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Singular CR structures of constant Webster curvature and applications

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Abstract

We consider the sphere $\mathbb{S}^{2n+1}$ equipped with its standard contact form. In this paper, we construct explicit contact forms on $\mathbb{S}^{2n+1} \setminus \mathbb{S}^{2k+1}$, which are conformal to the standard one and whose related Webster metrics have constant Webster curvature; in particular, it is positive if $2k < n-2$. As main applications, we provide two perturbative results. In the first one, we prove the existence of infinitely many contact forms on $\mathbb{S}^{2n+1} \setminus \tau(\mathbb{S}^1)$ conformal to the standard one and having constant Webster curvature, where $\tau(\mathbb{S}^1)$ is a small perturbation of $\mathbb{S}^1$. In the second application, we show that there exist infinitely many bifurcating branches of periodic solutions to the CR Yamabe problem on $\mathbb{S}^{2n+1} \setminus \mathbb{S}^1$ having constant Webster curvature.

KEYWORDS
conformal geometry, singular contact structures, singular Yamabe problem

1 | INTRODUCTION AND STATEMENTS OF THE RESULTS

For $n \geq 1$, we consider the sphere $\mathbb{S}^{2n+1}$ equipped with its standard contact form $\theta_n^{\mathbb{S}}$. The related Webster metric $g_{\theta_n^{\mathbb{S}}}$ has constant Webster scalar curvature $S_{\theta_n^{\mathbb{S}}} = 4n^2 + 4n$. The existence of conformal contact forms having constant curvature is the standard CR Yamabe problem on the sphere, which has been addressed by Jerison and Lee [12, 13] and many other authors [6, 7, 10, 11].

As in the Riemannian case, one is then interested in the existence of contact forms on noncompact manifolds, which carry a (complete) Webster metric having constant Webster curvature. In the Riemannian case, this question has been deeply studied. In fact, one finds two directions in the literature. The first one addresses the case of negative constant scalar curvature, see, for instance, [1–3, 15]. The second case addresses metrics of positive constant scalar curvature, starting with the pioneering works of Schoen and Yau [23] and Schoen [21, 22]. In particular, when considering a subset Λ on the standard sphere $\mathbb{S}^n$, it is proved that if $\mathbb{S}^n \setminus \Lambda$ carries a complete metric with positive scalar curvature, then a bound on the dimension of Λ holds, that is, $2dim(\Lambda) \leq n-2$; moreover, explicit examples are given of complete conformally flat metrics with constant positive scalar curvature on special sets Λ.

These results have been widely used and generalized in various directions; see, for instance, [4, 8, 17–19], and the references therein. In fact, one can prove the existence of complete conformally flat metrics with constant positive scalar curvature on $\mathbb{S}^n \setminus \Lambda$, where Λ is a perturbation of some special sets, namely, the equatorial spheres $\mathbb{S}^k \subseteq \mathbb{S}^n$ [19].

Moreover, by means of the theory of bifurcation, one can show the existence of periodic solutions to the standard Yamabe problem on $\mathbb{S}^n \setminus \mathbb{S}^1$ [4]: In these kind of results, the starting point is the knowledge of explicit complete conformally flat metrics with constant positive scalar curvature on the special manifolds $\mathbb{S}^n \setminus \mathbb{S}^k$.

In this paper, we will show the existence of explicit contact forms on $\mathbb{S}^{2n+1} \setminus \mathbb{S}^{2k+1}$, whose related Webster metrics are complete and have constant Webster curvature; in particular, it is positive if $2k < n-2$.

Our construction mimics the one in the Riemannian case. In fact, we first stereographically project (by means of the Cayley transform, which is a conformal CR diffeomorphism) the standard sphere $\mathbb{S}^{2n+1}$ to the Heisenberg group $\mathbb{H}^n$ endowed with its standard contact form $\theta_n^{\mathbb{H}}$, in such a way that a special equatorial sphere $\mathbb{S}^{2k+1}$ is mapped into the subgroup $\mathbb{H}^k$. Then, in the complementary set we use polar coordinates, so that (with some abuse of notation, which will be explained in details in the following sections), we have the product manifold $\mathbb{H}^n = \mathbb{H}^k \times \mathbb{R}^+ \times \mathbb{S}^{2N+1}$, endowed with the contact form $\theta_n^{\mathbb{H}} = 2r^2\theta_N^{\mathbb{S}} + \theta_k^{\mathbb{H}}$; here $n = k + N + 1$, and the polar coordinate r is the standard variable of $\mathbb{R}^+$. Then, we have the following:

Theorem 1.1. *Let us define the contact form* $\theta_{k,N} := \theta_N^{\mathbb{S}} + \frac{1}{2r^2}\theta_k^{\mathbb{H}}$ *on* $\mathbb{H}^n \setminus \mathbb{H}^k \simeq \mathbb{S}^{2n+1} \setminus \mathbb{S}^{2k+1}$. *It holds that* $\theta_{k,N}$ *is conformal to the standard CR contact form* $\theta_n^{\mathbb{S}}$ *of* $\mathbb{S}^{2n+1}$. *Moreover, the related Webster metric is complete and it has constant Webster scalar curvature* $\mathcal{S}_{\theta_{k,N}} = 4(n+1)(n-2k-2)$. *In particular,* $\mathcal{S}_{\theta_{k,N}}$ *is positive for* $2k < n-2$.

Now some remarks are in order. First, we notice that our construction works well for the odd-dimensional equatorial spheres $\mathbb{S}^{2k+1}$ that we will define in the next section; we are not able to handle the even-dimensional case with this strategy. Another interesting feature, which seems to be different from the Riemannian case, is the following. In the Riemannian case, one can see the product $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^+ \times \mathbb{S}^N = \mathcal{H}^{k+1} \times \mathbb{S}^N$, where $\mathcal{H}^{k+1}$ is the standard hyperbolic space, which in turn can be identified with the unit ball in $\mathbb{R}^{k+1}$ equipped with the Poincaré metric, having negative constant sectional curvature. For the CR case, in literature there exists a hyperbolic Heisenberg group $\mathbb{H}^k \times \mathbb{R}^+$, which can be seen as the Siegel domain in $\mathbb{C}^{k+1}$ or equivalently as the unit ball in $\mathbb{C}^{k+1}$ equipped with the Kähler Bergman metric, having negative constant holomorphic sectional curvature (see, for instance, [9]). Now, if one tries to write the product $\mathbb{H}^k \times \mathbb{R}^+ \times \mathbb{S}^{2N+1}$ endowed with the contact form $\theta_{k,N}$ as the product of a sort of hyperbolic Heisenberg group times the sphere $\mathbb{S}^{2N+1}$, this gives rise to a model in which the complex structure J associated to $\theta_{k,N}$ *mixes* vector fields from the Heisenberg group $\mathbb{H}^k$ and the sphere $\mathbb{S}^{2N+1}$; this will be clear from the explicit construction in Section 3. For similar results, see also [5, 14, 20]. With these explicit contact forms in hand, as applications we will prove two perturbative results. The first one is analogous to a result proved by Mazzeo and Smale in [19], which gives the existence of CR contact structures having constant Webster curvature by means of a small perturbation of the singular set. More precisely, we have the following:

Theorem 1.2. *Assume that* $n \geq 3$. *There exists a set of diffeomorphisms* $\mathcal{T}_T$ *such that if* $\tau : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ *is in* $\mathcal{T}_T$ *and close to the identity in the* $C^{3,\alpha}$ *topology, then there exists an infinite family of contact forms on* $\mathbb{S}^{2n+1} \setminus \tau(\mathbb{S}^1)$ *conformal to the standard one on* $\mathbb{S}^{2n+1}$ *and having complete Webster metric with constant Webster scalar curvature equal to* $\mathcal{S}_{\theta_{0,n-1}}$.

For the precise definition of $\mathcal{T}_T$ as subset of the diffeomorphisms, we refer the reader to Section 4.1.

The second application is about the existence of periodic solutions to the CR Yamabe equations, as in [4]. We recall here that we mean by a periodic solution, a solution obtained by lifting the structure on $\mathbb{C}P^{n-1} \times \mathcal{H}^2/\Gamma$, where Γ is a suitable group: We refer the reader to Section 4.2 for further details. These solutions are obtained by using the theory of bifurcation.

Theorem 1.3. *Assume that* $n \geq 3$. *There exist infinitely many branches of periodic solutions to the CR Yamabe problem on* $\mathbb{S}^{2n+1} \setminus \mathbb{S}^1$ *having constant Webster curvature, arbitrary close to* $\mathcal{S}_{\theta_{0,n-1}}$.

One can show that these solutions are nonisometric. Indeed, we refer the reader to Remark 4.1 in [4]: In particular, this applies to our setting, since after restricting our function spaces, we are led back to a similar Riemannian setting.

We want to point out here that there are crucial difficulties and technicalities that appear in our setting, compared to the Riemannian case. Indeed, the differential operator that we are dealing with is subelliptic, moreover, the sub-Laplacian does not transform well within a product structure as opposed to the Laplacian. The operator $\Delta_{\theta_{k,N}}$ obtained in Section 4 contains many cross terms, adding to the difficulty of the problem. That is why, in order to apply both the methods in [19] and [4], we will restrict our study to the case $k = 0$ and by choosing a specific space of functions as in Section 4.1, that allow us to be in a setting relatively similar to the Riemannian one. Due to the lack of symmetries in the CR setting, we are led to consider the action generated by the Reeb vector field, since, for instance, we need our set of diffeomorphisms to commute with this action.

To the best of our knowledge, in this setting these are the first results in this kind of direction.

2 | DEFINITIONS AND NOTATION

We recall here some well-known facts for further references and in order to fix our notations.

Let (M, θ) be a $(2n+1)$-dimensional contact manifold with contact form θ and Reeb vector field T (i.e., the unique vector field satisfying $\theta(T) = 1$ and $\mathrm{d}\theta(T, \cdot) = 0$). We set g_θ, the Webster metric, which is a Riemannian metric associated to θ, and a (1,1)-tensor ϕ satisfying:

$$g_\theta(T, X) = \theta(X), \quad \phi\phi X = -X + \theta(X)T, \quad \forall X \in T(M), \tag{1}$$

$$g_\theta(X, Y) = -\frac{1}{2} d\theta(X, \phi Y), \forall X, Y \in \ker(\theta).$$

We define $J = \phi|_{\ker(\theta)}$. If g_θ is a Riemannian metric associated to θ, then $(M, \theta, g_\theta, \phi)$ is called a contact Riemannian manifold. We denote by $\triangle_{g_\theta}$ the metric Laplacian and we consider the operator

$$\triangle_\theta = \triangle_{g_\theta} - T^2.$$

If $\{T, X_1, \dots, X_n, Y_1, \dots, Y_n\}$ is an orthonormal basis of the tangent space, such that $Y_i = JX_i$ for every $i = 1, \dots, n$, then the Webster scalar curvature $\mathcal{S}_\theta$ is given by (see [24], for instance)

$$\mathcal{S}_\theta = \sum_{j=1}^{n} \left(Ric_{g_\theta}(X_j, X_j) + Ric_{g_\theta}(Y_j, Y_j) \right) + 4n. \tag{2}$$

Here, we have denoted the Ricci tensor by Ric_{g_θ}. Let $(M, \theta, g_\theta, \phi)$ be a contact Riemannian manifold and let u be a positive smooth function on M, we consider a new manifold $(M, \tilde{\theta}, \tilde{g}_\theta, \tilde{\phi})$, where $\tilde{\theta}$ is the contact form defined by

$$\tilde{\theta} = u^{p-2}\theta, \qquad p = \frac{2n+2}{n},$$

with ϕ and $\tilde{\phi}$ acting in the same way on $\ker(\theta) = \ker(\tilde{\theta})$. The scalar curvatures $\mathcal{S}_\theta$ and $\mathcal{S}_{\tilde{\theta}}$ are related by the following identity (see [24]):

$$-\triangle_\theta u + \frac{n}{4(n+1)} \mathcal{S}_\theta u = \frac{n}{4(n+1)} \mathcal{S}_{\tilde{\theta}} u^{p-1}. \tag{3}$$

Now let $\mathbb{H}^n \simeq \mathbb{R} \times \mathbb{C}^n \simeq \mathbb{R} \times \mathbb{R}^{2n}$ be the Heisenberg group. We denote the coordinates by

$$w = (t, z) = (t, x_1, y_1, \dots, x_{2n}, y_{2n})$$

and the group law

$$w \cdot w' = (t, z) \cdot (t', z') = (t + t' + 2\mathrm{Im}(z\overline{z'}), z + z') \quad \forall\, w, w' \in \mathbb{H}^n,$$

where $\mathrm{Im}(\cdot)$ denotes the imaginary part of a complex number and $z\overline{z'}$ is the standard Hermitian inner product in $\mathbb{C}^n$. Left translations on $\mathbb{H}^n$ are defined by

$$\tau : \mathbb{H}^n \to \mathbb{H}^n \qquad \tau_w(w') = w \cdot w' \quad \forall\, w \in \mathbb{H}^n$$

and dilations are

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n \qquad \delta_\lambda(t,z) = (\lambda^2 t, \lambda z) \quad \forall\, \lambda > 0.$$

We denote by $Q = 2n+2$ the homogeneous dimension of $\mathbb{H}^n$ with respect to δ_λ. On $\mathbb{H}^n$ we consider the contact form

$$\theta_n^{\mathbb{H}} = 2\sum_{j=1}^{n}(y_j \mathrm{d}x_j - x_j \mathrm{d}y_j) - \mathrm{d}t.$$

The canonical orthonormal basis (with respect to $g_{\theta_n^{\mathbb{H}}}$) of left invariant vector fields on $\mathbb{H}^n$ is

$$X_j^{\theta_n^{\mathbb{H}}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\right), \quad Y_j^{\theta_n^{\mathbb{H}}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}\right), \quad T^{\theta_n^{\mathbb{H}}} = -\frac{\partial}{\partial t}, \quad j = 1,\dots,n.$$

We set for every $j = 1,\dots,n$

$$\begin{aligned}
\phi^{\theta_n^{\mathbb{H}}}\left(X_j^{\theta_n^{\mathbb{H}}}\right) &= Y_j^{\theta_n^{\mathbb{H}}},\\
\phi^{\theta_n^{\mathbb{H}}}\left(\frac{\partial}{\partial x_j}\right) &= \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t},\\
\phi^{\theta_n^{\mathbb{H}}}\left(\frac{\partial}{\partial y_j}\right) &= -\frac{\partial}{\partial x_j} - 2y_j\frac{\partial}{\partial t},\\
\phi^{\theta_n^{\mathbb{H}}}\left(\frac{\partial}{\partial t}\right) &= 0.
\end{aligned} \tag{4}$$

Now let $\mathbb{C}^{n+1}$ endowed with its standard complex structure J and $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ be the unit sphere

$$S^{2n+1} = \{\zeta \in \mathbb{C}^{n+1} : |\zeta| = 1\}.$$

We denote by $\theta_n^{\mathbb{S}}$ its standard contact form

$$\theta_n^{\mathbb{S}} = \sum_{j=1}^{n+1}(v_j \mathrm{d}u_j - u_j \mathrm{d}v_j), \qquad \text{with } \zeta_j = u_j + iv_j$$

and by $g_{\theta_n^{\mathbb{S}}}$ the related standard metric. Then the Reeb vector field is

$$T^{\theta_n^{\mathbb{S}}} = \sum_{j=1}^{n+1}\left(v_j\frac{\partial}{\partial u_j} - u_j\frac{\partial}{\partial v_j}\right)$$

and the Webster scalar curvature is

$$S_{\theta_n^{\mathbb{S}}} = 4n^2 + 4n.$$

The Cayley transform identifies the Heisenberg group with the unit sphere minus a point. More precisely, for $P_S \in \mathbb{S}^{2n+1}$, $P_S = (0,\dots,0,-1)$ the Cayley transform is $C : \mathbb{H}^n \to \mathbb{S}^{2n+1} \setminus \{P_S\}$

$$C(t,z) = (\zeta_1,\dots,\zeta_{n+1}) = \left(\frac{2z}{1+|z|^2 - it}, \frac{1-|z|^2+it}{1+|z|^2-it}\right)$$

or equivalently

$$C(t, x_1, y_1, \dots, x_n, y_n) = (u_1, v_1 \dots, u_{n+1}, v_{n+1})$$

with

$$u_j = 2\frac{x_j(1+|z|^2) - ty_j}{t^2 + (1+|z|^2)^2}, \quad v_j = 2\frac{tx_j + (1+|z|^2)y_j}{t^2 + (1+|z|^2)^2}, \quad j = 1, \dots, n,$$
$$u_{n+1} = \frac{1 - |z|^4 - t^2}{t^2 + (1+|z|^2)^2}, \quad v_{n+1} = \frac{2t}{t^2 + (1+|z|^2)^2}.$$

Then the contact forms $\theta_n^{\mathbb{H}}$ and $\theta_n^{\mathbb{S}}$ are related by the following identity:

$$C^*\theta_n^{\mathbb{S}} = \frac{2}{t^2 + (1+|z|^2)^2}\theta_n^{\mathbb{H}}. \tag{5}$$

In the sequel, we will need the inverse of C, that is, $C^{-1} : \mathbb{S}^{2n+1} \setminus \{P_S\} \to \mathbb{H}^n$

$$C^{-1}(\zeta_1, \dots, \zeta_{n+1}) = (t, z_1, \dots, z_n) = \left(\mathrm{Re}\left(i\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}}\right), \frac{\zeta_1}{1+\zeta_{n+1}}, \dots, \frac{\zeta_n}{1+\zeta_{n+1}}\right),$$

or equivalently

$$C^{-1}(u_1, v_1 \dots, u_{n+1}, v_{n+1}) = (t, x_1, y_1, \dots, x_n, y_n),$$

where

$$t = \frac{2v_{n+1}}{v_{n+1}^2 + (1+u_{n+1})^2}, \quad x_j = \frac{u_j(1+u_{n+1}) + v_jv_{n+1}}{v_{n+1}^2 + (1+u_{n+1})^2}, \quad y_j = \frac{v_j(1+u_{n+1}) - u_jv_{n+1}}{v_{n+1}^2 + (1+u_{n+1})^2}$$

with $j = 1, \dots, n$.

3 | EXPLICIT CONSTRUCTION OF THE SINGULAR CONTACT STRUCTURE

Here, we will construct an explicit contact form $\theta_{k,N}$ on $\mathbb{S}^{2n+1} \setminus \mathbb{S}^{2k+1}$, which will be conformal to the standard CR contact form $\theta_n^{\mathbb{S}}$ of $\mathbb{S}^{2n+1}$, having complete Webster metric and constant Webster scalar curvature.

First of all, we transform the problem on $\mathbb{S}^{2n+1}$ into a problem on $\mathbb{H}^n$ using the Cayley transform. In $\mathbb{C}^{n+1}$, we choose coordinates so that our equatorial sphere $\mathbb{S}^{2k+1}$ is defined by

$$\mathbb{S}^{2k+1} := \{\zeta \in \mathbb{C}^{n+1} \ : \zeta = (\zeta_1, \dots, \zeta_k, 0, \dots, 0, \zeta_{n+1}), \quad |\zeta| = 1\} \subseteq \mathbb{S}^{2n+1},$$

then we stereographically project $\mathbb{S}^{2n+1}$ using C^{-1}. We observe that not all the equatorial spheres can be written as in the previous formula: In particular, the spheres that we are considering are intersection of $\mathbb{S}^{2n+1}$ and a complex linear subspace in $\mathbb{C}^{n+1}$. Notice that, with this choice of coordinates, the sphere $\mathbb{S}^{2k+1}$ is projected down into $\mathbb{H}^k$, so now we consider $\mathbb{H}^n$ endowed with the standard contact form $\theta_n^{\mathbb{H}}$ and we split

$$\mathbb{H}^n \simeq \mathbb{R} \times \mathbb{C}^n \simeq \mathbb{R} \times \mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)} \simeq \mathbb{H}^k \times \mathbb{R}^{2(n-k)}$$

with coordinates

$$(t, z_1, \dots, z_n) \simeq (t, x_1, y_1, \dots, x_{2n}, y_{2n}) \simeq (t, x_1, y_1, \dots, x_k, y_k, \hat{z}),$$

where $z_j = x_j + iy_j$, $j = 1, \dots, n$. Then, let us set $n - k = N + 1$, and $M = \mathbb{H}^k \times \mathbb{R} \times \mathbb{S}^{2N+1} \subseteq \mathbb{H}^k \times \mathbb{R} \times \mathbb{R}^{2(N+1)}$ and the map $\varphi : \mathbb{H}^n \to M$

$$\varphi(t, x_1, y_1, \dots, x_k, y_k, \hat{z}) = (t, x_1, y_1, \dots, x_k, y_k, s, \xi_1, \eta_1, \dots, \xi_{N+1}, \eta_{N+1}), \tag{6}$$

which is the identity on t, x_i, y_i, for $i = 1, \dots, k$, and

$$s = \ln(|\hat{z}|), \quad \xi_j = \frac{x_{k+j}}{|\hat{z}|}, \quad \eta_j = \frac{y_{k+j}}{|\hat{z}|} \quad j = 1, \dots, N+1.$$

On M we consider the contact form

$$\theta_{k,N} := \theta_N^{\mathbb{S}} + \frac{e^{-2s}}{2}\theta_k^{\mathbb{H}}.$$

The following proposition shows the relationship between $\theta_{k,N}$, $\theta_n^{\mathbb{H}}$, and $\theta_n^{\mathbb{S}}$

Proposition 3.1. *Using the notation above, we have*

$$(\varphi^{-1})^*\theta_n^{\mathbb{H}} = 2e^{2s}\theta_{k,N}$$

and

$$(\varphi^{-1} \circ C)^*\theta_n^{\mathbb{S}} = \frac{4e^{2s}}{t^2 + \left(1 + \sum_{i=1}^k (x_i^2 + y_i^2) + e^{2s}\right)^2}\theta_{k,N}. \tag{7}$$

Proof. By straightforward computation, we find

$$(\varphi^{-1})^*\mathrm{d}x_{j+k} = e^s\xi_j \mathrm{d}s + e^s \mathrm{d}\xi_j, \qquad (\varphi^{-1})^*\mathrm{d}y_{j+k} = e^s\eta_j \mathrm{d}s + e^s \mathrm{d}\eta_j, \quad j = 1, \dots, N+1,$$

hence

$$(\varphi^{-1})^*\theta_n^{\mathbb{H}} = -\mathrm{d}t + 2\sum_{i=1}^{k}(y_i \mathrm{d}x_i - x_i \mathrm{d}y_i) + 2e^{2s}\sum_{j=1}^{N+1}(\eta_j \mathrm{d}\xi_j - \xi_j \mathrm{d}\eta_j) = \theta_k^{\mathbb{H}} + 2e^{2s}\theta_N^{\mathbb{S}}. \tag{8}$$

Then, equality (7) follows from (5) and the identity above. □

Remark 3.1. Let us explicitly note that one can see the contact form $\theta_{k,N}$ defined on $\mathbb{H}^n \setminus \mathbb{H}^k$ with the singularity along $\mathbb{H}^k$, just by letting $r = \sqrt{2}|\hat{z}|$ (see also formula (19) in the sequel). We chose the variable $s = \ln(|\hat{z}|)$ in order to make the computations easier.

From now on, we will consider the contact manifold $(M, \theta_{k,N})$, where

$$M = \mathbb{H}^k \times \mathbb{R} \times \mathbb{S}^{2N+1}$$

with coordinates

$$(t, x_1, \dots, x_k, y_1, \dots, y_k, s, \xi_1, \dots, \xi_{N+1}, \eta_1, \dots, \eta_{N+1}) = (t, x, y, s, \xi, \eta), \qquad |(\xi, \eta)| = 1$$

and contact form $\theta_{k,N}$. Moreover, we consider the metric $g = g_{\theta_{k,N}}$ defined by (1) and the associated (1,1)-tensor $\phi^{\theta_{k,N}}$. In particular, since $\theta_{k,N}$ and $\theta_n^{\mathbb{H}}$ are conformal, we have that

$$J^{\theta_{k,N}} = \phi|_{\ker(\theta_{k,N})} = \phi|_{\ker(\theta_n^{\mathbb{H}})} = J^{\theta_n^{\mathbb{H}}}. \tag{9}$$

Moreover, we notice that the metric $g = g_{\theta_{k,N}}$ is complete. We will show that the Webster scalar curvature $S_{\theta_{k,N}}$ is constant. In order to compute $S_{\theta_{k,N}}$, we choose a particular orthonormal basis for T_pM. Let us notice that, since $\theta_{k,N} = \theta_N^{\mathbb{S}} + \frac{e^{-2s}}{2}\theta_k^{\mathbb{H}}$, the Reeb vector field $T^{\theta_{k,N}}$ of $(M, \theta_{k,N})$ is the Reeb vector field of $(\mathbb{S}^{2N+1}, \theta^{\mathbb{S}}_N)$, so

$$T := T_{\theta_{k,N}} = \sum_{j=1}^{N+1} \left(\eta_j \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \eta_j} \right).$$

We consider the following vector fields in $\ker(\theta_{k,N})$:

$$X_0 = \frac{\partial}{\partial s}, \qquad Y_0 = -2e^{2s}\frac{\partial}{\partial t} - T, \qquad X_i = \sqrt{2}e^s X_i^{\theta_k^{\mathbb{H}}}, \qquad Y_i = \sqrt{2}e^s Y_i^{\theta_k^{\mathbb{H}}}, \qquad i = 1, \dots, k.$$

By straightforward computations, we have

$$d\varphi^{-1}(X_0) = \sum_{j=1}^{N+1} \left(x_{j+k} \frac{\partial}{\partial x_{j+k}} + y_{j+k} \frac{\partial}{\partial y_{j+k}} \right) = \sqrt{2} \sum_{j=1}^{N+1} \left(x_{j+k} X_{j+k}^{\theta_n^{\mathbb{H}}} - y_{j+k} Y_{j+k}^{\theta_n^{\mathbb{H}}} \right),$$

$$d\varphi^{-1}(Y_0) = 2|\hat{z}|^2 \frac{\partial}{\partial t} + \sum_{j=1}^{N+1} \left(y_{j+k} \frac{\partial}{\partial x_{j+k}} - x_{j+k} \frac{\partial}{\partial y_{j+k}} \right) = \sqrt{2} \sum_{j=1}^{N+1} \left(y_{j+k} X_{j+k}^{\theta_n^{\mathbb{H}}} + x_{j+k} Y_{j+k}^{\theta_n^{\mathbb{H}}} \right),$$

then, recalling the identities $J^{\theta_n^{\mathbb{H}}} X_j^{\theta_n^{\mathbb{H}}} = Y_j^{\theta_n^{\mathbb{H}}}$ for every $j = 1, \dots, n$, the above computations show that

$$J^{\theta_{k,N}} X_0 = Y_0. \tag{10}$$

Similarly, for $i = 1, \dots, k$, we have

$$d\varphi^{-1}(X_i) = |\hat{z}|\sqrt{2} X_i^{\theta_n^{\mathbb{H}}} \quad \text{and} \quad d\varphi^{-1}(Y_i) = |\hat{z}|\sqrt{2} Y_i^{\theta_n^{\mathbb{H}}}$$

so

$$J^{\theta_{k,N}} X_i = Y_i, \quad i = 1, \dots, k. \tag{11}$$

Now we notice that the metric and the endomorphism ϕ induced from $(M, \theta_{k,N}, g, \phi)$ on $\mathbb{S}^{2N+1} \subseteq M$ are the standard ones. Indeed

$$d\varphi^{-1}(T) = d\varphi^{-1}\left(-2e^{2s}\frac{\partial}{\partial t} - Y_0 \right) = -2|\hat{z}|^2 \frac{\partial}{\partial t} - \sqrt{2} \sum_{j=1}^{N+1} \left(y_{j+k} X_{j+k}^{\theta_n^{\mathbb{H}}} + x_{j+k} Y_{j+k}^{\theta_n^{\mathbb{H}}} \right),$$

$$W_j := d\varphi^{-1}\left(\frac{\partial}{\partial \xi_j} - \eta_j T \right) = |\hat{z}| \frac{\partial}{\partial x_{j+k}} - \frac{y_{j+k}}{|\hat{z}|} d\varphi^{-1}(T), \quad j = 1, \dots, N+1,$$

$$Z_j := d\varphi^{-1}\left(\frac{\partial}{\partial \eta_j} + \xi_j T \right) = |\hat{z}| \frac{\partial}{\partial y_{j+k}} + \frac{x_{j+k}}{|\hat{z}|} d\varphi^{-1}(T) \quad j = 1, \dots, N+1.$$

Thus, recalling (4),

$$\phi^{\theta_n^{\mathbb{H}}}(W_j) = Z_j \quad j = 1, \dots, N+1,$$

we have

$$\phi^{\theta_{k,N}} \left(\frac{\partial}{\partial \xi_j} - \eta_j T \right) = \left(\frac{\partial}{\partial \eta_j} + \xi_j T \right),$$

and as usual $\phi^{\theta_{k,N}}(T) = 0$. Since the metric and the endomorphism ϕ induced on $\mathbb{S}^{2N+1}$ from $(M, \theta_{k,N}, g)$ are the standard ones, locally, at each point $p \in M$, we can consider $2N$ orthonormal geodesic Killing vector fields for $(\mathbb{S}^{2N+1}, \theta_N^{\mathbb{S}})$

$$U_j, \qquad V_j, \qquad j = 1, \dots, N, \tag{12}$$

such that $J^{\theta_n^{\mathbb{H}}} d\varphi^{-1}(U_j) = d\varphi^{-1}(V_j)$ and $U_j, V_j \in \ker(\theta_N^{\mathbb{S}})$.

We define the set $\mathcal{B} := \{X_0, Y_0, X_1, \dots, X_k, Y_1, \dots, Y_k, T, U_1, \dots, U_N, V_1, \dots, V_N\}$.

Proposition 3.2. *The set $\mathcal{B}$ is an orthonormal basis for TM, and $J^{\theta_{k,N}} = \phi^{\theta_{k,N}}|_{\ker \theta_{k,N}}$ acts as follows:*

$$J^{\theta_{k,N}} X_0 = Y_0, \quad J^{\theta_{k,N}} X_i = Y_i, \quad J^{\theta_{k,N}} U_j = V_j. \tag{13}$$

Proof. Identities in (13) follow from (10), (11), and the definition of U_j and V_j. Now it is straightforward to check that $\mathcal{B}$ is orthonormal using the definition of g (see (1)):

$$g(Z, W) = -\frac{1}{2} d\theta_{k,N}(Z, \phi W), \qquad d\theta_{k,N} = d\theta_N^{\mathbb{S}} - e^{-2s} ds \wedge \theta_k^{\mathbb{H}} + \frac{e^{-2s}}{2} d\theta_k^{\mathbb{H}},$$

$Z, W \in \ker \theta_{k,N}$. We just compute $g(X_0, X_0)$ as an example:

$$g(X_0, X_0) = -\frac{1}{2} d\theta_{k,N}(X_0, Y_0) = -\frac{1}{2}(-e^{-2s}) 2e^{2s} = 1.$$

□

We will compute the Webster scalar curvature $\mathcal{S}_{\theta_{k,N}}$ with the aid of three lemmas. Let ∇ be the Levi–Civita connection on $(M, \theta_{k,N}, g)$, then we have the following:

Lemma 3.1. *For every $j = 1, \dots, N$, we have*

$$\begin{aligned} &\nabla_T T = 0 && \nabla_T U_j = V_j && \nabla_T V_j = -U_j \\ &\nabla_{U_j} T = -V_j, && \nabla_{U_j} U_j = 0, && \nabla_{U_j} V_j = T, \\ &\nabla_{V_j} T = U_j, && \nabla_{V_j} U_j = -T, && \nabla_{V_j} V_j = 0. \end{aligned}$$

Proof. Since T, U_j, and V_j are geodesic, we have $\nabla_T T = 0$, $\nabla_{U_j} U_j = 0$, $\nabla_{V_j} V_j = 0$ for every $j = 1, \dots, N$. Moreover, U_j is a Killing vector field on $(\mathbb{S}^{2N+1}, g_{\theta_N^{\mathbb{S}}})$, so

$$g(\nabla_X U_j, Y) + g(X, \nabla_Y U_j) = 0 \text{ for every } X, Y \in T\mathbb{S}^{2N+1}. \tag{14}$$

We denote by $\tilde{J}$ the complex structure on $\mathbb{C}^{N+1}$, by ν the outward unit normal to $\mathbb{S}^{2N+1}$, and by $\tilde{g}$ and $\tilde{\nabla}$ the standard metric and Levi–Civita connection of $\mathbb{C}^{N+1}$, respectively. We will use the same notation for the induced metric and connection on $\mathbb{S}^{2N+1}$. Then, on $T\mathbb{S}^{2N+1} \subseteq TM$, $\tilde{J}T = \nu$, and $\tilde{J}, J^{\theta_{k,N}}$ have the same actions on $\ker\theta_N^{\mathbb{S}} \subseteq \ker\theta_{k,N}$ and $\tilde{g} = g_{\theta_{k,N}}$. Also, we denote by $h(Z, W) = \tilde{g}(\tilde{\nabla}_Z W, -\nu)$, $Z, W \in T\mathbb{S}^{2N+1}$, the second fundamental form of M restricted to $\mathbb{S}^{2N+1}$. Notice that, with respect to the basis $\{T, U_1, V_1, \dots, U_N, V_N\}$, the second fundamental form h is the $(2N+1) \times (2N+1)$ identity matrix. Since $\tilde{g}$ is Kähler, we have

$$\tilde{g}(\cdot, \cdot) = \tilde{g}(\tilde{J}\cdot, \tilde{J}\cdot), \quad \tilde{\nabla}\tilde{J}\cdot = \tilde{J}\tilde{\nabla}\cdot \tag{15}$$

Then, for every $j, l = 1, \dots, N$, we have

$$\begin{aligned} g(\nabla_T U_j, U_l) &\overset{(14)}{=} -g(T, \nabla_{U_l} U_j) = -\tilde{g}(T, \tilde{\nabla}_{U_l} U_j) = \\ &= -\tilde{g}(\tilde{J}T, \tilde{\nabla}_{U_l} \tilde{J} U_j) \overset{(15)}{=} \tilde{g}(\nu, \tilde{\nabla}_{U_l} V_j) = h(U_l, V_j) = 0 \end{aligned}$$

and similarly

$$g(\nabla_T U_j, V_l) \overset{(14)}{=} -g(T, \nabla_{V_l} U_j) \overset{(15)}{=} h(V_l, V_j) = \delta_{jl},$$

$$g(\nabla_T U_j, T) \overset{(14)}{=} -g(T, \nabla_T U_j) \overset{(15)}{=} h(T, V_j) = 0.$$

Also,

$$g(\nabla_T U_j, X_i) = 0, \qquad g(\nabla_T U_j, Y_i) = 0 \quad \text{for every } i = 0, \dots, k.$$

Thus,

$$\nabla_T U_j = V_j \quad \text{for every } j = 0, \dots, N.$$

Recalling that $V_j's$ are geodesic Killing vector fields, the same argument gives

$$\nabla_T V_j = -U_j \quad \text{for every } j = 0, \dots, N.$$

Moreover,

$$g(\nabla_{U_j} T, U_l) \overset{(15)}{=} h(U_j, V_j) = 0,$$

$$g(\nabla_{U_j} T, V_l) \overset{(15)}{=} -h(U_j, U_l) = -\delta_{jl},$$

$$g(\nabla_{U_j} T, T) = g(\nabla_{U_j} T, X_i) = g(\nabla_{U_j} T, Y_i) = 0 \qquad \text{for } i = 0, \dots, k.$$

Hence,

$$\nabla_{U_j} T = -V_i.$$

Since U_js are geodesic, we have $\tilde{\nabla}_{U_j} U_j = -\nu$, from which we get

$$\tilde{\nabla}_{U_j} V_j = \nabla_{U_j} V_j = T.$$

Analogous computations give $\nabla_{V_j} T = U_j$ and $\nabla_{V_j} U_j = -T$. □

In the sequel, we will use the following formula to compute some covariant derivatives:

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big\{ X\big(g(Y, Z)\big) + Y\big(g(Z, X)\big) - Z\big(g(X, Y)\big) + g\big([X, Y], Z\big) - g\big([Y, Z], X\big) - g\big([X, Z], Y\big) \Big\}, \tag{16}$$

where $X, Y, Z \in TM$. So, first we compute the necessary commutators.

Lemma 3.2. *For every $i, l = 1, \dots, k$ and every $j = 1, \dots, N$, we have*

$$\begin{array}{llll}
[X_0, Y_0] = 2Y_0 + 2T, & [X_0, X_i] = X_i, & [X_0, Y_i] = Y_i, & [X_0, T] = 0, \\
[X_0, U_j] = 0, & [X_0, V_j] = 0, & [Y_0, X_i] = 0, & [Y_0, Y_i] = 0, \\
[Y_0, T] = 0, & [Y_0, U_j] = -2V_j, & [Y_0, V_j] = 2U_j, & [X_i, X_l] = 0, \\
[X_i, Y_l] = \delta_{il}(2Y_0 + 2T), & [X_i, T] = 0, & [X_i, U_j] = 0, & [X_i, V_j] = 0,
\end{array}$$

$$[Y_i, Y_l] = 0, \qquad [Y_i, T] = 0, \qquad [Y_i, U_j] = 0, \qquad [Y_i, V_j] = 0,$$
$$[U_j, T] = -2V_j, \qquad [V_j, T] = 2U_j.$$

Proof. Using Lemma 3.1, for every $j = 1, \dots, N$, we compute

$$[U_j, T] = \nabla_{U_j} T - \nabla_T U_j = -2V_j,$$
$$[V_j, T] = \nabla_{V_j} T - \nabla_T V_j = 2U_j,$$

from which we get

$$[Y_0, U_j] = [-T, U_j] = -2V_j,$$
$$[Y_0, V_j] = [-T, V_j] = 2U_j.$$

All the other commutators are computed using the explicit expression of the vector fields involved and the fact that M is a product manifold. □

Using (16) and Lemma 3.2, we compute the following covariant derivatives:

Lemma 3.3. *For every $i, l = 1, \dots, k$ and every $j = 1, \dots, N$, we have*

$$\nabla_{X_0} X_0 = 0, \qquad \nabla_{X_0} Y_0 = T, \qquad \nabla_{X_0} X_i = 0, \qquad \nabla_{X_0} Y_i = 0,$$
$$\nabla_{X_0} T = -Y_0, \qquad \nabla_{X_0} U_j = 0, \qquad \nabla_{X_0} V_j = 0, \qquad \nabla_{Y_0} X_0 = -2Y_0 - T,$$
$$\nabla_{Y_0} Y_0 = 2X_0, \qquad \nabla_{Y_0} X_i = -Y_i, \qquad \nabla_{Y_0} Y_i = X_i, \qquad \nabla_{Y_0} T = X_0,$$
$$\nabla_{Y_0} U_j = -2V_j, \qquad \nabla_{Y_0} V_j = 2U_j, \qquad \nabla_{X_i} X_0 = -X_i, \qquad \nabla_{X_i} Y_0 = -Y_i,$$
$$\nabla_{X_i} X_l = \delta_{il} X_0, \qquad \nabla_{X_i} Y_l = \delta_{il}(T + Y_0), \qquad \nabla_{X_i} T = -Y_i, \qquad \nabla_{X_i} U_j = 0$$
$$\nabla_{X_i} V_j = 0, \qquad \nabla_{Y_i} X_0 = -Y_i, \qquad \nabla_{Y_i} Y_0 = X_i, \qquad \nabla_{Y_i} X_l = -\delta_{il}(T + Y_0),$$
$$\nabla_{Y_i} Y_l = \delta_{il} X_0, \qquad \nabla_{Y_i} T = X_i, \qquad \nabla_{Y_i} U_j = 0, \qquad \nabla_{Y_i} V_j = 0,$$
$$\nabla_T X_0 = -Y_0, \qquad \nabla_T Y_0 = X_0, \qquad \nabla_T X_i = -Y_i, \qquad \nabla_T Y_i = X_i,$$
$$\nabla_{U_j} X_0 = 0, \qquad \nabla_{U_j} Y_0 = 0, \qquad \nabla_{U_j} X_i = 0, \qquad \nabla_{U_j} Y_i = 0,$$
$$\nabla_{V_j} X_0 = 0, \qquad \nabla_{V_j} Y_0 = 0, \qquad \nabla_{V_j} X_i = 0, \qquad \nabla_{V_j} Y_i = 0.$$

Proof. Since $\mathcal{B}$ is an orthonormal basis, formula (16) reduces to

$$g(\nabla_X Y, Z) = \frac{1}{2}\{g([X, Y], Z) - g([Y, Z], X) - g([X, Z], Y)\}, \quad \textit{for every } X, Y, Z \in \mathcal{B}.$$

Here, we compute $\nabla_{X_0} X_0$ as an example, the other covariant derivatives are computed similarly. Recalling Lemma 3.2, for every $i = 1, \dots, k$, and $j = 1, \dots, N$, we have

$$g(\nabla_{X_0} X_0, X_0) = 0,$$
$$g(\nabla_{X_0} X_0, Y_0) = -g([X_0, Y_0], X_0) = -g(2Y_0 + 2T, X_0) = 0,$$
$$g(\nabla_{X_0} X_0, X_i) = -g([X_0, X_i], X_0) = 0,$$
$$g(\nabla_{X_0} X_0, Y_i) = -g([X_0, Y_i], X_0) = 0,$$

$$g(\nabla_{X_0}X_0, T) = -g([X_0, T], X_0) = 0,$$
$$g(\nabla_{X_0}X_0, U_j) = -g([X_0, U_j], X_0) = 0,$$
$$g(\nabla_{X_0}X_0, V_j) = -g([X_0, V_j], X_0) = 0.$$

Thus, $\nabla_{X_0}X_0 = 0$. □

Now we are ready to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. We first note that the Webster metric associated to $\theta_{k,N}$ is complete. So, it remains to compute $\mathcal{S}_{\theta_{k,N}}$. For every $W \in \mathcal{B}$, we have

$$Ric_g(W, W) = \sum_{Z\in\mathcal{B}} g\big(\nabla_Z\nabla_W W - \nabla_W\nabla_Z W - \nabla_{[Z,W]}W, Z\big). \tag{17}$$

We explicitly compute $Ric_g(X_i, X_i)$ for every $i = 1, \dots, k$. By Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned}
Ric_g(X_i, X_i) &= \sum_{Z\in\mathcal{B}} g\big(\nabla_Z\nabla_{X_i}X_i - \nabla_{X_i}\nabla_Z X_i - \nabla_{[Z,X_i]}X_i, Z\big)\\
&= \sum_{Z\in\mathcal{B}} g\big(\nabla_Z X_0 - \nabla_{X_i}\nabla_Z X_i - \nabla_{[Z,X_i]}X_i, Z\big)\\
&= g\big(-\nabla_{X_i}X_i, X_0\big) + g\big(\nabla_{Y_0}X_0 + \nabla_{X_i}Y_i, Y_0\big) + \sum_{l=1}^{k} g\big(\nabla_{X_l}X_0 - \delta_{li}\nabla_{X_i}X_0, X_l\big) +\\
&\quad + \sum_{l=1}^{k} g\big(\nabla_{Y_l}X_0 - \delta_{li}\nabla_{X_i}(T + Y_0) + \nabla_{\delta_{li}(2Y_0+2T)}X_i, Y_l\big) +\\
&\quad + g\big(\nabla_T X_0 - \nabla_{X_i}Y_i, T\big) + \sum_{l=1}^{N} g\big(\nabla_{U_l}X_0, U_l\big) + \sum_{l=1}^{N} g\big(\nabla_{V_l}X_0, V_l\big)\\
&= g(-X_0, X_0) + g(-2Y_0 - T + T + Y_0, Y_0) + \sum_{l=1}^{k} g(-X_l + \delta_{li}X_i, X_l) +\\
&\quad + \sum_{l=1}^{k} g(-Y_l - 6\delta_{li}Y_i, Y_l) + g(-Y_0 + T + Y_0, T) + 0 + 0\\
&= -1 - 1 + (-k + 1) + (-k - 6) + 1 + 0 + 0\\
&= -6 - 2k.
\end{aligned}$$

Similarly,

$$Ric_g(Y_i, Y_i) = -1 - 1 + (-k - 6) + (-k + 1) + 1 + 0 + 0 = -6 - 2k,$$
$$Ric_g(X_0, X_0) = 0 - 7 - k - k + 1 + 0 + 0 = -6 - 2k,$$
$$Ric_g(Y_0, Y_0) = -7 + 0 - k - k + 1 + 0 + 0 = -6 - 2k,$$

here we have considered (17) with $W \in \mathcal{B}$, Z running in the ordered basis $\mathcal{B}$, and we have written, in the order, each of the terms in the sum in the right-hand side of (17). Moreover, since $M = \mathbb{H}^k \times \mathbb{R} \times \mathbb{S}^{2N+1}$ and $\{T, U_1, V_1, \dots, U_N, V_N\}$ is an orthonormal basis for $T\mathbb{S}^{2N+1}$ with respect to the metric $g_{\theta^{\mathbb{S}}_N}$, we have

$$\begin{aligned} Ric_g(U_j, U_j) &= Ric_{g_{\theta^{\mathbb{S}}_N}}(U_j, U_j) + \sum_{\substack{Z=X_0,Y_0,X_i,Y_i\\ i=1,\dots,k}} g\Big(\nabla_Z\nabla_{U_j}U_j - \nabla_{U_j}\nabla_Z U_j - \nabla_{[Z,U_j]}U_j, Z\Big)\\ &= Ric_{g_{\theta^{\mathbb{S}}_N}}(U_j, U_j) = 2N \end{aligned}$$

and

$$Ric_g(V_j, V_j) = 2N.$$

Hence, recalling (2) and the definition $N = n - k - 1$, we have

$$\begin{aligned} \mathcal{S}_{\theta_{k,N}} &= (2k+2)(-6-2k) + (N+N)2N + 4n\\ &= 4\Big((N-k)(N+k) + 2(N-k) - (N+k)\Big)\\ &= 4(N+k+2)(N-k-1), \end{aligned}$$

that is,

$$\mathcal{S}_{\theta_{k,N}} = 4(n+1)(n-2k-2).$$

In particular, we notice that $\mathcal{S}_{\theta_{k,N}}$ is positive for $k < \frac{n-2}{2}$. □

4 SINGULARITY ALONG A CIRCLE

Here, we will use the explicit contact structure that we found in order to obtain some existence results as applications.

We will need the explicit expression of $\triangle_{\theta_{k,N}}$, which is

$$\triangle_{\theta_{k,N}} = T^2 + \triangle_{\theta^{\mathbb{S}}_N} + 2e^{2s}\,\triangle_{\theta^{\mathbb{H}}_k} + 4e^{4s}\frac{\partial^2}{\partial t^2} - 4e^{2s}T\frac{\partial}{\partial t} + \frac{\partial^2}{\partial s^2} - 2(k+1)\frac{\partial}{\partial s}.$$

Indeed we have

$$\begin{aligned} X_0^2 &= \frac{\partial^2}{\partial s^2},\\ Y_0^2 &= T^2 + 4e^{4s}\frac{\partial^2}{\partial t^2} + 4e^{2s}T\frac{\partial}{\partial t},\\ X_i^2 &= 2e^{2s}\left(X_i^{\theta^{\mathbb{H}}_k}\right)^2,\\ Y_i^2 &= 2e^{2s}\left(Y_i^{\theta^{\mathbb{H}}_k}\right)^2 \quad \text{for } i = i, \dots, k, \end{aligned}$$

so

$$\sum_{i=1}^{k}(X_i^2 + Y_i^2) = 2e^{2s}\triangle_{\theta^{\mathbb{H}}_k},$$

and by Lemma 3.3,

$$\nabla_{X_0}X_0 = 0, \quad \nabla_{Y_0}Y_0 = 2\frac{\partial}{\partial s}, \quad \nabla_{X_i}X_i = \frac{\partial}{\partial s}, \quad \nabla_{Y_i}Y_i = \frac{\partial}{\partial s} \quad \nabla_T T = 0,$$

for $i = 1, \dots, k$. Hence,

$$\begin{aligned}\triangle_{\theta_{k,N}} &= \triangle_{g_{\theta_{k,N}}} - T^2 \\ &= X_0^2 - \nabla_{X_0}X_0 + Y_0^2 - \nabla_{Y_0}Y_0 + \sum_{i=1}^{k}(X_i^2 + Y_i^2) - \sum_{i=1}^{k}(\nabla_{X_i}X_i + \nabla_{Y_i}Y_i) \\ &\quad + \sum_{j=1}^{N+1}(U_j^2 + V_j^2) - \sum_{j=1}^{N+1}(\nabla_{U_j}U_j + \nabla_{V_j}V_j) - \nabla_T T \\ &= \frac{\partial^2}{\partial s^2} + T^2 + 4e^{4s}\frac{\partial^2}{\partial t^2} + 4e^{2s}T\frac{\partial}{\partial t} - 2\frac{\partial}{\partial s} + 2e^{2s}\triangle_{\theta_k^{\mathbb{H}}} - 2k\frac{\partial}{\partial s} + \triangle_{\theta_N^{\mathbb{S}}}. \end{aligned} \tag{18}$$

Next, we will need a kind of expansion of the Webster scalar curvature. So let us consider (6) with the additional change of variable $r = \sqrt{2}e^s$. We denote it by $\bar{\varphi}^{-1} : \mathbb{H}^n \to \mathbb{H}^k \times (0, +\infty) \times S^{2N+1}$. In these coordinates, the standard contact form of $\mathbb{H}^n$ is

$$\bar{\theta} = \bar{\varphi}^*\theta_n^{\mathbb{H}} = \theta_k^{\mathbb{H}} + r^2\theta_N^{\mathbb{S}} = r^2\theta_{k,N} \tag{19}$$

with $\bar{\phi}$ the related (1,1)-tensor. We consider smooth perturbations (let us say at least C^3) $(\hat{\theta}, \hat{\phi})$ as

$$\hat{\theta} = (1 + O(r^2))\bar{\theta}, \quad \hat{\phi} = \bar{\phi}, \quad \text{as } r \to 0. \tag{20}$$

We recall that the tensor field $\hat{\varphi}$ depends on the contact distribution and not the form. Hence, it is invariant under a conformal change of the contact form $\hat{\theta}$. We have the following:

Proposition 4.1. *Let $(\hat{\theta}, \hat{\phi})$ be as in (20) and consider $\tilde{\theta} = r^{-2}\hat{\theta}$. Then, the Webster scalar curvature of $(M, \tilde{\theta}, \hat{\phi})$ is*

$$\mathcal{S}_{\tilde{\theta}} = \mathcal{S}_{\theta_{k,N}} + O(r^2), \ \text{as } r \to 0.$$

Proof. Let f be a smooth function depending on r, by (19), we let

$$\tilde{\theta} = r^{-2}\hat{\theta} = r^{-2}(1 + f(r))\bar{\theta} = (1 + f(r))\theta_{k,N} = (u(r))^{\frac{2}{n}}\theta_{k,N},$$

where $u(r) = (1 + f(r))^{\frac{n}{2}}$. We will use formula (3)

$$-\triangle_{\theta_{k,N}} u + \frac{n}{4(n+1)}\mathcal{S}_{\theta_{k,N}}u = \frac{n}{4(n+1)}\mathcal{S}_{\tilde{\theta}}u^{\frac{2}{n}+1}.$$

Now, after the change of variable $r = \sqrt{2}e^s$ $(\partial_s = r\partial_r)$, by formula (18), we have

$$\triangle_{\theta_{k,N}}u(r) = r^2u''(r) - (2k+1)ru'(r).$$

We compute

$$u'(r) = \frac{n}{2}(1 + f(r))^{\frac{n}{2}-1}f'(r),$$

$$u''(r) = \frac{n}{2}(1+f(r))^{\frac{n}{2}-1} f''(r) + \frac{n}{2}\left(\frac{n}{2}-1\right)(1+f(r))^{\frac{n}{2}-2}(f'(r))^2,$$

therefore,

$$\triangle_{\theta_{k,N}} u(r) = \frac{n}{2}(1+f(r))^{\frac{n}{2}-1}\left[r^2 f''(r) - (2k+1) r f'(r) + \left(\frac{n}{2}-1\right)(1+f(r))^{-1} r^2 (f'(r))^2\right].$$

We obtain

$$\begin{aligned} S_{\tilde{\theta}} &= (1+f(r))^{-1} S_{\theta_{k,N}} \\ &\quad - 2(n+1)(1+f(r))^{-2}\left[r^2 f''(r) - (2k+1) r f'(r) + \left(\frac{n}{2}-1\right)(1+f(r))^{-1}(r f'(r))^2\right]. \end{aligned}$$

Now, if f is a smooth function such that $f(r) = O(r^2)$ as $r \to 0$, then for every $a \in \mathbb{R}$, we have $(1+f(r))^a = 1 + O(r^2)$, as $r \to 0$; also $r f'(r) = O(r^2)$ and $r^2 f''(r) = O(r^2)$, as $r \to 0$. Substituting in the last equation, we get

$$S_{\tilde{\theta}} = S_{\theta_{k,N}} + O(r^2), \text{ as } r \to 0. \qquad \square$$

4.1 | Existence by perturbation

In this subsection, we will follow closely the perturbation approach developed in [19]. First, let us set $L_\theta = \Delta_\theta - \frac{n}{4(n+1)} S_\theta$. We consider a smooth diffeomorphism τ that is close to the identity in the $C^{3,\alpha}$-topology. We will focus on the restriction of τ to the equatorial S^{2k+1} as detailed above. Hence, we view this restriction as an embedding $\tau : \mathbb{S}^{2k+1} \to \mathbb{S}^{2n+1}$ close to the identity, and we want to find a complete contact structures on $\mathbb{S}^{2n+1} \setminus \tau(\mathbb{S}^{2k+1})$ having constant Webster curvature. Namely, we want to solve on $\mathbb{S}^{2n+1} \setminus \tau(\mathbb{S}^{2k+1})$, the problem

$$L_{\theta_n^{\mathbb{S}}} v + \frac{n}{4(n+1)} S_{\theta_{k,N}} v^{p-1} = 0,$$

with v a positive function that blows up on $\tau(S^{2k+1})$. This is equivalent to solving the problem

$$L_{\theta(\tau)} v + \frac{n}{4(n+1)} S_{\theta_{k,N}} v^{p-1} = 0,$$

where $\theta(\tau) = u^{\frac{2}{n}} \tau^* \theta_n^{\mathbb{S}}$ and u is the function giving the conformal change from $\theta_n^{\mathbb{S}}$ to $\theta_{k,N}$. Since we plan to perturb the equation with respect to the diffeomorphism τ and around the constant solution 1, we introduce the functional

$$K(\tau, w) = L_{\theta(\tau)}(1+w) + \frac{n}{4(n+1)} S_{\theta_{k,N}} (1+w)^{p-1}.$$

We want then to solve $K(\tau, w) = 0$ via the implicit function theorem, after perturbation around $(\mathrm{id}, 0)$. So we start by linearizing with respect to w:

$$\partial_w K(\tau, w)_{|(\mathrm{id},0)} = \Delta_{\theta_{k,N}} + 2(n - 2k - 2).$$

Now, we will consider the case $\tau : \mathbb{S}^1 \to \mathbb{S}^{2n+1}$, that is, $k = 0, N = n-1$. Notice that the expression of $\Delta_{\theta_{0,n-1}}$ is quite complicated. But, if we restrict it to functions that are invariant under T, then its expression becomes more familiar. Though, this restriction on the function space needs to be preserved by $\Delta_{\theta(\tau)}$, and that is why one needs to choose the diffeomorphism τ carefully. For instance, it needs to commute with the action of T. We will provide below an explicit description of the set of considered diffeomorphisms. In this setting, the operator L takes the form

$$L = \Delta_{\mathbb{S}^{2n-1}} + 4e^{4s}\partial_t^2 + \partial_s^2 - 2\partial_s.$$

If one now uses the change of variable $r = e^{2s}$, one gets

$$L = \Delta_{\mathbb{S}^{2n-1}} + 4r^2\partial_t^2 + 4r^2\partial_r^2 = \Delta_{\mathbb{S}^{2n-1}} + 4\Delta_{\mathcal{H}^2}, \tag{21}$$

where $\mathcal{H}^2 = H\mathbb{R}^2$ is the standard hyperbolic space of dimension 2. The linearized equation becomes then

$$L_1 = \Delta_{\mathbb{S}^{2n-1}} + 4\Delta_{\mathcal{H}^2} + 2(n-2).$$

So we first investigate its kernel. For this purpose, we move to the unit disk model of the hyperbolic space with coordinates $x = (\sigma, \vartheta, y)$ where $\sigma \in [0,1]$, $\vartheta \in \mathbb{S}^1$, and $y \in \mathbb{S}^{2n-1}$. We introduce then the family of spaces $C^{\nu,\alpha,k}(\mathbb{S}^{2n-1} \times \mathcal{H}^2)$ that are adapted to the study of singular problems (see [16–19]) by

$$C^{k,\alpha,\nu}(\mathbb{S}^{2n-1} \times \mathcal{H}^2) := \{u \in C^{k,\alpha}_{loc}(\mathbb{S}^{2n-1} \times \mathcal{H}^2); \|u\|_{k,\alpha,\nu} < \infty\},$$

where

$$\|u\|_{k,\alpha,\nu} = \sup_{x_1,x_2 \in \mathbb{S}^{2n-1}\times\mathcal{H}^2} (\sigma_1+\sigma_2)^{-\nu}\left(\sum_{j=1}^{k}(\sigma_1+\sigma_2)^j|\nabla^j u| + (\sigma_1+\sigma_2)^{k+\alpha}[\nabla^k u]_\alpha\right).$$

Now, because of our restriction on the functions, we will be working on the space

$$C^{k,\alpha,\nu}_T(S^{2n-1} \times \mathcal{H}^2) := \{u \in C^{k,\alpha,\nu}(S^{2n-1} \times \mathcal{H}^2); Tu = 0\}.$$

In these coordinates, we can express the operator L_1 as follows:

$$L_1 = \left[(1-\sigma^2)^2\partial_\sigma^2 + \frac{(1-\sigma^2)^2}{\sigma}\partial_\sigma + \frac{(1-\sigma^2)^2}{\sigma^2}\Delta_{\mathbb{S}^1}\right] + \Delta_{\mathbb{S}^{2n-1}} + 2(n-2),$$

where $\sigma \in (0,1)$. We look for solutions of the form $u = \sum_{i,j} a_{i,j}(\sigma)\phi_i\psi_j$, where the ψ_j are T-invariant eigenfunctions of $\Delta_{\mathbb{S}^{2n-1}}$ with eigenvalue λ_j and the ϕ_i are the eigenfunctions of $\Delta_{\mathbb{S}^1}$ with eigenvalue μ_i (see [19], formula (2.13) with the squared eigenvalues). This yields the family of equations

$$A_{i,j}a_{i,j} = 0,$$

where

$$A_{i,j} = (1-\sigma^2)^2\left[\partial_\sigma^2 + \frac{1}{\sigma}\partial_\sigma - \frac{\mu_i}{\sigma^2}\right] - \lambda_j + 2(n-2).$$

This is a Bessel-type equation and the singularity at 0 and 1 is regular. Since we are looking for bounded solutions, there is only a unique regular solution to this equation corresponding to the indicial root $\gamma = i \in \mathbb{N}$, that is a function rotationally invariant. So, we move now to the singularity at 1. We set $\rho = 1 - \sigma^2$, then the operator $A_{i,j}$ becomes

$$A_{i,j} = 4\rho^2\left[(1-\rho)\partial_\rho^2 - \partial_\rho\right] - \frac{\rho^2}{1-\rho}\mu_i - \lambda_j + 2(n-2).$$

In this case, the indicial roots take the form

$$\gamma_j^\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \lambda_j - 2(n-2)}.$$

We recall that the surjectivity of the operator L_1 depends on the value ν_0^+ (using the notation of [19, Theorem 4.54]). The value of ν_0^+ depends on the real part of the indicial roots. Indeed, we notice that if $j > 0$, then $\gamma^\pm$ are real and in that case $\nu_0^+ > \frac{1}{2}$. But, if $j = 0$, and hence, $\lambda_j = 0$, then $\gamma^\pm$ is complex and its real part is $\frac{1}{2}$. Hence, we set $\nu_0 = \frac{1}{2}$ and the function space that we will take is $C^{2,\alpha,\nu}_T(\mathbb{S}^{2n-1} \times \mathcal{H}^2)$ where $\nu < \frac{1}{2}$. The kernel is then

$$\mathcal{K}(\alpha,\nu) = \{u \in C^{2,\alpha,\nu}_T; Lu = 0\}.$$

We recall now a result of Mazzeo–Smale [19, Theorem 4.54].

Lemma 4.1 [19]. *For $\nu < \frac{1}{2}$, the operator $L_1 : C^{2,\alpha,\nu}_T \to C^{0,\alpha,\nu}_T$ is onto.*

As suggested by the referee, we point out that there is a different way of finding the indicial roots and hence deducing the surjectivity with less technicalities, using the results in [16] .

We recall that a vector field X is said to be a contact vector field for a given contact form θ, if and only if there exists a smooth function f on M such that $\mathcal{L}_X\theta = f\theta$. The set of contact vector fields forms a Lie algebra that we denote by $CVect(M,\theta)$ and there exists a one-to-one correspondence between $CVect(M,\theta)$ and $C^\infty(M)$. The flow of a vector field $X \in CVect(M,\theta)$ generates a one-parameter family of contactomorphisms. We then consider the set

$$\mathcal{T}_T := \left\{\varphi_1^X(\cdot); X \in r^2 CVect(M,\bar{\theta}); [X,T] \in Span(T)\right\},$$

where φ_1^X is the flow generated by X at time one. Notice now that if $\tau \in \mathcal{T}_T$, then $\tau^*\overline{\theta} = (1 + O(r^2))\overline{\theta}$.

Proposition 4.2. *The map K is C^∞ from a neighborhood $\mathcal{N}$ of $(\mathrm{id}, 0) \in \mathcal{T}_T \times C_T^{2,\alpha,\nu}(\mathbb{S}^{2n-1} \times \mathcal{H}^2)$ to $C_T^{0,\alpha,\nu}(\mathbb{S}^{2n-1} \times \mathcal{H}^2)$.*

Proof. It is clear that $\mathcal{N}$ is mapped to $C^{0,\alpha}_{loc}$. Based on the construction above, we have $\theta(\tau) = (1 + O(r^2))\theta_{k,N}$. So by Proposition 4.1, we compute

$$K(\tau, w) - K(\mathrm{id}, 0) =$$

$$\Delta_{\theta(\tau)}(1 + w) - \Delta_{\theta_{k,N}}1 - \frac{n}{4(n+1)}(\mathcal{S}_{\theta(\tau)}(1 + w) - \mathcal{S}_{\theta_{k,N}}) + \frac{n}{4(n+1)}\mathcal{S}_{\theta_{k,N}}((1 + w)^{p-1} - 1).$$

Clearly, $\Delta_{\theta(\tau)}(1 + w) - \Delta_{\theta_{k,N}}1 \in C_T^{0,\alpha,\nu}$. Next, we have that $\mathcal{S}_{\theta(\tau)} = \mathcal{S}_{\theta_{k,N}} + O(r)$ hence, the second term also belongs to $C_T^{0,\alpha,\nu}$ and similarly for the third term. The higher order derivatives of K can be treated in a similar way. □

Theorem 4.1. *Let $0 < \nu < \frac{1}{2}$, then there exists a closed subspace W such that $C_T^{2,\alpha,\nu} = W \oplus \mathcal{K}(\alpha,\nu)$ and a smooth map $\Phi : \mathcal{N} \subset \mathcal{T}_T \times \mathcal{K}(\alpha,\nu) \to W$ such that $K(\tau, w) = 0$, where $w = (\Phi(\tau, w_1), w_1) \in W \oplus \mathcal{K}(\alpha,\nu)$.*

Proof. The proof is a direct corollary from the implicit function theorem and Lemma 4.1. □

As a corollary, we get our first application Theorem 1.2.

4.2 | Existence by bifurcation

In this last subsection, we will show the existence of another kind of solutions to the CR Yamabe problem on $S^{2n+1} \setminus S^1$, via bifurcation (in the sense of Definition 4.1 below), following the work [4]. We recall again that similarly to the previous section, the operator L, in (21), takes the form $L = \Delta_{\mathbb{S}^{2n-1}} + 4\Delta_{\mathcal{H}^2}$, when restricted to functions invariant under T and we propose to solve the problem

$$-Lu + \frac{n}{4(n+1)}\mathcal{S}_{\theta_{0,n-1}}u = \frac{n}{4(n+1)}\kappa u^{p-1}, \tag{22}$$

where κ is a positive constant. Notice now that the problem is purely Riemannian. That is, all the operators involved depend on the Riemannian metric defined on $S^{2n-1} \times \mathcal{H}^2$. A solution of (22) corresponds to a complete solution to our problem in $S^{2n+1} \setminus S^1$. Hence, now, we are just dealing with an analytical problem in a Riemannian setting rather than a geometrical problem. After taking the quotient of $\mathcal{H}^2$ by a Fuchsian group $\Gamma \subset PSL(2,\mathbb{R})$, we can reduce the study to the manifold $M = \mathbb{C}P^{n-1} \times \Sigma_\Gamma$, where $\Sigma_\Gamma = \mathcal{H}^2/\Gamma$ and $\mathbb{C}P^{n-1} = \mathbb{S}^{2n-1}/\mathbb{S}^1$ since the vector field T generates an $\mathbb{S}^1$ isometric action corresponding to the Hopf fibration. Notice that there is no need to verify that the action of Γ preserves the original contact structure since we are interested in solving a purely analytical problem. From now on, we will write Σ instead of Σ_Γ and we define the space $\mathcal{M}(\Sigma)$ of hyperbolic metrics on Σ. In this way, we can track the change of the hyperbolic structure by using the metrics g_Σ. Now, given $g_\Sigma \in \mathcal{M}(\Sigma)$, we define the Banach manifold

$$\mathcal{M}_{\Sigma,g_\Sigma} = \left\{u \in H^1(M); \int_M u^p\, dv_g = Vol_g(M); u > 0\ a.e.\right\},$$

where $g = g_{\mathbb{C}P^{n-1}} \oplus \frac{1}{2} g_\Sigma$, $H^1(M)$ is the classical Sobolev space on M with the metric g, dv_g is the volume form corresponding to the metric g, and $Vol_g(M)$ is the volume of the metric M with respect to g. Next, we introduce the functional defined on $\mathcal{M}_{\Sigma, g_\Sigma}$ by

$$\mathcal{A}_g(u) = \frac{1}{2} \int_M \left(|\nabla_{M,g} u|^2 + \frac{n}{4(n+1)} S_{\theta_{0,n-1}} u^2 \right) dv_g,$$

where $\nabla_{M,g} = \nabla_{\mathbb{C}P^{n-1}} \oplus 2\nabla_{g_\Sigma}$. Clearly, critical points of $\mathcal{A}_g$ lift to solutions to the problem (22). We notice also that 1 is always a solution to our problem with $\kappa = S_{\theta_{0,n-1}}$. We have then

$$\nabla \mathcal{A}_g(u) = L_M u + \frac{n}{4(n+1)} S_{\theta_{0,n-1}} u - \frac{n}{4(n+1)} \kappa u^{p-1},$$

where $L_M = -\Delta_{\mathbb{C}P^{n-1}} - 4\Delta_{\Sigma,g}$ and

$$J_{\Sigma,g} = \nabla^2 \mathcal{A}_g(1) = L_M - 2(n-2).$$

The operator $J_{\Sigma,g}$ is the Jacobi operator, corresponding to the functional $\mathcal{A}_g$ at the critical point $u = 1$. We refer the reader to [4, Section 3] for more details about the construction above. We want to investigate the negative eigenvalues of $J_{\Sigma,g}$, which correspond to the Morse index of $\mathcal{A}_g$ at the critical point 1. So we consider the number

$$n_t(g_\Sigma) := \max\{k \in \mathbb{N} \ : \ \lambda_k(g_\Sigma) \le t\},$$

where $0 < \lambda_1(g_\Sigma) \le \cdots \le \lambda_k(g_\Sigma) \le \cdots$ are the eigenvalues of the Laplacian on (Σ, g_Σ) repeated according to their multiplicities. The next two lemmas are in [4].

Lemma 4.2 [4]. *Let $t > \frac{1}{4}$, and fix $g_0 \in \mathcal{M}(\Sigma)$, then for any $k \in \mathbb{N}$, there exists $g_1 \in \mathcal{M}$ such that $n_t(\Sigma, g_1) \ge k + n_t(\Sigma, g_0)$.*

Lemma 4.3 [4]. *Given a hyperbolic surface Σ and $\lambda > \frac{1}{4}$, then the set $\mathcal{M}_\lambda(\Sigma) = \{g \in \mathcal{M}(\Sigma); \lambda \notin \sigma(-\Delta_{g_\Sigma})\}$ is open and dense in $\mathcal{M}(\Sigma)$.*

Now we notice that every eigenvalue λ_ℓ of $J_{\Sigma,g}$ takes the form

$$\lambda_\ell = 4\lambda_j(g_\Sigma) + \lambda_k(\mathbb{C}P^{n-1}) - 2(n-2),$$

for a certain $j, k \ge 0$.

Corollary 4.1. *Let $n \ge 3$, and let $d \in \mathbb{N}$. Then, there exists $g_\Sigma \in \mathcal{M}(\Sigma)$ such that $J_{\Sigma,g}$ has at least d negative eigenvalues.*

Proof. Indeed, we always have

$$1 < 2(n-2) < \lambda_1(\mathbb{C}P^{n-1}) = 4n.$$

Hence, one looks for eigenvalues of the form $\lambda_\ell = 4\lambda_j(g_\Sigma) - 2(n-2)$. Since $2(n-2) > 1$, we can always find $g_\Sigma \in \mathcal{M}(\Sigma)$ such that $\sigma(-\Delta_{g_\Sigma}) \cap (\frac{1}{4}, \frac{(n-2)}{2})$ is arbitrarily large, which proves the claim. □

In order to prove existence and multiplicity results for our problem, we will show the existence of bifurcation points while perturbing the metric. We will use the following definition of bifurcation [8]:

Definition 4.1. Given two Banach spaces B_0 and B_1 and a C^1-family of submanifolds $[0,1] \ni \lambda \mapsto D_\lambda \subset B_1$ and closed subspaces $[0,1] \ni \lambda \mapsto E_\lambda \subset B_0$, we define the fiber bundles $\mathcal{D} = \{(x,\lambda) \in B_1 \times [0,1]; x \in D_\lambda\}$ and $\mathcal{E} = \{(y,\lambda) \in B_0 \times [0,1]; y \in E_\lambda\}$. Let $F : \mathcal{D} \to \mathcal{E}$ be a C^1 bundle morphism. Let $\lambda \mapsto x_\lambda$ and $\lambda \mapsto y_\lambda$ be C^1 sections of $\mathcal{D}$ and $\mathcal{E}$, respectively, satisfying $F(x_\lambda, \lambda)$. We say that $\lambda_* \in [0,1]$ is a bifurcation point of the equation

$$F(x,\lambda) = (y_\lambda, \lambda)$$

for x_λ, if there exist a sequence $(\lambda_n)_{n\geq 1}$ and a sequence $x_n \in D_{\lambda_n}$ such that

(i) $\lim_{n\to\infty} \lambda_n = \lambda_*$,
(ii) $x_n \neq x_{\lambda_n}$,
(iii) $\lim_{n\to\infty} x_n = x_{\lambda_*}$,
(iv) $F(x_n, \lambda_n) = (y_{\lambda_n}, \lambda_n)$.

Now given a path of metrics $[0,1] : t \to g_{\Sigma,t} \in \mathcal{M}(\Sigma)$ and $g_t = g_{\mathbb{C}P^{n-1}} \oplus \frac{1}{2} g_{\Sigma,t}$, the manifold $\mathcal{M}_{\Sigma,g_t}$, will play the role of D_t, $F(u,t) = \nabla \mathcal{A}_{g_t}(u)$, in the definition above and $E_t = H^{-1}(M)$, the dual space of $H^1(M)$, carrying its canonical Hilbert space structure. We can see the constant solution 1 as a section of $\mathcal{D}$, that is, $[0,1] : t \mapsto 1_t$, and we have

$$F(1,t) = (0,t).$$

We want to show that we have a bifurcation point for F, which corresponds to a sequence of solutions to Equation (22) that are arbitrarily close to 1.

Theorem 4.2. *Assume that $n \geq 3$. Given $g_0 \in \mathcal{M}(\Sigma)$, then there exists $g_0', g_1' \in \mathcal{M}(\Sigma)$, with g_0' arbitrarily close to g_0 and a path $(g_t')_{t\in[0,1]}$ joining g_0' and g_1' such that F has at least one bifurcation point $t_* \in (0,1)$.*

Proof. We use the bifurcation theorem proved in [8, Theorem A.2]. First, we notice that for all metrics $g \in \mathcal{M}(\Sigma)$, the operator $J_{\Sigma,g}$ is symmetric and Fredholm of index 0. We consider now a metric $g_0 \in \mathcal{M}(\Sigma)$. If J_{Σ,g_0} is degenerate ($\ker J_{\Sigma,g_0} \neq 0$, so 1 is a degenerate critical point for $\mathcal{A}_{g_0}$), then by Lemma 4.3, we can choose $g_0' \in \mathcal{M}(\Sigma)$ arbitrarily close to g_0 and such that $J_{\Sigma,g_0'}$ is invertible (i.e., $\mathcal{A}_{g_0'}$ is Morse at 1), so we let $\mu(g_0)$ its Morse index. Using Lemma 4.3, we can choose yet another metric $g_1' \in \mathcal{M}(\Sigma)$ such that $\mathcal{A}_{g_1'}$ is Morse at the critical point 1 and $\mu(g_1') - \mu(g_0') \neq 0$. In order to conclude now, we consider a smooth path g_t' connecting g_0' to g_1' (such a path exists since $\mathcal{M}(\Sigma)$ is path connected). It is enough to notice now that $d_1F(\cdot,t) = J_{\Sigma,g_t'}$. It is important to point out that in the notation of [8, Theorem A.2], $(H_t, \langle\cdot,\cdot\rangle_t)$ is the space $H^{-1}(M)$ with its canonical Hilbert structure induced by the metric g_t. Hence, the assumptions of the bifurcation theorem [8] are satisfied and we have at least one bifurcation point $t_* \in (0,1)$. $\square$

Notice that since the bifurcation occurring in the previous theorem is from the constant solution 1, the new scalar curvature κ will also be close to $\mathcal{S}_{\theta_{0,n-1}}$. As a corollary, we get our second application Theorem 1.3.

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