Florida Tax Review

Volume 25 | Article 5

Why the Optimal Long-Run Tax Rate on Capital is Zero . . . or Very High: The Missing Explanation

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FLORIDA TAX REVIEW

Volume 25 *2021* *Number 1*

Why the Optimal Long-Run Tax Rate on Capital is Zero . . . or Very High: The Missing Explanation

by

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Abstract

Judd's (1985) finding that the optimal long-run rate of tax on capital is zero—even if equity is an important social objective—has exerted substantial influence in academic and policy circles over the last several decades. Only very recently has it become clear that Judd's zero-tax result rests on an implicitly adopted assumption about how savings responds to taxation. Working within the very same model structure, Straub and Werning (2020) demonstrate that the optimal long-term tax rate is positive and potentially large under an alternative, equally plausible assumption. This Article attempts to fill a remaining gap in the literature by providing a clear explanation of what is driving results in both variants of Judd's original model. Furthermore, it suggests that the real logical engine in both cases is an oddity in the mathematical conception of infinity that is of little consequence for actual tax policy.

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This and other papers by the author on capital income taxation are freely available at www.ssrn.com. The author wishes to thank Reed Shuldiner, John Vella, Stephen Bond, Michael Devereux, Martin Simmler, and seminar participants at Oxford University's Centre for Business Taxation.

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I. Introduction

The question of how and whether to tax capital income and wealth places in relief the tradeoff between equity and growth. On one hand, capital ownership is relatively concentrated, so that, all else the same, taxing capital is a powerful lever for progressivity. On the other, capital accumulation expands economic activity and boosts labor productivity, raising employment and wages. If taxing capital slows capital accumulation, the effect could be detrimental across the economic spectrum.

The theoretical economic literature on optimal capital taxation is a natural place to look for insight into these competing considerations. This Article analyzes one particular thread of that literature, the "Judd model."[1] In an often-cited article from the mid-1980s Judd presents the finding that the long-run tax rate on capital should be zero even if society places substantial weight on equity. Very recently Straub and Werning identify an implicit assumption in Judd's analysis and show how removing it produces contrary findings within the same model.[2] This Article sets out to provide a clear and accessible explanation of the results obtained in both model variants. That explanation in hand, it argues that

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1. Kenneth L. Judd, *Redistributive Taxation in a Simple Perfect Foresight Model*, 28 J. Pub. Econ. 59 (1985). Another important portion of the literature—which grows out of A. B. Atkinson & J. E. Stiglitz, *The Design of Tax Structure: Direct versus Indirect Taxation*, 6 J. Pub. Econ. 55 (1976)—is surveyed in Chris William Sanchirico, *Optimal Redistributional Instruments in Law & Economics*, *in* The Oxford Handbook of Law and Economics: Volume 1: Methodology and Concepts 321 (Francesco Parisi ed., 2017), *available at* https://ssrn.com/abstract=2956340 [https://perma.cc/YBY7-QHGU].

2. Ludwig Straub & Iván Werning, *Positive Long-Run Capital Taxation: Chamley-Judd Revisited*, 110 Am. Econ. Rev. 86 (2020). Straub and Werning also review several other articles that are often associated with Judd (1985), but are in fact based on different considerations: notably, Christophe Chamley, *Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives*, 54 Econometrica 607 (1986), and Kenneth L. Judd, *Optimal Taxation and Spending in General Competitive Growth Models*, 71 J. Pub. Econ. 1 (1999).

the Judd model has little to offer on the practical question of how optimally to tax capital.

The Judd model is populated with three kinds of actors who interact over an infinite number of periods. First, "capitalists"—who are all identical, infinitely lived and rational maximizers of their own intertemporal utility—enter the model endowed with all of the economy's resources and decide how to allocate those resources to their consumption in each period, over all periods stretching into the endless future. They make that decision based on perfect foresight regarding the infinite sequence of after-tax rates of return that they will be able to earn on their savings in each period. The resources they save in any period are—by an offstage market process—combined with labor to produce output in the immediately following period, out of which, *inter alia*, pre-tax returns are paid to the capitalist. Market forces are assumed to equate the pre-tax rate of return to the marginal product of capital.[3] Capitalists supply no labor.[4]

Second, "laborers"—who are identical and essentially inanimate—supply a fixed amount of labor in every period and always

3. The marginal product of capital includes the recovery of invested capital, and accordingly the pre-tax return mentioned here is also gross of principal.

Regarding how market forces might operate to equalize the (gross) marginal product of capital and the (gross) pre-tax return paid to capital, one can imagine a large number, *n*, of identical price-taking profit-maximizing "entrepreneurs" each employing *1/n*th of aggregate labor and *1/n*th of aggregate capital. In the Judd model, the production function is assumed to have constant returns to scale, which implies (not shown here) that each such entrepreneur would face the same marginal product of capital as applies in the aggregate economy. If it is also assumed that entrepreneurs are untaxed (or taxed on profits calculated with a full deduction for returns paid to capital), then each entrepreneur will only be maximizing profits if her marginal product of capital equals what she must pay capitalists for the marginal unit of capital, which is capitalists' pre-tax return. (Note that there is only one good in the economy, and its price is set to 1 without loss of generality, so that the marginal product of capital is also the marginal value product of capital.) Were the marginal product of capital greater than (less than) the pre-tax return, the entrepreneur could increase profit by hiring marginally more (marginally fewer) units of capital.

4. In this Article I describe and analyze the simplest form of the Judd model. Judd (1985) and Straub and Werning (2020) also consider extensions.

fully consume the sum of their wages and government transfers; they never save.

Third, an omniscient "government," before time begins, chooses, announces and credibly commits to the sequence of tax rates on capital that will apply for all time. It does so taking into account how this sequence will affect the prospective trajectory of capitalists' savings levels. Labor income is not taxed. The government cannot save or borrow, and in every period it must set aside a particular time-constant amount of revenue for non-transfer spending.

The tradeoff between equity and growth is starkly represented in the model. The government's policy objective is purposefully extreme to highlight the equity side of the balance: it cares only about the utility of the laborers. The growth counterweight is manifest in the government's chief constraint: the only way to increase laborers' consumption is to inspire capitalists to save more, and the only way to accomplish that is through altering the tax on savings.

Greater capitalist savings leads to greater laborer consumption through two avenues. First, the increase in productive capital increases labor productivity and thereby the wage; the wage rate is assumed to equal the marginal product of labor by virtue of offstage market forces. Second, when savings increases, the tax base for capital taxation increases. This increases tax revenue and thus also increases transfers to laborers.[5]

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5. As discussed in the online appendix, an equivalent and simpler way of conceiving of laborer consumption is that it is "residual output": output reduced by the after-tax returns paid to capitalists and by non-transfer government spending. The basic argument is as follows: first, the production function is assumed to exhibit constant returns to scale. Second, it is assumed that market forces equate the marginal product of capital with the pre-tax return paid to capital (see *supra* note 3) and the marginal product of labor with the wage rate. These two assumptions imply (not shown here) that total output, *OUTPUT,* equals the total amount of wages paid to labor, *WAGES* plus the total amount of pre-tax return paid to capital, *RENTS*. That is, *OUTPUT* = *WAGES* + *RENTS*. Furthermore, the portion of output that capital retains after tax is *CAPITAL'S PORTION* = *RENTS* – *TAX*. And the portion of TAX that is transferred to labor is TRANFLABOR = TAX – NTGS, where NTGS is non transfer government spending. Therefore, the portion of output that goes to labor is LABOR'S PORTION = WAGES + TRANFLABOR = (OUTPUT – RENTS) + (TAX – NTGS) = (OUTPUT – (CAPITAL'S PORTION + TAX)) + (TAX – NTGS) = OUTPUT – CAPITAL'S PORTION – NTGS.

In the context of this model—obviously artificial, but potentially informative—the following question is posed: how should the government set the infinite trajectory of tax rates on capital? Judd's original finding answers that question in part. The precise optimal infinite sequence of capital taxes is not determined, dependent as it is on unknowable parameter values that are left as variables in the model. Rather Judd's finding characterizes what the sequence must eventually look like whatever values such parameters take. In particular, Judd presents the result that the tax rate on capital must "converge to zero" over time. That is, the following must be true of the optimal sequence of tax rates: given any "collar" around zero, however tight, there is a point in time, however distant, after which the optimal sequence of tax rates stays within that collar.

Judd's article was, as noted, published in the mid 1980s. Over the course of the ensuing thirty years—through the restoration of preferential capital gains rates in the 1990s;[6] through the reduction of those preferential rates, their extension to dividends and the rollback of estate and gift taxation in the 2000s[7]; through to the end of the 2010s—Judd's finding remained an important part of the brief for lowering taxes on capital.

But there was always a nagging puzzle. The Judd model's Economics 501-level result did not seem to jibe with a rudimentary lesson from Economics 101. In the basic model of individual choice there is no general prediction that taxing the return to savings reduces savings. This ambiguity is not a matter of theoretical oddities that may be safely disregarded. It arises from something as elemental as the difference between how many units are purchased and how much is spent on those units.

It is generally understood that even though increasing the price of gasoline will cause individuals to decrease how many gallons of gas they buy (or so we may safely assume), this does not mean that it causes

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6. Omnibus Budget Reconciliation Act of 1990, Pub. L. No. 101-508, 104 Stat. 1388; Omnibus Budget Reconciliation Act of 1993, Pub. L. No. 103-66, 107 Stat. 312; Taxpayer Relief Act of 1997, Pub. L. No. 105-34, 111 Stat. 788.

7. Economic Growth and Tax Relief Reconciliation Act of 2001, Pub. L. No. 107-16, 115 Stat. 38; Jobs and Growth Tax Relief Reconciliation Act of 2003, Pub. L. No. 108-27, 117 Stat. 752; Tax Increase Prevention and Reconciliation Act of 2005, Pub. L. No. 109-222, 120 Stat. 345; *see also*, American Taxpayer Relief Act of 2012, Pub. L. No. 112-240, 126 Stat. 2313.

them to spend less on gas. What happens to spending is a race between buying fewer gallons and paying more for each. Spending on gas—which is not quantity Q, but P × Q, where P is price—decreases if and only if the percentage reduction in Q is greater than the percentage increase in P. In other words, spending decreases if and only if gasoline is sufficiently "price elastic."

Increasing the tax rate on the return to savings effectively increases the price of future consumption (or other future uses like bequests) in terms of forgone current consumption. If ATGR is the after-tax gross rate of return on savings—"gross" in the sense that it includes the return of principal—each unit of current consumption that is forgone enables ATGR more future consumption. Reciprocally, to "purchase" a *single* unit of future consumption one must "pay" P = 1/ATGR units of current consumption. A tax based on capital income or ownership lowers ATGR and so raises P = 1/ATGR. Even though it may be safe to assume that the individual "buys" less future consumption in response, this does not necessarily imply that she "spends" less on future consumption. Savings may be regarded as the amount of current consumption that is "spent on" future consumption. Savings (P × Q) decreases if and only if the percentage decrease in future consumption (Q) is greater than the percentage increase in its price (P = 1/ATGR)—that is, if and only if future consumption is sufficiently price elastic.[8]

The puzzle for Judd's zero-tax result was how it accommodated the possibility that taxing the return to savings might actually increase savings. How could it be true, one might have asked, that raising the long-run tax rate above zero never improves social welfare even though it might well increase savings, productive capital, wages and government transfers to labor all the way back to the beginning of time?

It would have been one thing had Judd been ruling out the possibility that savings would increase in response to greater taxes based on an assessment of the empirical literature on actual savings behavior. To be sure, there has never been clear empirical evidence regarding the elasticity of future consumption (not to mention other future uses like bequests).[9] But even if the evidence had always been clear, the puzzle

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8. Part VII describes how the same ambiguity in savings response can be equivalently viewed in terms of conflicting substitution and income effects on *current* consumption.

9. *See, e.g.*, Chris William Sanchirico, *Do Capital Income Taxes Hinder Growth?* (2013). Wharton Public Policy Initiative Issue Briefs. 39,

would remain that Judd's zero-tax result appears to apply without restriction. The Judd model makes no assumption about the price elasticity of future consumption—at least no explicit assumption.

The puzzle was resolved only very recently, and in an unexpected way. In an article published in 2020, Straub and Werning explain that Judd had implicitly, perhaps unknowingly, assumed that future consumption was indeed sufficiently price elastic to cause savings to fall in response to an increase in the tax rate on savings. Judd assumes fairly directly[10] that the optimal infinite sequence of tax rates on capital generates an infinite sequence of laborer consumption levels that does not converge to zero.[11] Straub and Werning prove mathematically that this downstream assumption works its way back up the logical flow to imply that Judd is effectively assuming that future consumption is sufficiently elastic.[12] Straub and Werning make the plausible claim that they are the first to point this out.

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available at https://repository.upenn.edu/pennwhartonppi/39 [https://perma.cc/8UA9-N8YJ]; Straub & Werning *supra* note 2 at 99 n.24 (asserting, though without citation, that "the case [in which future consumption is sufficiently price elastic] is widely considered more plausible empirically.").

10. *Technical note*: Judd explicitly assumes that the marginal social value, from the perspective of period *t*, of additional resources at period *t* (which is his "current value multiplier," q_1) converges in the optimal program. Judd, *supra* note 1 at 72 (equation (24c)). The marginal social value from the perspective of period *t* of additional resources available at period *t* is the laborer's marginal utility consumption at time *t*. Therefore, Judd is assuming that the laborer's marginal utility must converge. This rules out the possibility that laborer consumption goes to zero, because were that to happen, the laborer's marginal utility would diverge to infinity. Judd, *supra* note 1 at 61 (choice of laborer's utility function).

In their presentation of Judd's result, Straub and Werning explicitly assume that, inter alia, laborer consumption does not go to zero. Straub & Werning, *supra* note 2 at 93 (Theorem 1).

11. This may seem like a harmless assumption given that laborer consumption is the model's social welfare target. But it should be kept in mind that a single infinite-horizon optimizer in a simple model will send its own consumption toward zero if the rate of return on savings is consistently below the rate at which it discounts future utility. The issue is discussed further at *infra* Part II.H.

12. Straub & Werning, *supra* note 2 at 97 (Proposition 3). Straub and Werning show that were future consumption *not* sufficiently price elastic to cause savings to fall in response to an increase in the tax rate on savings, then laborer consumption would converge to zero in an optimal program.

Straub and Werning then go on to remove Judd's downstream assumption while adopting the opposite assumption upstream about the savings response to taxation. In their resulting variant of Judd's model, they find that laborer consumption does indeed converge to zero and that the optimal long-run rate on capital is not zero, but positive. Moreover, they find that the optimal long-run rate becomes infinitely large if, seemingly paradoxically, the model is rerun with ever smaller time-constant levels of non-transfer government spending.

What explains Judd's original result? What explains Straub and Werning's dramatic revision? One can gain a vague understanding of what is going in the two model variants without stepping beyond what has already been said. In Judd's original variant, in which the sequence of optimal tax rates converges to zero, lowering prospective tax rates on capital increases savings in periods leading up to the tax change. In Straub and Werning's variant, in which the long-run optimal tax rate is positive, *raising* prospective tax rates increases lead-up period savings.

Yet one should not be satisfied with this vague level of comprehension—for at least two reasons, one specific, one general.

In the first place, the story sheds no light on the main event for either model variant: the mechanism that sharpens long-run outcomes even in the absence of precisely specified parameter values. Pointing out that, in the original Judd variant, taxation has a detrimental effect in lead-up periods stops far short of explaining why the optimal long-run tax rate in that variant is always precisely zero—never slightly positive; never, for that matter, negative even though the model leaves open the possibility of subsidy. Pointing out that the effect of taxation in lead-up periods is beneficial in Straub and Werning's variant sheds little light on why the optimal rate of tax is unboundedly high with ever lower rates of non-transfer government spending—let alone why long-run laborer consumption is optimally zero.

Second, a deeper, more broadly applicable case can be made for being less easily satisfied when it comes to understanding model-based results. Exhibit A is the intellectual history of the Judd model itself. Throughout roughly three decades of theorizing (and policymaking), most who cited or deprecated Judd's zero-tax result apparently did not

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Therefore, by the contrapositive, to assume, as does Judd, that laborer consumption does *not* converge to zero is to assume that future consumption *is* sufficiently price elastic to cause savings to fall in response to an increase in the tax rate on savings.

realize that it rests on a rather significant assumption about savings response. During this time there were only two ways of understanding the model and its results. First, there were Judd's compact proofs. In retrospect, it seems safe to conclude that these did not speak clearly to even those—possibly few in number—who attempted to work through them. Second, there were the breezy "intuitions" provided by Judd and others.[13] In retrospect, one can see that those "intuitions" for the zero-tax result applied just as well on the substantial portion of the parameter space in which the result does not hold.

What was missing—and is unfortunately still missing[14]—is some kind of graspable middle ground. That middle ground might be

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13. Straub & Werning, *supra* note 2 at 87 ("[E]conomists have continued to take turns putting forth various intuitions to interpret [Judd's original finding], none definitive [and none] universally accepted."); *Id.* at 94 (criticizing Lansing (1999)'s interpretation of results for the case in which the price elasticity of future consumption is precisely one); *Id.* at 144 (discussing problems with a common intuition provided for Judd's original result: "Judd (1999) also offers an intuitive interpretation for the Chamley-Judd result based on the observation that an indefinite tax on capital is equivalent to an ever-increasing tax on consumption. . . . The equivalence between capital taxation and a rising path for consumption taxes is useful. It explains why prolonging capital taxation comes at an efficiency cost, since it distorts the consumption path. If the marginal cost of this distortion were increasing in T and approached infinity as T [approached infinity], this would give a strong economic rationale against indefinite taxation of capital. We now show that this is not the case: the marginal cost remains bounded, even as T [approaches infinity].")

14. This note recites the intuition provided by Straub and Werning, which the reader may wish to contrast with the story presented in this Article. Straub and Werning first provide some intuition in an introductory paragraph:

> The economic intuition we provide for this result is based on the anticipatory savings effects of future tax rates. When [future consumption is relatively price inelastic], any anticipated increase in taxes leads to higher savings today, since the substitution effect is relatively small and dominated by the income effect. When the day comes, higher tax rates do eventually lower capital, but if the tax increase is sufficiently far off in the future, then the increased savings generate a higher capital stock over a lengthy transition. This is desirable, since it increases wages and tax revenue. To exploit such anticipatory effects, the optimum involves an

> increasing path for capital tax rates. This explains why we find positive tax rates that rise over time and converge to a positive value, rather than falling towards zero.

Straub & Werning, *supra* note 2 at 88. An additional intuition then appears in a so-titled Subpart later in the Article:

> **Intuition: Anticipatory Effects of Future Taxes on Current Savings.** Why does the optimal tax eventually rise for s > 1 [i.e., future consumption is relatively price inelastic] and fall for s < 1 [i.e., future consumption is relatively price elastic]? Why are the dynamics relatively slow for s near 1?
>
> To address these normative questions it helps to back up and review the following positive exercise. Start from a constant tax on wealth and imagine an unexpected announcement of higher future taxes on capital. How do capitalists react today? There are substitution and income effects pulling in opposite directions. When s > 1, the substitution effect is muted compared to the income effect, and capitalists lower their consumption to match the drop in future consumption. As a result, capital rises in the short run and falls in the long run. When instead s < 1, the substitution effect is stronger and capitalists increase current consumption. In the logarithmic case, s = 1, the two effects cancel out, so that current consumption and savings are unaffected.
>
> Returning to the normative questions, lowering capitalists' consumption and increasing capital is desirable for workers. When s < 1, this can be accomplished by promising lower tax rates in the future. This explains why a declining path for taxes is optimal. In contrast, when s > 1, the same is accomplished by promising higher tax rates in the future; explaining the increasing path for taxes. These incentives are absent in the logarithmic case, when s = 1, explaining why the tax rate converges to a constant.
>
> When s < 1 the rate of convergence to the zero-tax steady state is also driven by these anticipatory effects. With s near 1, the potency of these effects is small, explaining why the rate of convergence is low and indeed becomes vanishingly small as s [approaches] 1.
>
> In contrast to previous intuitions offered for zero long-run tax results, the intuition we provide for our

called "explanation," to distinguish it from the extremes it lies between. Explanation is different from proof because it does not strive to be mathematically complete. It is also different from proof because it is not somehow motivated by saving space on the (often not to be) printed page, but rather by saving time-to-comprehension for a wide range of readers—accounting for the effort readers may need to unpack arguments levered on compact recipe-like techniques imported from applied mathematics.[15] Explanation is also different from "intuition", as that word is used—perhaps abused—in the theoretical economics literature. It aims to be something more than mnemonic. And it is offered not as a side-show to the theorems and proofs, but as a coequal event.

A bimodality of black-hole proofs and nebulous intuitions produces a special problem when the target of mathematical analysis is public policy. The Judd model, after all, is not like a model in physics or engineering wherein the rocket makes it to the moon or not, or the bridge collapses or not. No one reading a description of the Judd model would believe that it is somehow ready to be run on the real economy, to be

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> results—zero and nonzero long-run taxes alike, depending on s—is not about the desired level for the tax. Instead, we provide a rationale for the desired slope in the path for the tax: an upward path when s > 1 and a downward path when s < 1. The conclusions for the optimal long-run tax then follow from these desired slopes, rather than the other way around.
>
> Our intuition based on slopes has an interesting implication for the effects of limited commitment in this economy. Since the planner promises higher future taxation when s > 1, renegotiation by the planner might lead to lower rather than higher capital taxes. This is the polar opposite of the conventional wisdom, according to which limited commitment leads to higher capital taxation.

Id. at 99.

15. See, e.g., the infinite-horizon multiplier technique mentioned in *supra* note 10, recalling that seemingly innocuous assumptions placed directly on the multiplier's limiting value turned out to have strong implications for model results. (The costs and benefits of multiplier techniques are discussed in more detail in *Web Appendix, infra* note 20 at 4-5). See also the quoted discussion in *infra* note 16 referencing Laplace transforms, eigenvalues, and, more generally, the toolkit of applied-mathematical techniques used in analyzing dynamic systems.

verified or not. Rather the findings derived in the Judd model are policy arguments—nothing more, nothing less—and they should be evaluated as such. On the one hand, the too obvious objection that the Judd model is artificial just amounts to the assertion that the arguments made within its frame are not comprehensive. But no argument is. The comparatively self-evident incompleteness of the Judd model ought to be regarded as relative virtue. On the other hand, the fact that findings derived from the Judd model cite mathematical reasoning as authority cannot be the end of the conversation. Like any authority, the mathematical reasoning needs to probed, and the argument that is being made needs to be essentially comprehended.

* * *

Providing an explanation of Judd model findings is this Article's primary, and less contestable goal. Its secondary goal is more controversial.

The explanation that is offered in this Article calls into question the Judd model's relevance. It reveals that the findings of both model variants are driven by an oddity of infinite-horizon reasoning that cannot be regarded as connected to any real public policy consideration.[16]

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16. The closest the literature appears to come to making this point is found in Judd (1985) itself, in a notation- and technique-heavy discussion in a prior Part of his Article, in which, in fact, additional assumptions are imposed. Judd, *supra* note 1 at 69. This is related in its entirety below. The notation is as follows: beta is the "elasticity" of marginal "within-period" utility with respect to consumption level (the percentage reduction in such marginal utility per percentage increase in consumption level), rho is the common discount factor for capital and labor, H, is "the Laplace transform of . . . [h(t)]," which is in turn the trajectory of the marginal tax change (assumed "for technical reasons" to be "eventually constant"), and mu is one of the eigenvalues of J, which is a matrix whose cells depend on the first and second derivatives of the production function, the tax rate, and beta, as defined above. *Id.* at 65–66.

> The fact that mu > rho if and only if beta < 1 is important for our net gain calculation in this case since H(mu)H(rho)−1 goes to zero as the imposition of the tax is pushed into the future if mu > rho, whereas if mu < rho, then H(mu)H(rho)−1 diverges to infinity as the tax is delayed. These observations immediately lead to the determination of the desirability of imposing a tax which takes effect only in the very

Suppose a mason has two pallets of identical bricks, one with a finite number, one with an infinite number. Imagine that the mason starts taking bricks one at a time from the infinite pile and placing them in the finite pile. As the mason does so, the size of the finite pile increases. But the size of the infinite pile does not decrease—infinity minus one equals infinity. If the mason continues transferring bricks *ad infinitum*, the size of the finite pile becomes unboundedly greater than it was at the start, but the size of the infinite pile never changes.

Now consider the thought experiment in which, starting from a candidate optimal policy, one repeatedly pushes farther and farther into the future a potential single-period perturbation in the capital tax rate. As the tested perturbation is pushed ever farther out, the number of periods in the pile of those leading up to the change increases without bound, while the number of periods in the pile of those following the

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> distant future. First, if beta < 1, then mu > rho and for distant tax increases the H(mu)H(rho)−1 term becomes negligible. It follows from (18) that for large T, utility is unchanged if tau = 0 initially, and falls if tau > 0. We therefore see that if capitalists have a small elasticity of marginal utility, workers today will not want to impose a partially anticipated tax increase on the capitalists in the distant future, even if the revenues are distributed to the workers. Note that this is also the case where capitalists will increase current consumption in response to an increase in expected future taxation. This capital decumulation in response to future taxation leads to a decline in wages in the near term, offsetting the revenue gain of the tax increase.
>
> On the other hand, if beta > 1, then mu*beta>rho and workers will want anticipated redistributive taxation in the distant future. This can be seen from (18) by noting that for distant tax increases, H(mu)H(rho)−1 will be large and dominate (18), and utility will increase for distant tax increases if tau is initially zero but fall if we are in a steady state associated with a high tax rate. Note that beta > 1 is also the case where capitalists save in response to future tax increases, with the short-run immediate capital accumulation raising wages immediately. Hence, if we are in the untaxed steady state, this short-run wage effect is an additional benefit of the distant tax increase.

Id. at 69.

change remains the same. So for changes far enough in the future, lead-up periods become ever more dominant, all else the same.

Yet because the perturbations are, by hypothesis, acting on an optimal policy, the effects in both lead-up and follow-on phases must balance out no matter how early or late the tested perturbation. Thus, in terms of social welfare impact, the effect on lead-up savings levels, the effect on follow-on savings levels, and the mechanical, fixed-behavior effect on revenue in the period in which the perturbation occurs must all three balance out, even though the pile of lead-up periods is growing without limit.

The only way for this to happen is via an offsetting adjustment to the "filters" that translate savings responses into changes into social welfare. There are two such filters: the portion of marginal production from additional savings that actually goes to labor as opposed to capital; and laborers' marginal utility of consumption. These two filters must be either reining in the otherwise exploding pile of lead-up period effects or exploding the otherwise overwhelmed pile of follow-on and fixed-behavior effects.

Across the two model variants, savings responses differ directionally in lead-up periods. This produces different compensatory movements in social welfare filters, which with several more steps explains the differing results for long-run taxation. These steps are explained systematically in the body of the Article. But the basic ideas may be sketched out here.

In the original Judd model, the savings responses in lead-up and follow-on periods point in the same direction and are balanced against the fixed-behavior revenue effect in the perturbation period, which points in the opposite direction. For instance, increasing the tax in some future period reduces savings in all periods but increases the tax collected per savings dollar in that future period. In the original Judd model, the filter that adjusts in order to maintain social welfare balance for ever later perturbations is the portion of marginal production from additional savings that goes to labor. This portion converges to zero, and this dampens the otherwise exploding social welfare impact of the lead-up savings response. It also dampens the social welfare impact of the follow-on savings response. But it does not dampen the mechanical, *fixed-savings* revenue effect in the period of the perturbation—which points in the opposite direction—and this permits persistent balancing of social welfare effects. The convergence of labor's portion of the marginal production from additional savings to zero implies that the tax rate on savings also converges to zero: The pre-tax return is assumed to equal the marginal product of capital—that is, the *total* marginal production from additional savings. Therefore, the portion of marginal production from additional savings that goes to labor is equal to the tax on savings.

At first glance, then, one might think that pushing the perturbation ever farther into the future would create no particular problem for maintaining balance in the original Judd model between, on the one hand, lead-up and follow-on savings responses, and on the other, the fixed-behavior revenue effect. Stepping the perturbation back in time by one period just transfers one period from the follow-on phase to the lead-up phase. While it might increase the magnitude of the lead-up effect—by adding a lead-up period—it would also seem to decrease the magnitude of the follow-on effect—by subtracting a follow-on period. But the follow-on periods are always infinite in number, and even though a follow-on period is indeed being subtracted, there are no fewer follow-on periods.

In the Straub and Werning variant the savings responses in lead-up and follow-on periods point in opposite directions, and the fixed-behavior revenue effect in the perturbation period works together with the lead-up savings response. In particular, increasing the tax in some future period reduces savings in follow-on periods, while increasing savings in lead-up periods and increasing the tax collected per savings dollar in the period of the perturbation. In the Straub and Werning version of the model, as the hypothetical tax increase is pushed ever farther into the future, the social welfare impact of the increase in lead-up savings (plus that of the fixed behavior revenue effect) heads toward positive infinity.[17] To maintain zero-sum balance, it must then be that the social welfare impact of the decrease in follow-on savings heads toward *negative* infinity.

The fact that the social welfare impact of the decrease in follow-on savings heads toward *negative* infinity has two implications. First, the only way the social welfare impact of the follow-on savings response can become infinite is for laborers' marginal utility from consumption to become infinite, and the only way for that to happen is for laborer consumption to head toward zero. [18] Second, since follow-on savings levels always decrease in response to the hypothetical ever-later

17. This point, which is not obvious, is explained in detail *infra* Part V.

18. See *supra* note 11 regarding how this could be socially optimal.

tax increase, and since the social welfare impact of these decreases must be eventually negative (let alone unboundedly so), it must be that such hypothetical tax increases, however late in time they occur, always decrease laborer consumption in at least some follow-on periods. During such following periods, laborer consumption decreases because a decrease in savings reduces output more than it reduces capital's after-tax earnings. During such follow-on periods, therefore, the marginal product of capital, which equals the pre-tax rate of return, must exceed the after-tax rate of return. Thus, there must be periods stretching into the infinite future in which the tax rate is positive.

The preceding two implications—the convergence of laborer consumption to zero and the impossibility of the tax rate remaining at or below zero after some point in time—then come together in a simple diagram[19], which establishes that the tax rate not only is positive an infinite number of times, but indeed converges to some positive number. The diagram also makes clear why this positive number will be very large when non transfer government spending is very small.

As noted, these descriptions of the two models are merely sketches. The rest of the Article provides a fuller picture. It does so in two stages of increasing depth. The next Part of the Article provides, in eight steps, the basic story behind the findings in both model variants. Subsequent parts go deeper into several key points in that procession. An online appendix formalizes mathematically the arguments described in the text.[20]

II. Basic Explanation

This Part of the Article presents a novel explanation for both the zero-tax result in Judd's original Article (OJM) and the positive-tax result in Straub and Werning's recent variant (SW). The basic explanation is laid out in a sequence of eight steps. The key step is the fourth concerning how the Judd model relies on and exploits paradoxical features of the mathematical construct of infinity. Subsequent Parts of the Article take

19. *See infra* Figure 2.

20. Chris William Sanchirico, *Web Appendix to: Why the Optimal Long-Run Tax Rate on Capital is Zero . . . Or Very High: The Missing Explanation* (U. Penn Inst. for L. & Econ., Research Paper No. 20-34, 2020), https://ssrn.com/abstract=3589726 [https://perma.cc/65F9-X5LG][hereinafter Sanchirico, *Web Appendix*].

up important details. An online appendix provides mathematical understructure.[21]

A. Step 1: Testing an Optimum With Ever Later Single-Period Down-Pulses in the After-Tax Return to Savings

The first step is to fix in mind a particular two-dimensional thought experiment. Explained in detail in this Part, the experiment amounts to pushing an anticipated single-period down-pulse in the after-tax (gross) rate of return to savings (ATGR) ever farther into the future.

Thus suppose that it is the beginning of time and the government has just announced an infinite sequence of ATGRs, one for each period.[22] The sequence is optimal according to the government's social welfare calculus, which takes into account the sequence of savings levels that it will inspire and the sequence of redistributive transfers that it will allow.

Because the government's chosen sequence of ATGRs is optimal, hypothetically perturbing it in any manner cannot increase

21. *Id.*

22. Equivalently, and more naturally, the government announces a sequence of tax rates, which combined with the induced sequence of savings, implies the ATGR in each period. For reasons explained in the online appendix, however, it is much easier to proceed as if the government chooses the sequence of ATGRs directly. The rest of this note shows how to deduce the tax rate from the ATGR and the MPK, which is the gross-of-investment-return marginal product of capital that results from capitalists' prior-period savings choices: the model assumes that the pre-tax gross rate of return paid to the capitalist equals the MPK. Therefore, the capitalist's pre-tax wealth entering any given period is MPK × S, where S is last period's savings. The capitalists after-tax wealth is ATGR × S. Therefore, the total amount of tax owed by the capitalist in the current period is MPK × S – ATGR × S = (MPK – ATGR)S. Therefore, if the base is (unnaturally) last period's savings, the tax rate is [(MPK – ATGR)S]/S = MPK – ATGR. If the base is pre-tax wealth, the tax rate is [(MPK – ATGR)S] / (MPK × S) and the tax rate is (MPK – ATGR)/ MPK. If the base is pre-tax income, which is (MPK × S) – S = (MPK – 1)S, the tax rate is [(MPK – ATGR)S] / [(MPK – 1)S] = (MPK – ATGR) / (MPK – 1). Notice two things about these formula. First, the model is assuming a linear tax rate (that is, a single-rate, flat-tax structure)—whether the base be savings, wealth or income. Second, if MPK – ATGR = 0, then all three tax rates are zero (assuming that MPK is neither zero nor one).

anticipated social welfare. This fact can be used to characterize the optimal sequence.

Consider a very simple perturbation: reduce the ATGR in some single future period without affecting the ATGR in any other period.[23] The effects of this hypothetical down-pulse on prospective savings and redistributive transfers can then be determined. This is the first dimension of the thought experiment.

Now imagine repeatedly shifting the timing of the single-period down-pulse ever farther into the future. One can then calculate how these shifts would alter the down-pulse's effect on prospective savings and redistributive transfers. This is the second dimension of the thought experiment.

Throughout it will be assumed that the optimal sequence of ATGR's induces the capitalist to adopt a consumption plan that converges over time. The significance of assuming rather than proving convergence is discussed in Part VI.

B. Step 2: Dividing the Social Welfare Impact of a Single-Period Down-Pulse Into Three Parts

Given a single-period down-pulse in the ATGR for a particular period, say Period 100, divide into three parts the down-pulse's social welfare impact as follows:

First, go to the period of the down-pulse itself, Period 100, and hold the savings whose return is being reduced hypothetically constant. According to the model's time structure, that is savings in Period 99. Reducing the ATGR paid to the capitalist in Period 100 means collecting more tax revenue in that period and so increasing transfers to the laborer in that period. (For present purposes we can assume that there is a single capitalist and a single laborer, rather than many identical copies of each kind.[24])

Thus, if the capitalist saves $200 in Period 99, and if the ATGR paid on this savings in Period 100 is reduced by from 1.05 to 1.04, and

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23. Following on the discussion *supra* note 21, this will entail adjusting tax rates in all periods to offset changes in the sequence of MPKs due to responsive changes in savings.

24. Having a large number of each type justifies the competitive market assumption by which pre-tax returns and wages are equated with marginal products.

if the capitalist's savings in Period 99 were hypothetically to remain constant, then \$2 more (equal to $1.05 \times \$200 - 1.04 \times \$200$) would be collected in tax from the capitalist in Period 100. This always socially positive "fixed-behavior effect" is confined to the down-pulse period itself. In fact, it plays a limited role in the full explanation.

The second and third components are due to the capitalist's anticipated behavioral response, in savings levels, to the single-period down-pulse in the ATGR. They are divided by timing relative to the down-pulse period.

The second component is the impact on social welfare via the change in savings behavior in the periods leading up to the down-pulse period—hereafter the "lead-up effect."

The third component is the impact on social welfare via the change in savings behavior in the down-pulse period and thereafter—hereafter the "follow-on effect."

So if the down-pulse occurs in Period 100—which in the model would lower the return on what was saved in Period 99—then the lead-up effect is the social welfare impact of responsive changes in savings in Periods 0 through 99, and the follow-on effect is the social welfare impact of responsive changes in savings in Periods 100, 101, 102, . . . into the infinite future.

C. Step 3: Comparison of Savings Responses Across the Two Model Variants

The third step is to compare savings responses across the two model variants and across lead-up and follow-on periods. Savings responses in the two variants of the model are similar in the follow-on, but different in the lead-up.

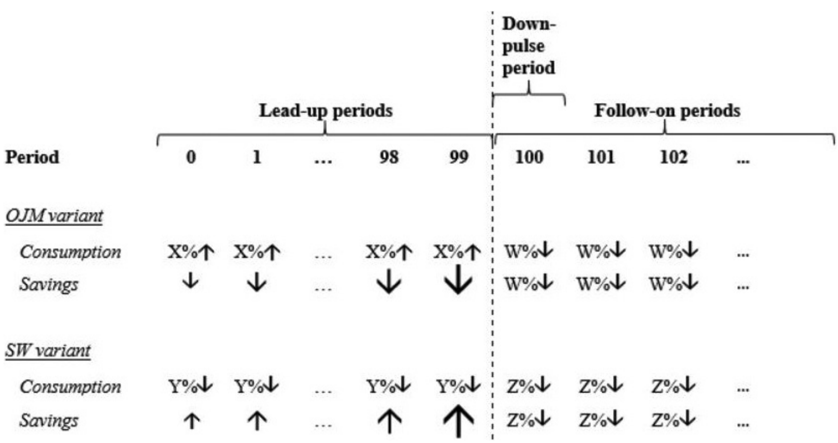
In a general model of intertemporal choice, many different patterns of savings response are possible. From the vast array of possibilities, the two model variants effectively pick out two particular patterns. To be sure, these patterns are not directly assumed. They are rather derived from upstream assumptions on model primitives—notably the capitalist's utility function. But different, no less valid primitives would have produced different patterns. And so for present purposes we may ignore the derivation and proceed as if the pattern itself were assumed. (Nevertheless, Part VII and the online appendix explain the derivation.)

Before discussing these patterns, it must be clarified that, in the context of the Judd model, the word "savings" is used in a cumulative sense, not in an incremental sense. The capitalist's "savings" in any given period is the portion of the total amount of after-tax resources available to the capitalist at the start of the period that the capitalist does not consume during the period. It is not unconsumed after-tax income flow; it is unconsumed after-tax wealth stock.

Thus, savings in any given period can be thought of as the "bank balance" that the capitalist carries into the next period. It is likewise the base upon which the capitalist earns the after-tax return payable at the beginning of that next period. And it is, correspondingly, the stock of productive capital employed in producing that next period's total economic output.

Now imagine a single-period down-pulse in the ATGR in some fixed future period, say 100. The impact in the two phases and the two model variants is summarized in Figure 1 and described in the following two Subparts.

Figure 1: Schematic of savings responses—lead-up v. follow-on, OJM v. SW



1. Lead-Up Periods

In the OJM variant, capitalist consumption levels in all lead-up periods increase—in fact they increase by the same percentage (X%), an artificial regularity whose source is of secondary importance. This causes an accumulating decrease in savings levels throughout the lead-up periods, as would happen over time to a depositor's bank balance were she to increase the amount she withdrew in some or all years.[25]

In the SW variant, on the other hand, capitalist consumption levels in all lead-up periods *decrease*, also by a uniform percentage (Y%).[26] This causes an accumulating *increase* in savings levels throughout the lead-up periods.

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25. *Technical note:* See Sanchirico, *Web Appendix*, *supra* note 19 for details. As derived in that appendix, the formula for the "percentage change" in lead-up period capitalist consumption levels per percentage *up-pulse* (not down-pulse, so as to follow calculus convention) in future return is as follows: $\forall t = 0,1,\ldots,T-1, \frac{dC_t}{dR_T}\frac{R_T}{C_t} = -\left(\frac{1}{\sigma}-1\right)\theta_T$, where C_t is consumption, R_t is the after-tax *gross* rate of return in period t (accounting for both "principal" and "interest"), $T \geq 1$ is the down-pulse period, $a_t = R_t s_{t-1}$ is initial after-tax wealth in period t, and $\theta_T = \frac{\frac{a_T}{R_T \times \ldots \times R_1}}{a_0} \in [0,1]$ is the portion of the capitalist's after-tax wealth entering Period 0 that is set aside for consumption in all follow-on periods. In the OJM variant, $\sigma \in (0,1)$ implying $\left(\frac{1}{\sigma}-1\right) > 0$, while in the SW variant, $\sigma \in (1,\infty)$ implying $\left(\frac{1}{\sigma}-1\right) < 0$. The change in savings can then be derived from the iterated capitalist intra-period budget constraint: $s_t = R_t \times \ldots \times R_1 a_0 - R_t \times \ldots \times R_1 C_0 - \ldots - R_t C_{t-1} - C_t$. Some manipulation leads to this: $\forall t = 0,1,\ldots, T-1$, $\frac{ds_t}{dR_T}\frac{R_T}{s_t} = \frac{a_T}{R_T \times \ldots \times R_{t+2} a_{t+1}}\left(1 - \frac{a_{t+1}}{R_{t+1} \times \ldots \times R_1 a_0}\right)\left(\frac{1}{\sigma}-1\right)$. (When applying this expression to $t = T-1$, use the convention $R_T \times \ldots \times R_{t+2} = 1$, so that it yields, $\frac{ds_{T-1}}{dR_T}\frac{R_T}{s_{T-1}} = \left(1 - \frac{a_T}{R_T \times \ldots \times R_1 a_0}\right)\left(\frac{1}{\sigma}-1\right)$.) The quantity $\frac{a_{t+1}}{R_{t+1} \times \ldots \times R_1 a_0}$ is the portion of initial after-tax wealth a_0 set aside for consumption following t, and so must be less than one.

26. *Technical note:* See *supra* note 24 for the formula.

2. Follow-On Periods

In the OJM variant the capitalist enters Period 100 with less to spend and save—that is, she has less initial after-tax wealth in Period 100. This is for two reasons: first, she saved less in lead-up Period 99, and, second, the after-tax gross rate of return (ATGR) on those savings, which is paid in Period 100, was reduced by the down-pulse. The resulting impact on her future plans is again artificially regular, and for unimportant reasons: Her consumption and savings levels in all follow-on periods fall, and all by the same percentage as the decline in her Period 100 initial after-tax wealth (W%).[27]

In the SW variant, consumption and savings levels in all follow-on periods also fall, and this is also by a uniform percentage (Z%). The uniform percentage decrease is smaller than in the OJM, consistent with the fact that the decline in after-tax wealth entering Period 100 is, in the SW variant, mitigated (but not overwhelmed) by increased savings in lead-up Period 99.

D. Step 4: Basic Engine—Infinity Paradoxes

The fourth step is to lay bare the core dynamic driving results in both model variants. The basic mechanism relies on oddities in the mathematical conception of infinity. Needless to say, infinity is a useful abstraction. But the translation of that abstraction into practical application has often proven treacherous, and the Judd model is a case in point.[28]

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27. *Technical note:* See Sanchirico, *Web Appendix*, *supra* note 19 for details. Using the notation from note 24, the percentage change in follow-on savings (and in follow consumption) per percentage change in the after-tax gross return in period T is a convex combination of $\frac{1}{\sigma} > 0$ and 1: $\forall t = T, T+1, \ldots, \frac{ds_t}{dR_T}\frac{R_T}{s_t} = \frac{1}{\sigma}(1-\theta_T) - 1\theta_T > 0$. Note again that this formula is for an up-pulse, not a down-pulse.

28. This is not uncommon in other areas of mathematical social science. See, for example, the literature on "merging of opinions" and "rational learning" and in particular, Ronald I. Miller & Chris William Sanchirico, *The Role of Absolute Continuity in "Merging of Opinions" and "Rational Learning"*, 29 Games & Econ. Behav. 170 (1999) and Ronald I. Miller & Chris

Recall from the Introduction the mason with two pallets of identical bricks—one finite, one infinite—who transfers bricks one by one from the infinite pile to the finite pile, the size of the finite pile becoming unboundedly greater than it was at the start, the size of the infinite pile remaining the same. In the two-dimensional thought experiment described in Step 1, as the down-pulse in the after-tax rate of return is pushed ever farther into the future, the number of lead-up periods grows without bound, while the number of follow-on periods remains the same: that is, infinite.

Part III below explains how the lead-up *savings response* exhibits the same unbounded pile-up as the raw number of lead-up periods, while the follow-on savings response exhibits the same constancy as the raw number of follow-on periods.

Yet no matter how far in the future the down-pulse occurs, the social welfare impact of the lead-up period savings response must always balance with the social welfare impact of the follow-on period savings response and the fixed-behavior effect in the single down-pulse period itself. Given that the thought experiment starts at a social optimum, the down-pulse can never produce an overall increase in social welfare. Nor can it produce an overall decrease, since then running it in reverse—that is, up-pulsing—would increase social welfare.

How the model maintains balance among lead-up, follow-on and fixed behavior effects in the face of the unbounded relative growth in the number of lead-up periods is central to understanding the findings of both model variants.

But before explaining how this balance is maintained, it is worth noting how divorced from reality the imperative to do so becomes as time proceeds. When the government arrives at Period 1000, the number of lead-up periods having grown from 0 to 999, it arrives there only in the process of making projections. It has never actually left Period 0; it is merely determining what tax rate to announce prospectively for Period 1000. In making that determination it takes into account how the announcement will affect savings in, inter alia, all the periods leading up to Period 1000, which have yet to occur.

William Sanchirico, *Almost Everybody Disagrees Almost All the Time: The Genericity of Weakly Merging Nowhere* (Columbia Econ. Dep't, Discussion Paper Series No. 9697-25, Aug. 1997), available for download at https://ssrn.com/abstract=34300 [https://perma.cc/76UW-RJ6T] or http://dx.doi.org/10.2139/ssrn.34300 [https://perma.cc/DEK5-8SUE].

But when the government actually does arrive at Period 1000, all the periods that preceded it have already happened. Short of reputational effects going forward, those preceding periods are irrelevant for social welfare optimization from that point forward. And while it may seem reasonable that such reputational effects would cause the government to find it worthwhile to honor commitments made 10 or even 100 periods prior, it borders on the absurd to imagine that Period 0 will still be a major consideration one thousand—let along one million, or one trillion—periods later. It would be as if current US tax policy were constrained by commitments inherited from the Roman empire and originally assumed for the purpose of shaping savings in the first century.

E. Step 5: The Multiplicative Filters That Translate Savings Responses into Marginal Social Welfare

The key to understanding how balance is maintained in the face of an ever growing pile of lead-up periods is understanding how the model translates changes in a given period's savings into changes in the next period's contribution to social welfare.

Ignoring time-discounting for a moment, the level of social welfare attributable to any given period, say Period 100, is, in the Judd model, equal to the level of the laborer's utility in Period 100. *Marginal* social welfare in Period 100 is thus marginal laborer utility in that period. In turn, marginal laborer utility in Period 100 is the product of the laborer's marginal utility of consumption (MU) in the period and the period's marginal increase in laborer consumption.

The marginal increase in laborer consumption from a marginal increase in the capitalist's prior period savings is the difference between two marginal quantities. The first is the marginal product of capital (MPK): the total amount of additional output produced in the economy as a result of additional prior period savings, where "output produced" includes the prior period savings itself to the extent it survives as undepreciated capital.[29]

The second component, which is subtracted from the first, is the ATGR: the additional amount of output that goes to capital rather than to labor. This is the additional after-tax return plus additional recovered

29. The model implicitly assumes that, if necessary, capital can be dismantled for consumption—or alternatively that there is a single good that, like a seed, can be either "eaten" or "planted."

principal that results, for the capitalist, from additional prior period savings.

The marginal increase in laborer consumption due to an increase in the capitalist's prior period savings is thus MPK − ATGR.[30] For instance, if the marginal product of capital in Period 100 is MPK = 1.30, then each additional unit of savings in Period 99 produces 1.30 more units of Period 100 output on the margin (gross of undepreciated capital). If the after-tax rate of return paid in Period 100 is 10%, so that the Period 100 ATGR is 1.10, then each additional unit of savings in Period 99 sends 1.10 more of Period 100 output to capital. So, per each additional unit of Period 99 savings, there is 1.30 more Period 100 output, 1.10 of which goes to capital, implying that MPK − ATGR = .20 more output goes to the laborer, who consumes it all.

Putting the last several paragraphs together, a marginal increase in single-period savings is translated into a marginal increase in the next period's social welfare through the multiplicative filter MU × (MPK − ATGR). For instance, if the laborer's marginal utility of consumption in Period 100 equals 2, then—continuing the numerical example from the last paragraph—the marginal increase in laborer utility is MU × (MPK − ATGR) = 2 × .20 = .40.

The value of the multiplicative filter MU × (MPK − ATGR) may vary from period to period. The time sequence of these multiplicative filters is generated by the optimal sequence of after-tax rates of return posited at the start of this discussion, which gives directly the sequence of ATGRs and indirectly—via savings levels and the resulting production—the sequence of MPKs and MUs.

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30. This explanation implicitly references the fact, as described *supra* note 5, that labor's portion of total output, which is delivered in the form of wages and government transfers, equals "residual output"—which is to say total output less the portion of output received by capital after tax, and less non-transfer government spending (which is constant). This makes sense: any marginal production not going to capital must work its way to labor given that non-transfer government spending is fixed and given that capitalists, labor and the government are the only actors in the model. (In connection with note 3, even if one also imagines that there are "entrepreneurs" who employ labor and capital and produce and sell output, another consequence (not shown here) of assuming that the production function has constant returns to scale is that such entrepreneurs make no profit and thus receive no portion of output.)

F. Step 6: Two Ways to Use the Multiplicative Filters

There are essentially only two possible ways in which the sequence of MU × (MPK − ATGR) filters can vary over time to maintain social welfare balance as the hypothetical single-period down-pulse is shifted ever farther into the future and the lead-up pile of savings responses grows unboundedly large.

The first is that MPK and ATGR can converge to each other over time quickly enough so that lead-up effects level off, as if the incremental brick added to the lead-up pile were shrinking so fast that the pile never grew beyond some given height.

The second way is that the laborer's consumption level can converge to zero so that the laborer's marginal utility of consumption grows infinitely large. The laborer's utility function is assumed to have the property that marginal utility not only diminishes, but diminishes from infinity, as if the first unit of consumption were the difference between life and death. The basic idea of this second way of using the filter is not to level off the lead-up effect as in the prior paragraph, but rather to blow up the follow-on effect so that it can keep up with the lead-up effect's unbounded growth. This is explained in greater detail in Step 8.

Part IV below explains that sending MPK-ATGR to zero and sending MU to infinity are the only two options for maintaining balance.

G. Step 7: Application to OJM Variant

The next step is to apply all of this to the OJM variant. From where the analysis now stands, this is fairly straightforward.

In that variant it is assumed that laborer consumption does not go to zero over time, thus ruling out the second method described in Step 6 based on MU's unbounded growth. Therefore, the lead-up effect must level off as the down-pulse is pushed ever farther into the future, and this must occur as a result of the mutual convergence of MPK and ATGR.

As the single-period down-pulse is pushed ever farther into the future, the mutual convergence of MPK and ATGR tempers the joint social welfare impact of lead-up and follow-on effects, which would otherwise explode as the pile of lead-up periods increased with no corresponding diminution in the pile of follow-on periods. On the other side of the balance, the mutual convergence of MPK and ATGR does not dampen the fixed-behavior effect, which is not run through the

MPK— ATGR filter, but flows directly into laborer consumption. In this way the mutual convergence of MPK and ATGR makes it possible for lead-up and follow-on effects to be fully offset in terms of social welfare impact by the fixed-behavior effect no matter how far in the future one tests a single-period down-pulse in the ATGR.

Finally, the mutual convergence of MPK and ATGR means that eventually the capitalist must be receiving as her after-tax rate of return essentially the full marginal product of capital. Under the assumption that market forces equate the MPK to the *pre*-tax gross return, this implies that the tax rate must converge to zero.

H. Step 8: Application to SW Variant

Application to the SW variant is more involved. There is an important preliminary result: in the SW variant, in which the down-pulse in ATGR increases, rather than decreases lead-up savings levels, the lead-up period effect (plus the fixed-behavior effect) cannot level off but rather must grow unboundedly, and toward positive infinity. The reasoning behind this preliminary result is explained in Part V.

Given that the lead-up plus fixed-behavior effects must be heading toward positive infinity, maintaining balance requires that the *follow-on* effect must be heading toward *negative* infinity. The follow-on effect of the down-pulse is driven by the capitalist's reduction in savings in follow-on periods. This reduction in savings filters into social welfare through MU × (MPK − ATGR). There are then two preliminary implications of the follow-on effect's divergence to negative infinity.

The first preliminary implication is that laborer consumption must be converging to zero. As discussed in detail in Part IV, MPK − ATGR cannot head toward positive or negative infinity. Thus, in order for the follow-on effect to go to infinity at all—positive or negative—MU must be going to infinity. MU must specifically be going to positive infinity since MU is always positive. In turn, the only way that MU can go to positive infinity is if laborer consumption converges to zero—contrary to the assumption imposed in the OJM.

Before moving on to the second preliminary implication, let us pause for a moment to address a question that naturally arises: Does it make sense that a government that is effectively maximizing laborer utility would choose a sequence of ATGRs that induces the laborer's consumption to go toward zero? The answer is yes—at least conditional on the premise that the government is optimizing over an infinite number of periods. Zero-limit consumption is a common result in simpler models in which a single actor is allocating initial wealth over an infinite future and the actor discounts future utility at a rate that is greater than the return she can obtain on her savings. The actor is induced to sacrifice later consumption for earlier consumption and the recursive persistence of this dynamic through infinite time sends consumption to zero. The Judd model is more complicated than a single-agent model with exogenous returns—savings must come through the private optimization decision of the capitalist, and the return is ultimately a matter of the marginal product of capital. But the same basic possibility arises. The government recursively and persistently sacrifices later laborer consumption for earlier.

The second preliminary implication of the fact that the social welfare impact of the follow-on effect heads toward negative infinity is that MPK − ATGR cannot be converging to a strictly negative number.[31] There are two steps. First, where MPK − ATGR is negative, less savings is better: when MPK is less than ATGR and savings decreases, the decrease in output is outstripped by the decrease in the payment to capital, so that laborer actually ends up with more. Second, a down-pulse in the ATGR reduces savings in follow-on periods.[32] Combining these two steps, were MPK − ATGR negative after some point, the savings reduction in follow-on periods caused by the down-pulse in ATGR would *increase* social welfare. This would contradict the premise that the social welfare impact of the follow-on effect heads toward *negative* infinity as the down-pulse in the ATGR is pushed ever farther into the future.

The preceding pair of preliminary implications—that the laborer's consumption is heading toward zero and that MPK − ATGR cannot be headed toward a strictly negative number—generate a sequence of strong conclusions regarding what the economy looks like in the long run.[33]

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31. Without the added assumption that the economy converges, this statement becomes: MPK − ATGR cannot be non-positive after some point in time. This weaker statement is indeed all that is needed for the argument that follows.

32. *See* Figure 1 and the discussion surrounding it.

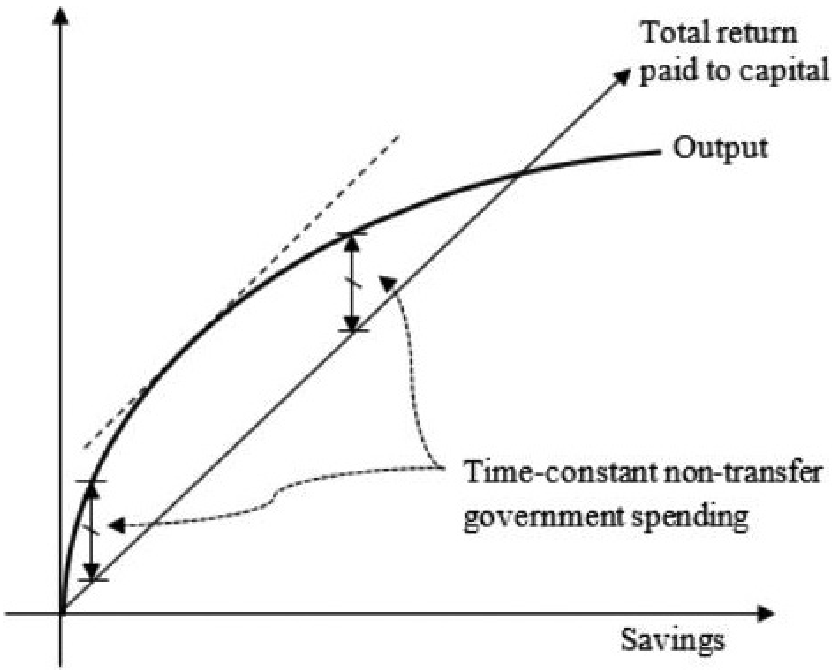
33. Unlike the explanation up to this point, the explanation from this point forward largely tracks the reasoning presented in SW.

First, given that laborer consumption is converging to zero, the capitalist's saving level must be settling down to a point where the resulting output, after paying the capitalist her long-run ATGR,[34] is just enough to finance the assumed time-constant amount of non-transfer government spending. Otherwise, resources would be wasted, contradicting the hypothesis that the policy being tested is optimal.

Second, there are two levels of savings at which this can happen. The reasoning behind this second assertion (as well as those to follow) is simple enough to reduce to a single familiar diagram, here shown as Figure 2. In that figure, the curved upward sloping line shows total output (y-axis) as a function of prior-period savings (x-axis). The slope of the curved line above any given level of savings is the MPK. As shown in the diagram, as the level of savings hypothetically moves along the x-axis from left to right, the MPK is assumed to start out at infinity, when savings is zero, and then diminish toward zero as savings increases to infinity. The straight upward sloping line shows the total return paid to capital as a function of savings: ATGR × S. Its slope is the posited level to which the ATGR converges, whatever that might precisely be. The amount by which the height of the curved line exceeds the height of the straight line, above any given level of savings, is the residual amount of output available for non-transfer government spending at that savings level (given that laborer consumption is zero). As indicated in Figure 2, there are exactly two savings levels where this residual equals the time-constant level of non-transfer government spending. And the path of savings must be converging to either one of these levels.

34. Given the assumption that the capitalist's consumption plan is converging, the ATGRs must be converging also—to the reciprocal of the capitalist's discount factor. *See* Sanchirico, *Web Appendix*, *supra* note 19.

Figure 2: Output and the total payment to capital, as functions of prior-period saving



Third, savings must in fact be converging to the lower of the two levels. At the lower level, the slope of the curved line (MPK) exceeds the slope of the straight line (that is, the ATGR). At the higher level of savings, on the other hand, MPK is less than ATGR. The higher level of savings is thus ruled out by the requirement that MPK— ATGR not converge to a negative number, as noted above. That is, only at the lower level of savings can the follow-on effect diverge to negative infinity so as to balance the positive infinity of the other two effects.

Fourth, at the lower level of savings, where MPK exceeds ATGR, the implied tax rate on savings (or on wealth or income)[35] is strictly positive.

Fifth, as is also easy gleaned from this diagram, were the model re-solved with ever lower time-constant levels of non-transfer government spending, the long run tax would increase toward infinity. The tax

35. See *supra* note 21 for the corresponding formula for the tax rate when the base is taken to be income or wealth rather than prior period savings.

rate (stated on a base of savings) is the difference between the slope of the production function (MPK) and the slope of the total return paid to capital (that is, of the ATGR). As government spending is lowered, the two vertical lines in the diagram are shortened, the two zero-waste savings levels move to the edges of the lens between the lines. In particular, the lower level goes to zero. As noted, the MPK is assumed to go to infinity as capital goes to zero. On the other hand, the long-run ATGR is the same no matter what the level of government spending, driven as it is by the convergence of capitalist's savings and consumption levels. (This is explained in Part III.B.1.) So as the level of government spending goes to zero, the long-run MPK grows unboundedly relative to the long-run ATGR, implying that the long-run tax rate grows unboundedly. Thus, what is happening is not that the after-tax return paid to capital is zeroing out—it is in fact staying the same—but rather that the pre-tax return (MPK) is going to infinity because there is vanishingly little capital in the economy. An incremental unit being extremely productive, the market is willing to pay exorbitant returns.

III. Why the Pile Metaphor is Apt for Unfiltered, Socially Discounted Savings Responses

Step 4, the basic engine driving results in both model variants, made use of a pile metaphor. It was suggested that pushing the single-period down-pulse in the ATGR ever farther into the future was like transferring bricks from an infinite pile to a finite pile. The number of periods leading up to the down-pulse—like the number of bricks in the finite pile—was increasing without bound, while the number of periods following the down-pulse—like the number of bricks in the infinite pile—was not decreasing. Despite this lopsided increase in lead-up periods, at any social optimum the down-pulse's effect on social welfare attributable to the lead-up periods had to remain in balance with the effect attributable to follow-on periods (along with the relatively stable fixed behavior effect in the down-pulse period). This then led to a discussion of what had to happen to the multiplicative filter MU × (MPK − ATGR) by which the capitalist's savings response in any given lead-up or follow-on period was translated into laborer utility. These required adjustments produced, fairly directly, the zero-tax result in the OJM variant, and also, with a few additional steps, the positive tax result in the SW variant.

However, an important link in this logical chain was only briefly mentioned. In order for the pile metaphor to be apt, the bricks must be associated with the unfiltered savings responses in each period, not the periods themselves. It is not enough to contrast the ever growing *number* of lead-up periods with the fixed number of follow periods. To get to the necessity of adjusting the multiplicative filters, which translate savings response into laborer utility, it is necessary to contrast an ever growing pile of lead-up *savings responses* to a fixed pile of follow-on period savings responses.

This leads to the following complication in applying the pile metaphor: the bricks sitting in both piles are changing size as bricks are transferred from the infinite pile to the finite pile. That is, when the down-pulse period is pushed one period into the future, and a single period is thus transferred from the follow-on pile to lead-up pile, the savings response in every period changes—in the single period that just went from follow-on to lead-up, in all "legacy" lead-up periods, and in all "legacy" follow-on periods.

But this complication can be dealt with. The pile metaphor remains apt when the role of the bricks is played by savings responses in each period. The purpose of the current Part is to explain why.

A. Counting Savings Responses After Socially Discounting Them/ Grossing Them Up to the Down-Pulse Period

A key preliminary step has to do with time-discounting. In brief: the savings response in each period will be measured from the social welfare perspective of the down-pulse period itself. The savings response will be discounted or grossed up, as the case may be, to the down-pulse period according to the social discount factor.

In more detail: The model assumes that both the capitalist and the laborer, in forming their overall level of intertemporal utility, aggregate intra-period utilities weighting later period utilities less than earlier period utilities. In particular, both actors are assumed to discount later intra-period utilities by a fixed factor. For instance, if the discount factor were .90, the laborer would consider a "util" in the next period to be the same as .90 "utils" in the current period. Likewise, the laborer would consider a "util" three periods hence to be the same as .90 × .90 × .90 "utils" in the current period.

Because social welfare equals the laborer's discounted utility in the model, the social welfare discount factor is the same as the laborer's. The capitalist's discount factor is assumed to be the same as the laborer's and so the same as the social discount factor.

The focus of analysis in this Part is not savings responses per se, but socially discounted savings responses. Applying the social discount factor to savings responses is, in fact, imperative given the rest of the analysis. The multiplicative filter to which the analysis is heading, MU × (MPK − ATGR), does not itself contain the social discount factor. Yet the effects that must remain in balance—the lead-up, follow-on and fixed behavior effects—must themselves be calculated in a manner consistent with social discounting.

More than this, socially discounted savings responses will be measured from the perspective of the down-pulse period rather than from the perspective of the beginning of time. That is, lead-up period savings responses will be grossed-up, and follow-on period savings responses discounted, each according to how many periods distant they are from the down-pulse period. For purposes of measuring the effects of a down-pulse in Period 100, for instance, a unit of savings response in Period 103 will be regarded as .90 × .90 × .90 units of savings response from the perspective of the down-pulse period, while a unit of savings response in Period 96 will be regarded as 1/(.90 × .90 × .90 × .90) units of savings response from the perspective of the down-pulse period.

Can this be done? If so, why do it? This can be done. All that matters is that the three effects—lead-up, fixed behavior, and follow-on—remain mutually offsetting as the down-pulse is shifted into the future. For purposes of determining whether this is true for any given down-pulse period, it does not matter whether one denominates all three effects in Period 38 social welfare units, Period 927 social welfare units, or—as shall be done—social welfare units in the down-pulse period itself.

The justification for the ever shifting denomination of savings responses is that it highlights what would otherwise be obscured. Were one always to discount savings responses back to the beginning of time, one would find that *all* three effects of the down-pulse—lead-up, fixed behavior, and follow-on—converge to zero in importance as the down-pulse period heads toward infinity. That secular decay obscures relative changes across the three effects, which is all that matters. In order to gain a clear view of relative changes, the analysis is in effect controlling for secular decay by discounting/grossing-up to the down-pulse period.

B. Lead-Up Periods

As the down-pulse shifts out in time by one period, a new period moves into the set of lead-up periods. This Part explains the following two points concerning, respectively, legacy lead-up periods and the newly added lead-up period: imagine that the down-pulse is already very far in the future. As the down-pulse period takes a further step forward in time, the socially grossed-up savings response in each legacy lead-up period does not decrease. Thus, the bricks already in the pile do not shrink. Moreover, the socially grossed-up savings response in the newly added lead-up period is the same as it was for the lead-up period that was newly added the last time the down-pulse was shifted out. Thus, the brick added to the lead-up pile is always of the same size. From these two points it follows that the lead-up pile—of savings responses discounted to the down-pulse period—grows in magnitude toward infinity—positive or negative infinity depending on the model variant.

1. Legacy Lead-Up Periods

Without social welfare gross-up to the down-pulse period the savings response in any fixed lead-up period fades to zero as the down-pulse period is pushed ever farther in the future. In fact, it fades in a very regular way: once the down-pulse has been pushed far enough out, each new step into the future reduces the savings response essentially by dividing by the starting level of the ATGR in the new down-pulse period. For instance, if one is looking at the un-grossed-up savings response in Period 1066, and if the down-pulse is being pushed from Period 1491 to Period 1492, and if the ATGR in Period 1492 starts out at 1.11 = 1/.90, then the saving response in Period 1066, whether positive or negative, is reduced in magnitude by multiplying it by 1/ATGR = .90.[36]

The reader may wish to know what causes this (eventually) very regular attenuation. But for present purposes it matters only *that* it

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36. *Technical note:* Recall from note 24 that $\frac{ds_t}{dR_T}\frac{R_T}{s_t} = \frac{a_T}{R_T \times \ldots \times R_{t+2}a_{t+1}}\left(1 - \frac{a_{t+1}}{R_{t+1} \times \ldots \times R_1 a_0}\right)\left(\frac{1}{\sigma} - 1\right)$. In this thought experiment, t is fixed and T is increased by one. The question is how $\frac{ds_t}{dR_T}R_T$ changes if down-pulses are considered "percentage changes". Note that, assuming convergence in the capitalist's consumption plan, a_T converges.

happens, not why.[37] Such attenuation, let alone its regularity, is an artifact of the model: it follows from the structure of the utility function assumed for the capitalist. In this analysis, it may be safely regarded as an assumption in its own right without understanding how it follows from assumptions made deeper in the model.

With the social welfare gross-up to the down-pulse period, the attenuation just described competes with a contrary dynamic: the fact that the process of grossing up from any given lead-up period to the down-pulse period covers an ever longer stretch of time as the down-pulse period is shifted into the future. Thus, in the example above, the savings response in 1066 is first grossed up to 1491, and then to 1492. Adding an additional period of gross-up effectively divides the grossed-up savings response by the social discount factor. For instance, if the social discount factor is .90, then the savings response in Period 1066 is divided by .90.

The reader will notice that in the numerical example used in the last two paragraphs, as the down-pulse was shifted from 1491 to 1492, attenuation of the un-grossed-up savings response meant multiplying by .90 and the additional grossing up meant dividing by .90. There was no net impact. This is in fact what happens, eventually, in the model. There are two reasons for this. First, the social discount factor and the capitalist's discount factor are assumed to be the same, as noted. Second, for the reason described in the next paragraph, because the capitalist's consumption level is assumed to be converging, 1/ATGR must be converging to the capitalist's discount factor, which is to say, the social discount factor.

Imagine that by Period 2000, the capitalist's consumption level has substantially converged and is essentially the same in Period 2000 as in Period 2001. Suppose, however, that the ATGR in Period 2001 is *not* equal to the reciprocal of the capitalist's fixed discount factor—that, while the discount factor is .90, the ATGR is, say, 1.20 rather than 1.11 = 1/.90. Then the capitalist cannot be privately optimizing. Given that consumption has essentially leveled off, intra-period marginal utility for the capitalist must be the same in Periods 2000 and 2001; denote this common value MUC. If the capitalist decreased consumption in Period 2000 by one unit, and instead saved that amount, she would lose MUC units of utility in Period 2000. However, she would render feasible 1.20 units of additional consumption in Period 2001, which would

37. The Web Appendix, *supra* note 19, explains why in detail.

generate 1.20 × MUC units of intra-period utility in Period 2001. Given a discount factor of .90, this would be equivalent to .90(1.20 × MUC) units of Period 2000 utility. Because 1.20 exceeds 1/.90, it follows that .90(1.20 × MUC), the gain in Period 2001, exceeds MUC, the loss in Period 2000. Thus, the capitalist's intertemporal utility would increase.[38]

In sum, if the down-pulse period is already far in the future, as it is pushed ever farther out, the savings response in any given legacy lead-up period, socially grossed-up to the down-pulse period, is essentially not decreasing. The bricks already stacked in the lead-up period pile are not shrinking.

2. Newly Added Lead-Up Period

What has just been said about the persistence of lead-up period savings responses is an important part of the reason that lead-up savings responses accumulate without bound. But it is not the whole story. The accumulation will still be bounded if the savings response in the newly added lead-up period is decreasing fast enough to zero. If, in each period, one moves half the remaining distance between oneself and a wall, one travels only a finite distance over infinite time even though the distance already traveled never shrinks.

But once the down-pulse is sufficiently far in the future, the saving response in the newly added lead-up period essentially does not decrease. This is not driven by the process of grossing up to the down-pulse period—the newly added lead-up period is always a single period earlier then the down-pulse. Rather it is driven by bankbook mechanics.

The capitalist's savings in any period, say Period 1491, is like the amount that she leaves in the bank as she exits that period—a bank that pays interest in every period according to that period's ATGR. Her original after-tax wealth going into Period 0 is her original (and only) deposit into this notional bank account. Her consumption in each period is a withdrawal. The negative impact of an additional withdrawal on the bank balance in any future period is the amount of the additional withdrawal multiplied by all the gross-of-principal rates of interest from the period of withdrawal to the period of the balance, reflecting both the missing principal and all the interest that was not earned.

38. Were the ATGR instead less than the capitalist's discount factor, the capitalist could gain by consuming one unit more, rather than less, in Period 2001.

Likewise, the negative impact of a change in consumption in, say, Period 1066 on savings in Period 1491 is the change in 1066 consumption multiplied by all the ATGRs between 1066 and 1491.

As the newly added lead-up period shifts into the future in tandem with the down-pulse period, two effects compete with each other and eventually become perfectly offsetting. First, the change in consumption in each earlier period deflates—due to the same kind of attenuation identified above in relation to lead-up savings. Second, each earlier period's consumption change must be accumulated over more periods to bring it forward to the now-later newly added lead-up period.[39] Eventually, as for savings, deflation in the consumption change amounts to dividing it by the down-pulse period's ATGR. At the same time, accumulating the effect of each consumption change up to the now-later newly added lead-up period means additionally multiplying by that newly added lead-up period's ATGR. Given convergence of ATGRs, dividing by the down-pulse period's ATGR while multiplying by the newly added lead-up period's ATGR has no net effect.

Suppose, for instance, that by Period 1490 the ATGR has essentially leveled off to 1.11. Consider first a down-pulse in Period 1491. The newly added lead-up period for a down-pulse in 1491 is Period 1490. The savings response in Period 1490 is the accumulation, according to the sequence of past ATGRs, of consumption responses in Periods 0 through 1490. For example, the consumption response in Period 476 is multiplied by the ATGRs in Periods 477 through 1490 in determining its impact on Period 1490 savings.

Now consider a down-pulse one period later, in Period 1492. The newly added lead-up period is now also one period later, Period 1491. Moving the down-pulse one period into the future means the consumption response in Period 476 is attenuated by dividing it by the ATGR in 1492, which is 1.11. However, in its impact on newly added lead-up periods savings, Period 476 consumption is now accumulated by multiplying by the ATGRs in Periods 477 through 1491 rather than just through 1490. Therefore, 476's consumption response is additionally multiplied by the ATGR in Period 1491, which is also 1.11.[40] Dividing and multiplying by 1.11 has no net effect.

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39. There is a third effect that eventually does not matter: a new "withdrawal" is added. *See infra* note 39.

40. Continuing the discussion *id.*, it is also true that, when the down-pulse is pushed from 1491 to 1492, Period 1491's consumption change

In sum, as the down-pulse is pushed one period farther in the future, the brick that is newly added to the pile of socially grossed-up savings responses is eventually of some fixed size. Combining this with the fact that the bricks already in the lead-up pile do not shrink, the pile must grow without bound.

C. Follow-On Periods

The complementary point concerns savings responses in follow-on periods: as the down-pulse is pushed one period farther into the future, the pile of socially discounted savings responses in the follow-on period stays the same size. This is a consequence of the fact that the whole model, inclusive of savings responses, is converging over time.

Thus suppose that savings responses in every period from Period 1492 on are essentially the same. Then the sequence of savings responses from Period 1493 on is essentially the same as the sequence of savings responses from 1492 on. And therefore, given the constant discount factor, the sum of savings responses from 1493 on, each discounted back to 1493, is the essentially the same as the sum of savings responses from 1492 on, each discounted back to 1492. This is for the same reason that, with a constant interest rate, a perpetuity neither depreciates or appreciates over time.

D. Summary

The bricks in the pile metaphor may be taken to be the unfiltered savings responses in each period grossed-up or discounted, as the case may be, to the down-pulse period. Eventually, as the down-pulse period is shifted ever farther into the future, the bricks already in the lead-up pile stop shrinking, while the size of the newly added brick levels off to some constant size—with the result that the lead-up pile grows infinitely large. On the other hand, the pile of follow-on bricks reaches a constant size.

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is added to the list of those impacting the newly added lead-up periods savings level. However, this individual, unaccumulated consumption change converges to zero, via the attenuation described in the text, and so its addition is eventually of no consequence.

IV. More on the Multiplicative Filter

In Step 6 it was asserted that there are only two possibilities for how the filter MU × (MPK − ATGR) can adjust to maintain balance in the face of the unbounded relative growth in the (aggregate socially grossed-up) lead-up savings response caused by pushing the down-pulse ever farther into the future. The first was that MPK − ATGR could go to zero fast enough to prevent the lead-up savings response from growing without bound in terms of its effect on laborer consumption. The second was that MU could grow toward infinity in such a way that the follow-on savings response keeps up with the lead-up savings responses in terms of its effects on laborer utility. This Part makes two clarifying points regarding the filter's role.

The first point is that the two options described in the preceding paragraph for the filter are indeed the only ones. In particular, a natural third candidate for maintaining balance—that MPK − ATGR heads toward positive or negative infinity—is ruled out. The reasoning behind this first point starts with the fact that production is both bounded from above and bounded above zero in the model. That is, there are two positive numbers, say 8 and 7,000,000, and output in every period must lie between 8 and 7,000,000. This two-sided containment arises chiefly from two sources. First, there is no technological change and there is a fixed rate of depreciation. This effectively bounds per-period output from above.[41] Second, there is the requirement that production in every

41. This note explains why production is bounded in the model. The production process in the model works as follows (ignoring the fixed amount of labor). A given amount of capital, say 100 units, is input in the preceding period. Output comes in the current period. It consists of the sum of two things: recovered depreciated capital and incremental output. The first addend is the capital that was input less a fixed fraction of that capital representing depreciation. If that fraction is 10%, for instance, this first addend would be 100(1 − .10) = 90. The second addend is determined by a fixed function that exhibits, inter alia, diminishing marginal returns that diminish all the way to zero as input capital is increased. Therefore, if the amount of capital in the economy is very large, incremental output is negligible. This implies that at very large levels of capital, depreciation dominates and total output is less than inputted capital. In this case, the current period's capital input, which comes out of current period production, must be less than last period's capital input, so that next period's output is less than the current period's. This, in turn, implies that there is some upper bound above which output cannot climb.

period be sufficient to meet the time-constant positive amount of non-transfer government spending. This bounds production above zero.

This two-sided boundedness of output then implies, as will be explained, that MPK and ATGR are each positive and bounded from above, and this immediately rules out that MPK— ATGR heads toward positive or negative infinity.

With regard to MPK: First, MPK is in fact definitionally positive (though diminishing). Second, the fact that production is bounded above zero means that MPK is bounded from above—despite the assumption in the model that MPK is infinite when capital is zero.

With regard to ATGR: First, note that the ATGR times savings last period, that is ATGR × S_{t-1}, is the capitalist's after-tax wealth entering this period. The capitalist's after-tax wealth entering this period must be positive: she must save a positive amount in this period so that there is enough production next period for non-transfer government spending.[42] Because ATGR × S_{t-1} is positive, ATGR must be positive. To show that ATGR is also bounded from above, one starts with the fact that S_{t-1} must be greater than some fixed positive amount, say 7, to be able to produce at least non-transfer government spending in the current period. This is combined with the fact that ATGR × S_{t-1} cannot be larger than some fixed amount, say 7,000,000, given the upper bound on output in each period. If S_{t-1} is no less than 7, and ATGR × S_{t-1} is no greater than 7,000,000, then ATGR cannot exceed 1,000,000.

The second point regarding the filter MU × (MPK – ATGR) fleshes out how the MU portion functions in the SW variant. The follow-on effect, which occurs over an infinite number of periods, is only finite in the first place because the model assumes (sufficiently fast) time-discounting. This means that the aggregate social welfare contribution of the follow-on periods, from the perspective of any given down-pulse period, is in effect a weighted average of the social welfare contribution in each follow-on period with most of the weight on follow-on periods near in time to the down-pulse, and with weights trailing off to zero (at an exponential rate). When the down-pulse is pushed ever farther into the future and when the laborer's marginal utility is going to infinity, the follow-on effect averages with ever greater weight on periods with ever greater MU's. Consequently, the follow-on effect grows ever larger.

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42. It is assumed in the model that no output can be produced with some capital.

It is true that the *lead-up* effect also grows as it takes in ever greater MU's. But it is in the nature of infinity that the follow-on effect can still be made to dominate. As the MU in the newly added lead-up period grows toward infinity, it is never any more difficult to find MU's for the periods that remain in the follow-on pile that are still greater—still greater to any degree. For example, the MU in Period 1214 may already be 1,000,000. But the MU in Period 1215 may be $1{,}000{,}000^{10}$, and in 1216, $1{,}000{,}000^{1{,}000{,}000}$, etc.

V. Why Sending MPK-ATGR to Zero Does Not Work in the SW Variant

As discussed in Step 5, savings responses are translated into social welfare via the multiplicative filter MU × (MPK – ATGR). MPK is the marginal product of capital, and so the additional output generated by additional savings. ATGR is after-tax gross rate of return to savings, and so the additional payout of such output to capital. The difference MPK – ATGR is thus the portion of the additional output generated by additional savings that goes to laborer consumption. MU then translates that consumption into laborer utility. In the OJM variant the march toward infinity of lead-up savings responses is damped by the concomitant convergence of MPK – ATGR to zero. A key assertion in Step 8 is that this device is unavailable in the SW variant. This Part explains why.

The outline of the argument is this: When down-pulses in ATGRs increase lead-up savings, as in the SW variant, it is possible to augment any given single-period down-pulse, say in Period 100, with specially calibrated down-pulses in all periods leading-up to Period 100 that accomplish the following:

1) they restore savings levels in Periods 0 through 99 to their original lower state; and

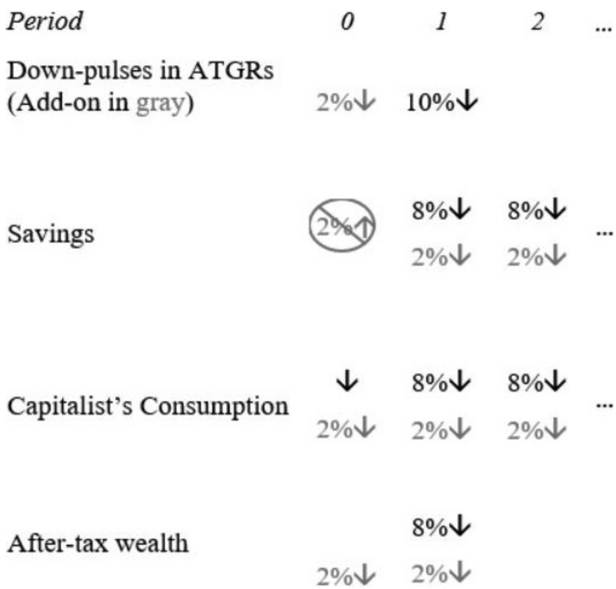
2) they produce an overall reduction in capitalist consumption in each of Periods 0 through 99 that is commensurate with such period's original increase in savings.

Because savings levels are now constant in the lead-up phase, MPK – ATGR is no longer a factor in the lead-up effect, and the decreases in capitalist consumption can be directly identified with increases in laborer consumption. As explained in this Part, this removes from the picture ambiguities regarding the sign and magnitude of MPK – ATGR (as noted in Part A, below) and enables the conclusion that, whatever is happening to MPK – ATGR, the lead-up effect (plus the fixed behavior effect) from ever later down-pulses heads toward positive infinity.

A. Add-On Perturbations Producing an Augmented Down-Pulse

Consider a 10% down-pulse in Period 1's ATGR, as depicted in black ink in Figure 3. (Ignore the gray ink for the moment.) The Period 1 ATGR is the after-tax gross rate of return paid to the capitalist in Period 1 per unit of her chosen level of savings in Period 0. For the Period 1 ATGR, Period 0 is the sole lead-up period and Periods 1, 2 etc. are the follow-on periods. In the SW variant the down-pulse in the Period 1 ATGR induces the capitalist to reallocate some Period 0 consumption toward Period 0 savings: the reallocation is zero-sum since after-tax initial wealth upon entering Period 0 is exogenous and so unchanged. Suppose that the 10% down-pulse in Period 1's ATGR causes Period 0 savings to increase by 2%, as indicated in the row labeled "Savings" in the figure (still ignoring the gray ink).

Figure 3: The add-on downpulse in Period 0 (in gray) neutralizes the lead-up savings increase and further reduces capitalist consumption.



With regard to follow-on Periods 1, 2, etc.: the capitalist enters Period 1 with an amount of after-tax wealth equal to Period 0 savings times Period 1's ATGR. The percentage change in × times Y is the percentage change in × plus the percentage change in Y. Therefore, a 10% decrease in the Period 1 ATGR combined with a mitigating 2% increase in Period 0 savings results in an 8% decrease in the capitalist's after-tax wealth entering Period 1, as indicated in the bottom row of the diagram. Thus, as the capitalist looks forward from the start of Period 1, no prospective ATGR's have changed, but she has 8% less after-tax wealth to allocate to consumption in Periods 1, 2, etc. In the SW variant (and the OJM variant) the impact of this wealth reduction across follow-up periods is artificially regular: it causes an 8% across-the-board deflation in consumption levels in Periods 1, 2, etc., and the same percentage deflation in the savings levels in those periods.

What do the three components of marginal social welfare look like for this 10% down-pulse in Period 1's ATGR?

- The fixed behavior effect is positive: holding savings levels fixed, reducing the return paid to capital in Period 1 increases the amount available for laborer consumption.

- But the lead-up and the follow-on effects—both of which derive from changes in savings—are each seemingly ambiguous. The reason is that no assumption can be made in any given period regarding whether savings is being taxed or subsidized, let alone to what extent. That is, no assumption can be made about whether MPK − ATGR is positive or negative, let alone how large it is in magnitude. Changes in savings translate into changes in social welfare through the MU × (MPK − ATGR) filter.

However, in the SW variant these ambiguities can be starkly resolved. This is done by coupling the 10% decrease in the Period 1 ATGR with whatever change in the *Period 0* ATGR restores Period 0 savings to its original level. This add-on perturbation in Period 0's ATGR, as well as its effects, are shown in gray ink in Figure 3.

If the goal is to restore Period 0 savings to its original level, must Period 0's ATGR be increased or decreased, and by how much? The answer is actually simple. Period 0 savings is follow-on savings for Period 0's ATGR. *Follow-on* savings and ATGRs move in the same direction (in both model variants). Therefore, to undo the 2% increase in Period 0 savings caused by the down-pulse in Period 1's ATGR, Period 0's ATGR must also be pulsed downward.

Indeed, Period 0's ATGR must be decreased by 2%, as shown in gray in the figure. A 2% decrease in Period 0's ATGR reduces after-tax wealth upon entering Period 0 by 2%. This causes a 2% across-the-board shrinkage in consumption levels and savings levels in follow-on periods with respect to Period 0's ATGR, which are 0, 1, 2, etc. (This is the same kind of model-artifactual across-the-board shrinkage of follow-on behavior that was seen above with respect to the reduction in Period 1's ATGR and Periods 1, 2, etc.)

Now focus on what this add-on downward perturbation to Period 0's ATGR does to the lead-up effect of the original down-pulse in Period 1's ATGR—that is to say, to the social welfare effect of changes in Period 0 savings. Before the add-on perturbation, there was an (assumed) 2% increase in Period 0's savings level which, filtered through MU × (MPK – ATGR) in Period 1, had some ambiguous impact on laborer utility in Period 1. By virtue of the add-on perturbation, this 2% increase in Period 0 savings has been replaced by a further 2% decrease in Period 0 consumption by the capitalist. The total decrease in Period 0's capitalist consumption—the initial decrease of unspecified size plus the add-on 2% decrease—produces an increase in Period 0 *laborer* consumption of equal magnitude. The increase in Period 0 laborer consumption is of equal magnitude because there is no overall change in 1) Period 0 output, which is exogenous, 2) the portion of Period 0 output going to Period 0 savings, which remain at their original level, and 3) non-transfer government spending, which, like Period 0 output, is also exogenous. This increase in laborer consumption filters into Period 0 social welfare via MU, with no mediation by MPK – ATGR.

Thus, the add-on perturbation in Period 0's ATGR has replaced the original lead-up effect with an unambiguous increase in social welfare driven by a reduction in Period 0 capitalist consumption that is, in percentage terms, larger than the original increase in Period 0 savings (which was assumed to be 2%).

It must also be considered whether adding the down-pulse in Period 0 changes the marginal social welfare attributable to laborer consumption in Periods 1, 2, . . . (it does). This is discussed below. But first there is more to say about marginal social welfare attributable to labor consumption in lead-up periods from the augmented down-pulse.

B. Lead-Up Effect of Augmented Perturbation Grows Toward Positive Infinity as Initial Down-Pulse is Pushed Ever Farther into the Future

In the SW variant, the technique of adding on earlier down-pulses to replace increases in lead-up savings with further decreases in lead-up capitalist consumption is available starting from any period. Had the original down-pulse been in Period 1066's ATGR, it would have been possible to design add-on down-pulses in all earlier periods—iterating backward from 1065 to 1064 to 1063, etc.—that would have replaced the savings increases in all periods leading up to 1066 with further capitalist consumption decreases. Moreover, such further consumption decreases in each of those periods—in Period 476 for instance—would have been of the same magnitude, in percentage terms, as the original increase in savings in Period 476 due to the original down-pulse in Period 1066. How and why this iterative replacement works starting from any period is described below in Part F and formalized in the online appendix.[43]

Given this replacement of savings increases with capitalist consumption decreases in all lead-up periods, it is possible to piggyback on what is already known from Part III about the unbounded increase in socially grossed-up lead-up savings as a single-period down-pulse is pushed ever farther into the future. That prior finding implies, as will now be discussed, that the socially grossed-up capitalist *consumption* decreases in lead-up periods due to the augmented down-pulse must also grow unboundedly. The reasoning combines the arithmetic of percentage changes with the intra-period boundedness of the Judd model.

It has been noted that the *percentage* decrease in each lead-up period's capitalist consumption level is at least as large as the original *percentage* increase in the same-period savings levels. What does this say about the relative size of unit changes? Suppose one knew that X and Y both changed by 2%. One cannot conclude from this that the unit change in X is as large in magnitude as the unit change in Y because X may be a much smaller base for calculating percentage change than is Y. However, if it were given that X was never less than, say ¼ of Y,

43. Sanchirico, *Web Appendix*, *supra* note 19.

then it would be possible to conclude that the unit change in X is never less than ¼ of the unit change in Y.[44]

What has just been imagined for X and Y is in fact true for savings and capitalist consumption. First, the amount the capitalist saves in any given period must always be greater than the minimum necessary to produce the time-constant level of non-transfer government spending in the next period. This implies that the capitalist's future consumption, which is what that savings finances, can never be below some fixed positive level, say 2.[45] Conversely, the output that is put aside as savings in any period cannot be *more* than the maximum amount producible in the economy in any single period. So savings is always less than some maximal level, say 8. Therefore, capitalist consumption is never less than $2/8^{\text{ths}}$ = ¼ of savings.

Putting the last two paragraphs together, it follows that the *unit* change in consumption in, say, Period 476, from the augmented multi-period down-pulse working back from Period 1066, is never less than, say, ¼ of the original *unit* change in savings in Period 476 from the original single-period down-pulse in Period 1066.

And then it is clear that the decreases in lead-up capitalist consumption from the augmented down-pulse—aggregated after grossing-up to the original down-pulse period—grow unboundedly just as do the lead-up savings increases from the original single-period down-pulse. For if the sum of Y_1, Y_2, Y_3, etc. explodes, then so too does the sum of $\frac{1}{4}Y_1$, $\frac{1}{4}Y_2$, $\frac{1}{4}Y_3$, etc.

It is now possible to conclude that, as the original down-pulse period is pushed ever farther into the future, the marginal social welfare attributable to laborer consumption in lead-up periods from the augmented perturbation must be heading toward positive infinity. As noted, the decreases in lead-up capitalist consumption, which are one-for-one increases in laborer consumption, filter into social welfare only through MU: the MPK – ATGR filter has been removed from consideration. Because laborer consumption can never be greater than the uniform maximum amount producible in the economy in any given period, laborer marginal utility MU is never smaller than some positive amount, say 1/5—despite the fact that MU diminishes. Therefore, at

44. That is, $\frac{X_1 - X_0}{X_0} = \frac{Y_1 - Y_0}{Y_0} \rightarrow X_1 - X_0 = \frac{X_0}{Y_0}(Y_1 - Y_0)$.

45. *Technical note:* This uses the assumption that capitalist consumption converges, as explained in Sanchirico, *Web Appendix*, *supra* note 19.

least 1/5 of each capitalist consumption decrease becomes a same-period increase in laborer utility. And if the sum of $\frac{1}{4}Y_1$, $\frac{1}{4}Y_2$, $\frac{1}{4}Y_3$, etc. explodes then so does the sum of $(1/5)\frac{1}{4}Y_1$, $(1/5)\frac{1}{4}Y_2$, $(1/5)\frac{1}{4}Y_3$, etc.

C. Fixed Behavior and Follow-On Effects

It remains to consider how the add-on perturbations change marginal social welfare attributable to laborer consumption in the follow-on periods including the period of the original down-pulse. For this purpose, return to the scenario in which the original down-pulse was in Period 1, as depicted in Figure 3.

First, consider Period 1. The add-on down-pulse in Period 0 (first row in gray) turns marginal social welfare attributable to laborer consumption in Period 1 into the original down-pulse's fixed behavior effect in Period 1. The fixed behavior effect becomes the entire social welfare effect attributable to laborer consumption in Period 1 because the add-on down-pulse restores Period 0 savings to its original level. Further, the fixed behavior effect in Period 1 is unchanged by the add-on down-pulse since the original Period 1 down-pulse remains intact.

Now consider Periods 2, 3, . . . As before the add-on down-pulse marginal social welfare attributable to laborer consumption in Periods 2, 3, . . . is a result solely of changes in savings in Periods 1, 2, . . . The add-on down-pulse does, however, change the magnitude of marginal social welfare attributable to laborer consumption in the periods that follow the original down-pulse period. Even so, the change in magnitude is limited in a particular way.

With the help of Figure 3, recall what happened under the original single-period down-pulse in Period 1: As the capitalist looked forward from the start of Period 1, having just received her return from Period 0's savings, no prospective ATGR's had changed, but she had 8% less after-tax wealth to allocate to consumption in Periods 1, 2, etc. That caused an 8% across-the-board shrinkage in all savings levels in Periods 1, 2, etc. The 8% deflation in after-tax wealth entering Period 1 was the net impact of two effects: the 10% reduction in Period 1's ATGR mitigated by the 2% increase in Period 0's savings level.

After the add-on down-pulse in Period 0, the 2% mitigating increase in Period 0 savings no longer exists. Therefore, the capitalist now enters Period 1 with 10% less initial after-tax wealth, rather than 8% less. This results in an across-the-board decrease of 10%, rather than 8%, in all follow-on savings levels starting with the current Period 1.

Therefore, after the add-on down-pulse, marginal social welfare attributable to laborer consumption in Periods 2, 3, . . . equals the follow-on effect of the original down-pulse inflated by a factor of 10/(10 – 2) = 5/4.

The key point is that the inflation factor by which the follow-on effect is multiplied (5/4 in the preceding paragraph) never rises above some uniform maximal level as the original single-period down-pulse is pushed ever farther into the future. This is because that factor can only grow unboundedly if the percentage increase in savings in the period immediately preceding the single-period down-pulse (2% in the example) becomes an ever larger fraction of the original down-pulse percentage (10% in the example).[46] In the Judd model, this does not happen. In fact, the mitigating percentage increase in savings in the period immediately preceding the single-period down-pulse converges to a fixed fraction of the down-pulse percentage as the single-period down-pulse is pushed ever farther into the future. Such convergence follows from underlying assumptions regarding the capitalist's intertemporal utility function.[47] But, consistent with the discussion surrounding Step 3, for present purposes it can be regarded as an assumption in its own right.

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46. X/(X – z) = 1/(1 – (z/X)). Given z < X, which holds in the Judd model, X/(X – z) only grows without bound if z approaches X.

47. *Technical note:* As explained in Sanchirico, *Web Appendix*, *supra* note 19, and consistent with *supra* note 24, $\frac{ds_{T-1}}{dR_T}\frac{R_T}{s_{T-1}} = \left(1-\theta_T\right)\left(\frac{1}{\sigma}-1\right)$,

where $\theta_T = \frac{\frac{a_T}{R_T \times \ldots \times R_1}}{a_0} \in [0,1]$ is the portion of the capitalist's after-tax wealth entering Period 0 that is set aside for consumption in all follow-on periods. (Recall that this derivative is for an *up-pulse* in R_T not a down-pulse in order to conform to calculus convention.) Because the capitalist is optimizing, θ_T converges to zero as $_T$ goes to infinity. Therefore, $\frac{ds_{T-1}}{dR_T}\frac{R_T}{s_{T-1}}$ converges to $\frac{1}{\sigma}-1$.

Thus, there is some number, say 6, such that, no matter how far in the future the single-period down-pulse, the add-on down-pulses in earlier periods multiply the magnitude of marginal social welfare from laborer consumption in periods following the original down-pulse—that is, multiply the follow-on effect—by no more than 6.

D. Summary and Implications for Lead-Up, Fixed Behavior, and Follow-On Effects

To summarize what happens to marginal social welfare attributable to laborer consumption in all periods as a result of the add-on down-pulses: first, in all periods leading up to the original down-pulse period, marginal social welfare from laborer consumption in such periods is driven entirely by decreases in contemporaneous capitalist consumption levels, and the aggregate social welfare impact, from the perspective of the original down-pulse period, grows unboundedly toward positive infinity as the down-pulse is moved ever farther into the future. Second, marginal social welfare attributable to laborer consumption in the original down-pulse period becomes equal to the fixed behavior effect of the original down-pulse. Marginal social welfare of the augmented perturbation in the original down-pulse period is thus positive. Third, marginal social welfare attributable to laborer consumption in periods after the original down-pulse period becomes equal to the follow-on effect multiplied by some positive number that is never more than, say 6.

Now, these three components of marginal social welfare cannot sum to a positive number. If they did, the augmented perturbation would increase social welfare, contradicting the stipulation that the starting point for the down-pulses was a social optimum. But the sum of the first two effects—accounting for laborer consumption in all periods up to and including the original down-pulse periods—heads toward positive infinity as the down-pulse is pushed ever farther into the future. Therefore, the marginal social welfare attributable to laborer consumption in periods after the original down-pulse period must be heading toward negative infinity. Therefore, 6 times the *original* follow-on effect, a product which is greater in magnitude than the transformed follow-on effect, must also be heading toward negative infinity. But then the *original* follow-on effect must be heading toward negative infinity: if $6Z_1$, $6Z_2$, $6Z_3$, etc. goes to negative infinity, then so does Z_1, Z_2, Z_3, etc.

Circling back, it then follows that the sum of the original lead-up and fixed behavior effects, which must offset the original follow-on effect, must be heading toward positive infinity.

E. Implications for Multiplicative Filter

In the SW variant, while it is not possible to deduce generally what is happening to the filter MPK − ATGR, it is possible to deduce that making MPK − ATGR go to zero does not, on its own, work as a way of keeping marginal social welfare at zero as the down-pulse is pushed ever farther into the future. Whatever might be happening to MPK − ATGR, the follow-on effect from a single-period down-pulse must be heading toward negative infinity. The follow-on effect is caused by future decreases in savings, which filter through MU × (MPK − ATGR) into laborer utility. This filter must be heading toward positive infinity, because, as discussed in Part III.C, the follow-on savings decreases themselves will not be. Because the factor MPK − ATGR is bounded from above, as discussed in Part IV, the only way for MU × (MPK − ATGR) to become infinitely large is for MU to become infinitely large.

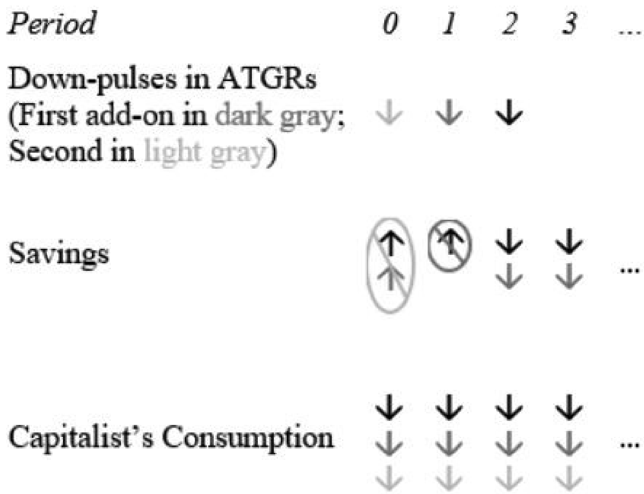
Moreover, since the change in follow-on savings is negative and the follow-on effect must be heading toward negative infinity, MU × (MPK − ATGR) must be converging to a positive number, implying that MPK − ATGR cannot be converging to a strictly negative number. This takes the analysis up to Figure 2 in Step 8.

F. Aside: How the Add-On Down-Pulses are Generally Constructed

To better understand how the add-on down-pulses are more generally constructed, start by considering a down-pulse in the ATGR in Period 2 with the aid of Figure 4. The add-on perturbations in Periods 1 and 0 are constructed working backwards from Period 1 to Period 0.

Starting with Period 1, one changes the ATGR in Period 1 to offset the increase in Period 1 savings caused by the original reduction in Period 2's ATGR. This is done under a hypothesis that will be fulfilled when Period 0 is addressed: that Period 0 savings have not changed. Because Period 1 savings is follow-on savings with respect to Period 1's ATGR, Period 1's ATGR must be reduced. That also causes Period 1 consumption to fall. These changes are shown in dark gray ink in the diagram.

Figure 4: The Add-On Downpulses (in dark gray and light gray) Conditionally Neutralize Same-Period Savings Responses and Further Reduce Capitalist Consumption



Now turn to Period 0. Period 0 is a lead-up period with respect to the ATGRs in both Period 1 and Period 2. Thus the original reduction in Period 2's ATGR increased Period 0 savings and decreased Period 0 consumption. Moreover, despite the hypothesis in the preceding paragraph, the add-on reduction in Period 1's ATGR further increased Period 0 savings while further decreasing Period 0 consumption. The target is the overall increase in Period 0 savings, and Period 0's ATGR is adjusted to offset it, as shown in light gray ink. Doing this requires reducing Period 0's ATGR because Period 0 is a follow-on period for Period 0's ATGR. This yet again reduces consumption in Period 0. It does not, however, change the fact that savings in *Period 1* remains at its original level since the reduction in Period 1's ATGR was designed to accomplish that under the hypothesis that Period 0 savings remains fixed at its original level and the change in Period 0's ATGR is merely making this so.

So looking over both lead-up periods, the two add-on reductions to the ATGR's in Periods 0 and 1 have replaced the increase in savings levels in those periods with further decreases in consumption.

A key additional point is this: the percentage reduction in lead-up period consumption after the add-on down-pulses is greater than the percentage increase in lead-up period savings from the original single-period down-pulse taken alone. This follows from a transitive chain that can be seen clearly by focusing on what the add-on adjustment in Period 0 must accomplish. The add-on adjustment in Period 0 must offset the percentage increase in Period 0 savings from both the original Period 2 down-pulse and the add-on Period 1 down-pulse, for both of which Period 0 savings is lead-up savings. So the add-on adjustment in Period 0 must at least offset the percentage increase in savings from the original Period 2 down-pulse. It does this by reducing the Period 0 ATGR, for which Period 0 savings is follow-on savings. Furthermore, it does this via lump-sum effect—according to which a percentage change in initial wealth produces the same percentage change in all follow-on savings and consumption levels. Therefore, the percentage reduction in the Period 0's ATGR must be at least the percentage increase in Period 0 savings from the original Period 2 down-pulse. The percentage reduction in Period 0's ATGR also produces the same percentage reduction in Period 0 consumption—because of its lump-sum impact. Therefore, the percentage reduction in Period 0 *consumption* due specifically to the percentage reduction in Period 0's ATGR must be at least the percentage increase in savings from the original Period 2 down-pulse. Now, the *actual* percentage reduction in Period 0 consumption after *all* add-ons is greater than that due solely to the reduction in Period 0's ATGR: Period 0 consumption is also reduced by the reduction in ATGR's in Periods 1 and 2. Therefore, the actual percentage reduction in Period 0 consumption after all add-ons must be at least the percentage increase in savings from the original Period 2 down-pulse.

If the down-pulse had been in a later period, say Period 100, the construction could have proceeded in essentially the same way starting with Period 99, under the hypothesis that Period 98 savings was fixed, then moving to Period 98 under the hypothesis that Period 97 savings was fixed—all the way back to Period 0 using the fact, rather than the hypothesis, that savings in Period – 1 ("minus one") is fixed in the model structure. The conclusions would be similar as well: in each lead-up period, say, Period 59, the increase in lead-up savings from the original single-period down-pulse in Period 100 would be replaced by a further decrease in Period 59 consumption.

The same transitive chain would apply to show that the overall decrease in Period 59 consumption, call it "Z", was larger in percentage terms than the original increase in Period 59 savings, call it "X": The original increase in Period 59 savings due to the single-period down-pulse in Period 100, that is X, would be less than the percentage increase in Period 59 savings caused by the down-pulses in all of Periods 60 through 100, call that Y. So Y > X. The remaining down-pulses to be constructed in Periods 0 through 59, would have to deflate Period 59 savings by Y. These earlier down-pulses would have a lump-sum effect on Period 59 saving, and would operate by deflating after-tax wealth entering Period 59 by Y. This would also deflate capitalist consumption in Period 59 by Y. That deflation in Period 59 consumption would be added on top of the deflation in consumption that has already occurred from down-pulses in Period 60 through 100 to produce the full percentage decrease in Period 59 consumption, which is Z. So Z > Y. Because Z > Y and Y > X, it follows that Z > X.

VI. Proving or Assuming Convergence

The explanation provided in this Article starts from the assumption that the model is converging. More precisely, the assumption is that the government's announced infinite sequence of ATGRs induces the capitalist to adopt an infinite-horizon consumption plan that converges to some given level. If, for instance, capitalist consumption levels converge to 125, this means that however tight a collar one draws around 125, the sequence stays within that collar after some point.

Convergence of the capitalist's consumption levels, combined with the intra-period boundedness of the model, implies convergence of the other key variables.[48] The sequence of ATGRs must itself be converging as explained in Part III.B.2. The savings level, which is the ATGR-discounted value of consumption going forward, must then be converging—and to a positive level to provide enough capital to at least support required non-transfer government spending. It follows that both output and the portion of output paid to capital (savings times the ATGR) must be converging to a positive number. This in turn implies that the portion of output going to laborer consumption must be converging. It also implies that MPK, and so MPK − ATGR, must be converging.

The assumption made in this Article that capitalist consumption converges is actually weaker than the assumption made regarding convergence in Judd's original Article. Judd assumes not only that capitalist consumption converges, but that the associated convergent sequence of laborer consumption converges in a particular way: to a positive number rather than to zero.

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48. *See* Sanchirico, *Web Appendix*, *supra* note 19.

SW, on the other hand, do not assume that capitalist consumption converges under the candidate optimal plan. Rather they offer a proof of convergence for the subset of parameters constituting the SW variant of the model. The proof appears in a lengthy online appendix and contains a dozen separately proven component propositions (lemmas). This Article makes no attempt to explain that proof. Nor does it take a position on whether or not the proof is complete and correct.

VII. A Note on Differences in Savings Responses Across Model Variants

This Part provides two ways of speaking about and conceiving of the differences in savings responses across the two model variants. The second relates to the introductory discussion contrasting quantity and expenditure. The link between the two conceptions is the fact that current savings may be viewed either as residual current assets (or income[49]) after removing current consumption or as spending on future consumption in terms of current consumption.

First, viewing current savings as a residual vis-à-vis current consumption, the differences in savings responses across the two model variants can be understood in terms of the relative strength of income and substitution effects on current consumption. Accessing the classic model of consumer choice, one may conceive of the capitalist as allocating Period 0's initial after-tax wealth to an infinite array of "goods" corresponding to the sequence of consumption levels in each period. To reduce the after-tax rate of return on savings in Period 100 is to increase the price, in terms of Period 0's initial wealth, of many "goods" at once—in particular, the price of consumption in every follow-on period. This multi-good price increase has a substitution effect and an income effect on consumption in every period. (These effects are very regular in the Judd model, as detailed in the online appendix.) The income effect drives down consumption in every period in both model variants, reflecting "the tightening of the capitalist's budget" due to the multi-price increase. The substitution effect in both model variants pushes consumption from now-more-expensive follow-on periods forward in time to lead-up periods. In both model variants, in follow-on periods, income and substitution effects work together to reduce

49. See the discussion *supra* Part 0 regarding different usages of the word "savings."

consumption. And in both model variants, in lead-up periods, income and substitution effects work against each other. In the OJM variant substitution effects dominate in the lead-up periods, while in the SW variant income effects dominate. Thus in the OJM variant, lead-up consumption increases, whereas in the SW variant lead-up consumption decreases.

The expenditure versus quantity explanation provided in the introduction (recall the gasoline analogy) emphasized the price elasticity of future consumption: the issue was whether desired future consumption went down at a faster or slower rate than the "price" of future consumption went up. This is another way of conceiving of the phenomenon discussed in the last paragraph, and it can be related to the differing substitution effects across the two model variants. In the OJM, the substitution effect is relatively strong (all else the same) and this works along with the income effect to cause a large enough decrease in units of future consumption to overwhelm the increase in the price of future consumption. The result is less spending of Period 0 initial after-tax wealth on future consumption, and so less saving. In the SW variant, the substitution effect is not strong enough to cause future consumption to drop by an amount sufficient to reduce spending of Period 0 initial after-tax wealth on future consumption.

VIII. Conclusion

Despite playing an important role in practical policy debates, the Judd model and the findings derived from it remain largely oracular. To the reader who reaches out to grasp why it is that the optimal long-run rate of tax on capital is zero—or potentially very positive—the literature offers only an unhappy choice between tightly knotted proofs and gauzy "intuitions."

This Article has attempted to provide something between those two extremes, an explanation. More than this, the Article argues that once the key moving parts of the Judd model are understood, it becomes clear that its findings, both new and old, are largely driven by quirks in the mathematical concept of infinity and are not properly included in any practical brief on optimal capital taxation, whether for or against.