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Abstract

Gauge theories on a space-time that is deformed by the Moyal-Weyl product are constructed by twisting the coproduct for gauge transformations. This way a deformed Leibniz rule is obtained, which is used to construct gauge invariant quantities. The connection will be enveloping algebra valued in a particular representation of the Lie algebra. This gives rise to additional fields, which couple only weakly via the deformation parameter θ and reduce in the commutative limit to free fields. Consistent field equations that lead to conservation laws are derived and some properties of such theories are discussed.

1   Introduction

The aim of this work is to construct and investigate gauge theories on deformed space-time structures that are defined by an associative but noncommutative product of C^∞ functions. Such products are known as star products; the best known is the Moyal-Weyl product [1, 2]. In this letter we shall deal with this product exclusively.

From previous work [3, 4, 5] we know that the usual algebra of functions and the algebra of vector fields can be represented by differential operators on the deformed manifold. The deformed diffeomorphisms have been used to construct a deformed theory of gravity. Here we shall show that along the same lines a deformed gauge theory can be constructed as well. The algebra, based on a Lie algebra, will not change but the comultiplication rule will. This leads to a deformed Hopf algebra. In turn this gives rise to deformed gauge theories because the construction of a gauge theory involves the Leibniz rule that is based on the comultiplication.

Covariant derivatives can be constructed by a connection. Different to a usual gauge theory the connection cannot be Lie algebra valued. The construction of covariant tensor fields (curvature or field strength) and of an invariant Lagrangian is completely analogue to the undeformed case. Field equations can be derived and it can be shown that they are consistent. This leads to conserved currents. It is for the first time that it is seen that deformed symmetries also lead to conservation laws; note that the Noether theorem is not directly applicable in the noncommutative context.

The deformed gauge theory has interesting new features. We start with a Lie(G)-valued connection and show that twisted gauge transformations close in Lie(G), however consistency of the equation of motion requires the introduction of additional, new vector potentials. The number of these extra vector potentials is representation dependent but remains finite for finite dimensional representations. Concerning the interaction, the Lie algebra valued fields and the new vector fields behave quite differently. The interaction of the Lie algebra valued fields can be seen as a deformation of the usual gauge interactions; for vanishing deformation parameters the interaction will be the interaction of a usual gauge theory. The interactions of the new fields are deformations of a free field theory for vector potentials; for vanishing deformation parameters the fields become free. As the deformation parameters are supposed to be very small we conclude that the new fields are practically dark with respect to the usual gauge interactions.

Finally we discuss the example of a $SU(2)$ gauge group in the two dimensional representation.

The treatment introduced here can be compared with previous ones. In [6] the noncommutative gauge transformations for $U(N)$ have an undeformed comultiplication. The action is the same as in (4.2) if we restrict our discussion, valid for any compact Lie group, to $U(N)$ in the n-dimensional matrix representation. In other terms we show that noncommutative $U(N)$ gauge theories have usual noncommutative gauge transformations and also twisted gauge transformations. In [7, 8, 9, 10, 11] the situation is different because we consider field dependent transformation parameters.

2 Algebraic formulation

A noncommutative coordinate space can be realized with the help of the Moyal-Weyl product [1, 2]. On such a space we are going to construct gauge theories based on a Lie algebra.

We start from the linear space of C^∞ functions on a smooth manifold $\mathcal{M}$, $Fun(\mathcal{M})$. To define an algebra $\mathcal{A}_\theta$ we shall use the associative but noncommutative Moyal-Weyl product. The algebra defined with the usual, commutative point-wise product we refer to as the algebra of C^∞ functions.

The Moyal-Weyl product is defined as follows

$$
\begin{aligned}
f, g &\in Fun(\mathcal{M}) \\
f \star g &= \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\} \\
\mu\{f \otimes g\} &= f \cdot g,
\end{aligned}
\tag{2.1}
$$

where $\theta^{\rho\sigma} = -\theta^{\sigma\rho}$ is x-independent. The $\star$-product of two functions is a function again

$$
\begin{aligned}
\mu_\star : \quad & Fun(\mathcal{M}) \otimes Fun(\mathcal{M}) \to Fun(\mathcal{M}), \\
& \mu_\star\{f \otimes g\} = f \star g.
\end{aligned}
\tag{2.2}
$$

Derivatives are linear maps on $Fun(\mathcal{M})$

$$
\begin{aligned}
\partial_\rho : \quad & Fun(\mathcal{M}) \to Fun(\mathcal{M}), \\
& f \mapsto \partial_\rho f.
\end{aligned}
\tag{2.3}
$$

The Leibniz rule extends these maps to the usual algebra of C^∞ functions

$$
(\partial_\rho(f \cdot g)) = (\partial_\rho f) \cdot g + f \cdot (\partial_\rho g).
\tag{2.4}
$$

This concept can be lifted to the algebra $\mathcal{A}_\theta$ [12]

$$
\begin{aligned}
\partial_\rho^\star : \quad & f \mapsto \partial_\rho^\star f \equiv \partial_\rho f \\
& \partial_\rho^\star(f \star g) = (\partial_\rho^\star f) \star g + f \star (\partial_\rho^\star g).
\end{aligned}
\tag{2.5}
$$

The last line is true because $\theta^{\mu\nu}$ is x-independent.

Analogously to differential operators acting on the usual algebra of functions we define differential operators on $\mathcal{A}_\theta$

$$
\mathcal{D}^\star \star f = \sum_n d^{\rho_1\dots\rho_n} \star \partial_{\rho_1}^\star \dots \partial_{\rho_n}^\star f.
\tag{2.6}
$$

This is well defined, $\star$ and $\partial_\rho^\star$ always act on functions. The product of such differential operators can be computed with the help of the Leibniz rule.

We can now define the $\star$-product as the action of a bilinear differential operator

$$
f \star g = \mu\{\mathcal{F}^{-1} f \otimes g\},
\tag{2.7}
$$

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with

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma}. \tag{2.8}$$

This differential operator can be inverted

$$f \cdot g = \mu_\star \{\mathcal{F} f \otimes g\}. \tag{2.9}$$

Equation (2.9) can also be written in the form [3]

$$f \cdot g = \Big(\sum_{n=0}^{\infty} \Big(-\frac{i}{2} \Big)^n \frac{1}{n!} \theta^{\rho_1\sigma_1} \dots \theta^{\rho_n\sigma_n} \Big(\partial_{\rho_1} \dots \partial_{\rho_n} f \Big) \star \partial^\star_{\sigma_1} \dots \partial^\star_{\sigma_n} \Big) \star g. \tag{2.10}$$

Equation (2.10) shows that the point-wise product $f \cdot g$ can also be interpreted as the $\star$-action of a differential operator $X^\star_f$ on g

$$f \cdot g = X^\star_f \star g = (X^\star_f \star g), \tag{2.11}$$

where

$$X^\star_f = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(-\frac{i}{2} \Big)^n \theta^{\rho_1\sigma_1} \dots \theta^{\rho_n\sigma_n} (\partial_{\rho_1} \dots \partial_{\rho_n} f) \star \partial^\star_{\sigma_1} \dots \partial^\star_{\sigma_n}. \tag{2.12}$$

From the associativity of the $\star$-product follows immediately

$$f \cdot g \cdot h = X^\star_{f \cdot g} \star h = X^\star_f \star X^\star_g \star h. \tag{2.13}$$

The differential operators $X^\star_f$ represent the usual algebra of functions

$$X^\star_f \star X^\star_g = X^\star_{f \cdot g}. \tag{2.14}$$

3 Gauge transformations

Ordinary gauge transformations are Lie algebra-valued

$$\alpha(x) = \alpha^a(x)T^a, \quad [T^a, T^b] = if^{abc}T^c. \tag{3.1}$$

The gauge transformation of a field is

$$\delta_\alpha \psi(x) = i\alpha(x)\psi(x) = i\alpha^a(x)T^a\psi(x), \tag{3.2}$$

i.e. $\delta_\alpha \psi = i\alpha \cdot \psi$. This can be viewed as a $\star$-action

$$\hat{\delta}_\alpha \psi = iX^\star_{\alpha^a} \star T^a\psi = iX^\star_\alpha \star \psi = i\alpha \cdot \psi. \tag{3.3}$$

When we deal with a gauge theory in physics we not only use the Lie algebra but also the corresponding Hopf algebra obtained from the comultiplication rule

$$\begin{aligned}
\Delta(\delta_\alpha)(\phi \otimes \psi) &= (\delta_\alpha \phi) \otimes \psi + \phi \otimes (\delta_\alpha \psi), \\
\Delta(\delta_\alpha) &= \delta_\alpha \otimes 1 + 1 \otimes \delta_\alpha.
\end{aligned} \tag{3.4}$$

3

The transformation of the product of fields is

$$\delta_\alpha(\phi \cdot \psi) = \delta_\alpha \mu\{\phi \otimes \psi\} = \mu\Delta(\delta_\alpha)(\phi \otimes \psi). \tag{3.5}$$

But there are different ways to extend a Lie algebra to a Hopf algebra. A convenient way is by a twist $\mathcal{F}$, that is a bilinear differential operator acting on a tensor product of functions. A well known example is

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma}. \tag{3.6}$$

It satisfies all the requirements for a twist [13, 14] and therefore gives rise to a new co-product (the dual description of twisted gauge transformations was already introduced in [15]; see also [16])

$$\Delta_\mathcal{F}(\hat{\delta}_\alpha)(\phi \otimes \psi) = i\mathcal{F}(\alpha \otimes 1 + 1 \otimes \alpha)\mathcal{F}^{-1}(\phi \otimes \psi) \tag{3.7}$$
$$= \sum_n (\frac{-i}{2})^n \frac{\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}}{n!}(\hat{\delta}_{\partial_{\mu_1}\cdots\partial_{\nu_n}\alpha} \otimes \partial_{\nu_1}\cdots\partial_{\nu_n} + \partial_{\mu_1}\cdots\partial_{\mu_n} \otimes \hat{\delta}_{\partial_{\nu_1}\cdots\partial_{\nu_n}\alpha})(\phi \otimes \psi) \, .$$

This coproduct defines a new Hopf algebra, the Lie algebra is extended by the derivatives, the comultiplication is deformed. This twist can also be used to deform Poincaré transformations [15, 12, 17, 3] respectively diffeomorphisms [3, 4, 5]. In [18] gauge theories consistent with twisted diffeomorphisms where constructed without deforming the coproduct for gauge transformations.

We now look at the transformation law of products of fields based on the deformed coproduct (3.7).

$$\hat{\delta}_\alpha(\phi \star \psi) = \mu_\star\{\Delta_\mathcal{F}(\hat{\delta}_\alpha)(\phi \otimes \psi)\}, \tag{3.8}$$

where $\mu_\star$ is defined in (2.2) and $\hat{\delta}_\alpha$ in (3.3). We obtain

$$\hat{\delta}_\alpha(\phi \star \psi) = iX^\star_{\alpha^a} \star \left((T^a\phi) \star \psi + \phi \star (T^a\psi)\right). \tag{3.9}$$

Note that the operator $X^\star_{\alpha^a}$ is at the left of both terms, this is due to the coproduct $\Delta_\mathcal{F}$. Formula (3.9) is different from

$$\hat{\delta}_\alpha(\phi \star \psi) = (\hat{\delta}_\alpha\phi) \star \psi + \phi \star (\hat{\delta}_\alpha\psi). \tag{3.10}$$

It is exactly the requirement that the $\star$-product of two fields should transform as (3.9) that leads to the twist $\mathcal{F}$. It is by the twisted coproduct that the $\star$-product of fields transforms like (3.3) again. The commutator of two gauge transformation closes in the usual way

$$\hat{\delta}_\alpha\hat{\delta}_\beta - \hat{\delta}_\beta\hat{\delta}_\alpha = \hat{\delta}_{-i[\alpha,\beta]} \, . \tag{3.11}$$

To construct an invariant Lagrangian we have to introduce covariant derivatives

$$D_\mu\psi = \partial_\mu\psi - iA_\mu \star \psi. \tag{3.12}$$

From

$$\hat{\delta}_\alpha\psi = iX^\star_{\alpha^a} \star (T^a\psi)$$

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we find

$$\hat{\delta}_\alpha(D_\mu\psi) = iX^\star_{\alpha^a} \star (T^a(D_\mu\psi)) \tag{3.13}$$

if we use the proper comultiplication for the term $A_\mu \star \psi$ in the covariant derivative and if the vector field transforms as follows

$$\hat{\delta}_\alpha A_\mu = \partial_\mu\alpha + iX^\star_{\alpha^a} \star [T^a, A_\mu]. \tag{3.14}$$

This can also be written in the familiar way [1]:

$$\hat{\delta}_\alpha A_\mu = \partial_\mu\alpha + i[\alpha, A_\mu]. \tag{3.15}$$

The transformation would take Lie algebra-valued objects to Lie algebra-valued objects. For reasons that will become clear in the following we will assume the hermitian field A_μ to be $n \times n$ matrix valued where n is the dimension of the Lie algebra representation. Formula (3.14) will still be true in that case.

The field-strength tensor can be obtained as usual

$$\begin{aligned} F_{\mu\nu} &= i[D_\mu \,\stackrel{\star}{,}\, D_\nu], \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \,\stackrel{\star}{,}\, A_\nu]. \end{aligned} \tag{3.16}$$

Using the deformed coproduct and the gauge variation of the potential we derive the following transformation law,

$$\begin{aligned} \hat{\delta}_\alpha F_{\mu\nu} &= iX^\star_{\alpha^a} \star [T^a, F_{\mu\nu}] \\ &= i[\alpha, F_{\mu\nu}]. \end{aligned} \tag{3.17}$$

4 Field equations

With the tensor $F_{\mu\nu}$ and the covariant derivatives we can construct invariant Lagrangians. Starting from the usual invariant Lagrangians we replace the point-wise product by the $\star$-product and the comultiplication (3.5) with (3.7). We convince ourselves that we can construct an invariant Lagrangian under the deformed Hopf algebra. The expression $F^{\mu\nu} \star F_{\mu\nu}$ transforms as follows

$$\begin{aligned} \hat{\delta}_\alpha(F^{\mu\nu} \star F_{\mu\nu}) &= iX^\star_{\alpha^a} \star [T^a, F^{\mu\nu} \star F_{\mu\nu}] \\ &= i[\alpha, F^{\mu\nu} \star F_{\mu\nu}]. \end{aligned} \tag{4.1}$$

This leads to an invariant and real action

$$S_g = c_1 \int \mathrm{d}^4x \, \mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}). \tag{4.2}$$

---

[1]Equations (3.15) and (3.17) were already introduced in a similar way in [16]. However, in [16] the gauge field was chosen to be Lie algebra valued which does not lead to consistent equations of motion.

The integral introduced in (4.2) has the trace property

$$\int \mathrm{d}^4x \, (f \star g) = \int \mathrm{d}^4x \, (f \cdot g) = \int \mathrm{d}^4x \, (g \star f). \tag{4.3}$$

Therefore we obtain the field equations by writing the varied field to the very left. Varying with respect to the matrix algebra-valued field A_μ leads to the field equations

$$(\partial_\mu F^{\mu\rho})_{AB} - i([A_\mu \overset{\star}{,} F^{\mu\rho}])_{AB} = 0. \tag{4.4}$$

Here A and B are matrix indices.

From the field equations and the antisymmetry of $F^{\mu\nu}$ in μ and ν follows the consistency requirement

$$\partial_\rho \Big(i[A_\mu \overset{\star}{,} F^{\mu\rho}] \Big) = 0. \tag{4.5}$$

To show (4.5) we have to use the equation of motion (4.4). We calculate

$$\partial_\rho \Big(i[A_\mu \overset{\star}{,} F^{\mu\rho}] \Big) = i[\partial_\rho A_\mu \overset{\star}{,} F^{\mu\rho}] + i[A_\mu \overset{\star}{,} \partial_\rho F^{\mu\rho}]. \tag{4.6}$$

In the second term we insert the field equation (4.4). In the first term we complete $\partial_\rho A_\mu$ to the tensor $F_{\rho\mu}$ by adding and subtracting the respective terms. We then use

$$[F_{\mu\rho} \overset{\star}{,} F^{\mu\rho}] = 0, \tag{4.7}$$

and obtain

$$+\frac{(i)^2}{2}[[A_\rho \overset{\star}{,} A_\mu] \overset{\star}{,} F^{\mu\rho}] + \frac{(i)^2}{2}[A_\mu \overset{\star}{,} [A_\rho \overset{\star}{,} F^{\mu\rho}]] - \frac{(i)^2}{2}[A_\rho \overset{\star}{,} [A_\mu \overset{\star}{,} F^{\mu\rho}]] = 0$$

for the right hand side of equation (4.6). That it vanishes follows from the Jacobi identity. Thus, we obtained a conservation law

$$J^\rho = i[A_\mu \overset{\star}{,} F^{\mu\rho}], \tag{4.8}$$
$$\partial_\rho J^\rho = 0.$$

From (3.16) follows that $F_{\mu\nu}$ is enveloping algebra valued if A_μ is. From the field equation follows that A_μ and $F_{\mu\nu}$ will remain enveloping algebra valued in the n-dimensional representation of the Lie algebra. Thus, we try to replace matrix algebra valued by enveloping algebra valued for A_μ. As an example we treat the case $SU(2)$ in the two-dimensional representation. In this representation the generators T^a of the Lie algebra satisfy the relations

$$[T^a, T^b] = i\epsilon^{abc}T^c \tag{4.9}$$

and

$$\{T^a, T^b\} = \frac{1}{2}\delta^{ab}. \tag{4.10}$$

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Note that (4.9) is valid for any representation. The anticommutator is representation-dependent. Equation (4.10) is only true in the two dimensional representation. In our example we can write A_μ as follows:

$$A_\mu = B_\mu + A_\mu^d T^d \,.$$

This is consistent with the gauge transformations; the field equations are a consequence of (4.9) and (4.10).

The tensor $F_{\mu\nu}$ is easy to calculate following (3.16):

$$F_{\mu\nu} = G_{\mu\nu} + \tilde{F}_{\mu\nu}^d T^d \,,$$

where

$$
\begin{aligned}
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu \stackrel{\star}{,} B_\nu] - \frac{i}{4}[A_\mu^d \stackrel{\star}{,} A_\nu^d] \\
\tilde{F}_{\mu\nu}^d &= \partial_\mu A_\nu^d - \partial_\mu A_\nu^d - i[B_\mu \stackrel{\star}{,} A_\nu^d] - i[A_\mu^d \stackrel{\star}{,} B_\nu] + \frac{1}{2}\{A_\mu^a \stackrel{\star}{,} A_\nu^b\}\epsilon^{abd} \,. \quad (4.11)
\end{aligned}
$$

Varying the Lagrangian (4.2) with respect to B_μ and A_μ^d leads to the field equations

$$
\begin{aligned}
\partial^\mu G_{\mu\nu} - i[B^\mu \stackrel{\star}{,} G_{\mu\nu}] - \frac{i}{4}[A^{\mu a} \stackrel{\star}{,} \tilde{F}_{\mu\nu}^a] &= 0 \\
\partial^\mu \tilde{F}_{\mu\nu}^d - i[A^{\mu d} \stackrel{\star}{,} G_{\mu\nu}] - i[B^\mu \stackrel{\star}{,} \tilde{F}_{\mu\nu}^d] + \frac{1}{2}\epsilon^{abd}\{A_\mu^a \stackrel{\star}{,} \tilde{F}_{\mu\nu}^b\} &= 0 \,. \quad (4.12)
\end{aligned}
$$

These field equations are consistent. They describe a triplet of vector fields A_μ^d as expected and a singlet B_μ. In the limit $\theta \to 0$, B_μ becomes a free field; it interacts only via θ and higher order terms in θ. The triplet A_μ^d satisfies the usual field equations of $SU(2)$ gauge theory in the limit $\theta \to 0$. For $\theta \neq 0$ both the triplet and the singlet couple to conserved currents but the current of B_μ has no θ-independent term.

We discover that the field equations (4.11) with four conserved currents also have a larger symmetry structure, i.e. the gauge transformations (3.1) can also be enveloping algebra valued.

Remarks

In [19] the authors misinterpreted our definition of the twisted coproduct (3.7) as it was given in the preprint version of this paper. This lead to the conclusion that twisted gauge transformations violate the gauge principle. In order to avoid further misunderstandings we added in this version a more explicit formula for twisted gauge transformations in (3.7). We also present some more details about coproducts in an appendix. It can now be seen from the explicit expression for gauge transformations that the gauge principle is not violated.

Note furthermore that we do not treat in this letter the quantum field theory aspects of our noncommutative gauge theories (with twisted coproduct) and therefore we have not referred to the relevant literature.

Acknowledgements

A   Notations on coproduct

We might rewrite the first part of Section 3 (up to formula (3.8)) using a more mathematically oriented language. We consider (the semidirect product of) the Lie algebra of the local gauge group $\alpha(x) = \alpha(x)^a T^a$ and of translations. The undeformed coproduct on the generators is

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha = \alpha_1 \otimes \alpha_2 \qquad (A.1)$$

$$\Delta(\partial_\nu) = \partial_\nu \otimes 1 + 1 \otimes \partial_\nu \qquad (A.2)$$

where $\alpha_1 \otimes \alpha_2$ is a convenient notation for $\alpha \otimes 1 + 1 \otimes \alpha$. In $\alpha_1 \otimes \alpha_2$ a sum is understood so that α_1 respectively assumes the values α and 1 (and similarly for α_2).

The action of α and ∂_ν on fields is given by the gauge transformation $\delta_\alpha \phi$ and by the usual derivative action $\partial_\nu \phi$. From the coproduct Δ we have the action of α on the product of fields

$$\delta_\alpha(\phi \cdot \psi) = \delta_\alpha \mu\{\phi \otimes \psi\} = \mu\{(\delta_{\alpha_1}\phi) \otimes (\delta_{\alpha_2}\psi)\} = (\delta_{\alpha_1}\phi)(\delta_{\alpha_2}\psi), \qquad (A.3)$$

and the usual Leibniz rule for partial derivatives.

We now deform the coproduct Δ by using the twist $\mathcal{F}$, and obtain the new coproduct

$$
\begin{aligned}
\Delta_{\mathcal{F}}(\alpha) &= \mathcal{F}(\alpha \otimes 1 + 1 \otimes \alpha)\mathcal{F}^{-1} \qquad &(A.4)\\
&= \sum_n \left(\frac{-i}{2}\right)^n \frac{\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}}{n!}(\partial_{\mu_1}\cdots\partial_{\mu_n}\alpha \otimes \partial_{\nu_1}\cdots\partial_{\nu_n} \\
&\qquad\qquad\qquad\qquad +\partial_{\mu_1}\cdots\partial_{\mu_n} \otimes \partial_{\nu_1}\cdots\partial_{\nu_n}\alpha)\\
&= \alpha_{1_{\mathcal{F}}} \otimes \alpha_{2_{\mathcal{F}}}
\end{aligned}
$$

where in the convenient notation $\alpha_{1_{\mathcal{F}}} \otimes \alpha_{2_{\mathcal{F}}}$ sum over n is understood.

Since $\mathcal{F}$ satisfies all the requirements for a twist, we obtain that the universal enveloping algebra generated by derivatives ∂_ν and gauge parameters $\alpha(x) = \alpha(x)^a T^a$ has been equipped with a new coproduct, the twisted coproduct $\Delta_{\mathcal{F}}$. This coproduct defines a new Hopf algebra.

We now consider the noncommutative action of α and ∂_ν on fields and on $\star$-products of fields. The noncommutative action of partial derivatives is undeformed,

see (2.5). Our noncommutative gauge principle is implemented by defining noncommutative gauge transformations as

$$\hat{\delta}_\alpha \psi = i X^\star_{\alpha^a} \star T^a \psi = i X^\star_\alpha \star \psi = i\alpha \cdot \psi \qquad (A.5)$$

and

$$
\begin{aligned}
\hat{\delta}_\alpha(\phi \star \psi) &= \mu_\star\{(\hat{\delta}_{\alpha_{1_\mathcal{F}}}\phi) \otimes (\hat{\delta}_{\alpha_{2_\mathcal{F}}}\psi)\}, \\
&= \mu_\star\{\sum_n \left(\frac{-i}{2}\right)^n \frac{\theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}}{n!}(\partial_{\mu_1}\cdots\partial_{\mu_n}\alpha)\phi \otimes (\partial_{\nu_1}\cdots\partial_{\nu_n}\psi) \\
&\qquad\qquad\qquad + (\partial_{\mu_1}\cdots\partial_{\mu_n}\phi) \otimes (\partial_{\nu_1}\cdots\partial_{\nu_n}\alpha)\psi\} \\
&= i X^\star_{\alpha^a} \star \left((T^a\phi) \star \psi + \phi \star (T^a\psi)\right). \qquad (A.6)
\end{aligned}
$$

where $\mu_\star$ is defined in (2.2). The Hopf algebra structure obtained with the twisted coproduct $\Delta_\mathcal{F}$ insures the consistency of the noncommutative gauge transformation $\hat{\delta}_\alpha$ on $\star$-products of fields.

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