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A boson approach to the structure of A=22 nuclei

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Abstract: We discuss a procedure to transfer the description of a fermion system from a subspace of the full shell model space built in terms of collective pairs onto a space of corresponding bosons. We apply the procedure to systems of six nucleons in the $1s0d$ major shell. We perform exact shell model calculations and compare them with calculations in the collective pair and boson approximations. The effects of the truncation of the boson Hamiltonian and of the consequent violation of the Pauli principle are examined.

Keywords: shell model, collective pair approximation, boson approximation, boson mapping of fermion systems, boson Hamiltonian, energy spectra of A=22 nuclei
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1 Introduction

The description of low-lying collective excitations in terms of the spherical shell model is still a central problem in the nuclear structure theory. The dimension of the shell-model space increases rapidly for the number of valence nucleons, becoming prohibitive

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for medium-heavy nuclei, and therefore making the shell-model calculations unfeasible. For instance, in relatively simple cases of medium weight nuclei weighing $10^{14} - 10^{18}$, shell model configurations are involved [1, 2]. Even if we could perform calculations in such a huge space the results would only have a limited value since the simple and regular features observed in nuclei would remain hidden in several million of expansion coefficients of an eigenstate. Therefore, irrespectively of whether the large-scale shell model calculation is feasible or not, a realistic truncation of the shell model space has to be found.

In the last two decades, many models have been developed to describe the collective motion in medium weight nuclei. These models can generally be classified into three categories: 1) fermion models which preserve the general philosophy of the shell model but in a truncated space [3-8], 2) boson models where the space is spanned by a quite small set of bosons [9-11], 3) boson-fermion models, where states are obtained from the coupling of a fermion with a bosonic core [12-15].

Among the fermion models, the collective pair approximation (CPA) [3, 4], and its recently extended version known as the nucleon-pair shell model (NPSM) [16, 17], are of great importance. In the CPA and NPSM approaches, nucleon collective pairs with various angular momenta are used as building blocks of the truncated shell model space.

Recently, we performed calculations within the CPA formalism for both even-A [18, 19] and odd-A [20] systems. In this paper, we investigate the possibility of transferring the description of even-A systems from the CPA space, which is a fermion space described by collective pairs, onto a space described by corresponding bosons. Besides establishing a link between phenomenological boson models and microscopic approaches, this mapping mechanism opens the way to a simplified (because only bosons are involved) but still microscopic (because no free parameters exist in the boson space) description of low-lying collective excitations. Hereafter we will define this approach based on the boson mapping as boson approximation (BA).

As a preliminary step, we will discuss a procedure to map all fermion operators of interest onto the boson space. As we will see more in detail below, in correspondence to a generic fermion operator, one can always construct a n-body boson operator which, for a system of n pairs, provides an exact image of the fermion operator in the boson space.

Such a boson operator therefore carries all information relating to the Pauli Exclusion Principle, which are missing in the definition of the boson space. Truncating this boson operator at a lower order inevitably introduces a violation of this principle whose effects are difficult to predict. Among the motivations which have inspired this work there is just that of investigating in detail how the truncation of the boson Hamiltonian at two-body terms affects the spectrum of a three boson system. Such a simplified Hamiltonian is suggested by practical reasons since the use of more sophisticated Hamiltonians, although desirable in principle, would make calculations rather complicated so making the advantages of the BA approach vanish. Answering this question will allow us to shed some light on the effectiveness of the BA formalism in realistic systems.

We will concentrate on the analysis of $A = 22$ systems. These nuclei, with six active nucleons in the $1s0d$ major shell, provide an ideal testing ground since, in this case, exact calculations are feasible without much effort. Our previous work on these nuclei within the CPA formalism [18] has served as a guide in the definition of the collective space.

The paper is organized as follows. The CPA approach is briefly reviewed in section 2. The mapping procedure, discussed in detail in reference [21], is outlined for reader's convenience in section 3. In section 4, we discuss an application of this procedure within the CPA formalism. Results are presented in section 5 and conclusions outlined in section 6. An Appendix is reserved for some mathematical details.

2 The collective pair approximation

In this section, we illustrate the formalism describing the spectra of nuclei with 2n ($n = 1, 2, ...$) fermions outside closed shells within the CPA.

Let us define the operator $\hat{Z}^{\dagger}_{\Gamma\Gamma'}(\lambda_1, \lambda_2)$ which creates a state of two nucleons occupying orbitals λ_1 and λ_2 and coupled to total spin and isospin angular momentum quantum numbers $\Gamma = (J, T)$ and projection $\Gamma' = (J', T')$

$$\hat{Z}^{\dagger}_{\Gamma\Gamma'}(\lambda_1\lambda_2) = (1 + \delta_{\lambda_1\lambda_2})^{-\frac{1}{2}}[a^{\dagger}(\lambda_1) \times a^{\dagger}(\lambda_2)]_{\Gamma\Gamma'}. \tag{1}$$

The operator $\hat{A}^{\dagger}_{\nu\Gamma\Gamma'}$ creating a collective pair of multipolarity Γ can be written as

$$\hat{A}^{\dagger}_{\nu\Gamma\Gamma'} = \sum_{\Gamma\Gamma'} c^{\nu}_{\Gamma}(\lambda_1\lambda_2)\hat{Z}^{\dagger}_{\Gamma\Gamma'}(\lambda_1\lambda_2), \tag{2}$$

where the index ν distinguishes different collective pairs with the same quantum numbers $\Gamma\Gamma'$. The coefficients $c^{\nu}_{\Gamma}(\lambda_1\lambda_2)$ are obtained from the diagonalization of the shell-model Hamiltonian in the complete space spanned by the two-nucleon states

$$|\lambda_1\lambda_2; \Gamma\Gamma'\rangle = \hat{Z}^{\dagger}_{\Gamma\Gamma'}(\lambda_1\lambda_2)|0\rangle. \tag{3}$$

The collective pairs defined by Eq. 2 serve as building blocks to construct a truncated shell-model space for a nucleus with $2n$ nucleons. The basis states spanning this space can be expressed as

$$|i\rangle = [\hat{A}^{\dagger}_{\nu_n\Gamma_n} \times \ldots \times [\hat{A}^{\dagger}_{\nu_3\Gamma_3} \times [\hat{A}^{\dagger}_{\nu_2\Gamma_2} \times \hat{A}^{\dagger}_{\nu_1\Gamma_1}]_{\Gamma_{12}}]_{\Gamma_{123}} \ldots]_{\Lambda\Lambda'}|0\rangle, \tag{4}$$

where square brackets indicate the order of spin-isospin angular momenta couplings, and quantum numbers $\Gamma_{12}, \Gamma_{123}, \Gamma_{123...}$ indicate intermediate spin-isospin angular momenta, while $\Lambda\Lambda'$ specify the total spin-isospin quantum numbers and their projections.

The states of Eq. 4 are neither normalized nor linearly independent. By constructing the overlap matrix $\langle i|j\rangle$ and diagonalizing it we find a new set of orthonormal states

$$|\Phi_\alpha\rangle = (N_\alpha)^{-\frac{1}{2}} \sum_{i=1}^{N} f_{i\alpha}|i\rangle, \quad \alpha = 1, 2, \ldots, \bar{N}. \tag{5}$$

In general the number $\bar{N}$ of orthonormal states $|\Phi_\alpha\rangle$, whose norm $N_\alpha > 0$, is less than the number N of states $|i\rangle$, i.e. $\bar{N} \leq N$. The states of Eq. 5 are used to diagonalize the standard shell-model Hamiltonian.

Eigenenergies and eigenstates obtained from this procedure approximate the eigenenergies and eigenstates of the shell-model Hamiltonian diagonalized in the full shell model space. The quality of the approximation depends on the set and the structure of the collective pairs, which have been introduced to expand the truncated shell model space 5.

3 The mapping procedure

In this section, assuming a formal correspondence between states spanning two vector spaces C and E, we describe a procedure to derive the image in E of a generic operator, acting within C (for the detailed discussion of this procedure see reference [21]).

Let C and E be the vector spaces, spanned by the N states $|1\rangle, |2\rangle, \ldots, |N\rangle$ and $|1), |2), \ldots, |N)$, respectively. We only assume that states of E are orthonormal, i.e.

$$(i|j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, N. \tag{6}$$

Having defined a generic operator $\hat{O}_C$, acting within C, we will search for an operator $\hat{O}_E$, acting within E, such that all eigenvalues of $\hat{O}_C$ in C are also eigenvalues of $\hat{O}_E$ in E. We will refer to $\hat{O}_E$ as the image operator of $\hat{O}_C$ in E. First, the image operator in nonhermitian form ${}^{nh}\hat{O}_E$ will be derived.

Let us construct the overlap matrix $\langle i|j\rangle$ and diagonalize it. We find N eigenstates

$$|\Phi_k\rangle = \sum_{i=1}^{N} f_{ik}|i\rangle, \quad k = 1, 2, \ldots, N. \tag{7}$$

Because of the diagonalization procedure, the coefficients f_{ik} satisfy the conditions

$$\sum_{i=1}^{N} f^*_{ij} f_{ij'} = \delta_{jj'} \quad \text{and} \quad \sum_{j=1}^{N} f^*_{ij} f_{i'j} = \delta_{ii'}. \tag{8}$$

Among the states $|\Phi_k\rangle$ of Eq. 7 only $\bar{N} \leq N$ states have a norm $N_\alpha \neq 0$ and we use them to define the orthonormal basis

$$|\Phi_\alpha\rangle = (N_\alpha)^{-\frac{1}{2}} \sum_{i=1}^{N} f_{i\alpha}|i\rangle, \quad \alpha = 1, 2, \ldots, \bar{N}. \tag{9}$$

Thus, the space C is $\bar{N}$-dimensional. The identity operator within space C is

$$\hat{I}_C = \sum_{\alpha=1}^{\bar{N}} |\Phi_\alpha\rangle\langle\Phi_\alpha| = \sum_{i,j=1}^{N} |i\rangle B(i,j)\langle j|, \tag{10}$$

where

$$B(i,j) = \sum_{\alpha=1}^{\bar{N}} (N_\alpha)^{-1} f^*_{i\alpha} f_{j\alpha}. \tag{11}$$

In general $\hat{O}_C|i\rangle \notin C$, but by defining the operator $\hat{\overline{O}}_C = \hat{I}_C\hat{O}_C$, we notice, that

$$\hat{\overline{O}}_C|l\rangle = \sum_{i=1}^{N}|i\rangle\{\sum_{j=1}^{N}B(i,j)\langle j|\hat{O}_C|l\rangle\} \in C \tag{12}$$

and

$$\langle i|\hat{\overline{O}}_C|l\rangle = \langle i|\hat{O}_C|l\rangle. \tag{13}$$

Let us now turn to the space E. Due to the orthonormality of the states $|i)$ $(i = 1, 2, \ldots, N)$, the identity operator is

$$\hat{I}_E = \sum_{i=1}^{N}|i)(i|. \tag{14}$$

Let ${}^{nh}\hat{O}_E$ be an operator acting within space E. Its action on a state $|l)$ is given by

$${}^{nh}\hat{O}_E|l) = \hat{I}_E\ {}^{nh}\hat{O}_E|l) = \sum_{i=1}^{N}|i)(i|{}^{nh}\hat{O}_E|l). \tag{15}$$

By comparing Eqs 12 and 15, one sees, that if ${}^{nh}\hat{O}_E$ is defined such that

$$(i|{}^{nh}\hat{O}_E|l) = \sum_{j=1}^{N}B(i,j)\langle j|\hat{O}_C|l\rangle, \tag{16}$$

its action on states of E is formally identical to that of $\hat{\overline{O}}_C$ on the corresponding states of C. As a result of that, if the state

$$|\Psi_\gamma\rangle = \sum_{\alpha=1}^{\bar{N}}c_{\alpha\gamma}|\Phi_\alpha\rangle = \sum_{i=1}^{N}(\sum_{\alpha=1}^{\bar{N}}(N_\alpha)^{-\frac{1}{2}}f_{i\alpha}c_{\alpha\gamma})|i\rangle = \sum_{i=1}^{N}a_{i\gamma}|i\rangle, \tag{17}$$

$$\gamma = 1, 2, \ldots, \bar{N},$$

is an eigenstate of $\hat{\overline{O}}_C$, corresponding to the eigenvalue λ_γ, then, the state

$$|\Psi_\gamma) = \sum_{i=1}^{N}a_{i\gamma}|i) \tag{18}$$

is also an eigenstate of ${}^{nh}\hat{O}_E$, with the same eigenvalue. Therefore, $\bar{N}$ of the N eigenvalues of ${}^{nh}\hat{O}_E$ in E are the same as the eigenvalues of $\hat{O}_C$ in C and the eigenstates of ${}^{nh}\hat{O}_E$ correspond to the eigenstates of $\hat{\overline{O}}_C$ in C. Thus, Eq. 16 defines the nonhermitian image operator of $\hat{O}_C$ in E.

In order to derive the hermitian form ${}^{h}\hat{O}_E$ of the image operator, let us introduce the $\bar{N}$ orthonormal states

$$|\Phi_\alpha) = \sum_{i=1}^{N}f_{i\alpha}|i), \quad \alpha = 1, 2, \ldots, \bar{N}, \tag{19}$$

corresponding to the $\bar{N}$ states of Eq. 9, spanning the space C. With the aid of states 19, we define the operators

$$\hat{B}^{\frac{1}{2}} = \sum_{\alpha=1}^{\bar{N}} |\Phi_\alpha)(N_\alpha)^{-\frac{1}{2}}(\Phi_\alpha|, \tag{20}$$

$$\hat{B}^{-\frac{1}{2}} = \sum_{\alpha=1}^{\bar{N}} |\Phi_\alpha)(N_\alpha)^{\frac{1}{2}}(\Phi_\alpha|, \tag{21}$$

which have the following properties

$$B(i,j) = (i|\hat{B}^{\frac{1}{2}}\hat{B}^{\frac{1}{2}}|j) = \sum_{\alpha=1}^{\bar{N}} f^*_{i\alpha}(N_\alpha)^{-1}f_{j\alpha}, \tag{22}$$

$$B^{\frac{1}{2}}(i,j) \equiv \left(i\middle|\hat{B}^{\frac{1}{2}}\middle|j\right) = \sum_{\alpha=1}^{\overline{N}} f^*_{i\alpha}(\mathrm{N}_\alpha)^{-\frac{1}{2}}f_{j\alpha}, \tag{23}$$

$$\hat{B}^{\frac{1}{2}}\hat{B}^{-\frac{1}{2}} = \sum_{\alpha=1}^{\overline{N}} |\Phi_\alpha)(\Phi_\alpha|. \tag{24}$$

From Eq. 24 it is evident that, if $\overline{N} = N$, $\hat{B}^{\frac{1}{2}}\hat{B}^{-\frac{1}{2}} = \hat{I}_E$ while, if $\overline{N} < N$ $\hat{B}^{\frac{1}{2}}\hat{B}^{-\frac{1}{2}}$ defines the identity operator in the subspace of E spanned by the states of Eq. 19 which correspond to states of Eq. 9 spanning the space C. If $|\Psi_\gamma)$ of Eq. 18 is an eigenstate of ${}^{nh}\hat{O}_E$ associated with the eigenvalue λ_γ then

$$\hat{B}_E^{-\frac{1}{2}}\, {}^{nh}\hat{O}_E\hat{B}_E^{\frac{1}{2}}\hat{B}_E^{-\frac{1}{2}}|\Psi_\gamma) = \lambda_\gamma\hat{B}_E^{-\frac{1}{2}}|\Psi_\gamma). \tag{25}$$

By defining

$${}^{h}\hat{O}_E \equiv \hat{B}_E^{-\frac{1}{2}}\, {}^{nh}\hat{O}_E\hat{B}_E^{\frac{1}{2}} \tag{26}$$

and

$$|\widetilde{\Psi}_\gamma) \equiv \hat{B}_E^{-\frac{1}{2}}|\Psi_\gamma), \tag{27}$$

we can rewrite Eq. 25 as

$${}^{h}\hat{O}_E|\widetilde{\Psi}_\gamma) \equiv \lambda_\gamma|\widetilde{\Psi}_\gamma), \tag{28}$$

what means that $|\ \widetilde{\Psi}_\gamma)$ is an eigenstate of ${}^{h}\hat{O}_E$ associated with the eigenvalue λ_γ. With the aid of Eqs 16, 19-24, 26 it can be proved that

$$(i|{}^{h}\hat{O}|j) = \sum_{kJ=1}^{N} B^{\frac{1}{2}}(i,l)\langle l|\hat{O}_C|k\rangle B^{\frac{1}{2}}(k,j), \tag{29}$$

from which one deduces that ${}^{h}\hat{O}_E$ is indeed hermitian. This equation defines the image operator of $\hat{O}_C$ in E in the hermitian form.

4 Boson mapping of fermion systems

Here, the mapping procedure presented in the previous section will be employed to describe nuclear systems in a space in which elementary bosons replace the collective pairs defined in Eq. 2. Let us call C^n the CPA space spanned by the states

$$|i\rangle = \hat{A}^{\dagger}_{\nu_1\Gamma_1\Gamma'_1}\hat{A}^{\dagger}_{\nu_2\Gamma_2\Gamma'_2}\ldots\hat{A}^{\dagger}_{\nu_n\Gamma_n\Gamma'_n}|0\rangle \ . \tag{30}$$

Similarly, let us call E^n the boson space spanned by the states

$$|i) = b^{\dagger}_{\nu_1\Gamma_1\Gamma'_1}b^{\dagger}_{\nu_2\Gamma_2\Gamma'_2}\ldots b^{\dagger}_{\nu_n\Gamma_n\Gamma'_n}|0) \ . \tag{31}$$

States 31 are formally obtained from states 30 by replacing pair creation operators $\hat{A}^{\dagger}_{\nu\Gamma\Gamma'}$ with boson creation operators $b^{\dagger}_{\nu\Gamma\Gamma'}$ and replacing the fermion vacuum state $|0\rangle$ with the boson vacuum state $|0)$.

Let us replace the abstract vector spaces C and E with the spaces C^n and E^n. Similarly, let a generic operator $\hat{O}_C$ acting within C be replaced with the standard shell-model Hamiltonian $\hat{H}$ acting within C^n. Then, following the mapping procedure of the previous section, we can find the image $\hat{H}_b$ of $\hat{H}$ acting within E^n. As a general result of the mapping procedure [21] we obtain

$$\hat{H}_b = \hat{H}^1_b + \hat{H}^2_b + \ldots + \hat{H}^n_b \tag{32}$$

i.e., the image $\hat{H}_b$ contains up to n-body terms even if the fermion operator $\hat{H}$ is at most two-body. In this case, by definition, its eigenvalues are the same as those of the operator $\hat{H}$. The presence of many-body terms results from the need to simulate complicated underlying nucleon exchange dynamics in the boson space. Therefore an important question is if terms of $\hat{H}_b$ higher than two-body can be considered as negligible higher order contributions when studying systems made of more than two bosons. In order to investigate this problem we have employed the image operator containing only one- and two- body terms, i.e.

$$\hat{H}_b \cong \hat{H}^1_b + \hat{H}^2_b, \tag{33}$$

where

$$\hat{H}^1_b = \sum_{\nu\Gamma\Gamma'} \varepsilon_\nu b^{\dagger}_{\nu\Gamma\Gamma'} b_{\nu\Gamma\Gamma'} \tag{34}$$

and

$$\begin{aligned}\hat{H}^2_b = \frac{1}{4} \sum_{\Gamma\Gamma'\nu_1\Gamma_1\Gamma'_1\nu_2\Gamma_2\Gamma_2}\ \sum_{'\nu_3\Gamma_3\Gamma'_3\nu_4\Gamma_4\Gamma'_4} & (1+\delta_{\nu_1\Gamma_1\nu_2\Gamma_2})(1+\delta_{\nu_3\Gamma_3\nu_4\Gamma_4}) \\ & \times E_\Gamma(\nu_1\Gamma_1\nu_2\Gamma_2;\nu_3\Gamma_3\nu_4\Gamma_4)(\Gamma_1\Gamma'_1\Gamma_2\Gamma'_2)|\Gamma\Gamma')(\Gamma_3\Gamma'_3\Gamma_4\Gamma'_4)|\Gamma\Gamma') \\ & \times b^{\dagger}_{\nu_1\Gamma_1\Gamma'_1} b^{\dagger}_{\nu_2\Gamma_2\Gamma'_2} b_{\nu_3\Gamma_3\Gamma'_3} b_{\nu_4\Gamma_4\Gamma'_4}\end{aligned} \tag{35}$$

In this work, we deal with systems of three bosons. We therefore introduce states of the form

$$|i) = |[(b^\dagger_{\nu_1\Gamma_1} \times b^\dagger_{\nu_2\Gamma_2})_{\Gamma_{12}} \times b^\dagger_{\nu_3\Gamma_3}]_{\Lambda\Lambda'}|0) \tag{36}$$

to expand three boson space. Since states 36 are neither orthogonal nor linearly independent we have first diagonalized the overlap matrix $(\bar{i}|i)$ to obtain an orthonormal set of states

$$|\alpha) = (N_\alpha)^{-\frac{1}{2}} \sum_{i=1}^{N} C_{i\alpha}|i), \qquad \alpha = 1, 2, \ldots, \overline{N} \tag{37}$$

expanding the three-boson space. The number $\overline{N}$ of states 37 having norm $N_\alpha > 0$ is, in general, less than the number N of states $|i)$. In the next step we have found the matrix representation of the boson Hamiltonian 33 in the boson space spanned by the states 37. These matrix elements can be written as

$$(\overline{\alpha}|\hat{H}_b|\alpha) = (N_\alpha N_{\overline{\alpha}})^{-\frac{1}{2}} \sum_{i\,\bar{i}} C^*_{\bar{i}\overline{\alpha}} C_{i\alpha}((\bar{i}|\hat{H}^1_b|i) + (\bar{i}|\hat{H}^2_b|i)) \tag{38}$$

The explicit expression of the matrix elements $(\bar{i}|\hat{H}^1_b|i)$ and $(\bar{i}|\hat{H}^2_b|i)$ are given in the Appendix.

5 Calculations and results

In this section, we will examine a series of calculations for systems of six nucleons in the $1s0d$ major shell (A=22 nuclei). We will compare exact shell model results for all values of total isospin T $(0 \leq T \leq 3)$ with results obtained within the CPA and BA approaches. Single-particles energies and two-body matrix elements have been taken from the work of Wildenthal [22] and have been employed to deduce, according to the mapping procedure of section 3, the one-boson energies and two-boson matrix elements of the boson Hamiltonian 33. Similarly to previous work [18-20], the collective pairs which define the CPA space have been fixed by diagonalizing the fermion Hamiltonian in the space of two-nucleon states. In the $1s0d$ major shell ($1s_{1/2}$, $0d_{3/2}$ and $0d_{5/2}$ orbits) one can form 28 two-nucleon states (14 $T = 0$ and 14 $T = 1$) with values of the total angular momentum J ranging from 0 up to 5. Following the standard notation, $T = 1$ pairs with $J = 0, 1, 2, 3, 4$ are denoted as S, P, D, F, G, respectively, while $T = 0$ pairs are denoted as Θ_J. Corresponding bosons are denoted as s, p, d, f, g and θ_J.

Before comparing the different approaches, it is appropriate to comment on the shell model results. In Table 1 we show energies and angular momenta of the lowest 10 eigenstates for each T. $T = 0$ and $T = 1$ states are in the same range of energy while large gaps occur between the $T = 1$ and $T = 2$ states as well as between the $T = 2$ and $T = 3$ states. $T = 0$ states only refer to a nucleus with an equal number of protons and neutrons. For the six particles systems under study, this means 3 protons and 3 neutrons, namely an odd-odd nucleus. The spectrum of a system with 2 protons and 4 neutrons (or viceversa), being characterized by an isospin projection $T_z = -1(+1)$, includes all eigenstates with $T = 1, 2, 3$. Due to the energy distribution evidenced in Table 1, however, we can state that the lowest eigenstates of these even-even systems are all $T = 1$ states. Therefore, it is easy to argue that all $T = 2$ states of Table 1 are the lowest eigenstates of systems with $T_z = \pm 2$ (1 proton and 5 neutrons or viceversa, i.e. odd-odd systems) while all $T = 3$ states of the same table are the lowest eigenstates of systems with $T_z = \pm 3$ (0 protons and 6 neutrons or viceversa, i.e. even-even systems). We will begin our analysis by discussing the $T = 1$ and $T = 3$ cases (even-even systems) and then proceed with the remaining cases (odd-odd systems).

$T = 1$ results are shown in figures 1 and 2. These figures refer to two different choices of the set of collective pairs (and corresponding bosons) defining the CPA and BA spaces: $S, S', D, D', G, \Theta_1, \Theta_3, \Theta_5$ (set (a)) in figure 1 and S, S', P, D, D', F, G (set (b)) in figure 2. Set (a) has already been used in our previous calculations [18].

For both sets, one observes a good agreement between the shell model (SM), and both CPA and BA results. Only at energies around 6-8 MeV, do some inversion occur in the approximate spectra (especially for set (a)) but this is, however, hard to be avoided due to the high density of states in this region. In any case, the difference between either CPA or BA results and the SM ones is never larger than about 200 KeV. We therefore conclude that both sets (a) and (b) provide a good description of the lowest $T = 1$ spectrum. From the boson point of view, this means that the Hamiltonian, which has been constructed with the procedure illustrated in the previous section, provides a good boson image of the fermion one and, therefore, that the violations of the Pauli principle that have been introduced by omitting higher order terms in this Hamiltonian have only limited effects on this spectrum.

The $T = 3$ spectrum is discussed in figure 3. In this case, only $T = 1$ pairs can contribute. Set (a) therefore reduces to S, S', D, D', G pairs only while set (b) remains unchanged. Already in this case of the former set, the CPA space for $J = 0, 2, 4$ states fully exhausts the SM space and so CPA and SM results are identical. For $J = 3$ states this is not fully true but CPA and SM results remain nevertheless very close. The identity is reached also in this case when we use set (b). In figure 3, together with the CPA spectrum (the results for sets (a) and (b) have been unified due to their undistinguishability in the figure), we show the corresponding BA calculations (BA1 for set (a) and BA2 for set (b)). The BA2 spectrum closely reproduces the CPA results with the only exception of the second $J = 4$ state which is calculated too high by about 700 KeV. This suggests that the inclusion of an additional g boson in the definition of the boson space might be appropriate. Such an effect cannot be seen in the CPA space since this already exhausts the SM space. A similar discrepancy is observed when comparing the CPA and BA1

$J = 3$ states but it disappears by turning to the set (b) (which also includes an f boson). Also for $T = 3$ states, we can therefore conclude that the overall agreement between CPA and BA results is reasonably good.

Let us now come to odd-odd systems and, in particular, to the $T = 2$ spectra. In figures 4 and 5, we show the results corresponding to sets (a) and (b), respectively. Concerning the latter case, we see that CPA results reproduce rather well the SM ones while some differences occur between the CPA and BA results. These differences become more evident in the case of set (a) (figure 4) where also the CPA results get worse with respect to set (b). Therefore, for $T = 2$, the boson approximation works less well than in the previous $T = 1$ and $T = 3$ cases. In other words, these numerical tests indicate that the BA formalism under study is more suitable for the treatment of even-even rather than odd-odd systems of the $1s0d$ shell. Such a deduction is confirmed by the analysis of the $T = 0$ results (still odd-odd). In this case, we have verified indeed that BA fails in providing a correct description of the CPA spectrum (which, in turn, also shows some discrepancies with respect to the SM one [18]).

6 Summary

In this paper, we have discussed a procedure to transfer the description of a fermion system from a subspace of the full shell model space built in terms of collective pairs onto a space of corresponding bosons. We have applied the procedure to systems of six nucleons in the $1s0d$ major shell. We have performed exact SM calculations and compared them with calculations in the CPA and BA formalism. With reference to the last ones, we have first constructed a two-body Hermitian boson Hamiltonian in correspondence to the fermion one.

CPA calculations (and the corresponding BA ones) have been made for two different sets of pairs (and bosons). The agreement between CPA and BA results has been found reasonably good for the lowest eigenstates with total isospin $T = 1$ and $T = 3$ (corresponding to even-even systems), less good for $T = 2$ and bad for $T = 0$ (the last two cases corresponding to odd-odd systems). Therefore, in these numerical tests, the BA approach has turned out to be more effective in the description of even-even systems. For these systems, the two-body boson Hamiltonian that we have constructed has provided a good boson image of the fermion one.

Of course a dependence of these results (i) on the choice of the interaction (of Wildenthal type in the present case) and (ii) on the definition of the collective pairs might be possible. These pairs have been fixed by diagonalizing our fermion Hamiltonian in a two-nucleon space and then keeping them "frozen" for all remaining calculations. It would certainly be of interest to see how the present results would be affected by the use of a different interaction and/or by a different (hopefully "dynamical") choice of the collective pairs. We hope to answer these questions in the near future.

7 Appendix

Matrix elements $(\bar{i}|\hat{H}_b^1|i)$ of the $\hat{H}_b^1$ defined by Eq. 34 read

$$(\bar{i}|\hat{H}_b^1|i) = (\varepsilon_{\nu_1} + \varepsilon_{\nu_2} + \varepsilon_{\nu_3})(\bar{i}|i) \tag{39}$$

where ε_{ν_i} $(i = 1, 2, 3)$ are the one-boson energies and the overlap

$$\begin{aligned}
(\bar{i}|i) = (&\delta_{1\bar{1}}\delta_{2\bar{2}}\delta_{3\bar{3}}\delta_{\Gamma_{12}\overline{\Gamma}_{12}} + \\
&+ (-)^{\overline{\Gamma}_{12}-\Gamma_{12}+\Gamma_2+\Gamma_3}\hat{\Gamma}_{12}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\Gamma_2 & \Gamma_1 & \Gamma_{12}\\ \Gamma_3 & \Lambda & \overline{\Gamma}_{12}\end{Bmatrix}\delta_{1\bar{1}}\delta_{2\bar{3}}\delta_{3\bar{2}} + \\
&+ (-)^{\Gamma_1+\Gamma_2-\overline{\Gamma}_{12}}\delta_{\Gamma_{12}\overline{\Gamma}_{12}}\delta_{1\bar{2}}\delta_{2\bar{1}}\delta_{3\bar{3}} + \\
&+ (-)^{\Gamma_2+\Gamma_3+\overline{\Gamma}_{12}}\hat{\Gamma}_{12}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_{12}\\ \Gamma_3 & \Lambda & \overline{\Gamma}_{12}\end{Bmatrix}\delta_{1\bar{3}}\delta_{2\bar{1}}\delta_{3\bar{2}} + \\
&+ (-)^{\Gamma_1+\Gamma_2-\Gamma_{12}}\hat{\Gamma}_{12}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\Gamma_2 & \Gamma_1 & \Gamma_{12}\\ \Gamma_3 & \Lambda & \overline{\Gamma}_{12}\end{Bmatrix}\delta_{1\bar{2}}\delta_{2\bar{3}}\delta_{3\bar{1}} + \\
&+ \hat{\Gamma}_{12}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_{12}\\ \Gamma_3 & \Lambda & \overline{\Gamma}_{12}\end{Bmatrix}\delta_{1\bar{3}}\delta_{2\bar{2}}\delta_{3\bar{1}}).
\end{aligned} \tag{40}$$

This formula has been obtained by employing the commutation relation

$$[b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta} \tag{41}$$

of the boson anihilation and creation operators, utilizing the orthonormality conditions of the C-G coefficients and relations between the 6j and C-G coefficients. The index $i(= 1, 2, 3)$ and $\bar{i}(= \bar{1}, \bar{2}, \bar{3})$ represents the set of quantum numbers $\nu\Gamma$ specifying state $|i)$ and $|\bar{i})$.

Formulae for the matrix elements $(\bar{i}|\hat{H}_b^2|i)$ of the $\hat{H}_b^2$ defined by Eq. 35 takes the form

$$\begin{aligned}
&(\bar{i}|\hat{H}_b^2|i) = (0|[(b_{\overline{\nu}_1\overline{\Gamma}_1}\times b_{\overline{\nu}_2\overline{\Gamma}_2})_{\overline{\Gamma}_{12}} \times b_{\overline{\nu}_3\overline{\Gamma}_3}]_{\overline{\Lambda}}|\hat{H}_b^2|[(b^\dagger_{\nu_1\Gamma_1} \times b^\dagger_{\nu_2\Gamma_2})_{\Gamma_{12}} \times b^\dagger_{\nu_3\Gamma_3}]_\Lambda|0) = \\
&= ((1 + \delta_{\nu_2\Gamma_2\nu_3\Gamma_3})^{\frac{1}{2}}\Big(\sum_\Gamma(E_\Gamma(\overline{\nu}_2\overline{\Gamma}_2\overline{\nu}_3\overline{\Gamma}_3; \nu_2\Gamma_2\nu_3\Gamma_3)(1 + \delta_{\overline{\nu_2}\overline{\Gamma}_2\overline{\nu_3}\overline{\Gamma}_3})^{\frac{1}{2}} \times \\
&\times(-)^{\overline{\Gamma}_2+\overline{\Gamma}_3+\Gamma_2+\Gamma_3}\hat{\Gamma}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\overline{\Gamma}_2 & \overline{\Gamma}_3 & \Gamma\\ \Lambda & \overline{\Gamma}_1 & \overline{\Gamma}_{12}\end{Bmatrix}\hat{\Gamma}\hat{\Gamma}_{12}\begin{Bmatrix}\overline{\Gamma}_1 & \Gamma_2 & \Gamma_{12}\\ \Gamma_3 & \Lambda & \Gamma\end{Bmatrix}\delta_{\nu_1\Gamma_1\overline{\nu}_1\overline{\Gamma}_1}\delta_{\nu_1\overline{\nu}_1} + \\
&+ E_\Gamma(\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_3\overline{\Gamma}_3; \nu_2\Gamma_2\nu_3\Gamma_3)(1 + \delta_{\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_3\overline{\Gamma}_3})^{\frac{1}{2}}(-)^{\overline{\Gamma}_2+\overline{\Gamma}_3+\Gamma_2+\Gamma_3+\overline{\Gamma}_{12}} \times \\
&\times \hat{\Gamma}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\overline{\Gamma}_1 & \overline{\Gamma}_3 & \Gamma\\ \Lambda & \overline{\Gamma}_2 & \overline{\Gamma}_{12}\end{Bmatrix}\hat{\Gamma}\hat{\Gamma}_{12}\begin{Bmatrix}\overline{\Gamma}_2 & \Gamma_2 & \Gamma_{12}\\ \Gamma_3 & \Lambda & \Gamma\end{Bmatrix}\delta_{\nu_1\Gamma_1\overline{\nu}_2\overline{\Gamma}_2} +
\end{aligned}$$

$$
\begin{aligned}
&+\ E_{\overline{\Gamma}_{12}}(\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_2\overline{\Gamma}_2;\nu_2\Gamma_2\nu_3\Gamma_3)(1+\delta_{\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_2\overline{\Gamma}_2})^{\frac{1}{2}}(-)^{\Gamma_2+\Gamma_3-\overline{\Gamma}_{12}}\ \times \qquad (42)\\
&\times\ \hat{\overline{\Gamma}}_{12}\hat{\Gamma}_{12}\begin{Bmatrix}\Gamma_3 & \Gamma_2 & \overline{\Gamma}_{12}\\ \overline{\Gamma}_3 & \Gamma & \Gamma_{12}\end{Bmatrix}\delta_{\nu_1\Gamma_1\overline{\nu}_3\overline{\Gamma}_3}\Big)+\\
&+(1+\delta_{\nu_1\Gamma_1\nu_3\Gamma_3})^{\frac{1}{2}}\Big(\sum_{\Gamma}(E_{\Gamma}(\overline{\nu}_2\overline{\Gamma}_2\overline{\nu}_3\overline{\Gamma}_3;\nu_1\Gamma_1\nu_3\Gamma_3)(1+\delta_{\overline{\nu}_2\overline{\Gamma}_2\overline{\nu}_3\overline{\Gamma}_3})^{\frac{1}{2}}(-)^{\Gamma_2+\Gamma_3+\overline{\Gamma}_2+\overline{\Gamma}_3+\Gamma_{12}}\\
&\times\ \hat{\Gamma}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\overline{\Gamma}_2 & \overline{\Gamma}_3 & \Gamma\\ \Lambda & \overline{\Gamma}_1 & \overline{\Gamma}_{12}\end{Bmatrix}\hat{\Gamma}\hat{\Gamma}_{12}\begin{Bmatrix}\Gamma_1 & \Gamma_3 & \Gamma\\ \Lambda & \overline{\Gamma}_1 & \Gamma_{12}\end{Bmatrix}\delta_{\nu_2\Gamma_2\overline{\nu_1}\overline{\Gamma}_1}+\\
&+\ E_{\Gamma}(\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_3\overline{\Gamma}_3;\nu_1\Gamma_1\nu_3\Gamma_3)(1+\delta_{\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_3\overline{\Gamma}_3})^{\frac{1}{2}}(-)^{\Gamma_3+\overline{\Gamma}_3+\Gamma_{12}+\overline{\Gamma}_{12}}\ \times\\
&\times\ \hat{\Gamma}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\overline{\Gamma}_1 & \overline{\Gamma}_3 & \Gamma\\ \Lambda & \overline{\Gamma}_2 & \overline{\Gamma}_{12}\end{Bmatrix}\hat{\Gamma}\hat{\Gamma}_{12}\begin{Bmatrix}\Gamma_1 & \Gamma_3 & \Gamma\\ \Lambda & \overline{\Gamma}_2 & \Gamma_{12}\end{Bmatrix}\delta_{\nu_2\Gamma_2\overline{\nu}_2\overline{\Gamma}_2})+\\
&+\ E_{\overline{\Gamma}_{12}}(\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_2\overline{\Gamma}_2;\nu_1\Gamma_1\nu_3\Gamma_3)(1+\delta_{\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_2\overline{\Gamma}_2})^{\frac{1}{2}}(-)^{\Gamma_{12}-\overline{\Gamma}_{12}+\Gamma_3+\overline{\Gamma}_3}\ \times\\
&\times\ \hat{\overline{\Gamma}}_{12}\hat{\Gamma}_{12}\begin{Bmatrix}\Gamma_3 & \Gamma_1 & \overline{\Gamma}_{12}\\ \overline{\Gamma}_3 & \Lambda & \Gamma_{12}\end{Bmatrix}\delta_{\nu_2\Gamma_2\overline{\nu}_3\overline{\Gamma}_3}\Big)+\\
&+(1+\delta_{\nu_1\Gamma_1\nu_2\Gamma_2})^{\frac{1}{2}}(E_{\Gamma_{12}}(\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_2\overline{\Gamma}_2;\nu_1\Gamma_1\nu_2\Gamma_2)(1+\delta_{\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_2\overline{\Gamma}_2})^{\frac{1}{2}}\delta_{\Gamma_{12}\overline{\Gamma}_{12}}\delta_{\nu_3\Gamma_3\overline{\nu}_3\overline{\Gamma}_3}+\\
&+\ E_{\Gamma_{12}}(\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_3\overline{\Gamma}_3;\nu_1\Gamma_1\nu_2\Gamma_2)(1+\delta_{\overline{\nu}_1\overline{\Gamma}_1\overline{\nu}_3\overline{\Gamma}_3})^{\frac{1}{2}}(-)^{\Gamma_3+\overline{\Gamma}_3+\overline{\Gamma}_{12}-\Gamma_{12}}\ \times\\
&\times\ \hat{\Gamma}_{12}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\overline{\Gamma}_3 & \overline{\Gamma}_1 & \Gamma_{12}\\ \overline{\Gamma}_2 & \Lambda & \overline{\Gamma}_{12}\end{Bmatrix}\delta_{\nu_3\Gamma_3\overline{\nu}_2\overline{\Gamma}_2})+\\
&+\ E_{\Gamma_{12}}(\overline{\nu}_2\overline{\Gamma}_2\overline{\nu}_3\overline{\Gamma}_3;\nu_1\Gamma_1\nu_2\Gamma_2)(1+\delta_{\overline{\nu}_2\overline{\Gamma}_2\overline{\nu}_3\overline{\Gamma}_3})^{\frac{1}{2}}(-)^{\overline{\Gamma}_2+\overline{\Gamma}_3-\Gamma_{12}}\ \times\\
&\times\ \hat{\Gamma}_{12}\hat{\overline{\Gamma}}_{12}\begin{Bmatrix}\overline{\Gamma}_3 & \overline{\Gamma}_2 & \Gamma_{12}\\ \overline{\Gamma}_1 & \Lambda & \overline{\Gamma}_{12}\end{Bmatrix}\delta_{\nu_3\Gamma_3\overline{\nu}_1\overline{\Gamma}_1})).
\end{aligned}
$$

This formula, similarly to formula 39 has been obtained by employing the commutation relation 41 and some relations of reference [23], connecting the $6j$ and C-G coefficients.

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J	T=0(MeV)	J	T=1(MeV)	J	T=2(MeV)	J	T=3(MeV)
3	-60.380	0	-59.788	4	-44.908	0	-35.402
1	-60.059	2	-58.278	3	-44.686	2	-31.864
4	-59.415	4	-56.238	2	-44.191	0	-30.566
5	-58.866	2	-55.188	5	-43.455	3	-30.311
1	-58.864	2	-54.679	2	-43.203	2	-28.555
3	-58.544	1	-54.048	1	-43.188	4	-28.210
2	-57.639	4	-54.038	3	-43.187	4	-27.853
3	-57.546	3	-53.997	1	-42.656	2	-27.258
6	-56.520	2	-53.277	0	-42.603	3	-26.833
1	-56.348	0	-53.155	4	-42.225	0	-26.227

Table 1 The lowest 10 shell model eigenstates of A=22 nuclei for all values of the total isospin T. States are also labeled with their angular momentum J.

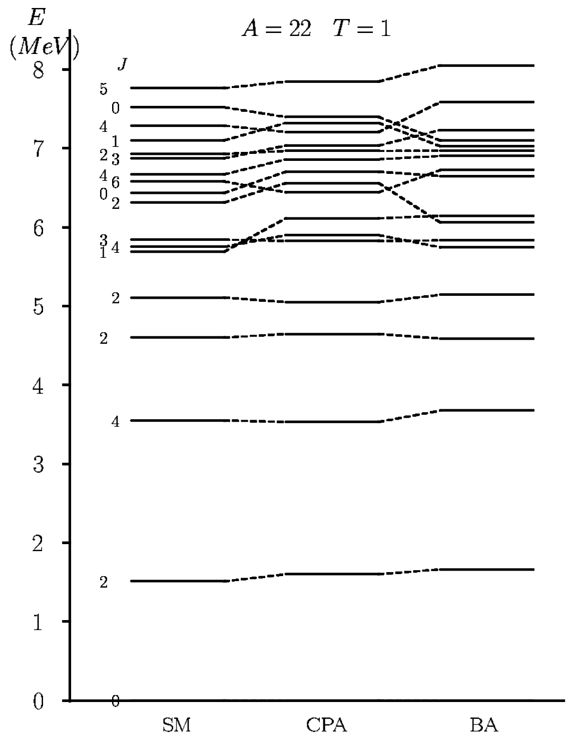

Fig. 1 Comparison of the $T = 1$ low-lying shell model spectrum (SM) of $A = 22$ nuclei with the collective pair approximation spectrum (CPA) calculated in the space spanned by the $T = 1$, S, S', D, D', G and $T = 0$, $\Theta_1, \Theta_3, \Theta_5$ pairs and with the boson approximation spectrum (BA) calculated in the corresponding space.

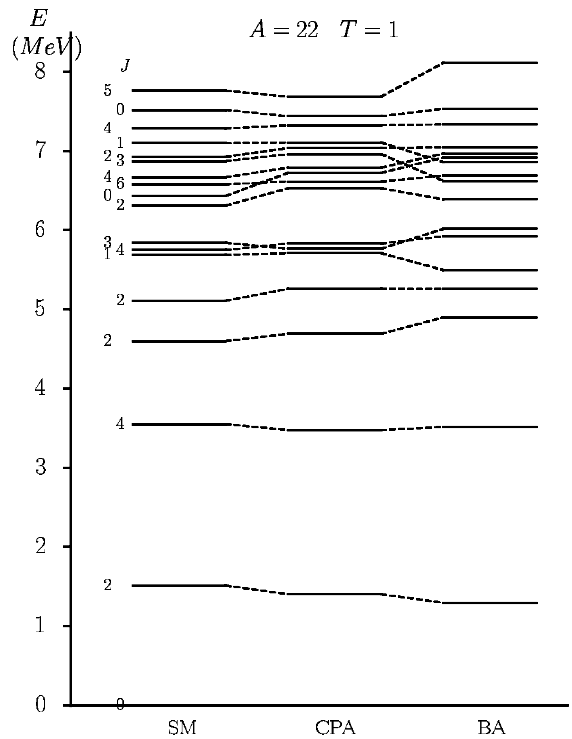

Fig. 2 Comparison of the $T = 1$ low-lying shell model spectrum (SM) of $A = 22$ nuclei with the collective pair approximation spectrum (CPA) calculated in the space spanned by the $T = 1$, S, S', P, D, D', F, G pairs and with the boson approximation spectrum (BA) calculated in the corresponding space.

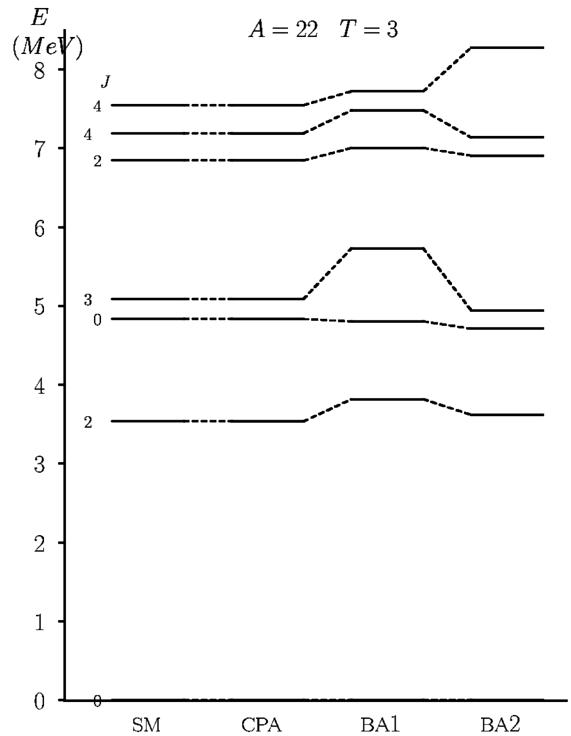

Fig. 3 Comparison of the $T = 3$ low-lying shell model spectrum (SM) of $A = 22$ nuclei with the collective pair approximation spectrum (CPA) calculated in the space spanned by the (1) $T = 1,\ S, S', D, D', G$ pairs or by the (2) $T = 1,\ S, S', P, D, D', F, G$ pairs (in both cases the low-lying CPA spectrum is the same) and with the boson approximation spectra calculated in the corresponding spaces (see also text).

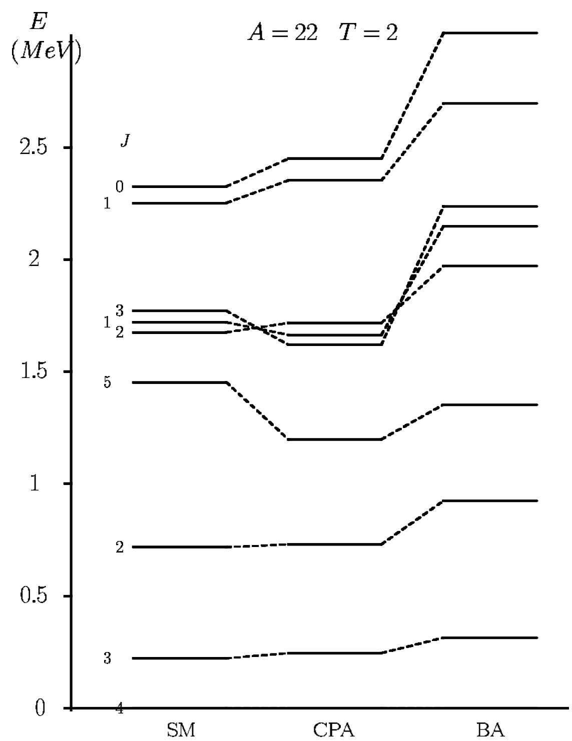

Fig. 4 The same as in figure 1 but for the $T = 2$ states of A=22 nuclei.

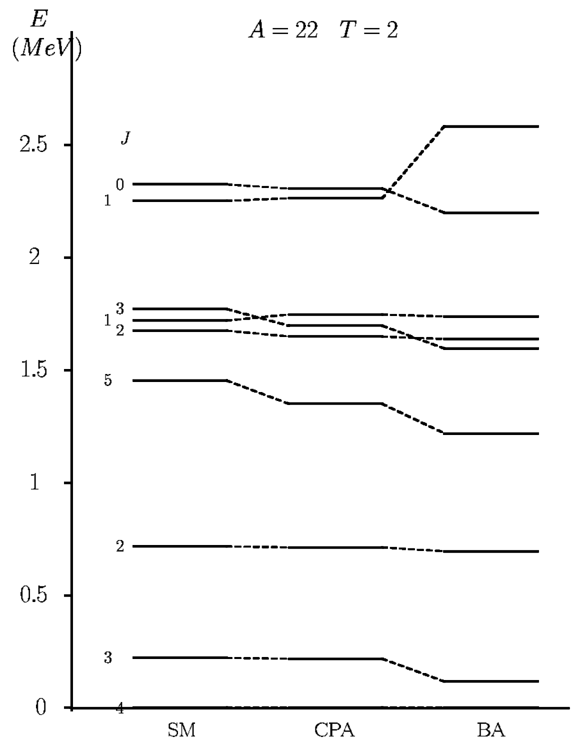

Fig. 5 The same as in figure 2 but for the $T = 2$ states of A=22 nuclei.