A note on the fluctuation of dissipative scale in turbulence

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We present an application of the multifractal formalism able to predict the whole shape of the probability density function (pdf) of the dissipative scale, η. We discuss both intense velocity fluctuations, leading to dissipative scales smaller than the Kolmogorov scale, where the formalism gives a pdf decaying as a superposition of stretched exponential, and smooth velocity fluctuations, where the formalism predicts a power-law decay. Both trends are found to be in good agreement with recent direct numerical simulations [J. Schumacher, "Sub-Kolmogorov-scale fluctuations in fluid turbulence," Europhys. Lett. **80**, 54001 (2007)].
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Turbulence is characterized by non-Gaussian velocity fluctuations on a wide range of scales and frequencies.[1] Much attention has been devoted in the past to the so-called inertial range, i.e., for scales much smaller than the stirring length and much larger than the typical dissipative scale, $\eta \ll r \ll L$. It is known and expected that inertial range statistics in isotropic and homogeneous turbulence is characterized by an anomalous power-law scaling, typically measured in terms of structure functions (SFs):

$$S^{(p)}(r) = \langle |v(x+r) - v(x)|^p \rangle = \langle |\delta_r v|^p \rangle \sim \left(\frac{r}{L}\right)^{\zeta(p)}, \qquad (1)$$

where, assuming isotropy, we can assume that all components of the velocity increments has the same statistics (possible different scalings between longitudinal and transverse increments are not addressed here, see conclusions).

The signature of an anomalous power law is in the deviations of the scaling exponents from the dimensional, Kolmogorov-like, prediction $\zeta^{K41}(p) = p/3$. Unfortunately, there are no rigorous results about anomalous scaling in three-dimensional Navier–Stokes (NS) equations. A powerful and simple phenomenological way to understand intermittency was proposed by "Parisi and Frisch,"[2] the so-called multifractal formalism (MF). According to the MF model, Eulerian velocity increments at inertial scales can be characterized by a local Hölder exponent h, i.e., $\delta_r v \sim (r/L)^h$, whose probability is $\mathcal{P}_h(r) \sim (r/L)^{3-D(h)}$, the function $D(h)$ being the fractal dimension of the set where h is observed. In terms of this description, anomalous scaling is easily recovered by a saddle-point estimate in the limit $r/L \to 0$ (but keeping $\eta \ll r$):

$$S^{(p)}(r) \propto \int dh \left(\frac{r}{L}\right)^{ph+3-D(h)} \sim r^{\zeta(p)}, \qquad (2)$$

where $\zeta(p) = \min_h [ph + 3 - D(h)]$. The ultimate goal would be to derive the $D(h)$ spectrum from the NS equations, a task which is still far from being achieved (see Ref. 3 for a recent attempt). Besides SF scaling, the MF formalism proved to be able to also reproduce multiscale correlation functions[4–7] and probability density functions (pdfs) of velocity differences and velocity gradients for both Eulerian and Lagrangian statistics.[8–12] Much attention has also been put on the "ideal" shape of the $D(h)$ spectrum which fits the experimental and numerical data better, many possibilities having been proposed (random beta model, p model, log-normal, log-Poisson, etc.).[13–17] From first principle, because we lack a derivation from NS equations, they are all on the same footing, except for log-normal models that are known to be affected by some shortcomings.[1] It is an open question whether there exists one preferable $D(h)$ functional form among those cited before: given the statistical limitations, they are all equivalently successful in fitting the measurable $\zeta(p)$ exponents. We are not interested in this problem here. For the sake of simplicity, when needed, we will specify the calculations using the log-Poisson as proposed in Refs. 17 and 18.

The goal of this letter is to use a simple argument, based on the same MF phenomenology, to predict the pdf of the dissipative scale. The idea is based on the well known assumption that dissipative scale is itself a fluctuating quantity, defined by the requirement that the *local* Reynolds number is of order 1:[19–23]

$$\frac{\delta_\eta v\, \eta}{\nu} \sim 1, \qquad (3)$$

where $\delta_\eta v = \delta_r v$, calculated at the dissipative scale. Such relation has been used in the past to predict the so-called intermediate-dissipative range[21] and the Reynolds dependence of both moments and pdfs of the velocity gradients.[8,20] Recently, using a different formulation based on the Mellin transform of the SF and the assumption that the velocity statistics is Gaussian at large scale, a prediction for the shape of the dissipative scale, $\phi(\eta)$ has been proposed.[22] Properly taking into account the fluctuations of the dissipative scale is important to control the effects of on lack of resolution of direct numerical simulations (DNSs) and the effects of filtering in an experimental apparatus not resolving the high frequency spectrum. Recent, highly resolved numerical simulations[24,25] have indeed shown the existence of nontrivial η fluctuations, which may locally become a fraction of the usual mean-field estimate, $\eta_{K41}=(\nu^3/\epsilon)^{1/4}$. Here, we show how one can predict the whole shape of the viscous dissipation pdf $\phi(\eta)$ by using the MF ansatz.

The argument is as follows. Suppose you have a large-scale Gaussian velocity field with d components, $\mathbf{v}_0$, with order 1 variance, $v_{\rm rms}=1$. Its amplitude must have the following pdf:

$$P(v_0)dv_0 = v_0^{d-1}\exp(-v_0^2/2)dv_0,$$

where we have neglected unessential order unity prefactors. Now, let us define the fluctuating dissipative scale by the Paladin–Vulpiani[19] argument [requiring $O(1)$ local Reynolds number],

$$\frac{\eta(h,v_0)|\delta_\eta v|}{\nu} \sim 1; \tag{4}$$

using the multifractal scaling,

$$|\delta_\eta v| = v_0\left(\frac{\eta}{L}\right)^h \tag{5}$$

and plugging it in Eq. (4), we get for the expression which connects η to v_0

$$v_0 = \mathrm{Re}^{-1}\left(\frac{\eta}{L}\right)^{-(1+h)}, \tag{6}$$

where the Reynolds number is defined as $\mathrm{Re}=L/\nu$, the typical amplitude of v_0 being $v_{\rm rms}=1$.

Let us also introduce a dimensionless dissipative scale,

$$\tilde{\eta} = \frac{\eta}{\eta_K},$$

where the Kolmogorov scale is defined as $\eta_{K41}=L\mathrm{Re}^{-3/4}$. In terms of a dimensionless variable, we have that relation (6) becomes

$$v_0 = \mathrm{Re}^{(3h-1)/4}\,\tilde{\eta}^{-(1+h)}. \tag{7}$$

Now, conservation of probability implies (for each given h exponent)

$$\phi(\tilde{\eta}|h) = P(v_0)(dv_0/d\tilde{\eta}). \tag{8}$$

Consider also the fluctuations of h and that the probability to measure an h exponent at scale $\tilde{\eta}$ is given by the multifractal rule $\mathcal{P}_h(\eta)=(\eta/L)^{3-D(h)}$. We have [plugging all together and considering relation (7)] that

$$\begin{aligned}\phi(\tilde{\eta}) &\propto \int dh\mathcal{P}_h(\eta)\phi(\tilde{\eta}|h)\\ &= \int dh\,\mathrm{Re}^{y(h)}\,\tilde{\eta}^{z(h)}\\ &\quad\times\exp(-0.5\,\mathrm{Re}^{(3h-1)/2}\tilde{\eta}^{-2(1+h)}),\end{aligned} \tag{9}$$

where we have defined $y(h)=[(3h-1)d-3(3-D(h))]/4$, and $z(h)=-d(1+h)-1+(3-D(h))$. Notice that for $K41$ scaling [i.e., the whole h support limited to $h=\frac{1}{3}$, with $D(\frac{1}{3})=3$], we recover that the pdf of a dimensional dissipative scale does not depend on Reynolds ($K41$ does not have any dependency on Reynolds for dimensionless variables).

This is the multifractal prediction, just like the one that worked well for gradients[8] and acceleration,[9] considering in the latter case the proper translation from Eulerian to Lagrangian domain.[26,27]

The requirement to have an order unity local Reynolds can be further relaxed by taking into account that velocity fluctuations are not an exact pure power law from the large scale down to the local dissipative scale. In other words, one can introduce a smooth transition between the inertial range behavior and the dissipative behavior (for each h exponent). This can be done using a generalization of the Batchelor parametrization:[12,28,29]

$$\delta_r v = v_0\frac{r/L}{((r/L)^2+c(\eta/L)^2)^{(1-h)/2}}, \tag{10}$$

where c is an $O(1)$ free parameter defining the matching between the two regimes. Relation (10) together with the corresponding generalization for the probability to observe a given h exponent: $\mathcal{P}_h(r)\propto((r/L)^{3-D(h)}+c(\eta/L)^{3-D(h)})$, leads to a consistent, simultaneous description of both inertial and dissipative range physics. Notice that the above receipt is also consistent with the requirement $\lim_{\mathrm{Re}\to\infty}\langle(\partial v)^2\rangle\sim\mathrm{Re}$, as requested by the existence of the dissipative anomaly.[1,8,20] By taking into account this extra degree of freedom, one gets a slightly modified version of Eq. (9) when using definition (10):

$$\begin{aligned}\phi(\tilde{\eta}) &= \int dh\mathcal{P}_h(\eta)\phi(\tilde{\eta}|h)\\ &= \int dhA^{x(h)}\,\mathrm{Re}^{y(h)}\,\tilde{\eta}^{z(h)}\\ &\quad\times\exp(-A^{2(1-h)}\,\mathrm{Re}^{(3h-1)/2}\,\tilde{\eta}^{-2(1+h)}),\end{aligned} \tag{11}$$

where $A=(1+c)^{1/2}$ and $x(h)=d(1-h)+3-D(h)$. In Fig. 1, we show the log-log plot of the pdf of $\tilde{\eta}$ as obtained from the above prediction compared with the data from a DNS.[25] The agreement is very good, with small deviations only for the very small viscous scales, where the DNS can have troubles in correctly resolving the flow. In the same figure (inset), we also show the multifractal prediction at changing Reynolds and at changing the free parameter c. We also superpose the result one gets for $K41$. The $D(h)$ spectrum used here is the one known to fit well the Eulerian SFs.[17] The range of h exponents is always $h\in[h_{\min},h_{\max}]$ where $h_{\min}=\frac{1}{9}$ for the particular choice[17] and $h_{\max}\sim0.38$ is defined as the exponent where the $D(h)$ attains its maximum $D(h_{\max}=3)$. Notice that the inclusion of also the exponent lying in the right side, with respect to the maximum, of the $D(h)$ spectrum would not lead to any important change in the pdf shape (not shown).

Exploring a value of h larger then $h_{\max}$ is not under control because we do not have experimental or numerical results of SFs of negative order. Let us notice also that other functional curves for $D(h)$ which would reproduce the measured scaling exponents, $\zeta(p)$, would give indistinguishable results. As one can notice, the interesting part of the curve is the left side with respect to the peak, the one corresponding to wild local intense fluctuations of $\delta_\eta v$, i.e., to local dissipative scales smaller then η_{K41}. The left part is weakly sen-

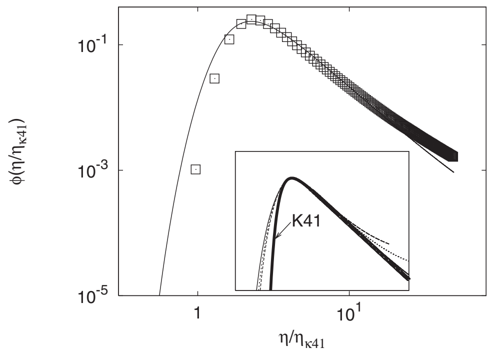

FIG. 1. (Solid line) The prediction for the dissipative scale from Eq. (11) compared with the measurement of the same quantity in a DNS (□, data from Ref. 25); both data have the same Reynolds numbers. The multifractal prediction is obtained with $A=8$. For the sake of convenience, we have used a log-Poisson expression for the $D(h)$ as proposed in Refs. 17 and 18: $D(h)=3(h-h_0)/\log\beta[\log(3(h_0-h)/d_0\log\beta)-1]+3-d_0$, with $\beta=\frac{2}{3}$, $h_0=\frac{1}{9}$, and $d_0=2$. Inset: (Dashed lines) Dependency of the MF prediction (11) as a function of the Reynolds numbers ($\mathrm{Re}=2.5\times10^3, 10^4, 2.5\times10^5$, from right to left), with the free parameter $A=6$ and $d=1$. Notice the small dependence on Re for the left tail. The right tails are calculated up to the maximum reachable scale for that given Reynolds number, $\tilde{\eta}_{\max}\sim\mathrm{Re}^{3/4}$. (Solid line) The $K41$ nonintermittent prediction.

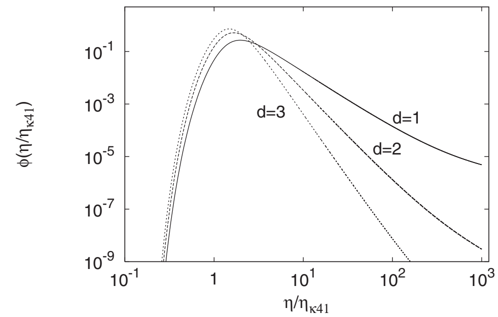

FIG. 2. log-log plot of the dissipative scale pdf for $\mathrm{Re}=10^4$ at changing the dimensionality $d=1,2,3$. The right tail is given by a superposition of power-law decay. The asymptotic, saddle-point limit of the prefactor in front of the exponential term in Eq. (11) is never reached when $\eta<L$.

sitive to Reynolds, as also demonstrated by the small deviations from the $K41$ nonintermittent shape (which is non-Gaussian), so it may be extremely difficult to have a quantitative benchmark of this prediction. Nevertheless, by comparing this result with the numerical measurements published in Ref. 25 one can notice a clear qualitative agreement.

The right part of the pdf is completely dominated by the power-law prefactor in Eq. (11). Therefore, it is strongly dependent on the dimensionality d, i.e., on the number of components used to define $|\delta_\eta v|$ when applying the $O(1)$ Reynolds requirement for the definition of the local dissipative scale, see Fig. 2. Moreover, it strongly depends on the shape of the velocity fluctuations around the peak, i.e., for very small velocity increments, where the local turbulent intensity is negligible.

Interestingly enough, the obvious bound $\eta<L$ (i.e., $\tilde{\eta}<\mathrm{Re}^{3/4}$) does not allow the exponent of $\tilde{\eta}$ in the power-law prefactor of Eq. (11) to reach the saddle-point limit $z(h_{\min})$ which would maximize the integrand in the limit $\tilde{\eta}\to\infty$ {the $z(h)$ function is monotonically decreasing in the range $h\in[h_{\min},h_{\max}]$}. As a consequence, the integrand turns out to be dominated by a superposition of all h exponents. The power-law decay for large $\tilde{\eta}$ is important for the normalization of the pdf if we want to compare curves with integral normalized to unity; there is a dependency on the maximum allowed $\tilde{\eta}$ value, which must be chosen to be of the order of $\tilde{\eta}_{\max}\sim\mathrm{Re}^{3/4}$. Such constraint in the normalization has no physical important consequences; it also exists in real data. Indeed, any cascade phenomenology is based on the assumption that the local Reynolds number is large, i.e., it cannot describe laminar regions where the velocity fluctuations are very small. Such laminar fluctuations are at the origin of the far tail in the power-law decay for large $\tilde{\eta}$, and are therefore out of interest here. It would be interesting to see if this sensitivity to the infrared cutoff is also shared by other cascade models such as the one proposed in Ref. 15.

In conclusion, we have presented a simple generalization of the MF formalism that is able to predict the fluctuations of the local dissipative scale defined using the Paladin–Vulpiani relation (4). The relation can be considered as a prediction for the local dissipative scaling as defined via Eq. (4) once given the spectrum of Eulerian Hölder exponent $D(h)$. The existence of a free parameter, c, is unavoidable in all approaches based on dimensional matching between inertial and viscous terms for the definition of the dissipative scale.[30] The same freedom is also present in the approach presented in Ref. 22, as discussed in Ref. 25. The large-scale velocity field is supposed to be Gaussian, as observed in all experiments and simulations of isotropic turbulence. We have discussed results for both intense velocity fluctuations, leading to small viscous scales, where the MF formalism predicts the pdf of η to decay as a superposition of stretched exponential [left tail of Eq. (11)], and for smooth velocity fluctuations, where the MF formalism predicts a superposition of power-law decay. Both trends are well reproduced in DNS.[25] Further simple refinements would be needed if one is interested in distinguishing between local dissipative scales of longitudinal or transverse fluctuations. There are claims that longitudinal or transverse SFs may have different scaling properties in isotropic turbulence[31] at the available Reynolds numbers (see also Ref. 32 for a critical review on a possible theoretical implication of this fact). In the latter case, different $D(h)$ functions fitting the two statistics would be necessary to describe the local dissipative scales. Recently, an attempt to refine the Batchelor parametrization (10) such as to match the Karman–Howarth relation has been proposed in Ref. 12. To do that, one has to pay the price of introducing further functions tracking the skewed part of the velocity statistics. Let us stress that the intermittency signature in the whole pdf shape is very weak: curves at different Reynolds are almost undistinguishable from the $K41$ prediction in Fig. 1 for the left tail. In order to enhance such intermittency trends, one should measure negative moments of the local dissipative scale, such as to have a more sensitive probe of the small differences seen in Fig. 1 for $\eta/\eta_{K41}\ll 1$.

The approach presented here can be easily generalized to treat the case of hyperviscosity, i.e., of NS evolution with a dissipative term $\propto \nu\Delta^{\alpha}v$, with $\alpha > 1$. It is easy to realize, for example, that the exponent in the stretched exponential superposition in Eq. (9) now becomes $\tilde{\eta}^{-2(1+h)-2(\alpha-1)}$. It would be interesting to test this relation against hyperviscous DNS.

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