Switching diffusions and stochastic resetting

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Abstract. We consider a Brownian particle that switches between two different diffusion states (D_0, D_1) according to a two-state Markov chain. We further assume that the particle's position is reset to an initial value X_r at a Poisson rate r, and that the discrete diffusion state is simultaneously reset according to the stationary distribution ρ_n, $n = 0, 1$, of the Markov chain. We derive an explicit expression for the non-equilibrium steady state (NESS) on $\mathbb{R}$, which is given by the sum of two decaying exponentials. In the fast switching limit the NESS reduces to the exponential distribution of pure diffusion with stochastic resetting. The effective diffusivity is given by the mean $\overline{D} = \rho_0 D_0 + \rho_1 D_1$. We then determine the mean first passage time (MFPT) for the particle to be absorbed by a target at the origin, having started at the reset position $X_r > 0$. We proceed by calculating the survival probability in the absence of resetting and then use a last renewal equation to determine the survival probability with resetting. Similar to the NESS, we find that the MFPT depends on the sum of two exponentials, which reduces to a single exponential in the fast switching limit. Finally, we show that the MFPT has a unique minimum as a function of the resetting rate, and explore how the optimal resetting rate depends on other parameters of the system.

1. Introduction

Advances in single-particle tracking (SPT) and statistical methods suggest that particles within the plasma membrane of living cells, for example, can switch between different discrete states with different diffusivities [7, 25, 28]. Such switching could be due to interactions between proteins and the actin cytoskeleton [18] or due to protein-lipid interactions [29]. Several theoretical studies of Brownian particles with switching diffusions have focused on the effects of space-dependent switching rates and how this provides a mechanism for generating multiplicative noise in the fast switching limit [3, 4], see also [15]. Space-dependent switching also appears to play a role in the formation of intracellular protein concentration gradient formation during the asymmetric division of the Caenorhabditis elegans (*C. elegans*) zygote [31, 5].

From a mathematical perspective, one can view a Brownian particle with switching diffusions as an example of a stochastic hybrid system, in which the piecewise dynamics of a continuous variable (particle position) is coupled with a discrete Markov chain that keeps track of the discrete state of the particle (diffusion coefficient). Stochastic hybrid systems are ubiquitous in cell biology [2]. Examples include membrane voltage fluctuations driven by ion channel noise, and protein concentration fluctuations driven by promoter noise in gene networks. A special class of stochastic hybrid system is a velocity jump process, whereby a particle switches between different motile states. Here the particle could represent a molecular motor on a filament track [20], the tip of a growing microtubule [8], or the position of a bacterium undergoing run-and-tumble during chemotaxis [16].

A run-and-tumble model has recently been analyzed in terms of a search process with stochastic resetting [13]. Resetting refers to the restarting of a stochastic process from a given initial condition. In the simpler case of pure diffusion, suppose that the position of the Brownian particle is reset randomly in time at a constant rate r (Poissonian resetting) to some fixed point X_r. Two characteristic features that are exhibited by such a process are as follows: (i) convergence to a nontrivial non-equilibrium stationary state (NESS), and (ii) the existence of an optimal resetting rate for minimizing the mean search time to find some target [10, 11, 12]. Such behavior has also been found for more general stochastic search processes with resetting, including non-diffusive processes such as Levy flights [17] and velocity jump processes [13, 6], and resetting in potential landscapes [21] and bounded domains [23, 9]. This is indicative of some form of underlying universality [26, 27, 22, 1]. (For a recent review see Ref. [14].) One additional requirement for a stochastic hybrid system with resetting is that one has to specify the resetting protocol for the discrete states. In the analysis of the run-and-tumble model [13], two different reset protocols were used: (i) velocity randomization that is independent of the current velocity state and (ii) velocity resetting that depends on the current state. Note that another example of a stochastic hybrid system with stochastic resetting was analyzed in Ref. [19]. These authors considered a continuous-time random walk with drift, with the latter randomly switching directions.

In this paper we investigate the NESS and optimal resetting rate of a one-dimensional Brownian particle with switching diffusions and stochastic resetting. For simplicity, we focus on the case of two discrete states with diffusivities D_0, D_1, respectively, and space-independent switching rates α, β. We introduce the model without resetting in section 2, and show how in the fast switching limit one obtains pure Brownian motion with an effective diffusivity given by $\overline{D} = \rho_0 D_0 + \rho_1 D_1$, where

$\rho_0 = \beta/(\alpha+\beta)$ and $\rho_1 = \alpha/(\alpha+\beta)$ are the steady-state probabilities of the two-state Markov chain. In section 3 we derive an explicit expression for the NESS in the case of position resetting and randomized diffusion resetting. This requires solving a fourth-order differential equation, which leads to an NESS consisting of a sum of two decaying exponentials. We also show how the latter reduces to the single exponential NESS of pure Brownian motion with resetting [10, 11] in the fast switching limit, with effective diffusivity $\overline{D}$. The extension to more than two discrete diffusion states is also discussed. In section 4, we turn to the first passage time problem of a particle with switching diffusions on the half-line and an absorbing target at the origin. We use a last renewal equation to express the survival probability with resetting in terms of the corresponding survival probability without resetting. We derive an explicit expression for the Laplace transform of the latter, which again requires solving a fourth-order ODE, and use this to express the MFPT in terms of a sum of two exponentials. (As with the NESS, this reduces to a single exponential in the fast switching limit.) Finally, we show that the MFPT has a unique minimum as a function of the resetting rate, and explore how the optimal resetting rate depends on other parameters of the system.

2. Stochastically switching diffusion

Consider a particle diffusing in one dimension that can switch between two different states $n \in \{0,1\}$ according to a two-state jump Markov process $N(t) \in \{0,1\}$, with

$$0 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} 1.$$

The diffusion coefficient is taken to depend on the state, that is $D = D_n$ when $N(t) = n$. For concreteness assume that $D_0 \geq D_1$. Define $p_n(x,t)$ to be the probability density that the particle is at position $x \in \mathbb{R}$ and in state n at time t. The stochastic dynamics of the particle is represented by the hybrid stochastic differential equation (SDE)

$$dX(t) = \sqrt{2D_n}dW(t), \quad \text{for } N(t) = n. \tag{2.1}$$

Given the joint Markov process $(N(t), X(t))$, introduce the probability density $p_n(x,t)$ that the particle is at $X(t) = x$ and in state $N(t) = n$ at time t. The probability density p evolves according to the forward differential CK equation

$$\frac{\partial p_n(x,t)}{\partial t} = D_n \frac{\partial^2 p_n(x,t)}{\partial x^2} + \sum_{m=0,1} A_{nm} p_m(x,t), \tag{2.2}$$

Here $\mathbf{A}$ is the matrix generator

$$\mathbf{A} = \left(\begin{array}{cc} -\alpha & \beta \\ \alpha & -\beta \end{array} \right), \tag{2.3}$$

We will assume the initial conditions

$$p_n(x,0) = \delta(x - x_0)\rho_n. \tag{2.4}$$

where ρ_n is the steady-state distribution of the Markov chain:

$$\rho_0 = \frac{\beta}{\alpha+\beta}, \quad \rho_1 = \frac{\alpha}{\alpha+\beta}. \tag{2.5}$$

Hence,

$$p_n(x,t) = \rho_0 p(x,n,t|x_0,0,0) + \rho_1 p(x,n,t|x_0,1,0),$$

where

$$p(x,n,t|x_0,m,0) = \mathbb{P}[X(t)=x, N(t)=n|X(0)=x_0, N(0)=m].$$

2.1. Fast switching limit

Consider the averaged diffusion coefficient

$$\overline{D} = \sum_{n=0,1} \rho_n D_n. \tag{2.6}$$

Intuitively speaking, one would expect the hybrid SDE (2.1) to reduce to the SDE

$$dX(t) = \sqrt{2\overline{D}}dW(t) \tag{2.7}$$

in the fast switching limit $\alpha, \beta \to \infty$. A heuristic argument is that the Markov chain then undergoes many jumps over a small time interval Δt, during which $\Delta x \approx 0$, and thus the relative frequency of the two discrete states n is approximately ρ_n. However, in order to define what one means by "fast", we need to have some fundamental time-scale τ of the system. Here we will take τ to be the observation time, which means that with high probability the particle will remain within a domain of size L with $\tau \equiv \min_n\{L^2/D_n\}$. We then fix the time-scale by setting $\tau = 1$ and rescale the transition rates according to $\alpha, \beta \to \alpha/\varepsilon, \beta/\varepsilon$, with $\alpha, \beta = O(1)$, so that $\varepsilon \to 0$ defines the fast switching limit. (If stochastic resetting is also present, see section 3, then we can define the fast switching limit relative to the resetting rate r.)

For small but non-zero ε, one can use an adiabatic approximation to reduce the CK equation (2.2) to a corresponding diffusion equation for the total probability density $p(x,t) = \sum_{n=0,1} p_n(x,t)$ [24, 20]. The basic steps are as follows. First, decompose the probability density p_n as

$$p_n(x,t) = p(x,t)\rho_n + \varepsilon w_n(x,t),$$

where $\sum_n w_n(x,t) = 0$. Substituting this decomposition into equations (2.2), summing both sides with respect to n, and using $\sum_n A_{nm} = 0$ yields an equation for p,

$$\frac{\partial p}{\partial t} = \overline{D}\frac{\partial^2 p}{\partial x^2} + \varepsilon \sum_{n=0,1} D_n \frac{\partial^2 w_n}{\partial x^2}. \tag{2.8}$$

Next we use equation (2.8) to eliminate $\partial p/\partial t$ in the expanded version of equation (2.2). Introducing the asymptotic expansion $w_n \sim w_n^{(0)} + \varepsilon w_n^{(1)} + O(\varepsilon^2)$ and collecting the $O(1)$ terms then yields an equation for $w_n^{(0)}$, which has a unique solution on imposing the condition $\sum_n w_n^{(0)}(x,t) = 0$. One can thus establish that there is an $O(\epsilon)$ correction to the diffusion equation of the form

$$\frac{\partial p}{\partial t} \sim \overline{D}\frac{\partial^2 p}{\partial x^2} + \epsilon\kappa \frac{\partial^4 p}{\partial x^2} \tag{2.9}$$

with an $O(1)$ coefficient κ, It follows that in the fast switching limit $\varepsilon \to 0$, the CK equation (2.2) reduces to a diffusion equation with effective diffusivity $\overline{D}$.

There is one subtle feature of the above adiabatic approximation that arises when considering boundary value problems. For example, in section 4 we will consider diffusion on the half line with an absorbing boundary at $x = 0$. Since the original system given by equations (2.2) involves two coupled diffusion equations, we need to impose two absorbing boundary conditions at $x = 0$. On the other hand, the reduced diffusion equation (2.9) has a single absorbing boundary condition. We thus have a singular perturbation problem, in which the solution to equation (2.8) represents an outer solution that is valid in the bulk of the domain, but has to be matched to an inner solution at the boundary using the fourth-order derivative term [30, 5]. However, this is a relatively small effect, and will not be considered further in this paper.

3. Position resetting and randomized diffusivity

Now suppose that the position of the particle is reset to a location X_r at a rate r and that the diffusion state is reset to n according to the stationary distribution ρ_n. This is analogous to the randomized velocity selection in the run-and-tumble model [13]. For simplicity we also set the initial position $x_0 = X_r$ so that resetting preserves the initial conditions (2.4). We can then work with the probability densities

$$p_{r,n}(x,t) = \rho_0 p_r(x,n,t|X_r,0,0) + \rho_1 p_r(x,n,t|X_r,1,0), \tag{3.1}$$

and equation (2.2) is modified according to

$$\frac{\partial p_{r,0}}{\partial t} = D_0 \frac{\partial^2 p_{r,0}}{\partial x^2} - \alpha p_{r,0} + \beta p_{r,1} - r p_{r,0} + \rho_0 r \delta(x - X_r), \tag{3.2a}$$

$$\frac{\partial p_{r,1}}{\partial t} = D_1 \frac{\partial^2 p_{r,1}}{\partial x^2} + \alpha p_{r,0} - \beta p_{r,1} - r p_{r,1} + \rho_1 r \delta(x - X_r). \tag{3.2b}$$

It follows that the total probability density $p_r(x,t) = p_{r,0}(x,t) + p_{r,1}(x,t)$ satisfies the last renewal equation

$$p_r(x,t) = \mathrm{e}^{-rt} p(x,t) + r \int_0^t d\tau \mathrm{e}^{-r\tau} p(x,\tau), \tag{3.3}$$

where $p(x,t)$ is the total probability density without resetting, see equations (2.2). The first term on the right-hand side represents the contribution from trajectories without any resetting, which occurs with probability $\mathrm{e}^{-r\tau}$, while the second term is the sum of contributions from trajectories whose last reset occurs at time $t - \tau$.

3.1. Non-equilibrium steady-state

One of the characteristic features of stochastic processes with resetting is that there exists a nonequilibrium steady-state (NESS) $p_{r,n} = p_n^*(x)$. (Equation (2.2) has a trivial equilibrium steady state that is point-wise zero.) One way to calculate the NESS is to Laplace transform the renewal equation (3.3) using the convolution theorem in order to express $\widetilde{p}_r(x,s)$ in terms of $\widetilde{p}(x,s)$, and then take the limit $p_n^*(x) = \lim_{s\to 0} s\widetilde{p}_{r,n}(x,s)$. Here it is more convenient to work directly with equations (3.2a) and (3.2b). Setting time derivatives to zero, and performing the change of variables $P_1(x) = p_1^*(x) - (\alpha + r)p_0^*(x)/\beta$ gives

$$-\rho_0 r \delta(x - X_r) = D_0 \frac{d^2 p_0^*}{dx^2} + \beta P_1, \tag{3.4a}$$

$$-\rho_1 r \delta(x - X_r) = D_1 \frac{d^2 P_1}{dx^2} - (\beta + r) P_1 + \frac{\alpha + r}{\beta} D_1 \frac{d^2 p_0^*}{dx^2} - \frac{r(r + \alpha + \beta)}{\beta} p_0^*. \tag{3.4b}$$

Substituting for P_1 in equation (3.4b) using equation (3.4a) leads to the fourth-order equation

$$-\rho_1 r \delta(x - X_r) = -\frac{D_1}{\beta}\left[D_0 \frac{d^4 p_0^*}{dx^4} + \rho_0 r \delta''(x - X_r)\right] + \frac{r + \beta}{\beta}\left(\rho_0 r \delta(x - X_r) + D_0 \frac{d^2 p_0^*}{dx^2}\right)$$

$$+\frac{r+\alpha}{\beta}D_1\frac{d^2p_0^*}{dx^2}-\frac{r(r+\alpha+\beta)}{\beta}p_0^*,$$

which can be rearranged to yield

$$\frac{\beta+r\rho_0}{D_0D_1}r\delta(x-X_r)-\frac{\rho_0 r}{D_0}\delta''(x-X_r)=\frac{d^4p_0^*}{dx^4}-\left[\frac{r+\beta}{D_1}+\frac{r+\alpha}{D_0}\right]\frac{d^2p_0^*}{dx^2}+\frac{r(r+\alpha+\beta)}{D_0D_1}p_0^*. \tag{3.5}$$

We solve equation (3.5) using Fourier transforms. Setting

$$\widehat{p}_0^*(k)=\int_{-\infty}^{\infty}\mathrm{e}^{ikx}p_0^*(x)dx,$$

we find that

$$[k^4+B(r)k^2+C(r)]\widehat{p}^*(k)=A(r,k)\mathrm{e}^{ikX_r},$$

where

$$A(r,k)=\frac{\beta+r\rho_0}{D_0D_1}r+\frac{k^2\rho_0 r}{D_0},\quad B(r)=\frac{r+\beta}{D_1}+\frac{r+\alpha}{D_0},\quad C(r)=\frac{r(r+\alpha+\beta)}{D_0D_1}.$$

Using the inverse Fourier transform, we have

$$\begin{aligned}p_0^*(x)&=\int_{-\infty}^{\infty}\mathrm{e}^{-ik(x-X_r)}\frac{A(r,k)}{k^4+B(r)k^2+C(r)}\frac{dk}{2\pi}\\&=\int_{-\infty}^{\infty}\mathrm{e}^{-ik(x-X_r)}\frac{A(r,k)}{(k^2+\lambda_+^2(r))(k^2+\lambda_-^2(r))}\frac{dk}{2\pi},\end{aligned} \tag{3.6}$$

where

$$\lambda_\pm^2(r)=\frac{1}{2}\left\{\frac{r+\beta}{D_1}+\frac{r+\alpha}{D_0}\pm\sqrt{\left[\frac{r+\beta}{D_1}+\frac{r+\alpha}{D_0}\right]^2-\frac{4r(r+\alpha+\beta)}{D_0D_1}}\right\}. \tag{3.7}$$

The integral on the right-hand side of equation (3.6) can be evaluated using contour integration. First, suppose that $x<X_r$ so that we can close the integral in the upper-half complex k-plane with poles at $k=i\lambda_\pm$. From the residue theorem,

$$\begin{aligned}p_0^*(x)&=\frac{1}{\lambda_+^2(r)-\lambda_-^2(r)}\left[A(r,i\lambda_-)\frac{\mathrm{e}^{-\lambda_-(r)|x-X_r|}}{2\lambda_-(r)}-A(r,i\lambda_+)\frac{\mathrm{e}^{-\lambda_+(r)|x-X_r|}}{2\lambda_+(r)}\right]\\&=\frac{A(r,0)}{\lambda_+^2(r)-\lambda_-^2(r)}\left[\frac{\mathrm{e}^{-\lambda_-(r)|x-X_r|}}{2\lambda_-(r)}-\frac{\mathrm{e}^{-\lambda_+(r)|x-X_r|}}{2\lambda_+(r)}\right]\\&\quad-\frac{\rho_0 r}{D_0}\frac{1}{\lambda_+^2(r)-\lambda_-^2(r)}\left[\lambda_-(r)\frac{\mathrm{e}^{-\lambda_-(r)|x-X_r|}}{2}-\lambda_+(r)\frac{\mathrm{e}^{-\lambda_+(r)|x-X_r|}}{2}\right].\end{aligned} \tag{3.8}$$

Now using equation (3.4a) and the definition of P_1 shows that

$$p_1^*(x)=-\frac{\rho_0 r}{\beta}\delta(x-X_r)-\frac{D_0}{\beta}\frac{d^2p_0^*(x)}{dx^2}+\frac{r+\alpha}{\beta}p_0^*(x), \tag{3.9}$$

with

$$\begin{aligned}\frac{D_0}{\beta}\frac{d^2p_0^*}{dx^2}&=\frac{D_0}{\beta}\frac{A(r,0)}{\lambda_+^2(r)-\lambda_-^2(r)}\left[\lambda_-(r)\frac{\mathrm{e}^{-\lambda_-(r)|x-X_r|}}{2}-\lambda_+(r)\frac{\mathrm{e}^{-\lambda_+(r)|x-X_r|}}{2}\right]\\&\quad-\frac{\rho_0 r}{\beta}\frac{1}{\lambda_+^2(r)-\lambda_-^2(r)}\left[\lambda_-(r)^3\frac{\mathrm{e}^{-\lambda_-(r)|x-X_r|}}{2}-\lambda_+(r)^3\frac{\mathrm{e}^{-\lambda_+(r)|x-X_r|}}{2}\right]\\&\quad-\frac{\rho_0 r}{\beta}\delta(x-X_r).\end{aligned} \tag{3.10}$$

Adding equations (3.8) and (3.9) finally yields an explicit expression for the NESS, which is given by a sum of two exponentials:

$$p^*(x) = p_0^*(x) + p_1^*(x) = \Lambda_-(r)\mathrm{e}^{-\lambda_-(r)|x-X_r|} + \Lambda_+(r)\mathrm{e}^{-\lambda_+(r)|x-X_r|}, \tag{3.11}$$

with

$$\Lambda_-(r) = \frac{1}{\lambda_+^2(r) - \lambda_-^2(r)} \left[\frac{r+\alpha+\beta}{\beta} - \frac{D_0\lambda_-^2(r)}{\beta}\right] \times \left\{\frac{\rho_0 r(r+\alpha+\beta)}{D_0D_1}\frac{1}{2\lambda_-(r)} - \frac{\rho_0 r}{D_0}\frac{\lambda_-(r)}{2}\right\} \tag{3.12a}$$

$$\Lambda_+(r) = -\frac{1}{\lambda_+^2(r) - \lambda_-^2(r)} \left[\frac{r+\alpha+\beta}{\beta} - \frac{D_0\lambda_+^2(r)}{\beta}\right] \times \left\{\frac{\rho_0 r(r+\alpha+\beta)}{D_0D_1}\frac{1}{2\lambda_+(r)} - \frac{\rho_0 r}{D_0}\frac{\lambda_+(r)}{2}\right\}. \tag{3.12b}$$

Some useful checks of the above calculation are as follows. First, when $D_0 = D_1 = D$, we recover the classical result for the NESS of a diffusing particle without switching diffusion [10, 11]. This follows from the fact that

$$\sqrt{\left[\frac{s+\beta}{D} + \frac{s+\alpha}{D}\right]^2 - \frac{4s(s+\alpha+\beta)}{D^2}} = \frac{\alpha+\beta}{D}, \tag{3.13}$$

and hence

$$\lambda_-^2(r) = \frac{r}{D}, \quad \lambda_+^2(r) = \frac{r+\alpha+\beta}{D}, \quad \Lambda_-(r) = \frac{1}{2}\sqrt{\frac{r}{D}}, \quad \Lambda_+(r) = 0.$$

Equation (3.11) thus reduces to the single exponential NESS

$$p_D^*(x) = \frac{1}{2}\sqrt{\frac{r}{D}}\mathrm{e}^{-\sqrt{r/D}|x-X_r|}. \tag{3.14}$$

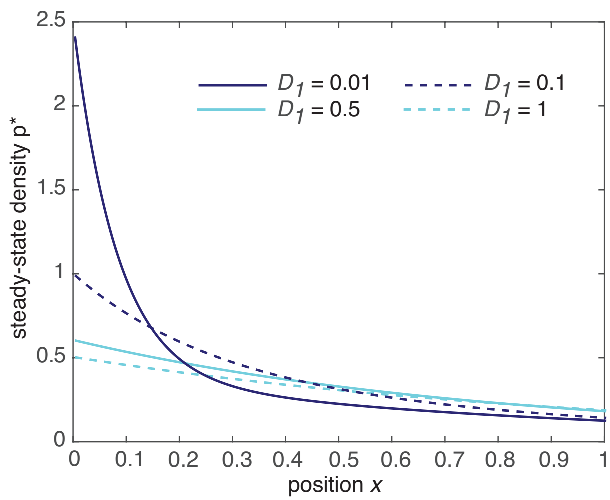

Figure 1: Plot of steady-state density $p^*(x)$ as a function of x for different diffusivities $D_1 \le D_0 = 1$. Other parameter values are $\alpha = \beta = 0.5$, $r = 1$ and $X_r = 0$.

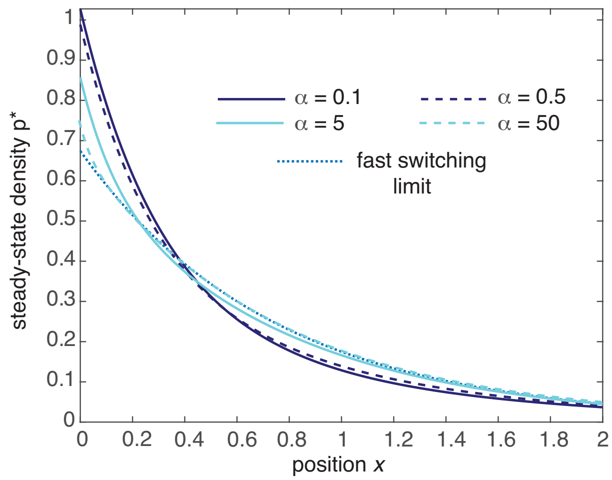

Figure 2: Plot of steady-state density $p^*(x)$ as a function of x for different switching rates α with $\alpha = \beta$. Other parameter values are $D_0 = 1$, $D_1 = 0.1$, $r = 1$ and $X_r = 0$.

In Fig. 1 we plot $p^*(x)$ as a function of x for different diffusivities $D_1 \leq D_0 = 1$. It can be seen that $p^*(x) \to p^*_D(x)$ as $D_1 \to D_0$ from below. Second, performing the rescalings $\alpha, \beta \to \alpha/\varepsilon, \beta/\varepsilon$ and carrying out a perturbation expansion in ε shows that $\lambda_+(r) = O(1/\varepsilon)$ while

$$\lambda_-^2(s) = \frac{1}{2D_0D_1\varepsilon}\{\varepsilon s(D_0 + D_1) + (\alpha D_1 + \beta D_0)\}$$
$$\times \left(1 - \left[1 - \frac{4\varepsilon s(\varepsilon s + \alpha + \beta)D_0D_1}{[\varepsilon s(D_0 + D_1) + (\alpha D_1 + \beta D_0)]^2}\right]^{1/2}\right)$$

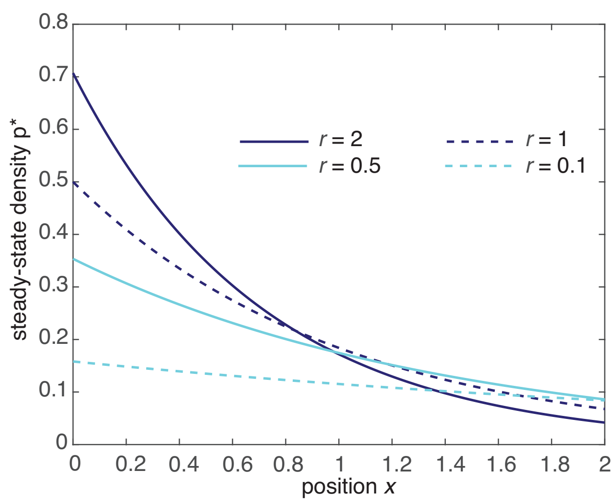

Figure 3: Plot of steady-state density $p^*(x)$ as a function of x for different resetting rates r. Other parameter values are $\alpha = \beta = 0.5$, $D_0 = 1$, $D_1 = 0.1$ and $X_r = 0$.

$$\approx \frac{s(\varepsilon s+\alpha+\beta)}{[\varepsilon s(D_0+D_1)+(\alpha D_1+\beta D_0)]}=\frac{s}{\overline{D}}+O(\varepsilon),$$

Hence, in the limit $\varepsilon \to 0$, we find that

$$p^*(x)=\frac{1}{2}\sqrt{\frac{r}{\overline{D}}}\mathrm{e}^{-\sqrt{r/\overline{D}}|x-X_r|}. \tag{3.15}$$

This is further illustrated in Fig. 2, which plots $p^*(x)$ for different switching rates α in the case $\alpha=\beta$. (Since $p^*(x)$ for $X_r=0$ is an even function of x, we only show plots for positive x. There is a cusp at $x=0$.) Finally, in Fig. 3, we show plots of $p^*(x)$ for different resetting rates r. As might be expected, reducing the rate of resetting flattens the steady-state density.

3.2. Matrix analysis

An alternative method for analyzing the NESS is to work directly with the generator $\mathbf{A}$ of the Markov chain. Since this equally applies to higher-order models, suppose that there now exist N discrete states with diffusivities D_n, $n=1,\ldots,N$. The steady-state master equation with resetting takes the form

$$\sum_{m=1}^N\left[(D_n\frac{d^2}{dx^2}-r)\delta_{n,m}+A_{nm}\right]p_n^*(x)=-\rho_n r\delta(x-X_r), \tag{3.16}$$

where $\mathbf{A}$ is the generator of the N-state Markov chain. (For $N=2$ the matrix $\mathbf{A}$ reduces to equation (2.3).) Fourier transforming the steady-state equation then yields the matrix equation

$$\sum_{m=1}^N\left[k^2\delta_{n,m}+\Sigma_{nm}\right]\widetilde{p}_n^*(k)=\frac{\rho_n}{D_n}r\mathrm{e}^{ikX_r}, \tag{3.17}$$

with

$$\Sigma_{nm}=\frac{1}{D_n}\left(r\delta_{n,m}-A_{nm}\right). \tag{3.18}$$

Denote the eigenvalues of $\mathbf{\Sigma}$ by λ_j^2, $j=1,\ldots,N$. We assume that the generator $\mathbf{A}$ is irreducible so that, from the Perron-Frobenius theorem, the eigenvalues μ_j of $\mathbf{A}$ can be ordered as

$$0=\mu_1>\mu_2\geq\mu_3>\ldots\geq\mu_N.$$

It then follows that the eigenvalues of $\mathbf{\Sigma}$ are positive, that is, λ_j is real for all j.

We now invert the matrix equation (3.17) by setting

$$\left[k^2\mathbf{I}+\mathbf{\Sigma}\right]^{-1}=\frac{\mathbf{M}(k^2)}{\prod_{j=1}^N(k^2+\lambda_j^2)}. \tag{3.19}$$

Taking the inverse Fourier transform then leads to the integral solution

$$p_n^*(x)=r\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ik(X_r-x)}}{\prod_{j=1}^N(k^2+\lambda_j^2)}\left(\sum_{m=1}^N M_{nm}(k^2)\frac{\rho_m}{D_m}\right)\frac{dk}{2\pi}. \tag{3.20}$$

Hence the eigenvalues of $\mathbf{\Sigma}$ determine the poles of the resulting contour integral. Depending on the sign of $x-X_r$, we close the contour in the upper-half or lower-half complex k-plane and use the residue theorem to obtain the following general expression for the NESS:

$$p_n^*(x) = r\sum_{j=1}^N \frac{\mathrm{e}^{-\lambda_j|x-X_r|}}{2\lambda_j}\frac{\left(\sum_{m=1}^N M_{nm}(-\lambda_j^2)\rho_m\right)}{\prod_{l\neq j}(\lambda_l^2-\lambda_j^2)}. \tag{3.21}$$

As in the two-state model, we should recover the NESS for pure diffusion when $D_n = D$ for all n. The simplest way to show this is to set $D_n = D$ in equation (3.17) and sum both sides with respect to n. Using the fact that $\sum_{n=1}^N A_{nm} = 0$ for the generator of a Markov chain and $\sum_{n=1}^N \rho_n = 1$(normalization), we find that

$$\widetilde{p}_n^*(k) = \frac{r\mathrm{e}^{ikX_r}}{Dk^2+r}, \tag{3.22}$$

whose inverse Fourier transform is given by equation (3.14). Similarly, in the fast switching limit we can rescale the generator so that equation (3.17) becomes

$$\sum_{m=1}^N \left[(D_nk^2+r)\delta_{n,m} - A_{nm}\right]\widetilde{p}_n^*(k) = \rho_n r\mathrm{e}^{ikX_r}, \tag{3.23}$$

Introducing the perturbation series expansion

$$\widetilde{p}_n^*(k) = \rho_n p^*(k) + \varepsilon\widetilde{p}_{n,1}^*(k) + \ldots, \tag{3.24}$$

we obtain the leading order equation

$$(k^2\overline{D}+r)p^*(k) = r\mathrm{e}^{ikX_r},$$

which thus recovers equation (3.15).

4. Survival probability and mean first passage time

We now turn to the FPT problem consisting of a particle diffusing on the half-line $x > 0$ with an absorbing target at $x = 0$ and initial position $x_0 = X_r$. This means that equations (3.2*a*) and (3.2*b*) are supplemented by the absorbing boundary conditions

$$p_0(0,t) = 0 = p_1(0,t). \tag{4.1}$$

A well-known property of diffusion on the half-line is that the MFPT to reach the origin is infinite for $x_0 > 0$. One way to render the MFPT finite is to include stochastic resetting. Let $\mathcal{T}$ denote the first passage time to be absorbed at $x = 0$ having started at position X_r (and in state m with probability ρ_m):

$$\mathcal{T} = \inf\{t>0;\ X(t)=0,\ X(0)=X_r\}. \tag{4.2}$$

One way to determine the mean first passage time (MFPT) $T_r = \mathbb{E}[\mathcal{T}]$ is to consider the survival probability of the particle, which is defined according to

$$Q_r(X_r,t) = \int_0^\infty p_r(x,t)dx, \quad p_r = p_{r,0}+p_{r,1} \tag{4.3}$$

with $p_{r,n}$ given by equation (3.1). Since resetting preserves the initial conditions, Q_r is related to the survival probability without resetting, Q, in terms of a last renewal equation [13]:

$$Q_r(X_r,t) = \mathrm{e}^{-rt}Q(X_r,t) + r\int_0^t Q(X_r,\tau)Q_r(X_r,t-\tau)\mathrm{e}^{-r\tau}d\tau. \tag{4.4}$$

The first term on the right-hand side represents trajectories with no resettings. The integrand in the second term is the contribution from trajectories that last reset at time $\tau \in (0,t)$, and consists of the product of the survival probability starting from X_r with resetting up to time $t-\tau$ and the survival probability starting from X_r without any resetting over the time interval τ. Since we have a convolution, it is natural to introduce the Laplace transform $\widetilde{Q}_r(X_r,s) = \int_0^\infty Q_r(X_r,t)\mathrm{e}^{-st}dt$. Laplace transforming the last renewal equation and rearranging shows that

$$\widetilde{Q}_r(X_r,s) = \frac{\widetilde{Q}(X,r+s)}{1-r\widetilde{Q}(X_r,r+s)}. \tag{4.5}$$

Finally, the MFPT to be absorbed at the origin is

$$T_r = \int_0^\infty Q_r(X_r,t)dt = \widetilde{Q}_r(X_r,0) = \frac{\widetilde{Q}(X_r,r)}{1-r\widetilde{Q}(X_r,r)}. \tag{4.6}$$

We wish to determine the MFPT under the combined effect of switching diffusions and stochastic resetting.

4.1. Survival probability without resetting

The survival probability without resetting for the initial condition $X(0) = x_0$ can be decomposed as

$$Q(x_0,t) = \rho_0 Q_0(x_0,t) + \rho_1 Q_1(x_0,t), \tag{4.7}$$

where $Q_n(x_0,t)$ is the survival probability when the initial state of the particle is n. The Q_n satisfy the backward CK equation

$$\frac{\partial Q_0}{\partial t} = D_0 \frac{\partial^2 Q_0}{\partial x_0^2} - \alpha Q_0 + \alpha Q_1, \tag{4.8a}$$

$$\frac{\partial Q_1}{\partial t} = D_1 \frac{\partial^2 Q_1}{\partial x_0^2} + \beta Q_0 - \beta Q_1, \tag{4.8b}$$

together with the boundary conditions

$$Q_0(0,t) = Q_1(0,t) = 0.$$

Laplace transforming the backward equation gives

$$-1 = D_0 \frac{\partial^2 \widetilde{Q}_0}{\partial x_0^2} - (s+\alpha)\widetilde{Q}_0 + \alpha \widetilde{Q}_1, \tag{4.9a}$$

$$-1 = D_1 \frac{\partial^2 \widetilde{Q}_1}{\partial x_0^2} + \beta \widetilde{Q}_0 - (s+\beta)\widetilde{Q}_1, \tag{4.9b}$$

It is convenient to introduce the rescaled survival probabilities $\widetilde{q}_n = \rho_n \widetilde{Q}_n$, with

$$-\rho_0 = D_0 \frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} - (s+\alpha)\widetilde{q}_0 + \beta \widetilde{q}_1, \tag{4.10a}$$

$$-\rho_1 = D_1 \frac{\partial^2 \widetilde{q}_1}{\partial x_0^2} + \alpha \widetilde{q}_0 - (s+\beta)\widetilde{q}_1. \tag{4.10b}$$

Following along analogous lines to section 3, we perform the change of variables $\widetilde{R}_1 = \widetilde{q}_1 - (s+\alpha)\widetilde{q}_0/\beta$ to obtain

$$-\rho_0 = D_0 \frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} + \beta \widetilde{R}_1, \tag{4.11a}$$

$$-\rho_1 = D_1 \frac{\partial^2 \widetilde{R}_1}{\partial x_0^2} - (s+\beta)\widetilde{R}_1 + \frac{s+\alpha}{\beta} D_1 \frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} - \frac{s(s+\alpha+\beta)}{\beta}\widetilde{q}_0. \tag{4.11b}$$

Substituting for $\widetilde{R}_1$ in equation (4.11*b*) using equation (4.11*a*) leads to the fourth-order equation

$$-\rho_1 = -\frac{D_0 D_1}{\beta}\frac{\partial^4 \widetilde{q}_0}{\partial x_0^4} + \frac{s+\beta}{\beta}\left(\rho_0 + D_0 \frac{\partial^2 \widetilde{q}_0}{\partial x_0^2}\right) + \frac{s+\alpha}{\beta} D_1 \frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} - \frac{s(s+\alpha+\beta)}{\beta}\widetilde{q}_0,$$

which can be rearranged to yield

$$\frac{\beta + s\rho_0}{D_0 D_1} = \frac{\partial^4 \widetilde{q}_0}{\partial x_0^4} - \left[\frac{s+\beta}{D_1} + \frac{s+\alpha}{D_0}\right]\frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} + \frac{s(s+\alpha+\beta)}{D_0 D_1}\widetilde{q}_0, \tag{4.12}$$

The solution is given by

$$\widetilde{q}_0(x_0, s) = \frac{\rho_0}{s}\left[1 - a_+(s)\mathrm{e}^{-\lambda_+(s)x_0} - a_-(s)\mathrm{e}^{-\lambda_-(s)x_0}\right], \tag{4.13}$$

where $\lambda_\pm^2(s)$ are given by equation (3.7), and $a_+(s) + a_-(s) = 1$ so that the boundary condition $\widetilde{q}_0(0, s) = 0$ holds. Finally, equation (4.11*a*) and the definition of $\widetilde{R}_1$ shows that

$$\widetilde{q}_1 = -\frac{\rho_0}{\beta} - \frac{D_0}{\beta}\frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} + \frac{s+\alpha}{\beta}\widetilde{q}_0 \tag{4.14}$$

with

$$\frac{\partial^2 \widetilde{q}_0}{\partial x_0^2} = -\frac{\rho_0}{s}\left[a_+(s)\lambda_+(s)^2\mathrm{e}^{-\lambda_+(s)x_0} + a_-(s)\lambda_-(s)^2\mathrm{e}^{-\lambda_-(s)x_0}\right].$$

Imposing the boundary condition $\widetilde{q}_1(0, s) = 0$ then gives

$$\begin{aligned}\frac{s}{D_0} &= a_+(s)\lambda_+(s)^2 + a_-(s)\lambda_-(s)^2 \\ &= \frac{1}{2}\left[\frac{s+\beta}{D_1} + \frac{s+\alpha}{D_0}\right] + \sqrt{\left[\frac{s+\beta}{D_1} + \frac{s+\alpha}{D_0}\right]^2 - \frac{4s(s+\alpha+\beta)}{D_0 D_1}}\left[\frac{a_+(s) - a_-(s)}{2}\right],\end{aligned}$$

that is

$$a_+(s) - a_-(s) = -\frac{\dfrac{s+\beta}{D_1} + \dfrac{\alpha - s}{D_0}}{\sqrt{\left[\dfrac{s+\beta}{D_1} + \dfrac{s+\alpha}{D_0}\right]^2 - \dfrac{4s(s+\alpha+\beta)}{D_0 D_1}}}. \tag{4.15}$$

Combining equations (4.13) and (4.14) thus gives the result

$$\begin{aligned}\widetilde{Q}(x_0, s) &\equiv \widetilde{q}_0(x_0, s) + \widetilde{q}_1(x_0, s) \qquad (4.16)\\ &= \frac{1}{s}\left[1 - \Gamma_+(s)a_+(s)\mathrm{e}^{-\lambda_+(s)x_0} - \Gamma_-(s)a_-(s)\mathrm{e}^{-\lambda_-(s)x_0}\right],\end{aligned}$$

with

$$\Gamma_\pm(s) = \rho_0\left[\frac{s+\alpha+\beta}{\beta} - \frac{D_0}{\beta}\lambda_\pm(s)^2\right]. \tag{4.17}$$

4.2. Mean first passage time with resetting

Substituting equation (4.16) into (4.6) yields the following expression for the MFPT for a particle with switching diffusions and stochastic resetting:

$$T_r = \frac{\left[1 - \Gamma_+(r)a_+(r)\mathrm{e}^{-\lambda_+(r)X_r} - \Gamma_-(r)a_-(r)\mathrm{e}^{-\lambda_-(r)X_r}\right]}{r\left[\Gamma_+(r)a_+(r)\mathrm{e}^{-\lambda_+(r)X_r} + \Gamma_-(r)a_-(r)\mathrm{e}^{-\lambda_-(r)X_r}\right]}. \tag{4.18}$$

Again we obtain known results in particular cases. First, suppose that $D_0 = D_1 = D$, Equation (3.13) then holds so that

$$\lambda_-^2(r) = \frac{r}{D}, \quad \lambda_+^2(r) = \frac{r+\alpha+\beta}{D}, \quad \Gamma_-(r) = a_-(r) = 1, \quad \Gamma_+(r) = 0 = a_+(r).$$

Equation (4.18) thus recovers the survival probability of a diffusing particle on the half-line [10, 11]:

$$T_r = \frac{1 - \mathrm{e}^{-\sqrt{r/D}X_r}}{r\mathrm{e}^{-\sqrt{r/D}X_r}} = \frac{1}{r}\left(\mathrm{e}^{\sqrt{r/D}X_r} - 1\right). \tag{4.19}$$

In the limit $r \to 0$, the MFPT diverges as $T_r \sim \sqrt{r}$, which recovers the result that the MFPT of a Brownian particle without resetting to return to the origin is infinite. T_r also diverges in the limit $r \to \infty$, since the particle resets to X_r so often that it never has the chance to reach the origin. One thus finds that the MFPT has a finite and unique minimum at an intermediate value of the resetting rate r [10, 11]. Similarly, performing the rescalings $\alpha, \beta \to \alpha/\varepsilon, \beta/\varepsilon$ and carrying out a perturbation expansion in ε recovers the fast switching limit with D replaced by $\overline{D}$ in equation (4.19).

In Fig. 4 we show plots of T_r as a function of the resetting rate for various switching rates α with $\beta = \alpha$, $D_0 = 1$ and $D_1 = 0.1$. It can be seen that the MFPT converges to the fast switching limit as $\alpha \to \infty$. Another feature of the plots is that for slow resetting the MFPT is a decreasing function of α, whereas the opposite holds for fast resetting. In Fig. 5 we show analogous plots for fixed $\alpha = 0.5$ and various

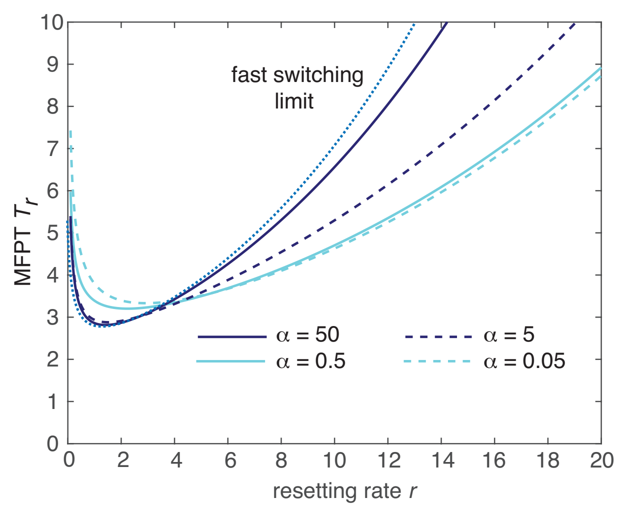

Figure 4: Plot of MFPT T_r as a function of resetting rate for different switching rates α and $\beta = \alpha$. Other parameter values are $X_r = 1$, $D_0 = 1$ and $D_1 = 0.1$.

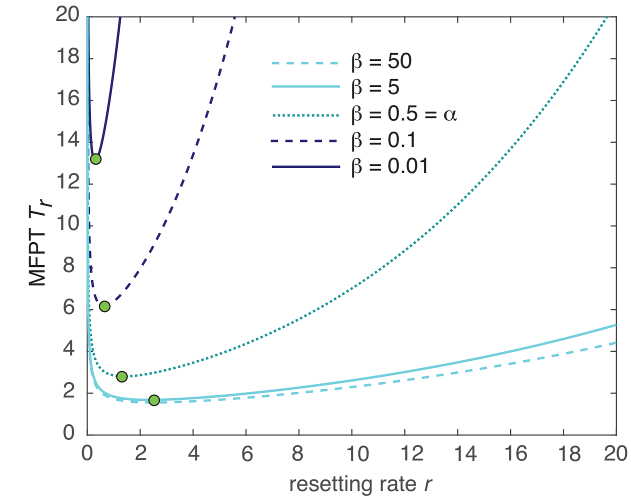

Figure 5: Plot of MFPT T_r as a function of resetting rate for different switching rates β. Other parameter values are $\alpha = 0.5$, $D_0 = 1$, $D_1 = 0.1$, and $X_r = 1$. Green dotes indicate the optimal resetting rate for each curve.

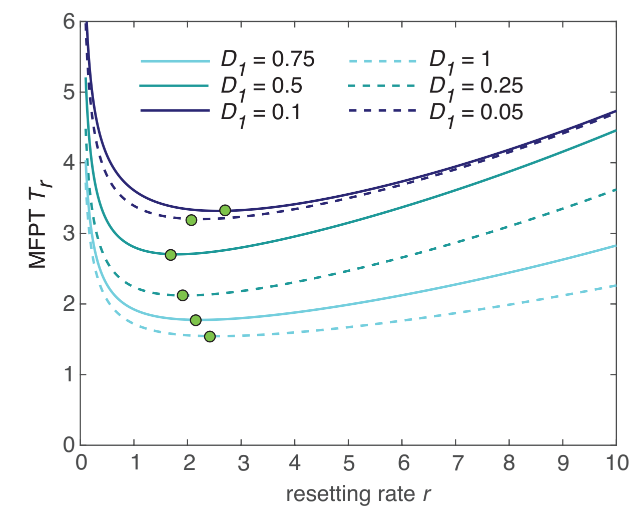

Figure 6: Plot of MFPT T_r as a function of resetting rate for different diffusivities D_1, $D_1 \leq D_0$. Other parameter values are $\alpha = \beta = 0.5$, $X_r = 1$, and $D_0 = 1$. Green dotes indicate the optimal resetting rate for each curve.

β. Since increasing β increases the amount of time spent in the fast diffusing state D_0, it follows that the MFPT is reduced. Finally, in Fig. 6 the effect of varying D_1 with $D_1 \leq D_0 = 1$ is explored. In this case we see that the optimal resetting rate is a non-monotonic function of D_1.

5. Conclusion

In this paper we analyzed the effects of stochastic resetting on a stochastic hybrid system consisting of a 1D Brownian particle with switching diffusions. We derived explicit expressions for the NESS on $\mathbb{R}$ and the MFPT on $\mathbb{R}^+$ with an absorbing boundary at $x = 0$. Consistent with previous studies of stochastic hybrid system with stochastic resetting [19, 13], the MFPT has a unique minimum as a function of the resetting rate r. Moreover, the optimal resetting rate and MFPT have a nontrivial dependence on parameters of the model. For example, increasing the relevant amount of time in the fast diffusing state reduces the MFPT. An analogous result was found in [19], where increasing the amount of time in the state drifting towards rather than away from a target reduced the MFPT. We also showed how one recovers the case of pure diffusion with resetting in the fast switching limit, with an effective diffusivity $\overline{D} = \sum_n \rho_n D_n$. One of the interesting issues concerning stochastic hybrid systems with resetting is how one specifies the reset protocol of the discrete state. In this paper we assumed that the diffusion state was simultaneously reset with particle position according to a randomization scheme based on the stationary distribution of the underlying Markov chain. This allowed us to use a simple renewal equation for the survival probability. As explored elsewhere for the run-and-tumble model [13], one could consider a more general resetting scheme in which the discrete state just after reset depends on the state just prior to reset. However, this significantly complicates the analysis of the renewal equation.

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