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What Knowledge do Teachers use in Lesson Study? A focus on Mathematical Knowledge for Teaching and Levels of Teacher Activity

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Abstract

This chapter combines the frameworks of Ball et al. (2008) and Margolinas et al. (2005) to demonstrate the elements of subject and pedagogical content knowledge utilized at varying levels of teacher activity in a cycle of lesson study. Qualitative data generated in a mathematics-based lesson study, conducted with eight primary school teachers in Switzerland, is analyzed and visualizations of the knowledge occurring at each phase of lesson study is provided. This fine-grained analysis of the mathematical knowledge incorporated by teachers in lesson study demonstrates that all forms of Mathematical Knowledge for Teaching, at each level of teacher activity, can occur across a cycle. In addition, the paper provides evidence that phases of lesson study do not necessarily occur in succession but can rather take place in a confluence of teachers' work across a full cycle.

Keywords

Lesson study, Professional Development, Mathematical Knowledge for Teaching, Level of Teacher Activity.

Acknowledgments

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Introduction

With increased attention on the impact of lesson study on teacher learning, there have been calls to deepen our understandings of the development of mathematics teachers' knowledge within this model (Lewis et al., 2009). Furthermore, research has highlighted the need to explore the theoretical underpinnings of teacher learning in lesson study (Miyakawa and Winsløw, 2009, Clivaz, 2015). In this chapter, we investigate the types of knowledge and activities employed by teachers in lesson study, incorporating both cognitive and situated perspectives of mathematics teachers' knowledge and skills (Ni Shuilleabhain and Clivaz, 2017). Based on a case study conducted with eight primary school teachers in Switzerland, we detail their participation across one cycle of lesson study utilizing a combination of the theoretical frameworks of Ball et al. (2008) and Margolinas et al. (2005). This fine-grained analysis demonstrates the constituents of both subject and pedagogical content knowledge employed by teachers, at varying levels of pedagogical activity, for each phase of the lesson study cycle. In this case study, teachers' pedagogical content knowledge, particularly related to their consideration of content, was the most utilized form of knowledge incorporated in their lesson study work. Teachers' values and considerations about teaching and learning was also apparent throughout their planning and reflection of the research lesson. These findings provide a detailed representation of the types of knowledge employed by teachers across the phases of lesson study and contribute to our understanding of teacher learning in lesson study. In addition, our analysis demonstrates that each phase of lesson study need not necessarily take place in succession, but rather occur in a confluence of teachers' conversations over one full cycle.

Theoretical Framework

There are a multitude of theoretical frameworks which particularize the knowledge and practices required to teach mathematics. While there is agreement within the research literature that both content

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knowledge and pedagogical content knowledge (PCK) are requirements in the teaching of mathematics (Rowland et al., 2005, Krauss et al., 2008, Hill, 2010, Schoenfeld, 2010, Speer et al., 2015), there is, at present, a separation between the cognitive and situated models of teacher knowledge (Rowland et al., 2011). Furthermore, differences exist between traditions in their considerations of the subject-related knowledge required to teach mathematics and the ways through which such knowledge can be developed in Anglo-American, Continental-European and East-Asian research (Depaepe et al., 2013). In an attempt to investigate the knowledge incorporated by mathematics teachers during their participation in lesson study, we have proposed an analytical framework of teachers' mathematical knowledge (Ni Shuilleabhain and Clivaz, 2017). This model combines the theoretical frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005) from differing perspectives and traditions. We refer to "combining" these frameworks in the sense of Prediger et al. (2008) and undertake this combination in order to gain further, multi-faceted insight into the knowledge incorporated by mathematics teachers in their participation in lesson study.

Mathematical Knowledge for Teaching

Ball and her colleagues (2008) developed a practice-based theory of the knowledge required "to carry out the work of teaching mathematics", presented as a framework of Mathematical Knowledge for Teaching (MKT) (p. 395). This model built on Shulman's theoretical suggestion of PCK as a specific type of knowledge unique to teachers and distinguished it from subject matter or content knowledge (1986, 1987). In their model, Ball and her colleagues highlighted particular categories of knowledge within the PCK and subject matter delineations:

Subject Matter Knowledge

- Common Content Knowledge (CCK)
- Horizon Knowledge (HK)
- Specialized Content Knowledge (SCK)

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Pedagogical Content Knowledge

- Knowledge of Content and Teaching (KCT)
- Knowledge of Content and Students (KCS)
- Knowledge of Content and Curriculum (KCC).

In their review of the conceptualization and evidence of PCK in mathematics education research, Depaepe et al. (2013) noted the MKT model as "probably the most influential re-conceptualizations of teacher PCK within mathematics education" (p. 13). However, Steinbring (1998) and Margolinas (2004) suggest that in Shulman's proposed framework of teacher knowledge (1986), on which the MKT framework is modelled, fixed categories of teacher knowledge are "not a good model for teacher's activity, which is more complicated" (Margolinas et al., 2005, p. 207). Indeed, Ball and her collaborators acknowledge that these categorizations of teacher knowledge can be interpreted as static and distinct (2008) and therefore difficult to incorporate within the active practices of teaching. Others have critiqued such consideration of teachers' knowledge for teaching as solely applied within a context of instruction, without consideration of the complex, social environment of a mathematics classroom (Hodgen, 2011, Putnam and Borko, 2000).
Davis and Renert (2013) suggest that understanding the relationship between teacher knowledge and student learning "will require more fine-grained analyses than large-scale assessments" (p. 264) in order to fully capture the sophisticated and enactive mix of knowledge utilized by mathematics teachers. The theory of didactical situations (Brousseau, 1997) can provide researchers with a tool to conduct such qualitative, fine-grained and mobile analyses (Winsløw et al., in print) and is described further below.

Levels of teacher activity

In 1970s, Brousseau's theory of didactical situations (1997) first modelled a learning situation where the teacher was largely absent from the analysis of student learning (Bloch, 2005). However, from the 1990s, the importance of the teacher's role became increasingly evident in the study and theorization of ordinary classroom situations

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(Bloch, 1999, Roditi, 2011, Dorier, 2012). This provided a platform through which to introduce a situated theory embedded in the context of the classroom, to analyze the various levels of practices, skills and knowledge required of mathematics teachers.

The concept of *milieu*[1] is central to the theory of didactical situations. The *milieu* is defined by "all of the pertinent features of the student's surroundings, including the space, the teacher, the materials and the presence or absence of other students" (Warfield, 2014, p. 66). Based on Brousseau's theory (1997), Margolinas (2002) developed a model of the mathematics teacher's milieu to describe a teacher's activity, both in and outside of the classroom. This model was designed to take into account the complexity of teachers' actions and capture the broad range of activities contained in teaching and learning (Margolinas et al., 2005). Centering on the action of the classroom, the model depicts the various levels at which a teacher must situate themselves within their pedagogical practices. In this model (see **Fig. 1**), level +3 refers to teachers' values and conceptions about learning and teaching, level +2 concerns teachers' actions and discourses about the global didactic project, level +1 pertains to the local didactic project, level 0 is the didactic action, and level –1 deals with the observation of pupils' activity. The teacher's point of view can be related to his or her considerations and reflections at different levels of generality. Observing students' work (including noticing student talk) relates to a more refined focus of the teacher on individual students and, hence, relates to level -1. Planning the local didactic project (about the lesson) relates to the content of the lesson as relative to the students in the classroom and, hence, is placed at level +1. At level +2, the teacher considers the didactic project in a more global sense (e.g. teaching a particular element of a topic as one lesson in a series of lessons). While at level +3, the teacher draws upon their beliefs about the teaching and learning of mathematics, which can be related to how the global and local projects may be constructed and to how they will engage with individual students. The model is not intended as a linear interpretation of teachers' work, but rather identifies the multidimensional tensions involved in teaching (Margolinas et al., 2005). At every level the teacher not

[1] Milieu is the usual translation for Brousseau's French term "milieu",. However, in French it refers not only to the sociological milieu, but is also used in biology or in reference to Piaget's work. A more accurate translation would be "environment".

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only has to deal with the current, most prescient, level of activity, but also with the levels directly before and after and, in some instances, with levels extending beyond.

As a situated model of teacher knowledge based within the context of teaching and learning practices, Margolinas and colleagues (2005) proposed this model as a way of delineating the multi-level knowledge required of teachers during varied stages of teaching - from the over-arching pedagogical values underpinning a lesson, to the didactic action within the classroom. However, while this model incorporates teacher values and acknowledges the pedagogical skills required to notice and interpret student thinking, it does not make explicit how a teacher's specific content or pedagogical content knowledge may be encompassed in such activities.

A proposed theoretical framework

Domains of MKT (Ball et al., 2008) have been shown to be incorporated and developed through teachers' participation in lesson study, at both in-service and pre-service levels (Leavy and O'Loughlin, 2016, Ni Shuilleabhain, 2016, Tepylo and Moss, 2011). However, considering the multitude of knowledge and practices incorporated within each phase of lesson study – studying the curriculum, planning, conducting or observing, and reflecting on a mathematics lesson – the MKT framework does not wholly capture the incorporation of teachers' beliefs nor the considerations involved in structuring content for a research lesson. Ni Shuilleabhain (2015) utilized the MKT framework to investigate teacher learning in lesson study and, in an attempt to capture the knowledge incorporated by teachers in their planning and reflection conversations, combined this with the idea of a critical lens relevant to student thinking (as suggested by Fernandez et al., 2003) as an additional layer of analysis of teacher learning in lesson study. This concept of a 'student lens' relates to the PCK a teacher utilizes in seeing mathematics "through the eyes of their students" (Fernandez et al., 2003, p. 179), but is separate to an action of the teacher noticing students' mathematical work in teaching (Jacobs et al., 2010). In our proposed theoretical framework, this 'student lens' is added to Margolinas et al.'s (2005) model as a level -2. This layered

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model relates to work by Clivaz (2014, 2017), who used the situated activity model (Margolinas et al., 2005) to observe teacher classroom practice and, in an effort to detail both the mathematical knowledge for and in teaching, aligned it with the cognitive framework of MKT. In our research, Ni Shuilleabhain and Clivaz (2017), we proposed a combination (Prediger et al., 2008) of these two existing frameworks of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005) to analyze the knowledge incorporated by teachers in two case studies. The graphical representation of this framework (**Fig. 1**) shows that the categorization of knowledge lies in one plane ("the egg"), while the levels of activity are characterized in a contrasting cross-sectional plane ("the cake"). In this chapter, we develop this work and employ the framework as a tool to further detail and analyze mathematics teachers' knowledge in various phases of planning, conducting, observing, and reflecting on teaching in one case-study cycle of lesson study (see **Fig. 1**).

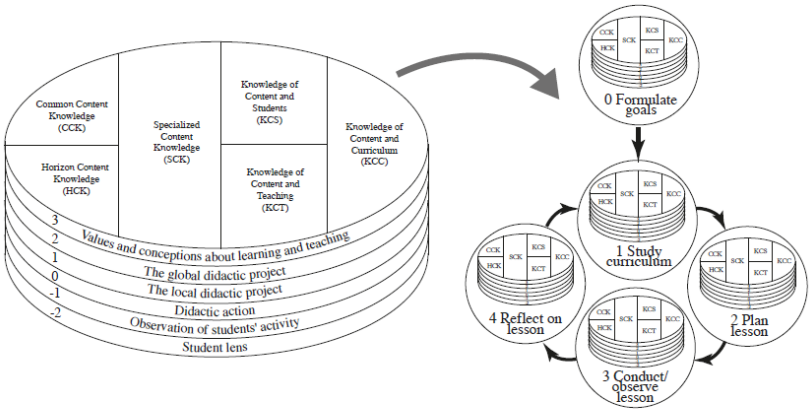

Fig. 1 MKT and levels of teacher activity in a cycle of lesson study.

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Context and methodology

Eight generalist grade 3-4 primary school teachers from the Lausanne region, French-speaking part of Switzerland, were introduced to lesson study and conducted four lesson study cycles over two school years. The group was facilitated by two university teacher educators, one specialist in teaching and learning and one specialist in mathematics didactics (first author of this chapter). All meetings (37 of an average of 90 minutes duration) and research lessons (8) were transcribed and coded in a qualitative analysis software (NVivo). Student work, teachers observations during lessons, and lesson plans were also recorded and coded (Clivaz, 2016).

In this chapter, our case study refers to the first lesson study cycle (Lewis et al., 2006) of the group, where teachers decided to focus on the topic of integer number and place value. Within this cycle of lesson study teachers met on nine occasions with the research lesson being taught, discussed, re-designed and taught in the redesigned form. Generated data, segmented into conversational utterances or section of note-taking, was coded according to its classification within each of the following categories: lesson study phase, level of teacher activity and form of MKT (see coding map in Appendix). Codes and subcodes were developed and revised through iterative phases of analysis (see Ni Shuilleabhain and Clivaz, 2017). The proportion of coded data within each lesson study meeting and research lesson varied from approximately 10% to 65%. All teachers' and students' names referred to throughout the chapter are pseudonyms. The lesson study facilitators, Anne and Stéphane (first author of this chapter), retain their real names

Analysis and findings

In this reporting of our analysis, we detail the different forms of Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005) that occur over each phase of the lesson study cycle. Through the use of quotes and graphical data, we explicate teachers' knowledge recorded in their participation in lesson study. While the data was originally generated and coded in

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French, translations included in this chapter were translated collaboratively by the two authors of this chapter, with an explicit effort made to keep the nuance and color of the spoken language as pertinent to teacher dialogue.

Levels of teacher activity incorporated in one cycle

In an attempt to identify the levels of activity at which teachers operated over the course of the lesson study cycle, we first focus specifically on the Levels of Teacher Activity (Margolinas et al., 2005) accounted for in our case-study. As might be anticipated, phase 2 (planning the research lesson) was mostly attributed to teachers' activities at level +1, focusing on the local didactic project within the research lesson (see **Fig. 2**). Similarly, phase 3 (conducting and observing the research lesson) was mostly related to level 0 of teacher activity, i.e. the didactic action within the classroom.

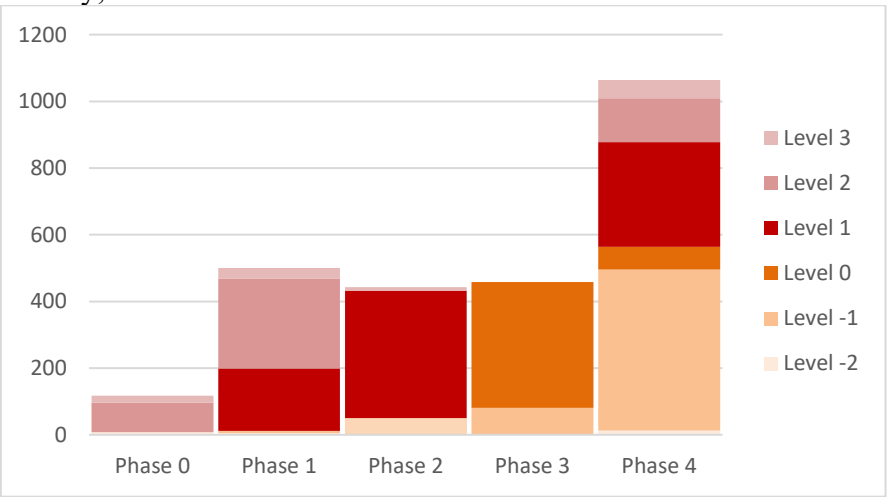

Fig. 2 Levels of teacher activity incorporated in one cycle.

However, during the initial phases of 0 and 1, where teachers formulate goals and study the curriculum, teachers articulated their values and conceptions about teaching and learning (level +3), while also considering the global didactic project relative to particular class group of students (level +2):

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Edith (T)	Our habits of doing things. Even having discussions with other people and hearing: “Bah, I don’t like that!” or “I don’t do it that way” … It makes me think: “oh, I didn’t think about that!” […] Because, to be honest, some stuff, you know, frankly…
Stéphane (F)	Of course.
Edith (T)	I mean, these cubes, I don’t like these manipulatives at all. To be honest, we just lose those little cubes!
Stéphane (F)	Yeah.
Edith (T)	And oh my god, that drives me crazy. I mean it. It drives me crazy. I don’t use them, I’ll do it another way. But, then… Sometimes it’s good to get a kick in the ass and say: “Well, give it a go.” Since, yeah, really, we are sometimes a little bit selfish! It annoys me, but if it is for the good of my students, well yeah, let’s see. Maybe it works and maybe it doesn’t!
Stéphane (F)	Hmm hmm.
Edith (T)	That’s it! It is true. In one’s teaching we are a little bit selfish. I mean, we cut our cloth to how it suits us. Because there are so many things, we make choices that are not… I mean, we suit ourselves.
Stéphane (F)	That’s important too!
Edith (T)	So, here we are! I’ll try it in another way and see.

Teachers’ values and conceptions impact their approaches to teaching and learning (Ni Shuilleabhain and Seery, 2017) and, in this context, teachers had opportunity to make their implicit practices explicit in their planning of the research lesson (Fujii, in press).

As demonstrated in **Fig. 2**, teachers also had opportunity to see the mathematics from the perspectives of the students in their planning of the lesson. As part of phase 2 of lesson study, teachers are encouraged to complete the tasks which might be incorporated within the research lesson. In our case study, teachers played a game which they intended as a key learning activity within the lesson and, in doing so, took part in the activity as if they themselves were the students.

Caroline (T)	For me, when I see that, my first reaction is: I can’t!
Anne (F)	But you have not ...
Caroline (T)	So, I did the exchange.
Anne (F)	And then?
Caroline (T)	Yes, but it's because I saw the ten, that’s why. But if I see a ten, do I always think to make an exchange to ten units?
Anne (F)	But that's what you just did. After that, why didn’t you do it?
Caroline (T)	I did it then. But after that, I don’t know.
Anne (F)	What should you do, ideally?
Caroline (T)	I think …
Valentine (T)	We need to start with the hundred.

This teacher activity, utilizing a student lens at level -2, provided teachers with further insight into student thinking and afforded them greater insight in deciding how to conduct the game within the research lesson.

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In addition to seeing the mathematics through the eyes of the student, our analysis also reveals a blend of levels of activity over the phases of the lesson study cycle. Teachers' participation in phases 0 to 2 focused largely on the local and global didactic projects (levels +2 and +1), while the work of phases 3 and 4 were mostly concerned with the action within the research lesson and focusing on students' thinking (levels 0 and -1). Teachers' values and conceptions of teaching and learning of mathematics (level +3) played a part in both their planning and reflection conversations, as did teachers' actions in seeing the mathematics through the eyes of the student (level -2). It is worthy to note that the post-lesson discussions included all levels of teacher activity (see **Fig. 2**). Such a finding demonstrates the significance of the reflection phase of lesson study, where teachers are provided with opportunity to articulate their pedagogical practices at all levels.

MKT expressed in one cycle of lesson study

Analyzing teachers' participation in lesson study according to the MKT framework (Ball et al., 2008), all categories of MKT were found to be expressed across each of the phases of lesson study (see **Fig. 3**). It is notable that all forms of MKT were incorporated in phase 1 of the cycle.

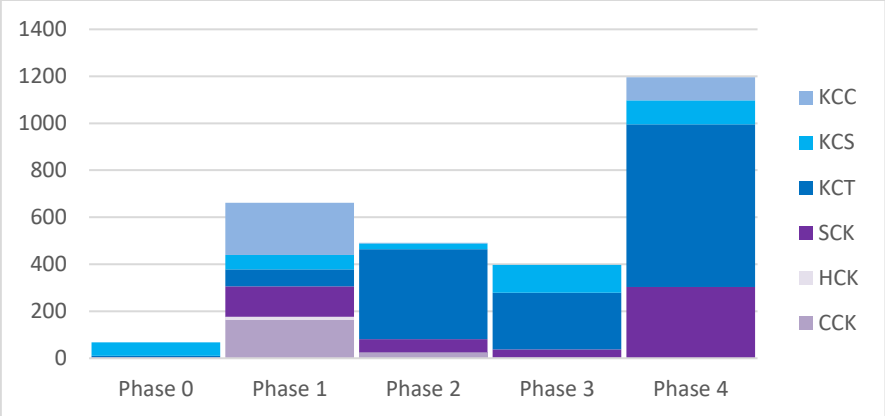

Fig. 3 MKT expressed in one cycle of lesson study.

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In phase 2, planning the research lesson, teachers regularly drew upon their KCT in designing the mathematical instruction. In this collaborative work teachers were obliged to deliberate and agree on the aims of the learning activity. Teachers considered whether the goal of the lesson should be for students to discover the idea of exchange or simply practice the exchange of hundreds, tens, and units.

Vanessa (T)	We thought about that, hmm. In fact, we are reflecting on the objectives of the task. Does the learning also lie in being familiar with the idea of exchange? I think so, but I don't know…
Edith (T)	Then, either we decide we don't want them [the students] to come to it by themselves and the goal of the lesson is really that they practice doing the exchanges. Because, if that's the case, we can explain the concept and after that it's OK!
Valentine (T)	Yeah.
Edith (T)	Or we decide that the goal is really to cause them to think, so that they find this solution of exchange, and, in that case, we must really define what our lesson goal is.

These discussions provided opportunity for teachers to select appropriate models, representations and examples which would support students' mathematical understanding of exchanging units, tens and hundreds.

Phase 3 included occurrences of KCT, KCS and SCK as relevant to the work of the conducting teacher and to the notes recorded by observers during the research lesson. For example, in **Fig. 4**, the observer noted a student's suggestion to change one one-hundred unit into nine tens and ten units. This observation note demonstrates the teacher's noticing of student thinking (level -1) and the incorporation of the teacher's interpretation of student work (KCS).

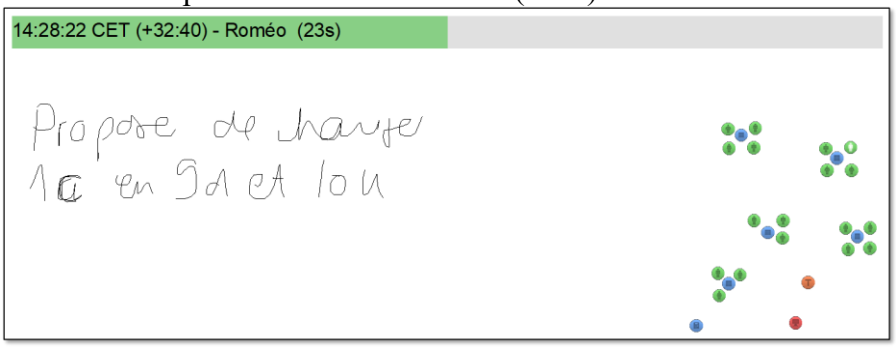

Fig. 4 Teacher's observation, recorded on Lesson Note (Lesson Study Alliance, 2012-).

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In our analysis, the occurrence of KCT in teachers' conversations was predominant across all of the phases, supporting the suggestion that participating in lesson study can develop teachers' PCK (Ni Shuilleabhain, 2016). As an example of a less frequently occurring knowledge from the data, in their study of the curriculum (phase 1) teachers had opportunity to utilize their CCK while testing various tasks to potentially include within the research lesson. Teachers undertook a task named 'Hit' from the textbook, which had the objective of displaying 387 on a calculator utilizing only 0 and 3). Drawing on their mathematical knowledge for common calculations, teachers made decisions on the appropriateness of the task for students.

Marius (T)	'Hit', on page 97.
Océane (T)	'Hit'? Oh yeah, it's with the calculator. […] They just have to do addition or …?
Marius (T)	They must do those …
Océane (T)	Okay yeah, agreed. Three and zero. Yeah, then …If they can do thirty times thirty… Thirty times three…
Marius (T)	It would make nine hundred divided into ...
Océane (T)	No, three times …it's thirty and then it makes ninety.
Marius (T)	Plus thirty times three.
Océane (T)	Yeah, yeah […]
Marius (T)	Subtract three. They have to see that already and see how we get to three hundred and how we get to ninety.

While the group thought the game could be useful, they decided it was not the most appropriate task for students and did not include it within the research lesson.

Participating in the lesson study cycle required teachers to draw on their various forms of MKT in their discussions and decisions around planning, conducting or observing, and reflecting on the research lesson. While the content of phase 0 work mostly related to students' learning (KCS), phase 4 required a far richer breadth of teacher knowledge related to teacher learning (KCT, SCK, KCS and KCC). These varying forms of knowledge were distributed across the phases of lesson study. Furthermore, our analysis demonstrates that teachers' PCK was particularly evident in their participation in lesson study, an important feature to consider in detailing the work that teachers do as part of this form of professional development.

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From phases to meetings

The visualizations often used to depict lesson study suggest a chronological order of phases, where the planning of the research lesson (phase 2) proceeds the study of curriculum (phase 1) and the reflection of the research lesson (phase 4) follows the conduction of that research lesson (phase 3). However, our analysis of the data demonstrates that phases of lesson study do not always occur in sequential order, but rather arise throughout the cycle as inter-connected and related phases during teachers' collaborative work (see **Fig. 5**).

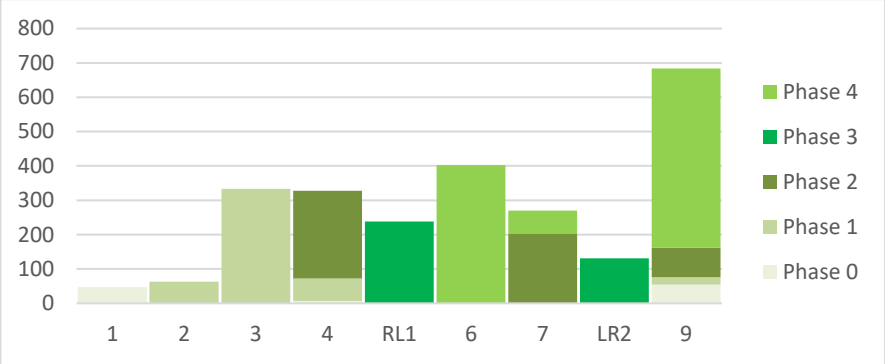

Fig. 5 Lesson study phases & meetings.

As demonstrated by our analysis, the planning of a research lesson might also include reflections of teaching, as depicted in meeting 7, **Fig. 5**. In the same manner, a post-lesson discussion might also include deliberations on the goal of the lesson study. Such conversations are depicted in meeting 9 of **Fig. 5**, where not only the most recent research lesson, but also the previous lesson were discussed along with the goals of the cycle. Our findings suggest that phases are not necessarily consecutive within lesson study. Such fluidity in the chronology of phases may be useful to highlight to participating teachers. It may be particularly relevant to make teachers aware that their work continuously evolves across a lesson study cycle in, for example, refining goals or considering elements of student thinking through each phase.

Analyzing the MKT across the meetings, (see **Fig. 6**) it is evident that teachers had more of a focus on the mathematical content of the lesson, as relative to student learning, after the first research lesson. This

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analysis across the lesson study cycle provides us with insight into the increased occurrences of KCT, KCS and SCK following the conduction and observation of research lessons, where teachers had increased focus on student learning in their lesson study work. This finding may be relevant when considering whether a research lesson should be redesigned and taught in a new form, since teachers may have increased focus on content as relevant to student learning.

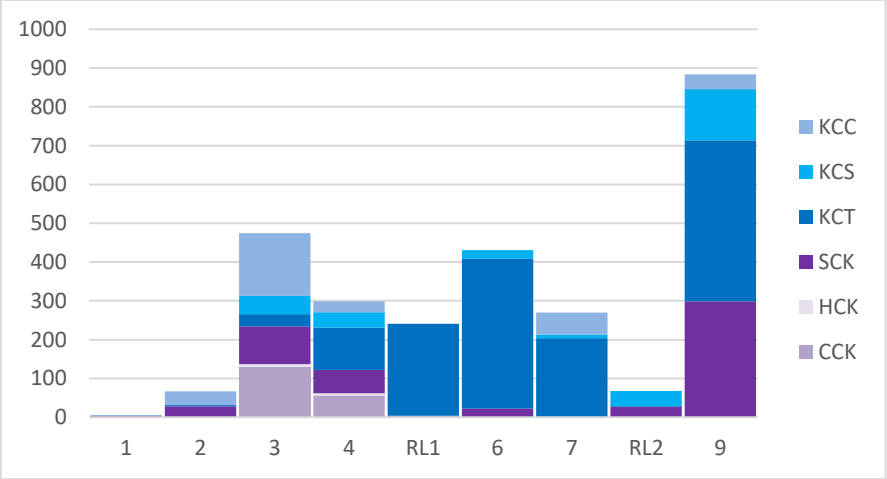

Fig. 6 MKT expressed across all meetings

MKT X Levels

Taking the lesson study cycle as a unit of investigation, we examined the occurrences both of MKT and Levels of Teacher Activity throughout (see **Fig. 7**). This analysis provides a holistic picture of the type of knowledge, at a particular level of teacher activity, which was evident in the case study cycle.

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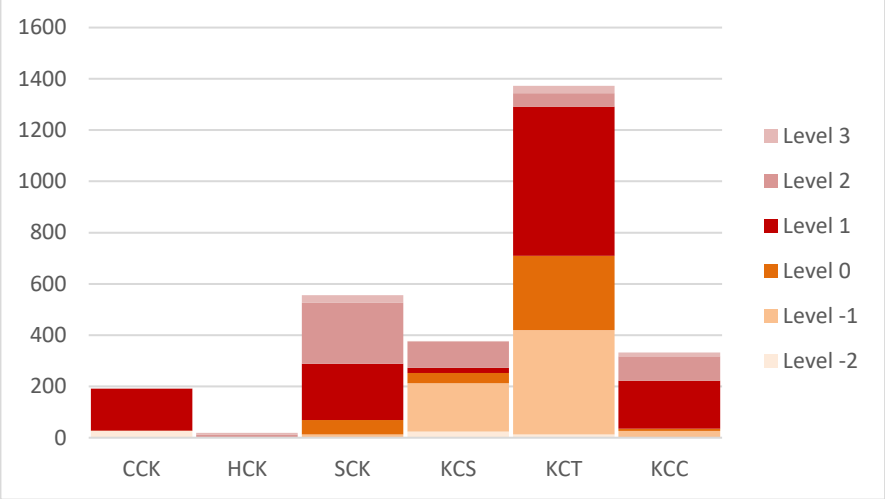

Fig. 7 MKT expressed according to Levels of Teacher Activity over one cycle of lesson study.

From **Fig. 7** it is clear that KCT was the most prevalent type of knowledge in teachers' collective participation in lesson study. It is interesting to observe that this type of knowledge was utilized at almost every level of teacher activity across the lesson study phases. As might be expected, teachers' noticing of student thinking (level -1) was prevalent in their utilization of both KCT and KCS during the cycle. It is also interesting to illustrate that teachers saw mathematics through the eyes of their students (level -2) when drawing on their KCS.

During their participation in lesson study, teachers reflected (unprompted) on how their participation in this collaborative form of professional development impacted on their pedagogical practices outside of lesson study. In the example below, Valentine, in phase 1, reflected on linking mathematical knowledge to a specific task (KCC) and indicated that she was employing changes to her choice of activities in her teaching practices outside of lesson study.

Valentine (T)	Maybe participating in this lesson study cycle, maybe it has changed my… I mean it has modified some approaches in my teaching, on reflection, in the subject. I'm thinking of other things.
Other teacher	Hum. Like what?
Valentine (T)	Well, for example, it's a bit like what Edith said, I think of different perspectives. I do extra activities. For example, "In Pieces", which is an activity in the book, I did it a second time. I'm doing my usual things, but I'm also trying to visualize more. I am aware… well… I am more attentive to some, ah, to some of the difficulties that I wouldn't have noticed before.

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Discussion & Conclusion

While lesson study is increasingly practised around the world, there have been calls within the mathematics education and lesson study research communities to develop the theoretical underpinnings of teacher learning in lesson study. In this chapter, drawing on a case study of eight generalist primary teachers participating in a cycle of lesson study, we have attempted to provide a detailed account of the knowledge incorporated by teachers in their work. Utilizing a framework combining Mathematical Knowledge for Teaching (Ball et al., 2008) and Levels of Teacher Activity (Margolinas et al., 2005) we have analyzed teachers' participation in the cycle, through their conversations and notes, and provided examples of the occurrences of these types of knowledge throughout the phases, meetings, and cycle of lesson study. In this work, we have built on previous analysis (Ni Shuilleabhain and Clivaz, 2017) and attempted to provide further fine-grained analysis of the knowledge incorporated by teachers in their lesson study work.

Several graphical representations in this chapter demonstrate the repartition of each of the components of MKT and Levels of Teacher Activity across a cycle of lesson study. Our research demonstrates that all levels of teacher activity, from the values and conceptions about learning and teaching to seeing mathematics through the eyes of the student, are afforded opportunities of articulation in teachers' participation in the collaborative work of lesson study (see **Fig. 2**). Analysis of the data also evidenced the presence of all categories of MKT across the phases of the cycle, particularly those of KCS and KCT (types of pedagogical content knowledge) and SCK (a form of subject matter knowledge) (see **Fig. 3**). Combining both frameworks, our data evidenced a prevalence of KCT within the lesson study cycle (see **Fig. 7**), which may support other research findings which have demonstrated changes to teachers' classroom practices as a result of their participation in lesson study (for example Goldsmith et al., 2014, Batteau, 2017, Ni Shuilleabhain and Seery, 2017, Takahashi and McDougal, 2017). Furthermore, **Fig. 2**, **Fig. 3** and **Fig. 7** depict the benefit of participation in lesson study, where teachers have opportunity to utilize almost all elements of their MKT across each phase of the cycle and across all levels of teacher activity. An advantage of

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participating in lesson study may be the fact that it encourages teachers to incorporate, draw on, and potentially develop their knowledge at various levels by explicitly articulating their knowledge through active participation across each of the phases.

Within our findings, it is interesting to note the distinct lack of occurrences of teachers' horizon content knowledge (HCK) within our case study data (see **Fig. 7**). This may be due to the fact that within this cycle of lesson study, there was no knowledgeable other (Takahashi, 2014) distinct to that of the facilitator, who joined in the lesson study cycle. The knowledgeable other often articulates future pathways of students' learning and guides teachers' thinking beyond that of the research lesson, thereby potentially highlighting future pathways of students' mathematical knowledge. Speer et al. (2015) suggest further research to investigate HCK as a distinct element of MTK. Such research is likely also necessary to explore the presence of HCK as part of teachers' participation in lesson study.

In analyzing the types of knowledge occurring across each of the lesson study meetings, our research demonstrates that the phases of lesson study do not necessarily occur in a strict chronological or sequential order, but rather take place at varied points throughout the cycle. This may be an important finding in facilitating and analyzing lesson study, where teachers can articulate goals, student learning, or subject topics at all point throughout their lesson study conversations. This articulation and the way lesson study propels and compels teachers to express this professional knowledge, is worth further theorization in order to sustain lesson study as a professional situation where teachers can develop their knowledge in and for teaching.

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Appendix

The indicators are *in italics*. The codes are in color, corresponding to Figures 2, 3, 5 and 7

MKT

SMK	CCK	*Performing mathematical task*
		Use of notations and vocabulary
		Determining if a solution, a definition, a representation… is correct
	HCK	*Considering other uses of a mathematical knowledge*
		Considering later purpose of a mathematical knowledge
	SCK	*Looking for patterns in student errors*
		Sizing up whether a nonstandard approach would work in general
		Unpacking of mathematics
		Understanding different interpretations of a concept/techniques appreciating the differences
		Talking explicitly about how mathematical language is used
		Choosing making and using mathematical representations effectively
		Explaining and justifying mathematical ideas
		Analyzing/building examples having mathematical characteristics
		Determining if a mathematical concept or rule is a convention or a mathematical necessity
PCK	KCT	*Sequencing mathematical content*
		Identifying or developing learning activities
		Selecting models, representations, examples, and procedures that support the development of mathematical understanding
		Anticipating/analyzing teacher's reaction to students' response or difficulties
		Anticipating/analyzing teacher's actions in relation to mathematical content

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	Sharing or comparing representations and procedures in teaching
	Selecting appropriate mathematical language, analogies and metaphors
KCS	*Identifying students' knowledge or learning*
	Identifying students' difficulties or misconceptions
	Anticipating students' mathematical responses
	Noticing and interpreting the mathematical meaning associated with students' responses
	Choosing an example that students will find interesting and motivating
	Selecting questions and tasks that seek out the presence of misconceptions
KCC	*Linking mathematical knowledge to the syllabus (maybe implicit)*
	Linking mathematical knowledge to a specific task available (in textbook or...)
	Lateral curriculum knowledge
	Vertical curriculum knowledge

LS phase

0 Consider issues and formulate general goals	*In/for student learning and development*	
	In/for teaching	
	In/for teacher's professional knowledge	
1 Study curriculum and formulate content specific goals	Consider learning of the topic	*Identify topic of interest*
		Formulate goals for student learning specific to the topic
		Discuss a learning trajectory related to the topic through grades
	Identify/analyze specific difficulties.	*In teaching*
		In student knowledge or learning
		In content
	Study curriculum, standards and material	*Study course of study, standards...*
		Study textbook, specific task, manipulative...
		Link topic to other topics
		Read and reference research literature

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2 Plan	Select (or revise) content	*Select (or revise) research lesson*
		Select (or revise) sequence of lessons
	Consider elements of the research lesson	*Long term goals*
		Learning objectives
		Model of learning trajectory
		Rationale for chosen approach
	Detail the conduction of the lesson	*Anticipated student thinking*
		Anticipated teacher actions
		Incorporating resources (tasks, material)
		Data collection plan
3 Do research lesson	*Conduct research lesson*	
	Observe and collect data	
4 Reflect	Use the data to illuminate	*Student actions*
		Student learning
		Teacher actions
		Disciplinary content
		Lesson and unit design
		Reflect on curriculum
		Broader issue in teaching-learning
	Documentation of cycle	*Consolidate and carry forward learning*
		New questions
	Reflect about other teachings of the research lessons	

Level of teacher activity

3 Values and conceptions about learning and teaching	*Educational project: educational values, conceptions of learning, conceptions of teaching*
2 The global didactic project	*The global didactic project, of which the planned sequence of lessons is a part: notions to study and knowledge to acquire*
1 The local didactic project	*The specific didactic project in the planned sequence of lessons: objectives, organization of work*
0 Didactic action and observation	*Observation of teacher's didactic action*
	Interactions with pupils, decisions during action

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-1 Observation of pupils' activity	*Perception of pupils' activity, regulation of pupils' work*
-2 Student critical lens	*Articulation of teaching and learning from a student's perspective*

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