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Random Walk in a Random Environment and $1/f$ Noise

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A simple model showing a $1/f$ behavior is proposed. It is argued, on the basis of a scaling argument, that is has $(\ln f)^k$ corrections. Numerical simulations confirm this picutre.

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The $1/f$ noise is a very common phenomenon which shows up in many physical systems of quite different kinds.[1] However, it is not simple to produce theoretical models which display such a behavior. Indeed when we study the power spectrum of a variable $x(t)$,

$$S(f)=\lim_{T\to\infty} T^{-1}\left|\int_0^T dt\, e^{ift}x(t)\right|^2, \tag{1}$$

if $x(t)$ is given by a Brownian motion we find for $S(f)$ a $1/f^2$ behavior, and if $x(t)$ satisfies simple deterministic equations, the power spectrum is not as singular as $1/f$ when f goes to zero.

In this Letter we want to point out a quite simple model which exhibits a $1/f$ power spectrum. We will use as a starting point the one-dimensional random walk in random environment[2] (random walk).

The model which has already been studied in a different context[2,3] is defined by the following equation:

$$\dot{x}=F(x)+\eta(x,t), \tag{2}$$

where $F(x)$ and $\eta(x,t)$ are independent random variables (which we may assume to be Gaussian) with covariance

$$\langle F(x)F(y)\rangle=\delta(x-y),$$
$$\langle \eta(x,t)\eta(y,t')\rangle=2\delta(x-y)\delta(t-t'), \tag{3}$$
$$\langle F(x)\rangle=\langle \eta(x,t)\rangle=0.$$

In other words we have a random walk with a drift $F(x)$ which is a randomly distributed, time-independent stochastic variable.

This model is very singular, and it is convenient to consider instead the lattice-regularized version of (2) and (3). Both the position $x(t)$ and the time t become integer variables; for each space point i we preassign a probability π_i^R of moving right and a probability π_i^L of moving left, such that $\pi_i^L + \pi_i^R = 1$.

Two extreme cases can be considered: (a) $\pi_i^R = \pi_i^L = \frac{1}{2}$; (b) π_i^R and π_i^L are random variables which may only assume the values 0 and 1 with probability $\frac{1}{2}$. Case (a) is the usual random walk, where $x^2(t) \sim t$ for large t. What happens in case (b) depends on the particular realization of the π probabilities: It is evident that, with probability one, the π_i will be such that $x(t)$ remains bounded uniformly in t [i.e., $x(t)$ cannot go outside of a given interval]. The corresponding power spectrum is $1/f^2$ in case (a), and in case (b) will not be divergent when f goes to zero.

It is remarkable that in the intermediate case [π_i^L and π_i^R having a symmetric distribution different from (a) and (b), e.g., a flat distribution on the interval (0, 1)] it has been proved by Sinai[2] that

$$x(t) \sim \ln^2 t \tag{4}$$

for $t \to \infty$, with probability 1.

Here we argue that in this case the power spectrum behaves as

$$\ln^4 f / f \tag{5}$$

for small f. The argument runs as follows: The random force acting at x derives from a random potential V such that potential differences scale like $\lambda^{1/2}$ when distances are multiplied by λ. The dynamics is dominated by the long time it takes to cross "mountains" (moutain passes if we were in several dimensions). When the particle is confined in a valley it has equilibrium distribution $\sim\exp(-V)$. When distances are multiplied by λ, the corresponding transition probabilities are thus changed from c to c to the power $\lambda^{-1/2}$ (roughly). Correspondingly, the times are changed from T to T to the power $\lambda^{1/2}$. Conversely, multiplication of the time by τ (for large times) corresponds to a multiplication of distances by a factor $\sim(\ln\tau)^2$, i.e., we recover Sinai's result. If this scaling relation is inserted in (1), a simple calculation yields $S(f) \sim |\ln f|^4/f$ asymptotically for small f.

A numerical experiment done with 30 000 different walks 4096 steps long and 100 walks 2^{19} steps long (each of them in a different random environment—for a detailed discussion see Marinari *et al.*[4]) gives results that are compatible with such a behavior (although, because of the finite T we have to use, power corrections of the form f^{-A} with $A < 0.4$ cannot be excluded).

What happens in higher dimensions? If the drift F is the gradient of a random potential V with a long-range correlation,

$$\langle [V(x) - V(y)]^2 \rangle \sim |x-y|^{2\alpha}, \tag{6}$$

with $\alpha > 0$, a generalization of the same argument[4] tells us that the power spectrum will have the form

$$S(f) \approx C(\ln f)^{2/\alpha}/f. \tag{7}$$

(To be more precise, we make a scaling assumption on the ensemble of the potential V; α is the scaling exponent equal to $\frac{1}{2}$ in Sinai's case.) If our random walk corresponds to thermal motion, the coefficient C in (7) will have $T^{2/\alpha}$ dependence on the absolute temperature.

An other interesting possibility is when F is not a gradient, but has short-range correlations. We investigated what happens on a two-dimensional square lattice if one chooses for the transition probabilities $P_i^{(r)}$ ($r = 1, \ldots, 4$) for going from the point i to the nearest neighbor in the direction r

$$P_i^{(r)}(K) = (a_i^{(r)})^K [Z_i(K)]^{-1}, \tag{8}$$

with

$$Z_i(K) = \sum_{r=1}^{4} (a_i^{(r)})^K,$$

where the $a_i^{(r)}$ are uniformly distributed in (0, 1).

Here also for $K = 0$ we recover the standard random walk, and in the limit $K \to \infty$ we get a deterministic process in which $x(t)$ is bounded. In this model we have studied numerically, using of the order of 10^3 walks of 2^{12}–2^{13} steps for each K, the behavior of $x^2(t)$ for large t as a function of K. For small values of K ($K = 1, 2$) we we find that our data are well described by the behavior $x^2(t) \sim t$. For higher K (i.e., $K = 7$) the data are better described by $x^2(t) \sim t^{2a}$, where a best fit for the power a is given by $a = 0.30 \pm 0.09$. For still higher values of K a behavior $\langle x^2(t) \rangle^{1/2} \sim \ln^2 t$ seems to be preferable: We should remark that the length of our walks is not large enough to resolve the difference between a small power and a logarithmic behavior, nor to distinguish between a preasymptotic regime and a truly asymptotic one. Analytic tools to investigate this model would be welcome.

It is clear that it is not easy to construct a microscopic model of a conductor in order to implement the mechanism that we discussed: Indeed already in a two-dimensional film with random structures a qualitative understanding of the conductivity itself, taking account of quantum localization effects, electron-electron interactions, and temperature effects, is not available.[5]

It is, however, satisfactory that we have succeeded to build up a simple model which displays the needed qualitative features.

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[3]B. Derrida and Y. Pomeau, Phys. Rev. Lett. 48, 627 (1982).

[4]E. Marinari, G. Parisi, D. Ruelle, and P. Windey, to be published.

[5]For reviews of recent efforts in this direction, see G. Parisi, in Proceedings of the 1982 Les Houches Summer School of Theoretical Physics (to be published); E. Fradkin, *ibid.*, and references therein.