JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 83 (1997) 127–130

Letter

Remarkable matrices and trigonometric identities

F. Calogero[a,b,*,1]

[a]*Dipartimento di Fisica, Università di Roma "La Sapienza", P. Aldo Moro, 1-00185 Roma, Italy*
[b]*Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy*

Received 10 April 1996; received in revised form 25 March 1997

Abstract

Two $(n \times n)$-matrices are exhibited, which have a simple expression in terms of trigonometric functions of n arbitrary angles and possess remarkably neat spectral properties, such as integral eigenvalues. Several related trigonometric identities are also exhibited.

Keywords: Matrices; Identities

AMS classification: 15A24; 33A10; 02.10.Sp; 02.30.Lt

1. Introduction and results

The purpose and scope of this paper is to exhibit two $(n \times n)$-matrices which have a simple expression in terms of trigonometric functions of n arbitrary angles and possess remarkably neat spectral properties, such as integral eigenvalues; these results might be useful to test the accuracy of computer codes which evaluate the eigenvalues of $(n \times n)$-matrices. Several related trigonometric identities are also exhibited.

In the following, the quantities θ_j, $j = 1, 2, \ldots, n$, are n *arbitrary* numbers (possibly complex), different from each other mod(π):

$$\theta_j \neq \theta_k \,\mathrm{mod}(\pi);\ j = 1, 2, \ldots, n;\ j \neq k. \tag{1}$$

* Correspondence address: Dipartimento di Fisica, Università di Roma "La Sapienza", P. Aldo Moro, 1-00185 Roma, Italy. E-mail. calogero@roma1.infn.it.

[1] On leave while serving as Secretary General, Pugwash Conferences on Science and World Affairs, Geneva, London, Rome.

PII S0377-0427(97)00101-5

Hereafter $(n \times n)$-matrices are denoted by upper-case boldface characters, and n-vectors by lower-case boldface characters; while n is an *arbitrary* positive integer, $n \geqslant 2$.

Proposition 1. *The* $(n \times n)$*-matrix* $\boldsymbol{C}$,

$$C_{jk} = \mathrm{i} \sum_{m=1, m \neq j}^{n} \operatorname{cotg}(\theta_j - \theta_m) \quad \text{if } j = k$$

$$C_{jk} = \mathrm{i}/\sin(\theta_j - \theta_k) \quad \text{if } j \neq k, \tag{1.2a}$$

has the n eigenvalues $\pm(n-1), \quad \pm(n-3), \quad \pm(n-5), \ldots, \pm 1$ or 0:

$$\boldsymbol{C}\boldsymbol{u}^{(m)} = m\boldsymbol{u}^{(m)}, \quad m = \pm(n-1), \quad \pm(n-3), \quad \pm(n-5), \ldots, \pm 1 \text{ or } 0, \tag{1.2b}$$

$$u_j^{(m)} = \exp(-\mathrm{i}m\theta_j) \Big/ \prod_{k=1, k \neq j}^{n} \sin(\theta_j - \theta_k). \tag{1.2c}$$

Throughout this paper i stands of course for the square root of -1.

Proposition 2. *The* $(n \times n)$*-matrix* $\boldsymbol{M}$,

$$M_{jk} = -\sum_{m=1, m \neq j}^{n} \cos\theta_j \sin\theta_m / \sin(\theta_j - \theta_m) \quad \text{if } j = k$$

$$M_{jk} = -\cos\theta_j \sin\theta_j / \sin(\theta_j - \theta_k) \quad \text{if } j \neq k, \tag{1.3a}$$

has the n eigenvalues $0, 1, 2, \ldots, n-1$:

$$\boldsymbol{M}\boldsymbol{v}^{(m)} = (m-1)\boldsymbol{v}^{(m)}, \quad m = 1, 2, \ldots, n, \tag{1.3b}$$

$$v_j^{(m)} = (\cos\theta_j)^{m-1}(\sin\theta_j)^{n-m} \Big/ \prod_{k=1, k \neq j}^{n} \sin(\theta_j - \theta_k). \tag{1.3c}$$

Remark 1. Note the possibility to generalize/reformulate these results by replacing the arbitrary quantities θ_j with $\theta_j + \theta$ (and then perhaps by setting $\theta = \pi/2$ to get a neater result). Also, note the "isospectral character" of the matrices $\boldsymbol{C}$ and $\boldsymbol{M}$, as manifested by the independence of their spectra from the n parameters θ_j.

Proposition 3. *In addition to the trigonometric identities implied by* (1.2b) *with* (1.2a, c) *and by* (1.3b) *with* (1.3a, c), *there also hold the following sum rules*:

$$\sum_{j,k,m=1, j \neq k, k \neq m, m \neq j}^{n} \operatorname{cotg}(\theta_j - \theta_k) \operatorname{cotg}(\theta_j - \theta_m) = -\tfrac{1}{3}n(n-1)(n-2), \tag{1.4a}$$

$$\sum_{j,k,m=1, j \neq k, k \neq m, m \neq j}^{n} \cos^2\theta_j \sin\theta_k \sin\theta_m / [\sin(\theta_j - \theta_k)\sin(\theta_j - \theta_m)] = \tfrac{1}{3}n(n-1)(n-2), \tag{1.4b}$$

$$\sum_{j,k,m=1,j\neq k,k\neq m,m\neq j}^{n} \sin(2\theta_j)\sin(\theta_k+\theta_m)/[\sin(\theta_j-\theta_k)\sin(\theta_j-\theta_m)] = -\tfrac{2}{3}n(n-1)(n-2), \tag{1.4c}$$

$$\sum_{j,k,m=1,j\neq k,k\neq m,m\neq j}^{n} \sin(\theta_j+\theta_k)\sin(\theta_j+\theta_m)/[\sin(\theta_j-\theta_k)\sin(\theta_j-\theta_m)] = \tfrac{1}{3}n(n-1)(n-2), \tag{1.4d}$$

$$\sum_{j,k,m=1,j\neq k,k\neq m,m\neq j}^{n} \sin(2\theta_j+\theta_k+\theta_m)/[\sin(\theta_j-\theta_k)\sin(\theta_j-\theta_m)] = 0. \tag{1.4e}$$

Remark 2. Other identities can be obtained from these by replacing θ_j with $\theta_j + \theta$ (perhaps with $\theta = \pi/2$ or $\theta = \pi/4$ or $\theta = \pi/8$). In this manner one can, for instance, replace the cosines with sines and the sines with cosines, in the numerators on the left-hand sides of (1.4b)–(1.4e). And, of course, other identities may be obtained by variously combining these.

2. Proofs and comments

The results of Proposition 1 are not new [2]. Analogous (but somewhat less neat) results exist for the matrix which has the same diagonal elements as $\boldsymbol{C}$, but has the off-diagonal elements $\mathrm{i}\,\mathrm{cotg}(\theta_j - \theta_k)$ instead of $\mathrm{i}/\sin(\theta_j - \theta_k)$ (see (1.2a)) [1].

The results of Proposition 2 are easily obtained from those of Section 5 of [3], via the following positions: $a = 1$, $b = 0$, $x_j = \cos\theta_j$, $y_j = \sin\theta_j$.

The trigonometric identity (1.4a) can be proven by evaluating the trace of $\boldsymbol{C}^2$, using Proposition 1. A more direct proof goes as follows [4]. Let

$$c = \sum_{j,k,m=1,j\neq k,k\neq m,m\neq j}^{n} \mathrm{cotg}(\theta_j-\theta_k)\,\mathrm{cotg}(\theta_j-\theta_m). \tag{2.1}$$

Then use the trigonometric identity

$$\mathrm{cotg}\,\alpha\,\mathrm{cotg}\,\beta = -1 - (\mathrm{cotg}\alpha - \mathrm{cotg}\beta)\,\mathrm{cotg}(\alpha-\beta) \tag{2.2}$$

to get (using the dummy character of the indices j, k, m)

$$c = \sum_{j,k,m=1,j\neq k,k\neq m,m\neq j}^{n} \{-1 - [\mathrm{cotg}(\theta_j-\theta_k) - \mathrm{cotg}(\theta_j-\theta_m)]\,\mathrm{cotg}(\theta_m-\theta_k)\} \tag{2.3a}$$

$$= -n(n-1)(n-2) - 2c \tag{2.3b}$$

This clearly implies (1.4a). □

The trigonometric identity (1.4b) can be proven by evaluating the trace of $\boldsymbol{M}^2$, using the results of Proposition 2. The diligent reader will have no difficulty to work out the details of the proof.

To prove (1.4c), we first use in its left-hand side the trigonometric identity

$$\sin(\theta_j+\theta_k)\sin(\theta_j+\theta_m) = \cos^2\theta_j\sin\theta_k\sin\theta_m + \sin^2\theta_j\cos\theta_k\cos\theta_m + \tfrac{1}{2}\sin(2\theta_j)\sin(\theta_k+\theta_m), \tag{2.5}$$

and then use (1.4b), its analogue with cosines and sines in the numerator interchanged (see the Remark 2), and (1.4c).

Finally, to prove (1.4e), we replace θ_j by $\theta_j + \delta$ in (1.4d) and expand in δ, namely we use the formula

$$\sin(\theta_j + \theta_k + \delta)\sin(\theta_j + \theta_m + \delta) = \sin(\theta_j + \theta_k)\sin(\theta_j + \theta_m) + \delta\sin(2\theta_j + \theta_k + \theta_m) + \mathrm{O}(\delta^2). \tag{2.6}$$

Insertion of this formula in (1.4d) yields (1.4e). □

Other trigonometric identities can be obtained by evaluating the trace of $\boldsymbol{C}^p$ or $\boldsymbol{M}^p$, with $p = 3, 4, \ldots$ This is left as an exercise for the diligent reader.

We finally note that the formulae written above become particularly neat for the special choice $\theta_j = 2\pi j/n$, $j = 1, 2, \ldots, n$; some of the corresponding identities for trigonometric functions of "rational angles" had been already noted in [5] and reported in [6].

Additional Note. The matrices $\boldsymbol{C}$ and $\boldsymbol{M}$ have been used by a referee in order to test some standard computer programs to evaluate the eigenvalues of matrices. The following statement and "warning" are quoted *verbatim* from the relevant referee report: "from extensive numerical experimenting it follows that indeed the results in the paper can be used to test computer codes for the calculation of eigenvalues and eigenvectors. If a computer code is able to obtain the eigenvalues of both the $(n \times n)$-matrices $\boldsymbol{C}$ and $\boldsymbol{M}$ for large values of n, one can conclude that this code is very good," "calculating the eigenvalues of the $(n \times n)$-matrix $\boldsymbol{M}$ for n larger than 80 is a very severe test." The second sentence is justified by having found that all the computer programs tested gave rise to considerable loss of accuracy in computing the intermediate eigenvalues of the matrix $\boldsymbol{M}$ for $n = 80$ (with the choice $\vartheta_j = j$). It is a pleasure to thank this unknown referee for the extra attention given to our paper and for suggesting and permitting that these findings, including the warning mentioned above, be brought to the attention of the readers of JCAM.

References

[1] F. Calogero, Interpolation, differentiation and solution of eigenvalue problems for periodic functions, Lett. Nuovo Cimento 39 (1984) 305–511.

[2] F. Calogero, Interpolation and differentiation for periodic functions, Lett. Nuovo Cimento 42 (1985) 106–110.

[3] F. Calogero, Interpolation in multidimensions, a convenient finite-dimensional matrix representation of the (partial) differential operators, and some applications, J. Math. Phys. 34 (1993) 4704–4724.

[4] F. Calogero, A.M. Perelomov, Properties of certain matrices related to the equilibrium configuration of the one-dimensional many-body problems with the pair potentials $V_1(x) = -\log|\sin x|$ and $V_2(x) = 1/\sin^2 x$, Commun. Math. Phys. 59 (1978) 109–116.

[5] F. Calogero, A.M. Perelomov, Some diophantine relations involving circular functions of rational angles, Linear Algebra Appl. 25 (1979) 91–94.

[6] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Section 15.823, Academic Press, 1980, pp. 1122–1123.