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On the cardinality of n-Urysohn and n-Hausdorff spaces

Research Article

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Abstract: Two variations of Arhangelskii's inequality $|X| \leq 2^{\chi(X)\cdot L(X)}$ for Hausdorff X [Arhangel'skii A.V., The power of bicompacta with first axiom of countability, Dokl. Akad. Nauk SSSR, 1969, 187, 967–970 (in Russian)] given in [Stavrova D.N., Separation pseudocharacter and the cardinality of topological spaces, Topology Proc., 2000, 25(Summer), 333–343] are extended to the classes with finite Urysohn number or finite Hausdorff number.

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Keywords: Urysohn number of a space • Hausdorff number of a space • $\mathrm{cl}^{\mathcal{H}}$-operator • θ-closure • $\mathrm{cl}_\theta^{\mathcal{H}}$-operator • Relative Lindelöf number • Almost Lindelöf degree of a space

Dedicated to Mikhail (Misha) Matveev

1. Introduction

In [4] the Hausdorff number (finite or infinite) $\mathrm{H}(X)$ of a topological space X is defined as the smallest cardinal τ such that for every subset $A \subset X$, $|A| \geq \tau$, there exist neighborhoods U_a, $a \in A$, such that $\bigcap_{a\in A} U_a = \emptyset$. A space X is said to be n-Hausdorff if $\mathrm{H}(X) = n$ (where $n \geq 2$ is finite). Of course, a space is 2-Hausdorff iff it is Hausdorff. For every finite n, n-Hausdorff implies $(n+1)$-Hausdorff, but there are $(n+1)$-Hausdorff spaces which are not n-Hausdorff [4]. The notion of Hausdorff number was also used in [10].

A space X is Urysohn if for any $a, b \in X$ with $a \neq b$ there exist neighborhoods $U_a \ni a$ and $U_b \ni b$ such that $\overline{U}_a \cap \overline{U}_b = \emptyset$. In [6] the Urysohn number (finite or infinite) $\mathrm{U}(X)$ was introduced as the smallest cardinal τ such that for every subset $A \subset X$ such that $|A| \geq \tau$ one can pick neighborhoods $U_a \ni a$ for all $a \in A$ so that $\bigcap_{a\in A} \overline{U}_a = \emptyset$. A space X is n-Urysohn (where $n \geq 2$ is finite) if $\mathrm{U}(X) = n$; of course, a space is 2-Urysohn iff it is Urysohn. The notion of Urysohn number was also used in [5–10].

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We use standard notation and terminology following [11, 13]: κ denotes infinite cardinal; ω is the first infinite cardinal, $L(X)$, $\psi(X)$, $\psi_c(X)$, $\chi(X)$ denote the Lindelöf number of X, the pseudocharacter of X, the closed pseudocharacter of X and the character of X, respectively. Further, if A is a subset of a space X, $[A]^{\leq m}$ denotes the family of all subsets of A whose cardinality $\leq m$.

The Hausdorff pseudocharacter of X, denoted $H\psi(X)$, is the smallest κ such that for each point x there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x \neq y$, then there exist $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \emptyset$ [14]. The cardinal function $H\psi(X)$ is defined only for Hausdorff spaces X. The following holds:

$$\psi(X) \leq \psi_c(X) \leq H\psi(X) \leq \chi(X).$$

In [17] Stavrova introduced the following cardinal invariant: the Urysohn pseudocharacter of X, denoted $U\psi(X)$, is the smallest κ such that for each point x there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x \neq y$, then there exist $\alpha, \beta < \kappa$ such that $\overline{V(\alpha, x)} \cap \overline{V(\beta, y)} = \emptyset$. The cardinal function $U\psi(X)$ is defined only for Urysohn spaces X. The following holds:

$$\psi(X) \leq \psi_c(X) \leq H\psi(X) \leq U\psi(X) \leq \chi(X).$$

In [4] Bonanzinga defined the n-Hausdorff pseudocharacter of X (where $n \geq 2$ is finite), denoted n-$H\psi(X)$, as the smallest κ such that for each point x there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x_1, \dots, x_n$ are distinct points from X, then there exist $\alpha_1, \dots, \alpha_n < \kappa$ such that $\bigcap_{i=1}^n V(\alpha_i, x_i) = \emptyset$. The cardinal function n-$H\psi(X)$ is defined only for n-Hausdorff spaces X. Of course, if X is a Hausdorff space, the 2-Hausdorff pseudocharacter of X is the Hausdorff pseudocharacter of X.

The following cardinal function n-$U\psi(X)$ is defined only for n-Urysohn spaces X.

Definition 1.1.

The *n-Urysohn pseudocharacter of* X (where $n \geq 2$), denoted n-$U\psi(X)$, is the smallest κ such that for each point x there is a collection $\{V(\alpha, x) : \alpha < \kappa\}$ of open neighborhoods of x such that if $x_1, \dots, x_n$ are distinct points from X, then there exist $\alpha_1, \dots, \alpha_n < \kappa$ such that $\bigcap_{i=1}^n \overline{V(\alpha_i, x_i)} = \emptyset$.

Of course $U\psi(X) \leq k$ implies n-$U\psi(X) \leq k$, for every $n \geq 2$. For every $n \geq 2$, $\omega \cup \{p\}$, $p \in \omega^*$, is a countable n-Urysohn space such that n-$U\psi(X) = \omega$ which is not first countable.

In [5] an $(n+1)$-Urysohn space which is not n-Urysohn, $n \geq 2$, is constructed. The same space can be used to construct the following example.

Example 1.2.

There is an $(n+1)$-Urysohn space such that $(n+1)$-$U\psi(X) = \omega$ and n-$U\psi(X)$ is not defined.

Proof. Denote $\mathbb{R}^n_0 = \{x = \langle x_1, \dots, x_n\rangle \in \mathbb{R}^n : x_1 = 0\}$. Put $X(n) = X_0 \cup X_+$, where $X_0 = \mathbb{R}^n_0 \cap \mathbb{Q}^n$ and $X_+ = \{x = \langle x_1, \dots, x_n\rangle \in \mathbb{Q}^n : x_1 > 0\}$. Pick $(n-1)$-dimensional hyperplanes $\pi_1, \dots, \pi_n \in \mathbb{R}^n$ such that (1) each of $\pi_1, \dots, \pi_n$ contains the origin, (2) normal vectors to $\pi_1, \dots, \pi_n$ are linearly independent, and (3) for every $x \in X(n)$ and $i \in \{1, \dots, n\}$, $(x + \pi_i) \cap X(n) = \{x\}$. Partition X_0 into n pairwise disjoint dense (in the Euclidean topology) subspaces X^i_0, $i \in \{1, \dots, n\}$.

Topologize $X(n)$ as follows. X_0 is open. Points of X^i_0 have neighborhoods as in the restriction of the Euclidean topology of $\mathbb{R}^n$ to X^i_0. Let $x = \langle x_1, \dots, x_n\rangle \in X_+$. For $i \in \{1, \dots, n\}$ and $\epsilon > 0$, denote

$$l_i(x) = (x + \pi_i) \cap \mathbb{R}^n_0, \quad i = 1, \dots, n, \qquad U_{i,\epsilon}(x) = \{q \in X^i_0 : d(q, l_i(x)) < \epsilon\},$$

where d is the Euclidean distance. The sets $U_\epsilon(x) = \{x\} \cup \{U_{i,\epsilon}(x) : 1 \leq i \leq n\}$ are basic neighborhoods of x. Since X is a first countable $(n+1)$-Urysohn space, we have that $(n+1)$-$U\psi(X) = \omega$. □

The following definition is a paraphrase and combination of [17, Definitions 2, 3, 6].

Definition 1.3.
Let X be a Hausdorff space (Urysohn space, resp.) and for each $x \in X$ let $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ be a collection of open neighborhoods of x which is closed under finite intersection and such that if $x \neq y$, there exist $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \emptyset$, resp. $\overline{V(\alpha, x)} \cap \overline{V(\beta, y)} = \emptyset$. Then we say that $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ *canonically realizes* $\mathrm{H}\psi(X) \leq \kappa$ (*realizes* $\mathrm{U}\psi(X) \leq \kappa$, resp.).

We can introduce the following generalization of the previous definition.

Definition 1.4.
Let $n \geq 2$, X be an n-Hausdorff space (n-Urysohn space, resp.) and for each $x \in X$ let $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ be a collection of open neighborhoods of x which is closed under finite intersection and such that if $x_1, \dots, x_n$ are distinct points from X, then there exist $\alpha_1, \dots, \alpha_n < \kappa$ such that $\bigcap_{i=1}^{n} V(\alpha_i, x_i) = \emptyset$, resp. $\bigcap_{i=1}^{n} \overline{V(\alpha_i, x_i)} = \emptyset$. Then we say that $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ *canonically realizes* n-$\mathrm{H}\psi(X) \leq \kappa$ ($\mathcal{H}$ *realizes* n-$\mathrm{U}\psi(X) \leq \kappa$, resp.).

Many variations of Arhangelskii's inequality $|X| \leq 2^{\chi(X) \cdot L(X)}$ for Hausdorff X [1] are known in the literature (see [15] for a survey). Some of these variations involve certain cardinal functions that are a priori less than or equal to the character and Lindelöf number.

The almost Lindelöf degree of X is $\mathrm{aL}(X) = \min\{\kappa : \text{for every open cover } \mathcal{U} \text{ of } X, \text{ there is a subfamily } \mathcal{U}_0 \subset \mathcal{U} \text{ such that } |\mathcal{U}_0| \leq \kappa \text{ and } \bigcup\{\overline{U} : U \in \mathcal{U}_0\} = X\}$ (see [15, 17]; in [3, 19] this function is denoted as $\mathrm{aL}(X, X)$). In [3] Bella and Cammaroto proved that if X is a Urysohn space, then $|X| \leq 2^{\mathrm{aL}(X) \cdot \chi(X)}$. In [17] Stavrova strengthen this result replacing $\chi(X)$ with $\mathrm{U}\psi(X)$.

Theorem 1.5 ([17]).
If X is a Urysohn space, then $|X| \leq 2^{\mathrm{aL}(X) \cdot \mathrm{U}\psi(X)}$.

Recently, relative versions of cardinal functions were considered by many authors (see, for example [2, 12, 16]). In [12] the following cardinal function was introduced: if Y is a subset of a space X, $\mathrm{L}(Y, X)$ is the minimum κ such that for every open cover $\mathcal{U}$ of X there exists a subfamily $\mathcal{U}_0 \subset \mathcal{U}$ covering Y such that $|\mathcal{U}_0| \leq \kappa$. For $Y = X$ this reduces to $\mathrm{L}(X)$.

Theorem 1.6 ([17]).
If X is a Hausdorff space and $Y \subseteq X$, then $|Y| \leq 2^{\mathrm{L}(Y,X) \cdot \mathrm{H}\psi(X)}$.

In this paper we extend Theorems 1.5 and 1.6 to the class of spaces with finite Urysohn number and finite Hausdorff number respectively.

2. Cardinality of $\mathrm{cl}^{\mathcal{H}}(\ )$ and $\mathrm{cl}_\theta^{\mathcal{H}}(\ )$

Recall that the θ-closure of a set A in the space X is the set $\mathrm{cl}_\theta(A) = \{x \in X : \text{for every neighborhood } U \ni x, \overline{U} \cap A \neq \emptyset\}$ [18]. A set A is called θ-closed if $A = \mathrm{cl}_\theta(A)$.

Let X be a space. Consider the family $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ where, for every $x \in X$, $\mathcal{H}(x)$ is a collection of neighborhoods of x. For every $A \subseteq X$ denote by $\mathrm{cl}^{\mathcal{H}}(A)$ the set

$$\mathrm{cl}^{\mathcal{H}}(A) = \left\{x \in X : U \cap A \neq \emptyset \text{ for every } U \in \mathcal{H}(x)\right\}$$

(see the proof of [14, Theorem II] where this set is denoted as A^* and [17, Definition 7] where it is denoted as $(A)_{\mathcal{H}}$) and by $\mathrm{cl}_\theta^{\mathcal{H}}(A)$ [17] the set

$$\mathrm{cl}_\theta^{\mathcal{H}}(A) = \left\{x \in X : \overline{U} \cap A \neq \emptyset \text{ for every } U \in \mathcal{H}(x)\right\}.$$

We have $\mathrm{cl}^{\mathcal{H}}(A) \subseteq \mathrm{cl}_\theta^{\mathcal{H}}(A)$ and $\mathrm{cl}_\theta(A) \subseteq \mathrm{cl}_\theta^{\mathcal{H}}(A)$. In [17, Lemma 2], it was proved that if A is a subset of an Urysohn space X and $\mathcal{H}$ realizes $\mathrm{U}\psi(X) \leq \kappa$, then $\left|\mathrm{cl}_\theta^{\mathcal{H}}(A)\right| \leq |A|^k$. The argument from [17, Lemma 2] needs only a slight modification to restate the result in terms of finite Hausdorff number and finite Urysohn numbers.

Proposition 2.1.

For a set X and $A \subseteq X$, if for each $x \in X$, $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ denotes a collection of subsets of X containing x which is closed under finite intersection and such that if $x_1, \dots, x_n$ are distinct points from X, then there exist $\alpha_1, \dots, \alpha_n < \kappa$ such that $\bigcap_{i=1}^n V(\alpha_i, x_i) = \emptyset$, we have that $\left|\mathrm{cl}^{\mathcal{H}}(A)\right| \leq |A|^\kappa$.

Proof. Let $x \in \mathrm{cl}^{\mathcal{H}}(A)$. Put $A_x = \{a(\alpha, x) \in V(\alpha, x) \cap A : \alpha \in \kappa\}$. Then $x \in \mathrm{cl}^{\mathcal{H}}(A_x)$. We have that $x \in \mathrm{cl}^{\mathcal{H}}(V(\alpha, x) \cap A)$ for every $\alpha \in \kappa$. Indeed, fix $\alpha \in \kappa$ and $V(\beta, x) \in \mathcal{H}$. Since $\mathcal{H}$ is closed under finite intersection, there is $\gamma \in \kappa$ such that $V(\gamma, x) = V(\beta, x) \cap V(\alpha, x)$. From $x \in \mathrm{cl}^{\mathcal{H}}(A)$ it follows that $\emptyset \neq V(\gamma, x) \cap A_x = (V(\beta, x) \cap V(\alpha, x)) \cap A_x = V(\beta, x) \cap (V(\alpha, x) \cap A_x)$. Put $\Gamma_x = \{V(\alpha, x) \cap A_x : \alpha \in \kappa\}$. Note that Γ_x is a centered family. Since $A_x \in [A]^{\leq\kappa}$, then $V(\alpha, x) \cap A_x \in [A]^{\leq\kappa}$, hence $\Gamma_x \in \left[[A]^{\leq\kappa}\right]^{\leq\kappa}$.

We claim that the mapping $x \to \Gamma_x$ is $(<n)$-to-one. Assume the contrary. Then there is a subset $K \subseteq \mathrm{cl}^{\mathcal{H}}(A)$ such that $|K| = n$ and Γ_x is the same for every $x \in K$. Call it just Γ. Pick $V(\alpha_x, x) \in \mathcal{H}$ for all $x \in K$ so that

$$\bigcap_{x \in K} V(\alpha_x, x) = \emptyset. \tag{$*$}$$

Then for every $x \in K$, $V(\alpha_x, x) \cap A_x \in \Gamma_x = \Gamma$. So by ($*$), Γ is not a centered family. On the other hand since $\Gamma = \Gamma_x$ for some x, Γ must be centered. A contradiction.

So the mapping $x \mapsto \Gamma$ from $\mathrm{cl}^{\mathcal{H}}(A)$ to $\left[[A]^{\leq\kappa}\right]^{\leq\kappa}$ is $(<n)$-to-one and thus $\left|\mathrm{cl}^{\mathcal{H}}(A)\right| \leq n \cdot |A|^\kappa = |A|^\kappa$. □

Corollary 2.2.

For a set A in a space X, if $H(X) = n \geq 2$ is finite and $\mathcal{H}$ canonically realizes n-$\mathrm{H}\psi(X) \leq \kappa$, then $\left|\mathrm{cl}^{\mathcal{H}}(A)\right| \leq |A|^\kappa$.

Proposition 2.3.

For a set X and $A \subseteq X$, if for each $x \in X$, $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ denotes a collection of subsets of X containing x which is closed under finite intersection and such that if $x_1, \dots, x_n$ are distinct points from X, then there exist $\alpha_1, \dots, \alpha_n < \kappa$ such that $\bigcap_{i=1}^n \overline{V(\alpha_i, x_i)} = \emptyset$, we have that $\left|\mathrm{cl}_\theta^{\mathcal{H}}(A)\right| \leq |A|^\kappa$.

Proof. Let $x \in \mathrm{cl}_\theta^{\mathcal{H}}(A)$. Put $A_x = \{a(\alpha, x) \in \overline{V(\alpha, x))} \cap A : \alpha \in \kappa\}$. Then $x \in \mathrm{cl}_\theta^{\mathcal{H}}(A_x)$. We have that $x \in \mathrm{cl}_\theta^{\mathcal{H}}\left(\overline{V(\alpha, x)} \cap A_x\right)$ for every $\alpha \in \kappa$. Indeed, fix $\alpha \in \kappa$ and $V(\beta, x) \in \mathcal{H}(x)$. Since $\mathcal{H}(x)$ is closed under finite intersection, there is $\gamma \in \kappa$ such that $V(\gamma, x) = V(\beta, x) \cap V(\alpha, x)$. From $x \in \mathrm{cl}_\theta^{\mathcal{H}}(A_x)$ it follows that $\emptyset \neq \overline{V(\gamma, x)} \cap A_x = \overline{V(\beta, x) \cap V(\alpha, x)} \cap A_x \subset \overline{V(\beta, x)} \cap \left(\overline{V(\alpha, x)} \cap A_x\right)$.

Put $\Gamma_x = \{\overline{V(\alpha, x)} \cap A_x : \alpha \in \kappa\}$. Note that Γ_x is a centered family. Since $A_x \in [A]^{\leq\kappa}$, then $\overline{V(\alpha, x)} \cap A_x \in [A]^{\leq\kappa}$; hence $\Gamma_x \in \left[[A]^{\leq\kappa}\right]^{\leq\kappa}$.

We claim that the mapping $x \mapsto \Gamma_x$ is $(<n)$-to-one. Assume the contrary. Then there is a subset $K \subset \mathrm{cl}_\theta^{\mathcal{H}}(A)$ such that $|K| = n$ and Γ_x is the same for all $x \in K$, call it just Γ. Pick $V(\alpha_x, x) \in \mathcal{H}(x)$ for all $x \in K$ so that

$$\bigcap_{x \in K} \overline{V(\alpha_x, x)} = \emptyset. \tag{$*$}$$

Then for every $x \in K$, $\overline{V(\alpha_x, x)} \cap A_x \in \Gamma_x = \Gamma$, so by ($*$), Γ is not centered. On the other hand, since $\Gamma = \Gamma_x$ for some x, Γ must be centered. A contradiction. So the mapping $x \mapsto \Gamma_x$ from $\mathrm{cl}_\theta^{\mathcal{H}}(A)$ to $\left[[A]^{\leq\kappa}\right]^{\leq\kappa}$ is $(<n)$-to-one, and thus $\left|\mathrm{cl}_\theta^{\mathcal{H}}(A)\right| \leq n \cdot (|A|^\kappa)^\kappa = |A|^\kappa$. □

Corollary 2.4.

For a set A in a space X, if $\mathrm{U}(X) = n$ (where $n \geq 2$ is finite) and $\mathcal{H}$ realizes n-$\mathrm{U}\psi(X) \leq \kappa$, then $\left|\mathrm{cl}_\theta^{\mathcal{H}}(A)\right| \leq |A|^\kappa$.

3. On cardinality of n-Hausdorff and n-Urysohn spaces, where $n \geq 2$ is finite

Now we show that Hodel's proof of [15, Theorem 3.3] works in case of *n-Hausdorff condition* instead of *Hausdorff condition*.

Theorem 3.1.

Let $n \geq 2$ be finite, X be a set, $Y \subseteq X$ and for each $x \in X$, $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ be a collection of subsets of X containing x which is closed under finite intersection. Assume the following:

(n-H) *if $x_1, \dots, x_n \in X$ are distinct, then there exist $\alpha_1, \dots, \alpha_n < \kappa$ such that $V(\alpha_1, x_1) \cap \dots \cap V(\alpha_n, x_n) = \emptyset$ (**n-Hausdorff condition**);*

(C) *for every function $f\colon X \to \kappa$, there exists $A \subseteq X$ with $|A| \leq \kappa$ such that $Y \subseteq \bigcup_{x \in A} V(f(x), x)$ (**cover condition**).*

Then $|Y| \leq 2^\kappa$.

Proof. By transfinite induction we shall define a family $\{H_\alpha : \alpha \in \kappa^+\}$ of subsets of X such that:

1. $|H_\alpha| \leq 2^\kappa$ for every $\alpha \in \kappa^+$.
2. For all $A \subseteq \bigcup_{\beta<\alpha} H_\beta$ such that $|A| \leq \kappa$,

   2a. $\mathrm{cl}^{\mathcal{H}}(A) \subseteq H_\alpha$,

   2b. if $f\colon A \to \kappa$ is a function, $W = \bigcup_{x \in A} V(f(x), x)$ and $Y \setminus W \neq \emptyset$, then $(H_\alpha \cap Y) \setminus W \neq \emptyset$.

Let $\alpha \in \kappa^+$ and $\{H_\beta : \beta \in \alpha\}$ already defined with properties 1.–2. Let

$$\mathcal{E}_\alpha = \left\{ \bigcup_{x \in A} V(f(x), x) \;:\; A \subseteq \bigcup_{\gamma<\beta} H_\gamma,\ |A| \leq \kappa,\ f\colon A \to \kappa,\ Y \setminus \bigcup_{x \in A} V(f(x), x) \neq \emptyset \right\}.$$

It easily follows that $|\mathcal{E}_\alpha| \leq 2^\kappa$ as $A \in \left[\bigcup_{\gamma<\beta} H_\gamma\right]^{\leq\kappa}$, $\left|\left[\bigcup_{\gamma<\beta} H_\gamma\right]^{\leq\kappa}\right| \leq 2^\kappa$, and $\left|\kappa^A\right| \leq 2^\kappa$.

For every $W \in \mathcal{E}_\alpha$, we choose a point $y_W \in Y \setminus W$ and let $\mathcal{C}_\alpha = \{y_W : W \in \mathcal{E}_\alpha\}$. Since $|\mathcal{E}_\alpha| \leq 2^\kappa$ we have that $|\mathcal{C}_\alpha| \leq 2^\kappa$. Finally put $H_\alpha = \left\{\mathrm{cl}^{\mathcal{H}}(\mathcal{C}_\alpha) \cup \bigcup\{\mathrm{cl}^{\mathcal{H}}(H_\beta) : \beta \in \alpha\}\right\}$. Using Proposition 2.1 we obtain that $|H_\alpha| \leq 2^\kappa$. It can be easily to see that properties 1.–2. are satisfied.

Let $H = \bigcup\{H_\alpha : \alpha \in \kappa^+\}$. Clearly $|H| \leq 2^\kappa$. Also $\mathrm{cl}^{\mathcal{H}}(H) = H$. To prove this it is sufficient to show that $\mathrm{cl}^{\mathcal{H}}(H) \subseteq H$. Let $x \in \mathrm{cl}^{\mathcal{H}}(H)$. For each $\gamma \in \kappa$ there exist $x_\gamma \in V(\gamma, x) \cap H$. By regularity of κ^+, there exists $\alpha < \kappa^+$ such that $\{x_\gamma : \gamma < \kappa\} \subset \bigcup_{\beta<\alpha} \mathrm{cl}^{\mathcal{H}}(H_\beta)$. Now $\bigcup_{\beta<\alpha} \mathrm{cl}^{\mathcal{H}}(H_\beta) \subset H_\alpha$ and so $V(\gamma, x) \cap H_\alpha \neq \emptyset$ for all $\gamma < \kappa$. It follows that $x \in \mathrm{cl}^{\mathcal{H}}(H_\alpha)$, hence $x \in H$.

It remains to prove that $Y \subseteq H$. Suppose there is $q \in Y \setminus H$. By Proposition 2.1, $\mathrm{cl}^{\mathcal{H}}(\{q\}) = \{q\}$. Then for every $x \in H$ we can choose $\gamma_x < \kappa$ such that $q \notin V(\gamma_x, x)$. From the other side for every $x \notin H = \mathrm{cl}^{\mathcal{H}}(H)$ we can choose $\gamma_x < \kappa$ such that $V(\gamma_x, x) \cap H = \emptyset$. Define $f\colon X \to \kappa$ by $f(x) = \gamma_x$. By the cover condition (C), there exists $B \subseteq X$ with $|B| \leq \kappa$ such that $Y \subseteq \bigcup_{x \in B} V(f(x), x)$. Put $A = B \cap H$. Then $A \subseteq H$ and $|A| \leq \kappa$. Further $\{V(f(x), x) : x \in A\}$ covers $H \cap Y$. Let $W = \bigcup\{V(f(x), x) : x \in A\}$. Note that $H \cap Y \subseteq W$ and $q \in Y \setminus W$. By regularity of κ^+, there exists $\alpha < \kappa^+$ such that $A \subseteq \bigcup_{\beta<\alpha} H_\beta$. By 2b., there exists $z \in (H_\alpha \cap Y) \setminus W$; a contradiction with $H \cap Y \subseteq W$. □

The next three results are consequences of Theorem 3.1. In particular, the following result is a generalization of Stavrova's result presented in Theorem 3.4 in terms of n-Hausdorff spaces. The proof shows that Hodel's proof of [15, Corollary 3.4] works in the case of n-Hausdorff spaces.

Theorem 3.2.
If X is an n-Hausdorff space, $n \geq 2$, and $Y \subseteq X$, then $|Y| \leq 2^{\mathrm{L}(Y,X)\cdot n\text{-}\mathrm{H}\psi(X)}$.

Proof. Let X be an n-Hausdorff space with $\mathrm{L}(Y,X)\cdot n\text{-}\mathrm{H}\psi(X) \leq \kappa$ and let $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ be a family canonically realizing $n\text{-}\mathrm{H}\psi(X) \leq \kappa$, where for every $x \in X$, $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ is a collection of open neighborhoods of x which is closed under finite intersection. Of course, V satisfies condition (n-H) of Theorem 3.1. To check that V satisfies the cover condition (C) of Theorem 3.1 let $f : X \to \kappa$ be a function. Then $\{V(f(x), x) : x \in X\}$ is an open cover of X. Since $\mathrm{L}(Y,X) \leq \kappa$ there is $A \subset X$ such that $|A| \leq \kappa$ and $\bigcup\{V(f(x), x) : x \in A\} \supseteq Y$. So, by Theorem 3.1, $|Y| \leq 2^\kappa$. □

In [15, Corollary 3.4], Hodel gives a relative version of Theorem 3.3. Recall the following relative version of aL. Let X be a space and let $Y \subseteq X$. The cardinal function $\mathrm{aL}(Y,X)$ is the smallest κ such that if $\mathcal{U}$ is an open cover of X, then there exists $\mathcal{U}_0 \subset \mathcal{U}$ such that $|\mathcal{U}_0| \leq \kappa$ and $\bigcup\{\overline{U} : U \in \mathcal{U}_0\} \supseteq Y$. For $Y = X$ this becomes $\mathrm{aL}(X)$. We have the following result.

Theorem 3.3.
If X is an n-Urysohn space, where $n \geq 2$ is finite, and $Y \subseteq X$, then $|Y| \leq 2^{\mathrm{aL}(Y,X)\cdot n\text{-}\mathrm{U}\psi(X)}$.

Proof. Let X be a n-Urysohn space with $\mathrm{aL}(Y,X)\cdot n\text{-}\mathrm{U}\psi(X) \leq \kappa$ and $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ be a family realizing $n\text{-}\mathrm{U}\psi(X) \leq \kappa$, where for every $x \in X$, $\mathcal{H}(x) = \{W(\alpha, x) : \alpha < \kappa\}$ is a collection of open neighborhoods of x which is closed under finite intersection. For $x \in X$ and for every $\alpha < \kappa$, put $V(\alpha, x) = \overline{W(\alpha, x)}$. Of course, V satisfies condition (n-H) of Theorem 3.1. To check that V satisfies the cover condition (C) of Theorem 3.1 let $f : X \to \kappa$ be a function. Then $\{W(f(x), x) : x \in X\}$ is an open cover of X. Since $\mathrm{aL}(Y,X) \leq \kappa$ there is $A \subset X$ such that $|A| \leq \kappa$ and $\bigcup\{\overline{W(f(x), x)} : x \in A\} \supseteq Y$, in other words, $\bigcup\{V(f(x), x) : x \in A\} \supseteq Y$. Then, $|Y| \leq 2^\kappa$. □

The next result represents a generalization of Stavrova's result presented in Theorem 3.3.

Theorem 3.4.
If X is an n-Urysohn space, where $n \geq 2$ is finite, then $|X| \leq 2^{\mathrm{aL}(X)\cdot n\text{-}\mathrm{U}\psi(X)}$.

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