Time Series Analysis of Coulomb Collisions in a Beam Dynamics Simulation

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Abstract. In this paper a time series analysis of collisional effects in a numerical simulation of a coasting beam transverse dynamics is presented. The simulation performs a numerical integration of the Hamilton's equations of a two-dimensional system of particles describing the transverse dynamics of the beam. Then an analysis of the time series generated has been applied in order to describe the dynamics of the system by means of the mean field equations with the addition of a stochastic process in order to model Coulomb collisions.

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INTRODUCTION

Intra Beam Scattering (IBS), the study of the collisional effects in beam dynamics, is getting object of many researches in recent years, because of the importance of this matter in new accelerators of high intensity beams at moderate energy. The standard theory consists in introducing collisional effects as a Wiener stochastic process, which adds to the mean field contribution to the dynamics of the particles (see [1]) and leads to a Fokker-Plank equation for the single particle distribution function in the phase space.

In our approach the study of collisional effects is carried out by studying the dynamics of a two-dimensional system of interacting wires which can be shown to be equivalent, under longitudinal coherence hypothesis, to the transverse dynamics of a coasting beam [2]. Considering the beam composed by particles moving along the z axis at constant velocity, satisfying the relation $|\mathrm{v}_x|,|\mathrm{v}_y| \ll |\mathrm{v}_z| = \mathrm{v}_0$, makes it possible to assume the curvilinear abscissa s to be equal to the z coordinate, according to $\mathrm{d}s \simeq \mathrm{v}_0 \mathrm{d}t$ and taking s as the time variable of the dynamical system. In this paper we study the dynamics of a system of wires, assuming the equations of motion to be the mean field equations plus the contribution of a stochastic process to the momentum derivative representing the collisional effects of the system. Our aim is to identify the probability distribution of the stochastic terms and write a stochastic equation describing from a statistical point of view the dynamics of the system. The analysis of the time series has been accomplished according to the standard theory presented in any book of statistics (*e.g.* [5],[6]).

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HAMILTONIAN OF THE SYSTEM

The (adimensional) total Hamiltonian describing the transverse dynamics of a longitudinal coherent coasting beam with a linear confining force, in a constant focusing channel (see [2]), can be written in the form :

$$H_{TOT} = \sum_{i=1}^{N} \frac{\mathbf{p}_i{}^2}{2} + \frac{\omega_0^2}{2}\mathbf{r}_i{}^2 - \frac{\xi}{N} \sum_{1\le i<j\le N} \log(|\mathbf{r}_i - \mathbf{r}_j|) \quad (1)$$

where ξ is the perveance, ω_0 is the phase advance per unit length and N is the number of wires (from now on we will refer to them as particles).

Note that $\mathbf{r}_i$ and $\mathbf{p}_i$ are the position and momentum vectors of the i-th particle in the two-dimensional transverse plane.

The Hamilton's equations read:

$$\begin{cases} \dot{\mathbf{r}}_i &= \mathbf{p}_i \\ \dot{\mathbf{p}}_i &= -\omega_0{}^2\,\mathbf{r}_i + \frac{\xi}{N} \sum\limits_{\substack{j=1 \\ j\neq i}}^{N} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \end{cases} \quad (2)$$

Let we consider the center of mass of the system

$$\mathbf{r}^{\mathrm{CM}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{r}_i\,, \qquad \mathbf{p}^{\mathrm{CM}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{p}_i\,,$$

we immediately recognize

$$\begin{cases} \dot{\mathbf{r}}^{\text{CM}} &= \mathbf{p}^{\text{CM}} \\ \dot{\mathbf{p}}^{\text{CM}} &= -\omega_0{}^2\, \mathbf{r}^{\text{CM}}\,. \end{cases} \tag{3}$$

If we take the coordinates and momenta of the particles in the center of mass frame:

$$\tilde{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}^{\text{CM}}, \qquad \tilde{\mathbf{p}}_i = \mathbf{p}_i - \mathbf{p}^{\text{CM}}\,,$$

we have the equations of motion of the i-th particle in the form :

$$\begin{cases} \dot{\tilde{\mathbf{r}}}_i &= \dot{\tilde{\mathbf{p}}}_i \\ \dot{\tilde{\mathbf{p}}}_i &= -\omega_0{}^2\, \tilde{\mathbf{r}}_i + \frac{\xi}{N} \sum\limits_{\substack{j=1 \\ j\neq i}}^{N} \frac{\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j}{|\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j|^2}\,. \end{cases} \tag{4}$$

Thus, we can limit ourselves to study the total Hamiltonian :

$$\tilde{H}_{TOT} = \sum_{i=1}^{N} \frac{\tilde{\mathbf{p}}_i^2}{2} + \frac{\omega_0^2}{2}\tilde{\mathbf{r}}_i^2 \quad - \frac{\xi}{N} \sum_{1\leq i<j\leq N}^{N} \log(|\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_j|) \tag{5}$$

MEAN FIELD THEORY

In mean field approximation the electric field produced by the (charged) particles of the beam is assumed to be described by the field generated by a continuous charge distribution in the single particle phase space:

$$\rho(\mathbf{r},\mathbf{p},t), \qquad \int \rho(\mathbf{r},\mathbf{p},t)\, d^2\mathbf{p}\, d^2\mathbf{r} = 1. \tag{6}$$

The particle distribution of the beam results to be :

$$\rho_s(\mathbf{r},t) = \int \rho(\mathbf{r},\mathbf{p},t)\, d^2\mathbf{p},$$

and the electric potential is taken to be :

$$\phi(\mathbf{r},t) = -\int \log(|\mathbf{r}-\mathbf{r}\prime|)\, \rho_s(\mathbf{r}\prime,t) d^2\mathbf{r}\prime. \tag{7}$$

Note that $\phi(\mathbf{r},t)$ verifies the Poisson's equation in two dimensions $\Delta\phi(\mathbf{r},t) = -2\pi\rho_s(\mathbf{r},t)$.

The single-particle Hamiltonian becomes:

$$H = \frac{\mathbf{p}^2}{2} + \frac{\omega_0^2}{2}\mathbf{r}^2 + \xi\,\phi(\mathbf{r},t), \tag{8}$$

which gives the following Hamilton's equations :

$$\begin{cases} \dot{\mathbf{r}} &= \mathbf{p} \\ \dot{\mathbf{p}} &= -\omega_0{}^2\, \mathbf{r} + \xi \int \frac{\mathbf{r}-\mathbf{r}\prime}{|\mathbf{r}-\mathbf{r}\prime|^2} \rho_s(\mathbf{r}\prime,t) d^2\mathbf{r}\prime. \end{cases} \tag{9}$$

Finally, we have the Vlasov equation for the evolution of the particle density:

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial\mathbf{r}}\frac{\partial\mathbf{r}}{\partial t} + \frac{\partial\rho}{\partial\mathbf{p}}\frac{\partial\mathbf{p}}{\partial t} = \frac{\partial\rho}{\partial t} + [\rho,H] = 0. \tag{10}$$

THE KV PARTICLE DISTRIBUTION

Consider the following space particle distribution:

$$\rho_{\text{s}}(\mathbf{r}) = \begin{cases} \frac{1}{\pi R^2} & r \leq R, \\ 0 & r > R, \end{cases} \tag{11}$$

with $R > 0$. By inserting (11) in (7) we get the following electric potential (changing the additive constants) :

$$\phi(\mathbf{r}) = \begin{cases} -\frac{r^2}{2R^2} & r \leq R, \\ -\frac{1}{2}\log\frac{r^2}{R^2} - \frac{1}{2} & r > R; \end{cases}$$

and the single particle Hamiltonian can be written in the form :

$$H = \begin{cases} \frac{p^2}{2} + \frac{\omega_0^2}{2}r^2 - \frac{\xi}{2}\frac{r^2}{R^2} & r \leq R, \\ \frac{p^2}{2} + \frac{\omega_0^2}{2}r^2 - \frac{\xi}{2}\left[2\log\left(\frac{r}{R}\right) + 1\right] & r \geq R. \end{cases} \tag{12}$$

Now define the values :

$$\omega = \sqrt{\omega_0^2 - \frac{\xi}{R^2}}, \qquad E = \frac{\omega^2}{2}R^2 = \frac{\omega_0^2}{2}R^2 - \frac{\xi}{2},$$

and take the particle distribution in the phase space as follows :

$$\rho_{KV}(x,y,p_x,p_y,t) = \frac{1}{2\pi^2R^2}\,\delta\left(\frac{p^2}{2} + \frac{\omega^2}{2}r^2 - E\right). \tag{13}$$

Note that the distribution (13) produces the space particle distribution (11) with all the particles having energy E. In virtue of the definitions of E and R, we can write the phase space density as a function of the single particle Hamiltonian (12):

$$\rho_{KV}(x,y,p_x,p_y,t) = \frac{1}{2\pi^2R^2}\delta(H-E)\,.$$

and then ρ_{KV} is a stationary distribution :

$$\frac{\partial\rho_{KV}}{\partial t} + [\rho_{KV},H] = 0. \tag{14}$$

The distribution (13) is known as Kapchinskij-Vladimirskij particle distribution (see [3]).

SIMULATION

We performed a numerical simulation of a two-dimensional system evolving according to the Hamiltonian (1), then we have analyzed the time series of the differences between the momenta of the particles calculated from the Hamilton's equations and the momenta obtained from the mean field ones. The initial distribution of the particles is the ρ_{KV} in (13) and the values of the other parameters are listed below.

Parameters of the simulation :

- Number of particles : $N = 8192$
- $\omega = 6.4 \cdot 10^{-3} rad/cm$
- $\omega_0 = 1 \cdot 10^{-2} rad/cm$
- Radius of the beam : $R = 1.8405 \cdot 10^{-1} cm$
- Perveance : $\xi = 2 \cdot 10^{-6}$
- Temporal step : $\Delta s = 2 \cdot 10^{-1} cm$
- Length of simulation in temporal steps: $L = 49087$

Note that the total length of the simulation corresponds to 10 betatron oscillations.

We indicate the position and momentum of the i-th particle at the k-th temporal step obtained by the simulation in the following manner:

$$\begin{cases} \mathbf{r}_{i,k} = \mathbf{r}_i(k\Delta s) = (x_i(k\Delta s), y_i(k\Delta s)), \\ \mathbf{p}_{i,k} = \mathbf{p}_i(k\Delta s) = (p_{x_i}(k\Delta s), p_{y_i}(k\Delta s)); \end{cases}$$

$$\text{for} \quad k = 0, 1, \ldots, L \quad \text{and} \quad i = 1, \ldots, N.$$

MEAN FIELD APPROXIMATION WITH STOCHASTIC CORRECTION

According to the previous notation we take :

$$\begin{cases} \tilde{\mathbf{r}}_{i,k} = \mathbf{r}_{i,k} - \mathbf{r}^{\text{CM}}_{i,k}, \\ \tilde{\mathbf{p}}_{i,k} = \mathbf{p}_{i,k} - \mathbf{p}^{\text{CM}}_{i,k}, \\ i = 1, \ldots, N; \qquad k = 0, 1, \ldots, L. \end{cases}$$

From the single particle Hamiltonian (12) it is easy to see that the equations of motion of a particle in the beam in the mean field approximation are :

$$\begin{cases} \frac{d\tilde{\mathbf{r}}}{ds}_{\text{MF}} & = \tilde{\mathbf{p}}_{\text{MF}}, \\ \frac{d\tilde{\mathbf{p}}}{ds}_{\text{MF}} & = -\omega^2 \tilde{\mathbf{r}}_{\text{MF}}. \end{cases}$$

We assume the single particle equations of motion to be:

$$\begin{cases} d\tilde{\mathbf{r}} & = \tilde{\mathbf{p}}\, ds, \\ d\tilde{\mathbf{p}} & = -\omega^2 \tilde{\mathbf{r}}\, ds + d\mathbf{B}, \end{cases} \tag{15}$$

where $\mathbf{B}$ is a stochastic process. Equations (15) are the mean field equations with the addition of a stochastic term to the interaction with the other particles of the beam. The finite difference equations for the motion of the i-th particle turn out to be:

$$\begin{cases} \tilde{\mathbf{r}}_{i,k+1} = \tilde{\mathbf{r}}_{i,k} + \tilde{\mathbf{p}}_{i,k}\Delta s, \\ \tilde{\mathbf{p}}_{i,k+1} = \tilde{\mathbf{p}}_{i,k} - \omega^2 \tilde{\mathbf{r}}_{i,k}\Delta s + \{\mathbf{B}_{\Delta s}\}_{i,k}, \\ i = 1, \ldots, N; \quad k = 0, 1, \ldots, L. \end{cases} \tag{16}$$

where $\{\mathbf{B}_{\Delta s}\}_{i,k} = \{(B_{\Delta s,x}, B_{y_{\Delta s}})\}_{i,k}$ are random vectors which distributions depend on Δs. In fig. 1 we show the sample paths of the processes $B_{\Delta s,x}$ and $B_{y_{\Delta s}}$ for the particle $i = 500$.

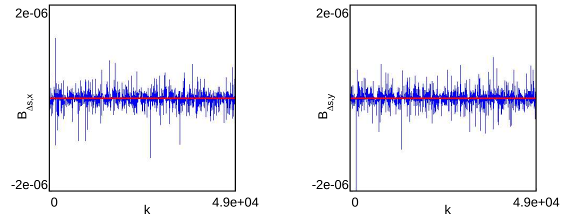

FIGURE 1. The time series $\{B_{\Delta s,x}\}_{500,k}$ and $\{B_{\Delta s,y}\}_{500,k}$, for $k = 0, 1, \ldots, L$.

In order to identify the distributions of $B_{\Delta s,x}$ and $B_{\Delta s,y}$, we have assumed the following hypotheses :

- The process $\mathbf{B}$ is produced by the Coulomb collisions between the particles.
- Every large value of $\{\mathbf{B}_{\Delta s}\}_{i,k}$ is due to a hard collision between only two particles.
- For simplicity, we consider the random variables $\{B_{\Delta s,x}\}_{i,k}$ and $\{B_{\Delta s,y}\}_{i,k}$ to be independent and identically distributed.

First of all we have plotted the histograms of the time series $\{B_{\Delta s,x}\}_{i,k}$ and $\{B_{\Delta s,y}\}_{i,k}$ in semilogarithmic scale; the plots suggest to fit a probability density of the form :

$$p(b) = \begin{cases} c_1 \exp\left(-\frac{b^2}{2\sigma^2}\right) & |b| \le b_{\text{th}} \\ c_2 |b|^{-\alpha} & |b| > b_{\text{th}}\,; \end{cases} \tag{17}$$

in order to have a normalized distribution the following condition must be satisfied:

$$1 = \sqrt{2\pi}\sigma \text{Erf}\left(\frac{b_{\text{th}}}{\sqrt{2}\sigma}\right) c_1 + 2\frac{b_{\text{th}}^{1-\alpha}}{\alpha - 1} c_2\,.$$

We have chosen the values of the parameters in (17) that minimize the χ^2 value of the fit. As it can be seen, we have reached a good agreement between the fitted density and the simulated data, which differ for less than 10% in each bin.

A distribution of the form of (17) has been found independently of us by Chavanis (see [4]) in calculation of the total force acting on a particle in the center of mass of a two-dimensional system of uniformly distributed charged particles.

In fig. 2 and fig. 3 we show the histograms of the processes $\{B_{\Delta s,x}\}$ and $\{B_{\Delta s,y}\}$ with the fitted distributions obtained from the density (17) giving to the parameters the values in table below.

Values of parameters in the distributions

Parameter	$B_{\Delta s,x}$	$B_{\Delta s,y}$
α	3.95	3.95
b_{th}	$7.5 \cdot 10^{-8}$	$7.5 \cdot 10^{-8}$
σ	$5.01 \cdot 10^{-8}$	$4.76 \cdot 10^{-8}$
c_1	$7.55 \cdot 10^{7}$	$7.89 \cdot 10^{7}$
c_2	$2.35 \cdot 10^{-21}$	$2.16 \cdot 10^{-21}$

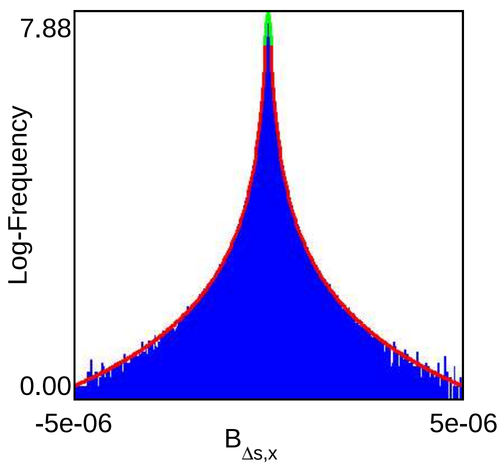

FIGURE 2. Histogram of the time series $\{B_{\Delta s,x}\}_{i,k}$ compared with the fitted distribution (17).

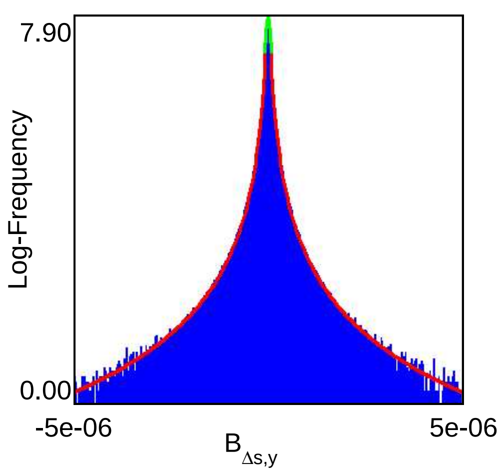

FIGURE 3. Histogram of the time series $\{B_{\Delta s,y}\}_{i,k}$ compared with the fitted distribution (17).

The density distribution (17) can be interpreted in the following way: each particle in the beam moves subjected to the mean field produced by the continuous distribution (11) plus the contribution from Coulomb collisions with the other particles. Particles far away contribute with small perturbations $|B_{\Delta s,\cdot}| \leq b_{\text{th}}$, with approximately Gaussian distribution while at certain times a hard Coulomb collision with a very close particle takes place, giving contribute $|B_{\Delta s,\cdot}| > b_{\text{th}}$ with power-law distribution.

DISTRIBUTION OF LOCAL MAXIMA OF $B_{\Delta s,x}$ AND $B_{\Delta s,y}$

According to our assumptions, large values of $\{\mathbf{B}_{\Delta s}\}_{i,k}$ (in modulus bigger than b_{th}) are mostly due to the interaction, at the k-th time step, of the particle i with the closest particle (say j), according to the equation:

$$\{\mathbf{B}_{\Delta s}\}_{i,k} \approx \frac{\xi}{N} \frac{\mathbf{r}_{i,k} - \mathbf{r}_{j,k}}{|\mathbf{r}_{i,k} - \mathbf{r}_{j,k}|^2} \Delta s \,.$$

As a consequence, each string $\{B_{\Delta s,\cdot}\}_{i,k}, k = n, n+1, ..., n+m$ of consecutive values of $\{B_{\Delta s,x}\}_{i,k}$ or $\{B_{\Delta s,y}\}_{i,k}$ such that $|\{B_{\Delta s,\cdot}\}_{i,k}| > b_{\text{th}}$ for all k , is produced by a single hard collision process (which takes several time steps to complete) between the particle i and the particle j. The local maximum of each string occur at the minimum distance $r^{\text{MIN}}_{i,j;k}$, :

$$r^{\text{MIN}}_{i,j;k} \leq \frac{\xi}{N} \Delta s \left(B^{\text{MAX}}_{\Delta s,\cdot \quad i,k} \right)^{-1} .$$

Hence, the study of the distribution of the local maxima of the processes $B_{\Delta s,x}$ and $B_{\Delta s,y}$ gives us information about the minimum distance at which a particle can be found from the others along its motion.

We have considered each subsequence of the time series $\{B_{\Delta s,x}\}_{i,k}$ and $\{B_{\Delta s,y}\}_{i,k}$ formed by consecutive values obeying the condition $\{B_{\Delta s,\cdot}\}_{i,k} > b_{\text{th}}$ and we have calculated their local maxima $B^{\text{MAX}}_{\Delta s,\cdot \quad i,k}$.

Following the same procedure as before, we have plotted the histograms of these local maxima in semilogarithmic scale, then we have fitted the (truncated) function :

$$f(b) = c \left(1 + \frac{(kb)^2}{n} \right)^{-\left(\frac{n+1}{2}\right)}, \qquad \text{for} \quad b > b_{\text{th}} \tag{18}$$

$$c = 1.74 \cdot 10^{6}, \quad k = 1.06 \cdot 10^{7}, \quad n = 3.$$

The values of the parameters in (18) have been calculated by minimizing the χ^2 value. The results are shown in fig. 4 and fig. 5 where we show the histograms of the series $B^{\text{MAX}}_{\Delta s,\cdot \quad i,k}$. The continuous line is the function (18),

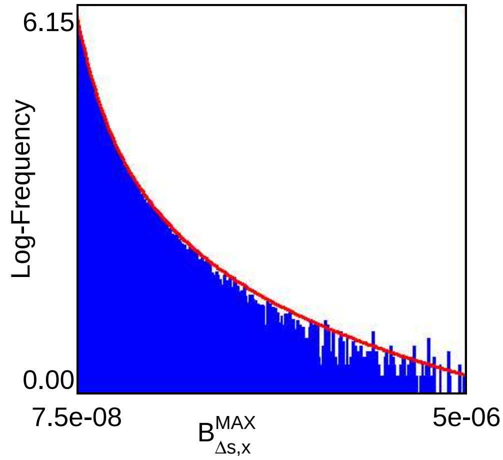

FIGURE 4. Histogram of $B^{\mathrm{MAX}}_{\Delta s,x}$ with the fitted distribution

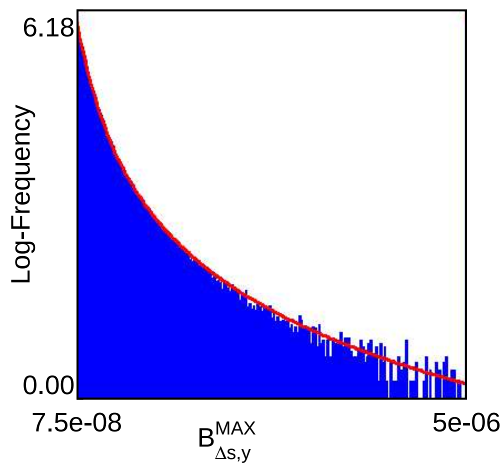

FIGURE 5. Histogram of $B^{\mathrm{MAX}}_{\Delta s,y}$ with the fitted distribution

which is related to the Student's t distribution density with $n = 3$ degrees of freedom :

$$t_n(b) = \Gamma\left(\frac{n+1}{2}\right)\frac{1}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{b^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}.$$

CONCLUSIONS

In this paper the dynamics of a 2D system of charged particles confined by a linear restoring force in the constant focusing case has been studied. A simulation of the system evolution has been carried out by direct integration of the Hamilton's equations, then the data of the simulation have been compared with the mean field dynamics, considering the differences as a time series of a stochastic process describing the Coulomb collisions between the particles. The distribution of the time series has been identified and an explanation of its properties has been given. The distribution of the local maxima of the process has also been identified, providing a starting point for the study of the collisions process of the particles in the system. Further research are needed to find a full theory of the stochastic process describing Coulomb collisions in more general cases and to write a general stochastic differential equation for the system.

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