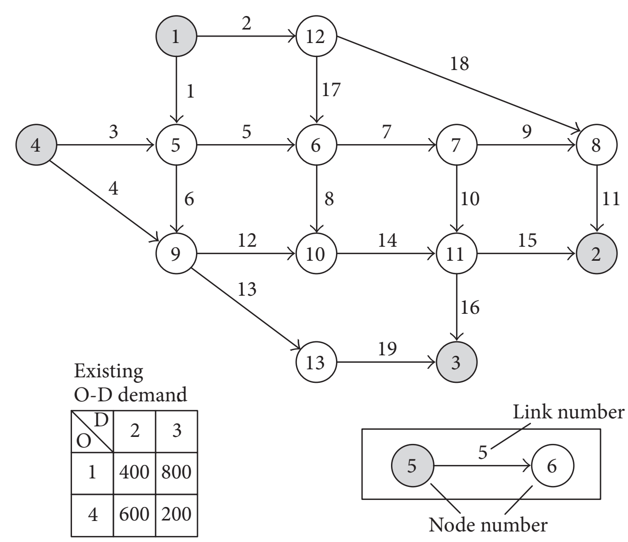


Figure 2: An example network.

Bureau of Public Road link performance function was used in the experiments:

$$t_a(v_a) = t_a^0\left[1 + 0.15 \cdot \left(\frac{v_a}{C_a}\right)^4\right], \tag{18}$$

where t_a^0 is the free-flow travel time for link a.

We applied the proposed approach for robust network spare capacity estimation with the three typical uncertainty sets which are described in this paper. Assume that the intervals for the O-D travel demands are [300, 600], [600, 1050], [400, 950], and [100, 375] for O-D pairs 1-2, 1-3, 4-2, and 4-3, respectively.

Table 2: Robust reserve capacities with ellipsoidal region and polyhedral region.

	Ellipsoid			Polyhedron	
Parameter θ	Max μ	Robust capacity	Parameter γ	Max μ	Robust capacity
0.0	0.1795	392.63	0.0	0.1944	388.89
0.2	0.1727	391.03	0.25	0.1892	378.38
0.5	0.1633	388.71	0.5	0.1843	368.54
1.0	0.1498	385.46	0.75	0.1795	359.01
2.0	0.1286	376.68	1.0	0.1753	350.54

Table 3: Robust reserve capacity estimations and the performances.

Network: Nguyen-Dupuis	Nominal	Conservative	Robust-e	Robust-p
Reserve capacity estimation	388.89	363.42	385.46	368.54
Travel demand multiplier	0.1944	0.1346	0.1498	0.1843
Percentage of meeting capacity constraints (%)	17.2	100.0	98.4	72.4
Percentage of above-robust-estimation (%)	51.8	100.0	54.6	98.8

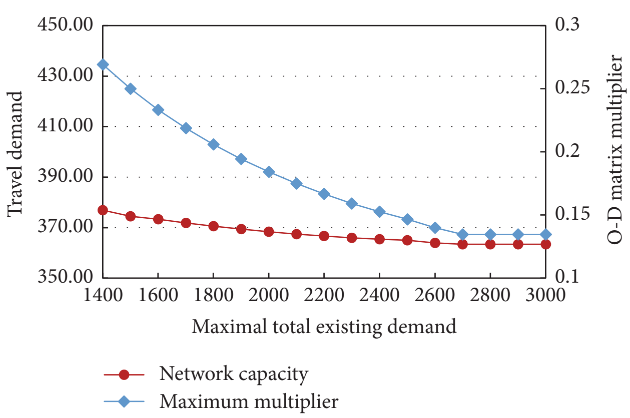


Figure 3: Robust reserve capacities with interval constraints under different maximal total existing demand.

(1) Interval Region. Theoretically, under the interval constrains for each O-D pair, the total of the existing O-D demand can vary from 1400 to 2975, which covers a wide range. However, according to our practical experience, the travel demands between the O-D pairs may not reach their maximum simultaneously. Thus, an additional constraint $\sum_{i,j} q_{ij} \leq D$ was introduced to give an upper bound of the total existing demand. To be consistent with the interval constraints, D was set to change from 1400 to 3000. Figure 3 illustrates how uncertainty of the existing travel demand affects the reserve capacity results of the example network. The reserve capacity value is calculated by $\mu^* \sum_{i,j} q_{ij}^*$, where μ^* is the robust solution of the RRNC problem and q_{ij}^* is the corresponding travel demand pattern. Note that the reserve capacity value and the μ^* decrease synchronously along with the growth of D. Because when D is increased, the vector of travel demand pattern, $\mathbf{q}$, will have more space to change its spatial distribution, and this makes the existing demand easier to archive a demand pattern whose corresponding multiplier can reach a smaller value. Besides, when D exceeds 2700, the value of μ^* cannot get worse. At this value $\mu^* = 0.1346$, the corresponding existing travel demand is $\mathbf{q}^* = (600, 1050, 950, 100)$, which produces a smallest reserve capacity value 363.42 (referred to as the *conservative solution* in the rest part of this section). Furthermore, for the robust solutions when $D > 2700$, the change of q_{43} will have no effect on the value of μ^*. Here, we use the lower bound of q_{43} to conduct the worst-case results of the reserve capacity. According to the properties of the robust optimization, the solutions of the existing O-D demand may not be unique, so no specific solution of the existing demand is presented in this paper.

(2) Ellipsoidal Region. For the ellipsoidal uncertainty set, the parameter θ is set to be 0, 0.2, 0.5, 1.0, and 2.0. The center of the ellipsoid is decided by the boundary for each O-D demand (not the nominal value for this example). The computational results in Table 2 show that the robust solution to μ changes between 0.1795 and 0.1286 with a descending trend, as well as the corresponding network capacity value.

(3) Polyhedral Region. For the robust network capacity estimation with the polyhedral set described in previous section, the robust results at $\gamma = 0, 0.25, 0.5, 0.75$, and 1.0 are computed and presented in Table 2. The nominal demand (given in Figure 2) is used in this type of uncertainty. The value of μ declines from 0.1944 to 0.1753. The same trend has been observed on the robust values of network capacity, which varies from 388.89 to 350.54.

Table 3 reports the solutions of the reserve capacity value from two RNC problems with $\mathbf{q}^0 = (400, 800, 600, 200)$ and $\mathbf{q} = (600, 1050, 950, 100)$ separately and two RRNC problems with Q defined in the ellipsoid region ($\theta = 1.0$) and the polyhedral region ($\gamma = 0.5$). We refer to the first two solutions as "nominal" and "conservative" estimation, respectively, and the latter two as "robust-e" and "robust-p" separately. For the results, the robust solution for the O-D demand multiplier indicates that all the values of the existing demand within the uncertainty set can be applied by a multiplier larger than this robust value without violating the capacity constraints. A

Table 4: Network capacity estimations on Sioux-Falls.

Network: Sioux-Falls	Nominal	Conservative	Robust-e	Robust-p
Reserve capacity estimation	209,029.1	195,065.8	201,366.7	176,850.3
Travel demand multiplier	0.5797	0.4291	0.5588	0.4904

corresponding maximum travel demand is produced as the robust network capacity solution for the same uncertainty set. Since the same maximum travel demand may produce by different combinations of the multiplier and feasible existing demand, the proposed algorithm only finds the smallest value of the maximum multiplier. Besides, if the shape of the uncertainty set changes, different solutions for the robust multiplier and maximum demand will be produced. Table 3 shows these features.

To evaluate the four estimations of the network capacity, we randomly generate 500 samples of O-D demand as the possible realizations of the existing travel demand pattern. The samples are from the normal distribution, $N(\overline{q}_{ij}, 0.25\overline{q}_{ij}^2)$, within its interval $[q_{ij}^L, q_{ij}^U]$ for all O-D pairs. Then, for each sample, one has the following.

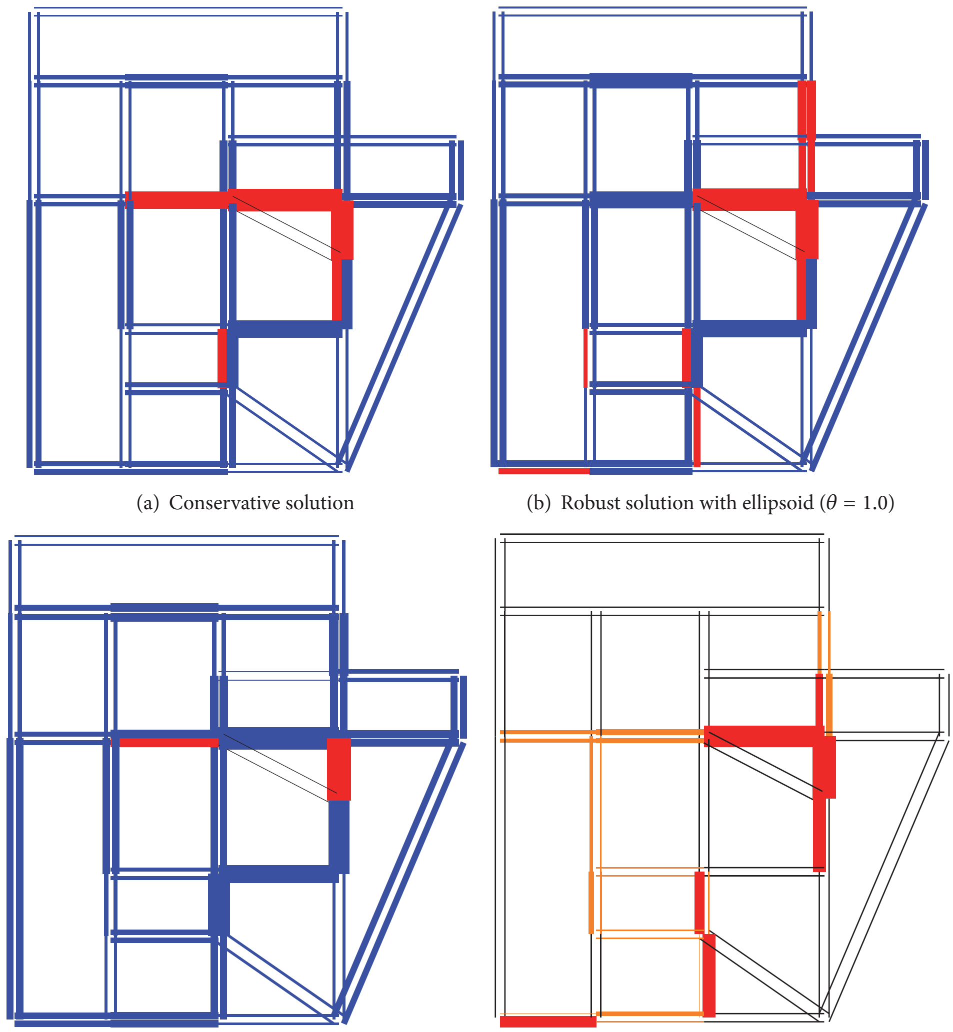
Firstly, the user equilibrium assignment associated with each reserve network capacity estimation (i.e., the largest μ) is computed. The number of failures is counted whenever the link flow exceeds its capacity at each sample. In the end of the simulation, the successful rates are computed as [1 − total number of failures/(number of samples × number of links)]. The results are referred to as "percentage of meeting capacity constraints" in Table 3. Obviously, the conservative estimation gives the worst value of the network capacity; the nominal estimation produces a medium level of result. By comparison, the robust results from the ellipsoid and polyhedron are more reasonable. Note that all samples generated in the above simulations belong to a box region comprised of the intervals for O-D demands. The aforementioned uncertainty sets are used as replacements of the box regions, so as to prevent overly conservative robust resolutions. The results in Table 3 indicate that the ellipsoid approximate the box region better than the polyhedron, because more random samples will not exceed the network capacity after they are scaled by the robust multiplier $\mu = 0.1498$. On the other hand, although $\mu = 0.1843$ is the worst case in the polyhedral set, when the boundary is extended to the box region, only 72.4% of the random samples can be covered.

Furthermore, considering that the demand multiplier in essence is a relative value, the reserve capacity results are derived. Therefore, the reserve capacity problem with the every random existing O-D demand is solved in our test. For each estimation value of the reserve capacity: the number of successes is counted whenever the reserve capacity value of the sample exceeds the robust capacity estimation. Consequently, the successful rates are computed as [total number of successes/number of samples]. These results are also presented in Table 3 as "percentage of above-robust-estimation." From this aspect, the polyhedron provides a more robust estimation of the network capacity, which can be met by most realizations of the existing travel demand pattern (98.8% versus 54.6% compared with the ellipsoid). We may suppose that the worst cases of network capacity values exist in the corner area of the box-shaped uncertain region, where it is not easy to be covered by the corresponding ellipsoidal set. Consequently, the choice of the robust results depends on the decision-makers' attitude to risk and the usage of the robust results. From the computational perspective, the robust optimizations with polyhedral uncertainty sets have more advantages. More effective solution methods could be developed in future studies.

5.2. Experiment 2: Sioux-Falls Network. Experiments are further presented on the Sioux-Falls network [25]. The network contains 24 nodes, 76 links, and 528 O-D pairs. The characteristics of the links and travel demands are also provided in Bar-Gera [25]. In this experiment, we used the default values of the travel demand in as the nominal value of the existing demand in the network. If $q_{ij} \geq 1000$, then the demand for O-D pair ij is uncertain and its upper limit and lower limit are set to $q_{ij}^L = 0.5q_{ij}$ and $q_{ij}^U = 1.5q_{ij}$, separately. Hence, there are 104 O-D pairs with uncertain existing demands. According to the previous section, four estimation methods are employed to evaluate the network capacity of the Sioux-Falls.

Table 4 reports the solutions of the reserve capacity from the four estimation methods. Note that the robust-e solution gives a moderate robust result compared to the others. The robust-p solution provides the lowest estimation of network capacity. One may use this lowest value as the worst-case performance that the network can serve the travel demand. Besides, note that the lowest estimation of the network capacity was not derived from the conservative solutions, in which the uncertain O-D demands are set to its upper limits. Although the conservative solution is corresponding to the lowest demand multiplier, it may not reflect the most unfavorable situation which is possible to be resulted from the changes of the network demands.

We selected the conservative, robust-e, and robust-p solutions to further inspect the link flow patterns at the maximum travel demand situations (i.e., the reserve capacity). The link flow patterns are shown in Figures 4(a), 4(b), and 4(c). The width of the line indicates the traffic volume through the link. The red lines show the links whose V/C (volume/capacity) ratio is greater than 0.9, and the black lines denote the links with zero flow. As a reference, we also randomly generate 500 samples of O-D demand which are from the normal distribution, $N(\overline{q}_{ij}, 0.25\overline{q}_{ij}^2)$, within its interval $[q_{ij}^L, q_{ij}^U]$, $\forall i, j$. For each sample, the reserve capacity problem is solved. Then, at each link, the highly saturated number is counted whenever its V/C ratio is greater than



(a) Conservative solution

(b) Robust solution with ellipsoid ($\theta = 1.0$)

(c) Robust solution with polyhedron ($\gamma = 0.5$)

(d) Simulation results

FIGURE 4: Link flow pattern associated with different maximum travel demands (a, b, and c) and probabilities that links are congested (d).

0.9. In consequence, the link saturation rates are computed as [total the saturated number/number of samples]. The results are also shown in Figure 4(d). The width of the line indicates the saturation rate of the link. The red lines indicate links with saturation rates greater than 50%. The orange lines denote links with saturation rates lower than 50% but greater than zero. The black lines mean the links have no chance to be blocked. From Figure 4, we conclude that the solution of the robust-e will produce a link flow pattern most likely to be realized when the network reaches its capacity (compared to the simulation results). The robust-p solution provides an extreme result that the network is congested only because of a very few links. These links could be considered as the most critical links which restrict the capacity of the entire road network. By contrast, the conservative solution seems not to fit the simulation results very well. In conclusion, the robust estimation to the network capacity problem could be much more practical than the results from any specific scenario.

6. Conclusions and Perspectives

In this study, a robust network capacity model with uncertain demand has been proposed. The robust optimization is formulated using the min–max model with a bounded uncertainty set of the existing O-D travel demands. With the uncertainty set, the low-probability realizations of the travel demand pattern are excluded, and thus the robust model can produce a proper estimation of network capacity which can be achieved with a large probability. Then, a heuristic algorithm has been proposed for the proposed robust model. It solves two inner problems iteratively: one is the worst-case scenario problem; and the other is the relaxed robust optimization, namely, the standard reserve capacity model. At each iteration, the cutting plane method has been adopted to generate the worst-case demand scenario, and the sensitivity analysis based approach has been developed for the solution of the worst-case model and the reserve capacity model. The validity and performance of the proposed robust model have been demonstrated in the computational experiments. Different results under three typical uncertainty sets, say interval, ellipsoidal, and polyhedral region, have been conducted and compared. The interval set is simple but easy to produce too conservative results; the ellipsoidal set is a good approximation to the uncertain region and produces results with moderate robustness, but its solution is more complicated due to the nonlinear constraints; the polyhedral set is considered if a high level of robustness is required, and its linear formulation makes the robust model easier to solve. Furthermore, by conducting computational experiments on the Sioux-Falls network, robust solutions shown can provide more practical results of the link flow patterns. In applications, these uncertainty sets and their parameters should be selected according to the desired level of robustness.

The robust model based on the reserve capacity model has been proposed and explored in this study. Future researches should focus on more efficient solution approaches for the robust problem with the min–max model. The experiments on large-scale networks are also needed. Besides, the demand uncertainties existing in other network capacity models are also expected to be detected and discussed. Alterative traffic assignment model, such as the stochastic user equilibrium, could also be discussed for the network capacity problems. The robust solution of network capacity gives a lower bound to the possible schemes of the maximum demand in a transportation network. These possibilities constitute a range where the robust solution can be most likely to be reached in reality. Therefore, the robust solution to network capacity problems needs to receive more attentions in transportation planning applications.

Disclosure

An initial version of this paper was presented at the Transportation Research Board (TRB) 96th Annual Meeting. It therefore also appears in the proceedings of the TRB 96th Annual Meeting Compendium of Papers.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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