International Journal of Science and Mathematics Education
https://doi.org/10.1007/s10763-023-10420-8

Low-Attaining Secondary School Mathematics Students' Perspectives on Recommended Teaching Strategies

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Received: 13 March 2023 / Accepted: 24 September 2023

Abstract

Recent research syntheses have identified several potentially high-leverage teaching strategies for improving low-attaining secondary school students' learning of mathematics. These strategies include the structured use of representations and manipulatives and an emphasis on derived facts and estimation. This paper reports on 70 semi-structured interviews conducted with low-attaining students in Years 9–10 (ages 13–15) in England. The interviews addressed the students' perceptions of learning mathematics and the teaching strategies that they experienced and believed were most helpful. Many students reported rarely using number lines, not spontaneously estimating answers and being unfamiliar with derived facts. During the interviews, with minimal direction, students often showed that they were well able to make use of these strategies; however, they did not report making spontaneous use of them independently. We conclude that many of the most well-evidenced and recommended strategies to support low-attaining students in mathematics appear to be unfamiliar and unvalued, and we discuss how this might be addressed.

Keywords Derived facts · Estimation · Low attainment · Manipulatives · Representations · Research-informed practice · Student perspectives

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Published online: 12 October 2023

Introduction

Despite numerous attempts over recent decades to address the issue of low attainment in lower secondary school mathematics, it remains a serious and even growing problem (e.g. Hodgen et al., 2022; Marshall, 2013; Organization for Economic Co-operation and Development [OECD], 2013; Shayer & Ginsburg, 2009). Low attainment has been a longstanding concern of government policy in England (Foster 2022; Hodgen et al., 2022), particularly because of the importance of mathematics to the economy, as well as to individuals (e.g. Layard et al., 2002; Vignoles et al., 2011). Since the 1970s, the percentage of very low-attaining students at age 14 has approximately doubled, and this group now constitutes about one-sixth of the Year 9 (ages 13–14) cohort (Hodgen et al., 2010, 2012). These students have difficulty with questions based on concepts fundamental to the *primary* (i.e. elementary) school mathematics curriculum, and during the first 3 years of secondary school (ages 11–14) the gap between the lowest- and highest-attaining students in mathematics increases.

Recent research syntheses (e.g. Hodgen et al., 2018) have identified several specific teaching strategies that could be potentially high leverage for improving the learning of low-attaining secondary school mathematics students, which have been communicated in a high-profile guidance report for teachers (Education Endowment Foundation, 2017). These 'best bets' include the structured use of representations and manipulatives (e.g. Carbonneau et al., 2013), derived facts (e.g. Dowker, 2009) and estimation (e.g. Dowker, 2015). The role of the teacher, and the particular teaching strategies that they choose to adopt, is always critical in enabling students to learn mathematics effectively (e.g. Ball et al., 2008; Watson et al., 2003), but this is especially so for lower-attaining students. While in recent years there seems to have emerged an increasing desire among schools and teachers to adopt evidence-informed practices in the classroom (e.g. Christodoulou, 2014; Willingham, 2010), what this might mean in practice, and how it is experienced by low-attaining students in mathematics, remains largely unknown.

In this paper, we report on 70 semi-structured interviews which we conducted with low-attaining students in Years 9–10 (ages 13–15) in England as part of the *Low attainment in mathematics project*. In the interviews, we asked the students about their perceptions of learning mathematics in school, what they thought they found easy or difficult and the particular teaching strategies that they experienced. We specifically asked students about their experiences of using the potentially high-leverage strategies identified in the review and promoted in the guidance report: representations and manipulatives, derived facts and estimation. Our analysis of this interview data addresses the question: *What are low-attaining students' perceptions of learning mathematics, and in particular how do they perceive teaching strategies that involve representations and manipulatives, derived facts and estimation?* Understanding students' perspectives on their experiences of these strategies is important in considering how the strategies might be effectively implemented in classrooms, so as to improve low-attaining students' experiences of school mathematics and progress in the subject.

Teaching Strategies for Low-Attaining Students in Mathematics

There is limited detailed evidence in the research literature concerning how low-attaining students in England are currently taught, particularly regarding specific teaching strategies, and many schools continue to focus their teaching of low-attaining students on remedial mathematics presented procedurally (Connolly et al., 2019). Scherer et al.'s (2016) review of the research literature on interventions directed at addressing mathematical learning difficulties found consistent evidence for the benefits of direct and explicit instruction, but direct instruction was defined in different ways by different authors. To understand in more depth how and why particular approaches may be more or less effective than others, it is necessary to focus on particular strategies that teachers employ in their teaching of low-attaining students and to examine how they operate, as well as how they are experienced in the classroom. A recent research synthesis of evidence concerning effective strategies for teaching low-attaining students in mathematics indicated the importance of addressing students' specific mathematical weaknesses, through appropriate use of explicit instruction, concrete manipulatives, representations and worked examples (Hodgen et al., 2018). This synthesis was developed into a high-profile guidance report, presenting these strategies practically for teachers and recommending their use (Education Endowment Foundation, 2017).

In this paper, we focus on three recommended strategies presented in these documents (Education Endowment Foundation, 2017; Hodgen et al., 2018) that show promise as potentially high-leverage strategies for teaching low-attaining students in mathematics: derived facts, computational estimation and the use of representations, particularly number lines. These are more than simply 'calculation strategies': they are strategies that the teacher can use to help students build a deeper and more memorable understanding of concepts. Based on the best-available research, these reports recommended these strategies as among the 'best bets' for improving the learning of low-attaining students in mathematics. They are likely to support *threshold concepts*, meaning concepts that dramatically transform how a student thinks, and consequently are key to further progression (Meyer & Land, 2006a, b). We briefly summarise here the evidence base behind these recommendations.

Gersten et al.'s (2009) review indicated that improving arithmetic (fact) retrieval is essential to supporting students who struggle with mathematics (see also Geary, 2011). However, arithmetic (fact) retrieval is not simply a matter of a set of associations that are learned by rote, and Baroody (1999) highlighted the importance of developing what he termed 'covert non-retrieval strategies'. Key to this is the use of derived facts (i.e. using a known fact, such as $8+8=16$, to solve a related problem, such as $8+9=16+1=17$; Dowker, 2009; Gaidoschik et al., 2017; Gray & Tall, 1994). Star et al. (2009) argued that computational estimation is critical to the development of number sense and fluency (see also Baroody, 1999; Dowker, 1992, 2015; LeFevre et al., 1993). Ruthven (1998) described how:

> pupils had been encouraged to develop and refine informal methods of mental calculation from an early age; they had been explicitly taught mental methods based on 'smashing up' or 'breaking down' numbers; and they had been expected to behave responsibly in regulating their use of calculators to complement these mental methods. (Ruthven, 1998, pp. 39-40)

By seeing how one number fact is related to—and may be derived from—another, students build a deeper understanding of numbers and their properties.

Hodgen et al.'s (2018) review supported the benefits of using visual representations (e.g. Gersten et al., 2009) and concrete manipulatives (e.g. Carbonneau et al., 2013) with low-attaining students. However, to be effective, teaching needs to explicitly highlight the mathematical ideas represented by these tools. Particular attention has focused on the use of number lines to help students to develop their understanding of numerical magnitude (Schneider et al., 2018) and the sense that rational numbers are 'numbers' that extend the whole number system (Siegler et al., 2010; see also Bartelet et al., 2014). Number lines, whether hand-drawn or printed on the page, or embodied in concrete, physical manipulatives (cubes, Cuisenaire rods or washing lines), offer a powerful way to visually embody both cardinal and ordinal aspects of number (Hodgen et al., 2018).

Consequently, in our semi-structured interviews with low-attaining students, as well as discussing their broader perceptions of learning mathematics, we focused specifically on students' experiences and awareness of these particular strategies and their use. We sought to gain insight into if and how low-attaining mathematics students experience these strategies, while aiming to keep the interviews open enough to gain broader insight into their perspectives on learning mathematics, so as to provide them with the space to give voice to what they perceived to be important.

Method

We conducted 70 semi-structured interviews (Kvale, 2008) with low-attaining students in Years 9–10 (ages 13–15), in parallel with a set of interviews with their teachers (Foster et al., under review). We sought to discover the students' perceptions of learning mathematics, what they found easy or difficult and the teaching approaches that they reported finding helpful. More specifically, and emanating from the literature outlined above, we were interested in the extent to which the students would recognise, and report using, different representations, particularly number lines and arrays. We also wanted to know whether students would show awareness of derived facts, an ability to use them and an understanding of when these might be useful; in particular, whether they might use them spontaneously or with prompting. Finally, we wanted to know what understandings students might display regarding ideas of estimation or approximation, and how they might handle requests to give estimated answers.

Participants

We approached five secondary schools, all of which were willing to participate, and asked them to identify low-attaining students from Years 9–10 (ages 13–15) for the interviews. We defined low-attaining as the lower 40% of the school cohort, which although broad represents the group of students who typically do not go on to obtain the gatekeeping 'level 4' at General Certificate of Secondary Education (GCSE) mathematics, the standard national qualification taken at age 16.

We asked schools to select students available and willing to speak to a researcher. We chose to interview older students within compulsory secondary schooling (ages 11–16), as we expected these students to have more experience of secondary school mathematics to reflect on, and perhaps to be more confident speaking to an unfamiliar interviewer. We did not ask for students from Year 11 (age 15–16), as this is the year of the GCSE public examination, and schools would be reluctant to release students in this year group from their normal activities.

We interviewed a total of 70 students (see Table 1). Each interview lasted about 25 min and was held in a quiet room on the school site, with just the student and the interviewer present, and each interview was audio recorded. Any student written work produced during the interview was collected and scanned. None of the interviewers, the authors of this paper, had a pre-existing relationship with either the school or with any of the individual teachers or students. The head teacher, mathematics teachers, students and their parents were provided with information sheets describing the research project, which they read before we sought their informed consent to participate. Ethical approval for the project was obtained from the University of Nottingham Research Ethics Committee.

All of the participating schools were mixed-gender comprehensive schools in England (i.e. non-selective and non-fee-paying), and all of the schools used setting for mathematics. All of the schools also provided *teachers* for interview, who were currently teaching a lowest set mathematics class within the same age range, and were experienced in doing so. The analysis of these teacher interviews is reported elsewhere (Foster et al., under review) and the labelling of the schools in Table 1 matches that used there.

Interview Protocol

The protocol was devised to elicit students' perspectives chiefly on four issues: three of the key areas arising from our literature review (representations and manipulatives, derived facts and computational estimation), together with students' perceptions of learning mathematics. We asked what students thought that they found easy or difficult and the teaching approaches that they reported finding helpful. Beginning

Table 1 Interview participants

School	Age range	School type	Location	Number of students interviewed
A	11–16	Community college	Midlands	41
B	11–16	Church of England academy	Midlands	2
C	11–16	Academy	Midlands	13
D	11–18	Free school	London	11
E	11–18	Academy	London	3
			TOTAL	*70*

with these broader questions was intended to help settle the students at the start of the interview and enable them to feel comfortable.

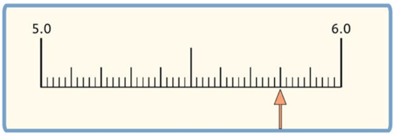
Each student was interviewed individually by one of the authors, who collaboratively developed the protocol in detail together across a series of planning meetings. The authors began the interview process by observing several of each other's interviews, in order to ensure a consistent approach. Not all aspects of the protocol could be covered with every student, due to time constraints, but the interview materials used were structured in the four sections described below. The complete interview protocol is provided in the Appendix, and the mathematics questions used for discussion are given in Fig. 1. We did not use all of these questions with all students, partly due to lack of time, but also to ensure that the interview experience was a positive one for each student and students did not risk experiencing repeated 'failure'. In particular, we ensured that each interview ended with student success and that the student felt comfortable about the experience when leaving.

The interview was semi-structured, with an overall structure comprising four sections:

Perceptions of Learning Mathematics

After asking students for their consent to take part in the interview, we stated that we were interested in how they learn mathematics, what they found easy or hard and what helped them to learn. A pen and blank paper were available throughout the interview for the student to write on whenever they wished. We had available some mathematics questions (Fig. 1), which we sometimes drew on in the later sections of the interview.

1. What number is the arrow pointing to?



2. What is 10 more than 3597?

3. Look at this calculation: $86 + 57 = 143$

Find a quick way to work out the answer to $85 + 57 = ?$

4. Look at this calculation: $15 \times 24 = 360$

Find a quick way to work out the answer to $16 \times 24 = ?$

5. Which number is nearest in size to 2.9 multiplied by 7?

0.002   0.02   0.2   2   20   200   2000

Fig. 1 Mathematics questions for discussion

We began by asking students what they were currently doing in mathematics and how it was going. If students had their mathematics notebook with them, they were invited to show us. Additional questions were used to encourage them to talk about their broad perceptions of learning mathematics:

> Can you tell me about a maths lesson where you feel that you really learned something? What was it? Why was that lesson successful? What did the teacher do?

The intention was to capture at the start of the interview what students felt to be important in their mathematics lessons and elicit their perceptions and what approaches they felt were more or less effective for them.

Representations and Manipulatives

We asked students to talk about different representations of number that they might have used in mathematics lessons. Since the word 'representation' might not be clear or familiar to them, we asked questions such as:

> If I asked you to draw me a picture to show me the number 8, what would you draw? Can you tell me about it? Is there anything else you could draw to show the number 8?

We specifically asked about arrays and number lines, as well as practical equipment such as cubes, asking in what topics they had been used and for students to give us examples of how they were used. We were particularly interested in whether students ever made autonomous *choices* to use these representations or manipulatives for themselves, without being explicitly directed to do so by the teacher. We asked some students to answer questions 1 and 2 from Fig. 1, talking aloud about how they approached them.

Derived Facts

We also asked students about derived facts. Again, we did not necessarily expect students to be familiar with this term. We dealt with this in two ways. First, by asking students to answer questions 3 and 4 from Fig. 1, again inviting them to talk aloud about how they approached them. Second, and in order to address *multiplicative* derived facts, we used the context of multiplication tables, asking:

> Which times tables do you know?

Students were invited to indicate by pointing on a blank 12×12 multiplication grid. We then asked:

> Do you just know them or do you work them out? If you work them out, how do you do it?

For students who did not report using derived fact approaches in this context, we selected one multiplication table fact that they said that they knew, and asked:

> Can you work out any other times tables – ones you don't know – by starting with this one?

If the student was hesitant in responding, we would select an example that we did not expect them to be able to answer immediately. For example, if they had said that they knew $5\times7=35$, we might ask them how they could work out 6×7, or 15×7, or 50×7, starting from $5\times7=35$.

We asked students whether they remembered being taught to do this kind of thing, or whether they reported ever choosing to use derived facts for themselves, without being directed to do so by the teacher.

Computational Estimation

We asked students what they understood by the words 'estimate' or 'estimation', and whether they ever estimate when the main topic being studied is *not* 'estimating'; for example, to predict or check their answers. We sometimes asked students to answer question 5 (Fig. 1) in order to probe further their understanding of estimation.

Analysis

All of the audio recordings were transcribed in full, and the student written work was scanned and filed. The analysis was conducted following what Braun and Clarke (2022) described as a *deductive* approach, in which the coding and themes are largely directed by existing concepts and ideas.

Results

We now report our findings under the four themes that comprised the organising structure of our interview protocol, three of which were based around the strategies which emerged from the literature as potentially high leverage (Hodgen et al., 2018) and led to recommendations for teachers (Education Endowment Foundation, 2017). Throughout our presentation of results, we include specific quotes from students to illustrate the themes, referencing the individual students quoted by using the letter of the school (as in Table 1) followed by a number to distinguish the particular student at that school (e.g. 'C2' to indicate school C and the 2nd student we refer to).

Perceptions of Learning Mathematics

Many students reported a generally positive outlook on their school mathematics lessons, although they mostly described finding mathematics hard. Most seemed to like their mathematics teacher, and they were particularly appreciative of the large amount of one-to-one support available in their generally small (low set) classes. However, a few complained that it was difficult to obtain as much of the teacher's time as they would have liked, because the teacher was frequently occupied with other students.

Many students said that the most useful aspect of individual help was the provision of longer, more detailed explanations than those that the teacher had provided to the whole class, and some students said that they found support from a teaching assistant or peers to be more helpful than that from the teacher. Many students said that they liked methods to be broken down into small steps and said that they wanted to be provided with lots of examples:

> C1: Erm so like our teacher is like teaching us in steps, like how to do it in like easier ways.
> INT: And does that help, having it in steps?
> C1: Yes.
> INT: Why does that help?
> C1: Because like instead of doing it like in a complicated way, and take forever in just doing it in an easy way like easy.
> INT:So you like things to be as easy as possible, teacher make it as easy as possible?
> C1: Yes.

However, other students said that they were eager to get on with doing the task and did not want too much teacher talk:

> D1: But all I like is like as soon as we get into the work. Like when the teacher like explains it a little bit, and like not too much... I don't want her talking too. I just want a little bit of explaining so we can get onto it and then get on from there.

When asked which topics were difficult, students' responses seemed to be influenced by topics that had been encountered recently. However, as would be expected from the literature, many students highlighted algebra as being especially difficult and, in some cases, they saw the topic as pointless.

Several students expressed frustration with forgetting:

> INT: And so I'm interested in which topics and which things in maths are tricky and difficult … What do you think of as being quite hard in maths?
> A1: It depends. Erm ... sometimes just like I get it at first and then when you don't go over it I completely forget like factors, or stuff like that.
> INT: And what is it that you would forget about something like factors?
> A1: How you like separate the factors, like... it's hard to explain.
> …
> A1: Like I would get it at first, and then like if you don't go over it probably like in about two weeks or something I'll completely forget about it.
> INT: Yes. And so what works for you in terms of dealing with that?
> A1: Probably repeating.

The issue of forgetting, which emerged from our analysis of these perceptions, seemed important to students, given that it was mentioned frequently. Indeed, it may be noted that the other three themes, in their different ways, address strategies that teachers could use in order to attempt to make learning more meaningful and, consequently, memorable for students (Education Endowment Foundation, 2017).

Representations and Manipulatives

When asked how they might represent the number 8, some students drew 8 dots or used tally marks, and one student drew a front door to a house with a number 8 on it. When asked for examples of diagrams or pictures that they might use in mathematics to represent numbers, many suggested area representations for fractions, either using circles or rectangles. Some mentioned common methods of layout for standard algorithms, such as the 'bus stop' for division, and others mentioned bar charts, Venn diagrams and line graphs. Several were positive in general about the value of representations:

INT: Are the pictures useful? Are they helpful?
C2: Yes they help me to … to understand a problem … Because you see on the … the diagram you see it … when you see a picture you understand.
A2: I find them [pictures] helpful when I am struggling with the work.

We specifically asked the students about number lines and arrays, and we also report below our observations on students' use of fingers for calculation, which is something we noticed repeatedly during the interviews and their analysis.

Number Lines

Most students reported that they did not use number lines much. Many students said that they did not like them, either finding them too difficult and confusing, and too slow to draw, or else believing that they were too simple, and that they no longer required them. Many reported using number lines at primary school, and at secondary school only using them in the context of directed (positive and negative) numbers.

C3: I don't really like to use them [number lines].
INT: You don't like to use them, why is that? Are they…?
C3: No, I don't know because I find it kind of confusing.
INT: Right you don't find them helpful?
C3: Yes I don't find it helpful.

Many students struggled to answer question 1 (Fig. 1) or did so only with considerable assistance. Very few students approached question 2 (Fig. 1) by means of a number line and bridging through 10, with most opting for a standard column addition algorithm.

Arrays

Few of the students were familiar with the word 'array'. On showing them an example, some said that they had not seen such things; others that they had used them "a really long time ago … in Year 3" (A3), generally for division. Most said that they did not use arrays, other than for calculating areas by counting squares. Reasons given were that they take too long to draw and were perceived to be unnecessary, but it seemed that some students were unfamiliar with possible uses of arrays for visualising distributivity.

Fingers

Many students were observed to use their fingers during the interviews for various calculations, often with an apparent attempt to conceal this from the interviewer. (Andres and Persenti (2015) indicated the benefits of finger counting.) When asked about using fingers, students often stated that they preferred using their fingers to alternative representations, such as a number line. Occasionally, 'fingers' referred to shortcut methods, such as an instant way to see the tens and ones digits of the multiples of 9 by folding down a finger corresponding to which multiple of 9 was required. However, generally the finger strategies observed were coded as counting-all, counting-on or counting-back. Some students were very reliant on fingers to carry out most of the calculations undertaken in the interview, and saw fingers as a way to be *sure* of the correct answer:

INT: But how would you kind of work it out so you were sure what it was?
A3: Erm I would do it on my fingers.

It seemed that several students had become stuck in inefficient methods from the past, which were perhaps never intended to have remained as their principal method:

INT: If you have got that method why do you bother with your fingers?
D2: I don't know … It is just ... I used to do it in primary school.
INT: Yeah sure.
D2: I just keep doing it.

Derived Facts

When presented with question 3 (Fig. 1), most students proceeded to calculate $85+57$, usually by using a standard column algorithm, without reference to the *given* answer to $86+57$. When their attention was drawn to the instruction to 'Find a quick way', or students were asked explicitly whether they could use the given result to help them, some students were then able to do so, with increasing confidence as subsequent, similar problems were presented. However, even with considerable prompting, many students found it very difficult to see that they could modify the given result by subtracting 1. This appeared to be an unfamiliar style of question for most students, with students occasionally seeming unsure about whether they could assume that the given statement was correct. On being reassured about this, several students quickly made the intended deduction:

A1: You do 80 + 50, and then you add the 2, 5 and 7 to it.
INT: Okay. Yeah you could do that. What about this calculation they give you the answer to, why do you think that's there?
A1: To check if we've done the process right.
INT: How does it help you check?
A1: Well you could do 86 + 57 and see if that answer's right.
INT: I think they want you to *assume* that's right. They're telling you that and they want you to use that to help you get the answer to this one.
A1: Oh… Yeah, you just add one more. It'd be 144.
INT: So why are you adding 1?
A1: Wait, you take away 1. You take away 1.
INT: Okay. Why are you taking away 1?
A1: Because that's 86 and that's 85.

As would be expected, question 4 (Fig. 1), incorporating distributivity, was considerably more difficult for students than question 3, and very few were able to succeed with this without considerable assistance.

When asking students about the multiplication facts that they knew, and how they would work out ones that they did not know, most students resorted to repeated addition, skip-counting up from zero, instead of starting from a nearer known fact. Overall, students showed a lack of flexibility, and while their repeated addition methods were sometimes accurate, they were slow. Some students acknowledged this but stated that they had more confidence in their slow methods:

A4: I don't know, it's just easier for people to do [count up in 6s, 12 times], than do it the way that they might get it wrong … But at least they get it right than doing it the way they don't know.

Computational Estimation

Students displayed different understandings of the term 'estimating'. Some equated this with rounding to a particular degree of accuracy, such as 1 decimal place:

INT: Can you give me an example of when you have done it [estimating]?
D3: Erm we count rounding as estimating … we are doing rounding, erm doing that instead of fully working the answer out.
INT: Can you give me an example?
D3: Like if we had 47 for example, we have to estimate how.... we have to round it up or down and since it is above 5 we will go up, but anything below 5 we go down.
INT: So that's rounding 47. Do you do anything with that or is that it
D3: That gives you 50.

In many cases, students' difficulties with rounding and place value meant that the answers that they obtained from rough calculations were several orders of magnitude away from the correct value. When tackling question 5 (Fig. 1), for example, one student responded that the answer to 2.9×7 would be about 0.002, because "It had a dot in it" (A5):

INT: So you picked it because it had a dot in it?
A5: Yeah.
INT: Mhm. Is it the only one with a dot in it?
A5: No. These two as well.
INT: Okay. So why did you pick this one?
A5: Probably has something to do with two point nine?
INT: Why?
A5: Because… I'm not sure.
INT: Okay! And why did you want to pick one with a dot in it?
A5: Because it has like more zeros in it. And it probably has something to do with the question.
INT: Because the number... One of the numbers in the question has a dot in it?
A5: Yeah. And it probably has more zeros… with the final answer.

Other students saw estimating as making a 'wild guess' or using everyday knowledge of typical sizes, checking an answer and having a gut feeling as to whether a value was too big or too small for a given calculation. One student related estimating to probability, where answers were perceived as uncertain.

Several students said that they did *not* estimate, or did not like to, and those that said that they did estimate were sometimes apologetic about it, seeing it as inferior to an exact calculation, and something to be done only as a last resort:

D3: Just when I'm lazy I will estimate … I'll attempt the calculation but then I'll probably end up estimating.
INT: Why would you do that?
D3: Because I might end up getting the calculation wrong and then to understand why I had then ... I will just go to the conclusion, so you just estimate … As I see it is better than not leaving a question blank than having nothing at all there, because there is a chance that you still get it right.

There was a strong preference among many students for attempting to calculate exactly rather than estimating, and many students did not seem to believe that the answer to a rough calculation should necessarily be at all close to the exact answer.

Discussion

Most of the 70 low-attaining students interviewed in this study reported enjoying their mathematics lessons and valuing their mathematics teacher and teaching assistants. The students mostly displayed a positive affect for the learning of mathematics, despite the difficulties which they appeared to experience across all of the areas of mathematics addressed in the interviews.

Many of the most well-evidenced strategies recommended for use with low-attaining students, such as derived facts and estimation (see Dowker, 2009, 2015; Gray & Tall, 1994) and representations and manipulatives (e.g. Carbonneau et al., 2013), did not appear to be familiar to the students. It seems that the students were not making use of many approaches and strategies which research suggests would be potentially beneficial to them in their learning of mathematics. For example, most students reported that they did not use number lines very often, and many expressed a dislike for them, sometimes suggesting that they viewed such representations as more suitable for younger-age students than for themselves. Instead of performing calculations by visualising empty number lines, the students interviewed tended to rely on standard written algorithms, which they did not always perform accurately, and which were always slow. Sometimes the students' strategies led to answers that were orders of magnitude different from the correct answer, without them noticing. This might be less likely to happen with approaches emphasising computational estimation (e.g. Baroody, 1999) and/or visual approaches involving number lines, in which the approximate magnitude of the numbers is more salient (Schneider et al., 2018; Siegler et al., 2010).

In a similar way, many of the students did not appear to recognise the power of using derived facts to obtain an unknown answer from a related known fact (Dowker, 2009; Gaidoschik et al., 2017; Gray & Tall, 1994). In the interviews, when students did not appear to be familiar with this process, it was briefly explained to them, and this often resulted in them immediately being able to use it successfully. It is not possible to say whether the students had been previously exposed to this approach, and had forgotten it, or whether this was something new to them. We do not have data on whether the students would have or should have encountered these strategies before in school. The students we interviewed had been taught by multiple different teachers over their secondary school years and at a wide variety of different feeder primary schools, meaning that it would not be possible to ascertain the range of different strategies and practices that had been introduced and used at different times. Therefore, we can report that students do not appear to recognise a strategy, or that they report never having seen it, but we cannot speculate on whether they might indeed have encountered it previously in some form.

It may be that the students' experience is that they had been shown over many years a multitude of different strategies, but many of these strategies were not processes that they had available to call on and use reliably in any particular circumstance. It might be the case that focusing teaching for low-attaining students on fewer, high-leverage strategies, thereby allowing these to be experienced more deeply and rehearsed to a high level of performance, would provide the students with a much greater overall level of power to make sense of and tackle mathematical problems (Foster, 2022).

Several students seemed highly reliant on using their fingers for number calculations, or on slow, skip-counting-up or counting-on procedures, and some expressed a preference for doing this in high-stakes contexts, where they wished to be sure that their answers were *definitely* correct. The extra time which low-attaining students are often entitled to in high-stakes examinations may result in these inefficient approaches not being penalised, and perhaps therefore this has the unintended consequence of trapping students in laborious methods that may not be conducive to their subsequent progress. Helping students to move to more efficient strategies, and supporting them to move on from approaches that they may wish to cling to, because they have previously experienced success with them, would seem essential if students are to succeed with more demanding mathematical tasks. This may be difficult for the teacher who wishes to be sensitive to their students and to value the methods that they bring and appear to be comfortable with. Challenging students to move away from these methods and onto more efficient ones can be a fraught process that teachers might wish to avoid. The tendency to 'support' and 'scaffold' low-attaining students' learning may sometimes unintentionally lead to the dominance of strategies that students might be better served by 'moving on' from.

Finally, it is noteworthy that estimation did not appear to be a strategy that students found powerful in supporting their arithmetic work. In our parallel interview study with teachers of low-attaining students (Foster et al., under review), we found that teachers believed estimation to be an extremely valuable skill for their students, and something that they reported emphasising in their lessons. However, the teachers also reported that their students (both high-attaining and low-attaining) were very reluctant to estimate in calculations, perceiving an estimate to be inferior to an exact answer. We also see that phenomenon in this student interview data, and yet it would seem that the ability to perform estimates would be of potentially *particular* benefit for low-attaining students. Low-attaining students often struggle to obtain accurate answers to mathematical questions, and efficient self-checking and self-correcting mechanisms could be highly advantageous. Devising effective ways of teaching estimation skills to low-attaining students would seem to be an urgent priority, both for success in school but also for helping students to develop robust number sense for everyday life.

Conclusion

We have seen in the interviews with low-attaining students reported in this study that many of the most well-evidenced teaching strategies that are recommended for use with low-attaining students (Education Endowment Foundation, 2017; Hodgen et al., 2018), such as representations and manipulatives, and derived facts and estimation (e.g. Carbonneau et al., 2013; Dowker, 2009, 2015; Gray & Tall, 1994), do not appear to be familiar to them, and certainly are not automated to a level where they are reliable tools to use. Consequently, it is not surprising that students did not express a fondness for these strategies or a tendency to use them autonomously.

There would seem to be an urgent need for carefully designed classroom interventions to be developed which explicitly teach these strategies in ways that enable low-attaining students to achieve high fluency and success with them. A strategic focus on a small number of potentially high-leverage strategies, such as derived facts and estimation, and use of key representations, such as the number line (Foster, 2022), could enable low-attaining students to access powerful ways of handling numbers and develop enhanced number sense. Associated appropriate teacher professional development would of course be equally necessary.

It is noteworthy that many of the students in this study expressed a disliking of manipulatives and number lines, despite both of these being well-evidenced strategies (e.g. Carbonneau et al., 2013; Siegler et al., 2010) that are recommended in high-profile guidance reports for teachers (Education Endowment Foundation, 2017; Hodgen et al., 2018). It may be helpful for teachers to know that students may be likely to resist these approaches, for a variety of reasons. This may help teachers to be prepared *not* to assume that when they encounter students' dislike of these strategies that this indicates that they are not appropriate, and are unsuitable for their students. It may be that until students have become very fluent with some of these tools they do not derive the kind of benefit from them that would encourage them to view these strategies favourably and make spontaneous use of them. There may be a need to persevere with these teaching strategies for longer than teachers might suppose.

It is not sufficient to address the problem of low attainment in mathematics merely by identifying high-leverage teaching strategies from the research literature and then recommending them for use. Strategies which require a long-term investment in teaching time and do not offer an immediate 'quick win' for students may be perceived as ineffective, but to support students' progress in the long term it may be necessary to prioritise the recommended strategies and persevere with them. Explaining and justifying this rationale to students may enable them to become invested in the process and supportive of the approach.

Appendix. Interview protocol

We're interested in how you learn maths and anything that makes maths hard or anything that helps you to learn in maths.

Can you tell me about a maths lesson where you feel that you really learned something? What was it? Why was that lesson successful? What did the teacher do? Can you show me your book? Can you tell me what this was about?

Representations

If I asked you to draw me a picture to show me the number 8, what would you draw? Can you tell me about it? Is there anything else you could draw to show the number 8? Could you show it on a number line?

Do you use number lines in lessons? When? Can you give me an example? What do you do with the number lines? Do you find them useful? For what? Why/why not?

Do you ever decide *for yourself* to draw a number line without your teacher telling you to?

Do you know what I mean by an 'array'? I mean some dots or squares or some other object all lined up in rows and columns [draw an example]. Do you use arrays in maths? When? Can you give me an example? What do you do with them? Do you find them useful? For what? Why/why not?

Do you ever decide to draw something like an array to help you in maths without your teacher telling you to?

Are there any other things that you can draw that help you to work things out in maths? When do you use them? Why?

Derived facts

[use items from Fig. 1 – *"How would you do this?"].*

Which times tables do you know? [Show a blank tables square.] Do you just know them, or do you work them out? If you work them out, how do you do it?

Let's pick one times table that you know – for example $5 \times 6 = 30$ [or use their example].

Can you work out any other times tables – ones you *don't* know – by starting with this one? [If unsure] For example, can you work out 5×7, or 15×7 or 5×17? How would you do it?

Estimation

[use items from Fig. 1 – *"How would you do this?"].*

Do you sometimes *estimate* the answer to a question, rather than work out an exact answer? When do you do that? Why?

Do you ever do that without the teacher telling you to, for example to check your answer? Why / why not?

Acknowledgements We are grateful to the teachers and students who took part in the study.

Funding This research was supported by the Nuffield Foundation (EDU/42015).

Declarations

Conflict of Interest The authors declare no competing interests.

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