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Microscopic gain mechanism in the high-gain free electron laser

B.W.J. McNeil *, P. Pierini and R. Bonifacio [1]
INFN - Sezione di Milano, Via Celoria 16, 20133 Milano, Italy

Invited paper

In this article we present a detailed investigation of the microscopic electron/radiation evolution in a high-gain free electron laser (FEL). Both the steady-state and in particular the superradiant regimes are investigated. Solutions of the one-dimensional FEL evolution equations are represented dynamically by animated computer graphics to produce "movies" of the electrons/radiation evolution. We believe this form of representation of the FEL process provides a clearer, intuitive picture of the underlying mechanisms involved. This has been of particular importance in the understanding of more complex FEL mechanisms such as superradiant phenomena.

1. Introduction

This article describes by means of animated computer graphics (subsequently referred to as "movies") the microscopic gain mechanisms in a high-gain free electron laser amplifier in the Compton regime. The main body of this text describes the movies as presented at the 12th International Free Electron Laser Conference. As such, a copy of the video tape containing the movies should be obtained. This can be done by following the instructions at the end of this article.

No new analytical results are presented in this article. We simply provide "visual solutions" to the governing 1D FEL equations and present our understanding of these solutions in the hope that a more intuitive understanding of the FEL mechanisms may be gained by the reader/viewer. As such, we refer the reader to the literature for a full description of the underlying theory. (For single particles see ref. [1], for steady state see ref. [2], for superradiance see refs. [3–4]).

We begin with a look at the single-particle model of the Maxwell–Lorentz equations. With no coupling between particle and field there is no field evolution and the equation of motion of the particle is that of a simple pendulum [1]. With field/particle coupling (no matter how weak) the field may evolve and the particle evolution is not that of a pendulum, indeed it may have very un-pendulumlike trajectories.

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[1] Also: Dipartimento di Fisica dell'Università Statale di Milano, Italy.

We then introduce many particles and look at the familiar high-gain steady-state regime [2]. In particular we investigate the region around the detuning threshold above which there is no exponential steady-state instability.

The introduction of pulse effects (slippage) changes the coupled ordinary differential equations describing the steady-state FEL evolution into a set of coupled partial differential equations in two independent variables [3]. In describing the length of an electron pulse we introduce the cooperation length l_c, where l_c is the slippage distance between the radiation and a resonant electron in one gain length. An electron pulse is then defined to be long or short with respect to this cooperation length [3].

We first look at the short electron pulse, which gives rise to weak superradiance (superradiant behaviour is defined when the radiation intensity scales as n_e^2, the electron density squared) and then look at long electron pulses which give rise to steady-state behavior and strong superradiance.

2. The single-particle model

The Maxwell–Lorentz equations for a single particle may be written, with the usual approximations [2], in the form

$$\frac{\mathrm{d}a}{\mathrm{d}\bar{z}} = \sin(\theta+\phi) \tag{1}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\bar{z}} = \frac{1}{a}\cos(\theta+\phi), \tag{2}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\bar{z}^2} = -2a\,\sin(\theta+\phi), \tag{3}$$

where the field and particle variables appear here as dimensionless variables scaled with respect to the fundamental FEL parameter, ρ, of ref. [2].

If we decouple the field evolution from the set of eqs. (1)–(3) then a and ϕ are constants and the equation of motion of the particle has the same form as that of a pendulum [1]. The force on the particle may be described in terms of its position within a ponderomotive potential $\Phi(-\mathrm{d}\Phi/\mathrm{d}\theta \equiv \mathrm{d}^2\theta/\mathrm{d}\bar{z}^2)$, which is, from eq. (3),

$$\Phi = -2a\cos(\theta+\phi). \qquad (4)$$

We show in movies M1–M3 this pendulum evolution of the uncoupled system to familiarize the viewer with the movie layout.

In all pendulum movies to be shown the angles θ and ϕ are measured from the downward vertical, anticlockwise. The green arrow points in the direction of the force of the effective gravitational field as determined by ϕ (this is also the position in θ of the bottom of the ponderomotive well). The height of the bar at the right-hand side of the screen gives the effective gravitational field strength (the field intensity).

The initial conditions of the first three movies (M1–M3) are:

M1: $\theta_0 = 1;\ \dot{\theta}_0 = 0;\ a = 2;\ \phi_0 = 0,$

M2: $\theta_0 = 2;\ \dot{\theta}_0 = 0;\ a = 2;\ \phi_0 = 5,$

M3: $\theta_0 = 0;\ \dot{\theta}_0 = 0;\ a = 2;\ \phi_0 = 0.$

where $\dot{\theta} \equiv \mathrm{d}\theta/\mathrm{d}\bar{z}$.

In M1 we see a single pendulum oscillating, the direction of gravity pointing down. By changing the field phase ϕ we change the direction of the gravitational field, as seen in M2. In the third movie M3 we see the pendulum in a state of stable equilibrium, simply hanging down. We show this as, as will be seen in the next movie M4, with these initial conditions the pendulum is unstable if we couple the pendulum evolution to the field.

The source of this instability is seen immediately from eqs. (1)–(3). As $\theta_0(\bar{z}\equiv 0) = \phi_0(\bar{z}=0) = 0$, we see that $\sin(\theta_0+\phi_0) = 0$, there is no initial field amplitude evolution or angular force on the pendulum (the particle is at the bottom of the potential well). However, $\cos(\theta_0 + \phi_0) = 1$, so that the rate of change of the field phase ϕ is maximum. For small values of $a(\bar{z}=0)$ the phase of the field can change rapidly until $\theta+\phi \approx \pi/2$, where $\cos(\theta+\phi) \approx 0$. The pendulum now falls towards the direction of gravity losing energy to the field as it does so. The pendulum then begins to reabsorb the radiation energy until the field phase changes rapidly again. (The reader will recognize this emission and reabsorption of the field as the synchrotron oscillatory regime of the single-particle system.)

In order to demonstrate further the un-pendulumlike motion of the coupled system we now show two movies (M5 and M6) with the following initial conditions:

M5: $\theta_0 = \pi;\ \dot{\theta}_0 = 5;\ a = 10^{-1};\ \phi_0 = 0,$

M6: $\theta_0 = \pi;\ \dot{\theta}_0 = 1.8;\ a = 10^{-1};\ \phi_0 = 0.$

If both sets of initial conditions for M5 and M6 were assigned to a normal uncoupled pendulum, it would simply librate. When coupled to the field, we see in M5 that the pendulum drives the phase in such a way that it oscillates about the "top" of a normal pendulum swing – i.e. it "stands on end" in the rotating gravitational field. In terms of the potential well the particle is oscillating about the top of the well. This motion is around the additional elliptic point in the coupled-pendulum phase space as described in ref. [5].

The initial conditions used for M6 are different from those of M5, namely there is a reduction in the initial condition $\dot{\theta}(\bar{z}=0)$. The resulting pendulum/field evolution is very different: the pendulum now has insufficient angular velocity ($\dot{\theta}(\bar{z}=0)$) to drive the field phase/amplitude in a way which allows it to perform an oscillation about the top of the well. Instead the pendulum becomes trapped by the well, falling into it and giving energy to the field. The pendulum then goes on to perform synchrotron oscillatory-type motion as in movie M4.

These two coupled-pendulum evolutions correspond to the evolution above (M5) and just below (M6) the critical value of the detuning as defined in ref. [2] ($\dot{\theta} = \sqrt[3]{27/4}$). This is also the threshold value of the electron detuning parameter below which there exists the high-gain exponential instability in the many-particle model which we will now go on to discuss.

3. The many-particle model in the steady state

The 1D evolution equations of the field and electron distribution in a FEL with the steady-state approximation have the same form as eqs. (1)–(3) except that now we must follow the evolution of many particles and average the rhs of the field evolution eqs. [2]:

$$\frac{\mathrm{d}a}{\mathrm{d}\bar{z}} = \langle\sin(\theta+\phi)\rangle, \qquad (5)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\bar{z}} = \frac{1}{a}\langle\cos(\theta+\phi)\rangle, \qquad (6)$$

$$\frac{\mathrm{d}^2\theta_j}{\mathrm{d}\bar{z}^2} = -2a\sin(\theta_j+\phi), \qquad (7)$$

where $j = 1,\cdots,N_e$, and $\langle\cdots\rangle = (1/N_e)\Sigma_{j=1}^{N_e}\cdots$.

The solution to these equations in the linear regime [2] gives rise to an exponentially growing field amplitude with maximum growth rate $a(\bar{z}) \approx a_0 \exp(\sqrt{3}\,\bar{z}/2)$ for $\bar{z} > 1$ and the detuning $\delta \equiv \dot{\theta}(\bar{z}=0) \approx 0$.

Numerical simulations show saturation (for $\delta = 0$) of the dimensionless field intensity $a_{sat}^2 \approx 1.4$. The emitted power scales as $n_e^{4/3}$.

Increasing δ from zero, the dimensionless length of the wiggler $\bar{z}$ required to saturate increases (lethargy), however, the saturated field intensity also increases. This increase in the lethargy and saturated field intensity continues with increasing δ until $\delta > \delta_T = \sqrt[3]{27/4}$ where the exponential instability stops and there is no appreciable exchange of energy between radiation and electrons.

We begin with movie M7, which shows the salient features of steady-state evolution with zero electron detuning. The bottom left-hand image shows the electron phase-space evolution, and the right-hand images show from top to bottom the intensity $a^2(\bar{z})$, the average electron detuning $\langle p(\bar{z}) \rangle$ and the bunching parameter b. $p \equiv \dot{\theta}$ is proportional to the average electron energy [2]. Each electron in phase space is represented by a green cross (there is one larger blue reference electron) and is plotted in the phase space θ, $\dot{\theta}$, with $-\pi < \theta < \pi$. The instantaneous separatrix is plotted in red. This would define the phase-space trajectories of each electron if the field evolution were to stop at that instant (each electron would then evolve as a simple pendulum). The two green horizontal lines define the maximum height of the separatrix at saturation for $\delta = 0$ $(= \pm 2\sqrt{2a_{sat}})$. The initial conditions of the electrons spread them uniformly in phase θ, the initial field amplitude is $a_0 = 10^{-2}$ and the initial field phase $\phi = 0$.

After a period of lethargy the field intensity rises exponentially until nonlinear effects dominate and the evolution of the system saturates, with intensity $a_{sat}^2 \approx 1.4$. The system then enters the synchrotron oscillatory phase of evolution in which the electrons reabsorb energy from the field. Notice the conserved quantity $a^2 + \langle p \rangle$ corresponding to energy conservation.

We now look in more detail at the electron phase-space evolution for the three cases of detuning:

M8: $\delta = 0$ (maximum growth rate),

M9: $\delta < \sim \delta_T$ (maximum saturation intensity/long lethargy),

M10: $\delta > \sim \delta_T$ (outward the steady-state exponential instability regime).

In M8–M10 the $\langle p \rangle$ marker on the right-hand axis gives the average of p_j.

The three quantities given at the top of M8–M10 are the dimensionless distance down the wiggler, $\bar{z}$, the dimensionless field intensity a^2 and the field phase ϕ.

We shall discuss the electron trajectories in terms of their motion in the potential well. The bottom of the well is at the elliptic point (at the center of the separatrix) while the hyperbolic points of the separatrix correspond to the top of the potential well.

An increase/decrease in the field phase ϕ will cause the separatrix to move to the left/right.

For zero detuning, $\delta = 0$, (M8) the electrons fall into the potential well and begin to bunch around $\theta + \phi = 0$. This bunching, as can be seen from eqs. (5)–(7), does not initially drive the field amplitude as $\langle \sin(\theta + \phi) \rangle = 0$. The quantity $\langle \cos(\theta + \phi) \rangle$, however, is positive, and for small initial field amplitude the phase is increasing $(\mathrm{d}\phi/\mathrm{d}\bar{z} > 0)$. The separatrix/potential well then moves to the left in the $(\theta, \dot{\theta})$ space allowing the bunching electrons to give energy to the field as they fall into the potential well $(\mathrm{d}a/\mathrm{d}\bar{z} = \langle \sin(\theta + \phi) \rangle > 0)$. The electrons then begin to reabsorb the energy from the field and perform synchrotron oscillations in the well. Notice that the phase $\mathrm{d}\phi/\mathrm{d}\bar{z}$ is always greater than zero (for $\delta = 0$).

In the next movie, M9, we can see the phase-space evolution for an initial electron detuning just below the threshold value δ_T. Here we see the electrons moving to the right and a small bunching occurring in the region just before the top of the potential well. It may be seen from eqs. (5)–(7) that a bunching around this part of the well causes $\mathrm{d}\phi/\mathrm{d}\bar{z}$ to be negative, so that the separatrix also moves to the right. It may be easily shown that in this driving of the radiation phase, if we assume a constant $\dot{\phi}$, the radiation frequency may be rewritten as $\omega' \approx \omega + \dot{\phi}$. In this way we see that it appears that the electrons are trying to drive the radiation frequency to that with the maximum growth rate. This bunching also allows energy exchange to the field, so that the field amplitude grows. This growth, however, is slower than in the case of zero detuning. Eventually the bunched electrons become trapped within the well and are seen to be trapped by the separatrix whence they lose energy rapidly to the field. Note also that the phase evolution has effectively stopped (the separatrix is stationary). The electrons then go into synchrotron oscillatory motion.

The next movie, M10, shows the evolution for $\delta \gtrsim \delta_T$. The picture, at the beginning, looks very similar to M9, but this time the electron detuning is too great to allow sufficient bunching near the top of the well to drive either the phase or the amplitude of the field sufficiently to allow trapping of the bunched electrons. Hence in this case there is no exponential growth of the field.

We note the obvious similarities in the dynamics of the single-particle evolutions with the many-particle model through the movies M4 and M7/8; M5 and M10; M6 and M9.

4. Slippage and superradiance

The steady-state (s.s.) model discussed in the previous section does not take into account the slippage of the electrons and radiation due to their relative velocity difference. In the s.s. model the electrons within one radiation wavelength of the beam evolve self-consistently with the radiation they emit. Inclusion of the effect of slippage allows electrons in one part of the beam to interact with the radiation emitted by other electrons from another part of the beam many radiation wavelengths distant. If these electrons (and the intermediate electrons) have evolved identically we retain the s.s. picture of evolution. If, however, they have not evolved identically (for example if they evolved at the trailing edge of the electron pulse into which no radiation is propagating) then there is no s.s. evolution and a different evolution of both electrons/radiation is expected. This type of evolution leads to superradiant processes where the intensity of the radiation scales as n_e^2. The coupled Maxwell–Lorentz equations now take on a partial differential form:

$$\frac{\partial a(\bar{z}_1, \bar{z}_2)}{\partial \bar{z}_1} = \langle \sin(\theta(\bar{z}_1, \bar{z}_2) + \phi(\bar{z}_1, \bar{z}_2)) \rangle, \quad (8)$$

$$\frac{\partial \phi(\bar{z}_1, \bar{z}_2)}{\partial \bar{z}_1} = \frac{1}{a(\bar{z}_1, \bar{z}_2)} \langle \cos(\theta(\bar{z}_1, \bar{z}_2) + \phi(\bar{z}_1, \bar{z}_2)) \rangle, \quad (9)$$

$$\frac{\partial^2 \theta_j(\bar{z}_1, \bar{z}_2)}{\partial \bar{z}_2^2} = -2a(\bar{z}_1, \bar{z}_2) \sin\left(\theta_j(\bar{z}_1, \bar{z}_2) + \phi(\bar{z}_1, \bar{z}_2)\right). \quad (10)$$

We refer the reader to ref. [3] for further discussion and method of numerical solution of this system of equations and to ref. [4] for solutions of the linearised system.

The fundamental parameter governing the effects of slippage between the electron/radiation in the high-gain FEL is the "cooperation length", l_c, which may be defined as *the distance a photon will travel with respect to a resonant electron in one gain length*. A gain length is the distance corresponding to the dimensionless distance through the wiggler of $\bar{z} = 1$. The cooperation length is then the minimum distance between which electrons may interact cooperatively within the beam. The cooperation length is easily calculated from the definition of $\bar{z}$:

$$\bar{z} = 2k_w \rho z = 4\pi\rho N_w,$$

where N_w is the number of wiggler periods. The number of wiggler periods required to produce a slippage of one cooperation length is then $N_{wc} = 1/4\pi\rho$. From the resonance condition follows

$$l_c = \lambda N_{wc} = \lambda/4\pi\rho.$$

We define electron pulses to be long or short with respect to the cooperation length. For short electron pulses we are in the regime of weak superradiant phenomena where there is no s.s. evolution and the radiation intensities scale as n_e^2 (as opposed to $n_e^{4/3}$ in the s.s.). For long electron pulses we may have both s.s. and superradiant effects. The superradiance arises from the radiation emitted by the trailing region of the electron pulse (which has short pulse dynamics), which, we hypothesise, is subsequently amplified on propagating through the electron pulse to produce spikes of strong superradiance with peak intensities much greater than the saturated s.s. value and which scale as n_e^2. We now use the movies to look at both short- and long-pulse evolution.

The first of the pulse movies, M11, shows a typical short-pulse evolution of the radiation intensity with parameters

length of electron pulse (in units of radiation wavelength) = 3;
total $\bar{z}$ of wiggler = 30.0;
number of wiggler periods = 180;
initial dimensionless intensity $a_0^2 = 10^{-2}$.

The movie shows a window traveling at velocity c of the radiation. At the top of the window we show the relative position and dimensions of the electron pulse (the x-axis is in units of radiation wavelength). As the velocity of the electrons $v_z < c$, the electron pulse is seen to slip relative to the radiation window from right to left. As the electron pulse slips, the radiation emitted by the pulse escapes and propagates in vacuum. It is clear that the maximum distance in which the radiation may interact with the electron pulse is the slippage distance through the pulse, and this corresponds to an interaction length $\bar{z} < 1$ as the length of the electron pulse is smaller than l_c. Thus there are no cooperative effects within the electron pulse and the electrons effectively radiate spontaneously, but coherently, and without any saturation mechanism – if we increase the length of the wiggler the electron pulse continues to radiate at the average electron spontaneous frequency. The radiation intensity scales as n_e^2 and with peak intensities less than the s.s. saturated intensity, hence we term this type of dynamics "weak superradiance".

In the next movie, M12, we show a similar short-pulse evolution in more detail. The form is the same as that of M7 with the additional top left-hand image of the electron pulse in a frame traveling at the resonant electron velocity. The radiation intensity is plotted as it escapes from the pulse. The blue region of the electron pulse is the position within the pulse at which all other quantities in the movie are plotted. The saliant points to notice are that the electrons are not trapped by the separatrix and continue to fall in phase space. Also note that the quantity $a^2(\bar{z}_1, \bar{z}_2) + \langle p(\bar{z}_1, \bar{z}_2)\rangle$ is no longer a conserved quantity as it is in the s.s. [3].

We now look at the evolution of radiation and electrons in the long pulse limit. We show two sets of four movies: M13–M16 and M17–M20. The first set is for zero detuning of the electrons and the other set for detuned electrons such that $\delta > \delta_T$, i.e. the electrons are outward the exponential instability regime of the s.s.. Each set is shown in the following order:

a) radiation intensity in c velocity window,
b) radiation phase, ϕ, in c velocity window,
c) electron phase-space evolution,
d) general overview movie.

The parameters common to both sets were:
length of electron pulse (in units of radiation wavelength) = 100;
total $\bar{z}$ of wiggler = 30.0;
number of wiggler periods = 100;
initial dimensionless intensity $a_0^2 = 10^{-2}$.

We begin with the first set of four for zero electron detuning. We see in M13 that as the electron pulse/radiation evolves, it is clear there are three distinct regions of evolution: the radiation which propagates forward out of the electron pulse into vacuum; the flat region whose evolution is that of the s.s.; and the slippage region where there is no s.s. evolution. It is in the latter region in which we see the evolution of the spiking which we have defined as strong superradiance.

In order to better understand this spiking mechanism, it is instructive to look at the phase evolution of the radiation (M14). We use the same movie format as the previous and plot the phase of the radiation in the limits $0 < \phi \leqslant 2\pi$. As discussed in the previous section we observe the monotonically increasing phase in the s.s. region. Note, however, the phase evolution in the slippage region. As the electrons enter this region from the s.s. they experience a radiation field whose phase is increasing at a rate greater than that of the s.s.. We have already seen that this increase in the radiation phase corresponds to a lower frequency of radiation, implying a type of self-tapering of the radiation frequency in the slippage region.

This phase mechanism may be seen in the next movie, M15, as the increase in the separatrix drift velocity to the left as the electrons enter the slippage region from the s.s. at $\bar{z} \approx 10.0$. Note also the large increase in the separatrix height as the electrons pass through the spike. After passing through the spike the separatrix decreases and the electrons become de-trapped.

A general overview of the above processes is shown in the next movie, M16. Note the nonconservation of the quantity $a^2(\bar{z}_1, \bar{z}_2) + \langle p(\bar{z}_1, \bar{z}_2)\rangle$ once the electrons have left the s.s. and entered the slippage region.

Finally we show the same set of movies as the last four but with $\delta = 2.0$ ($> \delta_T$).

We see from the first movie, M17, that there is now no s.s.-type evolution of the field intensity as the electrons are detuned above the exponential threshold. Notice, however, the growth of the large spike in the slippage region. The fact that this spiking exists above the detuning threshold of the s.s. and its intensity scales as n_e^2 implies that this is a type of exponential instability distinct from that of the s.s.

As with the previous resonant case the clue to the spiking mechanism lies in the evolution of the radiation phase, which is shown in M18.

As expected from the s.s. results from the previous section we see that the phase is decreasing in the s.s. (flat) region of evolution. In the slippage region the electrons see a further decreasing phase due to the spatial profile of the phase. This further decrease is equivalent to an increase in the radiation frequency – the electrons can then become resonant with the radiation leading to the exponentially growing field.

This decreasing phase is seen clearly in the electron phase-space movie M19, where on entering the slippage region the radiation separatrix moves more rapidly to the right allowing the radiation to trap the electrons within the separatrix. Again this process is a type of self-tapering of the radiation field – this time to a higher frequency to allow resonance with the electrons.

A summary of the detuned case is shown in the last movie, M20.

5. Conclusions

We have tried in this article to give an intuitive "feel" for the microscopic gain mechanisms in a high-gain FEL by using movie representations of the numerical solutions to the 1D FEL equations.

It is clear that the phase evolution of the radiation field is crucial to the understanding of both the single-particle, steady-state and superradiant gain mechanisms.

In the superradiant regime the extra degree of freedom introduced by the relative slippage of the electron/radiation pulses allows a "self-tapering" of the radiation field. This self-tapering is seen over the spiked regions of the radiation pulses in strong superradiant phenomena.

6. Video ordering

In order to obtain a video of the movies described in this paper, please send a blank VHS videocassette to the authors, with the *full* return address *within six months* of the publication date of these Proceedings.

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