The Maximum Clique Problem in Spinorial Form

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Abstract. We present a formulation of the Maximum Clique problem of a graph as a geometrical problem of null vectors in complex space. This problem is then translated in spinorial language.

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THE MAXIMUM CLIQUE PROBLEM

Given a graph of size n and adjacency matrix A, a clique is a subgraph with pairwise adjacent vertices and the Maximum Clique (MC) problem asks for the *size* $\omega(A)$ of the largest clique. It is an old, well studied, NP-complete problem and there are reviews with hundreds references [1].

Any symmetric matrix like A may be expressed in the form

$$A = B'B = BB = B^2 \tag{1}$$

where B is a complex, symmetric matrix that we can think as formed by n complex vectors $\mathbf{z}_i \in \mathbb{C}^n$ called *null vectors* since they have zero length: $\mathbf{z}_j^2 = a_{jj} = 0$.

It is well known [1] that the MC problem of A is identical to the problem of the maximum independent set of the conjugate graph $\bar{A} = J - \mathbb{1} - A$ and a formulation with appealing properties is

$$\max_{\mathbf{x} \in \{\{0,1\}^n : \mathbf{x}'\bar{A}\mathbf{x} = 0\}} \mathbf{x}'\mathbf{x} = \omega(A) \ . \tag{2}$$

This problem has the following geometrical interpretation: the null vectors $\bar{\mathbf{z}}$ span the space $V \subseteq \mathbb{C}^n$ and any couple of linearly independent vectors $\bar{\mathbf{z}}_j$ and $\bar{\mathbf{z}}_k$ span a two-dimensional space contained in V. If $\bar{\mathbf{z}}_j'\bar{\mathbf{z}}_k = \bar{a}_{jk} = 0$ it is easy to verify that this space has the property that all of its elements are null vectors and are all mutually orthogonal: this space is called a Totally Null Plane (TNP). If $\bar{A}$ contains at least one nondiagonal zero element, then V contains at least one two dimensional TNP.

The solution of the MC problem provides the largest subset $\bar{\mathbf{z}}_{j_1}, \bar{\mathbf{z}}_{j_2}, \ldots, \bar{\mathbf{z}}_{j_k}$ of $\bar{\mathbf{z}}$'s which define a TNP contained in V; [2].

SPINORS, WITT AND FOCK BASIS

Cartan was the first to show [3] that the geometry of null vectors can be treated elegantly with spinors; we will follow this road. A spinor Φ is a vector belonging to the spaces S of endomorphism of $Cl(2n) = \mathrm{End}S$ and is defined by the Cartan's equation:

$$v\Phi = \left(\sum_{j=1}^{2n} v_j \gamma_j \right) \Phi = 0 \tag{3}$$

where $v\Phi$ is a Clifford product $v\Phi = v \rfloor \Phi + v \wedge \Phi$.

Let us define the null, or Witt, basis of $Cl(2n)$:

$$p_j = \frac{1}{2}\left(\gamma_{2j-1} + i\gamma_{2j}\right) \quad \text{and} \quad q_j = \frac{1}{2}\left(\gamma_{2j-1} - i\gamma_{2j}\right) \tag{4}$$

for $j = 1, 2, \ldots, n$ with the properties

$$\left[p_j, p_k\right]_+ = \left[q_j, q_k\right]_+ = 0 \quad \text{and} \quad \left[p_j, q_k\right]_+ = \delta_{jk}\mathbb{1} \ .$$

With this basis $\mathbb{C}^{2n}$ is easily seen as the direct sum of two maximal TNP P and Q spanned by null vectors $\{\mathbf{p}_j\}$ and $\{\mathbf{q}_j\}$ respectively:

$$\mathbb{C}^{2n} = P \oplus Q \ ,$$

since $P \cap Q = \emptyset$ each vector $\mathbf{v} \in \mathbb{C}^{2n}$ may be expressed in the form $\mathbf{v} = \sum_{i=1}^{n} (\alpha_i \mathbf{p}_i + \beta_i \mathbf{q}_i)$ with $\alpha_i, \beta_i \in \mathbb{C}$.

A spinor $\Phi \in S$, defined by Cartan equation (3), may be represented by Minimal Left Ideals (MLI) of $Cl(2n)$ [4]. Consider the 2^n MLI that form the Fock basis in spinor space [5]

$$\begin{aligned}
&\omega_0 = p_1 p_2 \ldots p_n; \\
&\omega_1 = q_1\omega_0, \quad \omega_2 = q_2\omega_0, \ldots, \quad \omega_{2^n-1} = q_n\omega_0; \\
&\omega_3 = q_1 q_2 \omega_0, \quad \omega_5 = q_1 q_3 \omega_0, \quad \ldots; \\
&\ldots\ldots \\
&\omega_{2^n-1} = q_1 q_2 \ldots q_n \omega_0
\end{aligned} \tag{5}$$

When we write the Cartan equation (3) in the basis, defined in (4) and (5), we get

$$v\Phi = \left(\sum_{i=1}^{n} \alpha_i p_i + \beta_i q_i \right) \left(\sum_{s=0}^{2^n-1} \xi_s \omega_s \right) = 0 \tag{6}$$

and this equation can be read in two ways depending on wether v or Φ plays the role of the unknown. In [6] we have proved the

Proposition 1 *Given $v := Span(x_1, \ldots, x_k)$, there exists a spinor Φ, satisfying the Cartan equation (6) if, and only if, v is a TNP.*

MC PROBLEM IN SPINOR LANGUAGE

We are now ready to give a spinorial formulation of the MC problem and start by introducing new vectors $\bar{\mathbf{z}}_i$ in the Witt basis of $Cl(2n)$:

$$\bar{\mathbf{z}}_i = \mathbf{p}_i + \sum_{j=1}^{n} \bar{a}_{ij}\mathbf{q}_j \qquad i = 1, 2, \ldots, n\,.$$

These vectors have the properties (immediate to prove):

- belong to $\mathbb{C}^{2n}$ and are linearly independent;
- are null, i.e. $\bar{\mathbf{z}}_i'\bar{\mathbf{z}}_i = 0$ since $\bar{a}_{ii} = 0$;
- satisfy $\bar{\mathbf{z}}_i'\bar{\mathbf{z}}_j = \bar{a}_{ij}$.

To fully exploit the spinorial formulation we will consider the $\bar{\mathbf{z}}_i$ vectors as representative of the subspace they induce i.e. $Span(\mathbf{p}_i, \bar{a}_{i1}\mathbf{q}_1, \ldots, \bar{a}_{in}\mathbf{q}_n)$ of dimension $\sum_{j=1}^{n} \bar{a}_{ij} + 1$. We do this introducing in the definition arbitrary coefficients α

$$\bar{\mathbf{z}}_i = \alpha_i\mathbf{p}_i + \sum_{j=1}^{n} \bar{a}_{ij}\alpha_j\mathbf{q}_j \qquad i = 1, 2, \ldots, n \tag{7}$$

The equation $\mathbf{x}'\bar{A}\mathbf{x} = (\bar{B}\mathbf{x})^2 = 0$, representing the constraints of the MC problem in (2), may be linearized formulating the problem in spinorial form:

$$\bar{B}\mathbf{x}\Phi = \left(\sum_{i=1}^{n} x_i\bar{\mathbf{z}}_i\right)\Phi = 0 \tag{8}$$

or with $\bar{\mathbf{z}}_i$ from (7)

$$\left[\sum_{i=1}^{n} x_i \left(\alpha_i\mathbf{p}_i + \sum_{j=1}^{n} \alpha_j\bar{a}_{ij}\mathbf{q}_j\right)\right]\Phi = 0 \tag{9}$$

of the form (6). In this equation, in general, x_i must be interpreted as complex variables, restricted to values in $\{0,1\}$ in formulation (2) of the MC problem .

We thus have a set of n vectors $\bar{\mathbf{z}}_i$ defining an n-dimensional subspace of $\mathbb{C}^{2n}$ and we will look for the spinors Φ that satisfy (8).

We study solutions of (9) with an example: let us suppose that $\bar{a}_{12} = 0$, this means that $\bar{\mathbf{z}}_1$ and $\bar{\mathbf{z}}_2$ form a TNP and, setting $x_3 = x_4 = \ldots = x_n = 0$, with (7) we get

$$\left(x_1\bar{\mathbf{z}}_1 + x_2\bar{\mathbf{z}}_2\right)\Phi(\bar{\mathbf{z}}_1\bar{\mathbf{z}}_2) = 0$$

where $\Phi(\bar{\mathbf{z}}_1\bar{\mathbf{z}}_2)$ represent the TNP $Span(\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2)$. This example shows that it is simple to get particular solutions to (9) the real problem being to find the *set of all solutions*, for which in [6] we proved the following

Proposition 2 *The set of nonzero spinors that solve the Cartan equation (9) is isomorphic to the set of cliques of A.*

We can now reformulate our initial MC problem (2): it will correspond to that solution of (9) with the maximum intersection with P, i.e.

$$\omega(A) = \max_{\Phi:\left(\sum_{i=1}^{n} x_i\bar{\mathbf{z}}_i\right)\Phi=0} \dim\left(P \cap M(\Phi)\right)$$

where $M(\Phi)$ represent the TNP corresponding to Φ.

CONCLUSIONS

We have thus shown that:

- the MC problem can be formulated in spinorial language;
- the problem of finding all possible spinor solutions of (9) is NP-complete since, given the set of all solutions, one gets also the solution of the MC problem.
- With respect to the MC formulation (2) we remark two main differences:
  - the demanding restriction $\mathbf{x} \in \{0,1\}^n$ can be relaxed since all solutions of (9) necessarily have binary x_i.
  - The quadratic constraint $\mathbf{x}'\bar{A}\mathbf{x} = 0$ of (2) is linearized here to $\bar{B}\mathbf{x}\Phi = 0$.

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