DE GRUYTER Depend. Model. 2021; 9:43–61

Research Article Open Access

Monica Billio*, Lorenzo Frattarolo, and Dominique Guégan

Multivariate radial symmetry of copula functions: finite sample comparison in the i.i.d case

https://doi.org/10.1515/demo-2021-0102
Received January 5, 2021; accepted April 7, 2021

Abstract: Given a d-dimensional random vector $\mathbf{X} = (X_1, \dots, X_d)$, if the standard uniform vector $\mathbf{U}$ obtained by the component-wise probability integral transform (PIT) of $\mathbf{X}$ has the same distribution of its point reflection through the center of the unit hypercube, then $\mathbf{X}$ is said to have copula radial symmetry. We generalize to higher dimensions the bivariate test introduced in [11], using three different possibilities for estimating copula derivatives under the null. In a comprehensive simulation study, we assess the finite-sample properties of the resulting tests, comparing them with the finite-sample performance of the multivariate competitors introduced in [17] and [1].

Keywords: Copula, reflection symmetry, radial symmetry, empirical process

MCS: 62G10, 62G30, 62H05, 62H15

1 Introduction

Let $\mathbf{X} = (X_1, \dots, X_d)$ be a continuous random vector with marginal cumulative distribution functions (CDFs) $F_i(x)$, $i = 1, \dots, d$. Let $\mathbf{U}$ be the standard uniform random vector obtained by using the component-wise probability integral transform on $\mathbf{X}$:

$$\mathbf{U} = (U_1, \dots, U_d) = (F_1(X_1), \dots, F_d(X_d)).$$

Let $\mathbf{1}_d$ be the d-dimensional vector with all components equal to one. The hypothesis of copula radial symmetry is equivalent to the following distributional identity:

$$\mathcal{H}_0 : \mathbf{U} \overset{d}{=} \mathbf{1}_d - \mathbf{U} \tag{1.1}$$

This relationship was introduced, in the bivariate context, by [20] and [21], under the name of radial symmetry of the copula function. Under copula radial symmetry, $\mathbf{U}$ has the same distribution of its point reflection through the center of the unit hypercube, and [26] and [16] call it copula reflection symmetry.

As the name suggests, the symmetry can be framed in the copula context. Following [28], the joint CDF $F(\mathbf{x})$ of $\mathbf{X}$ could be expressed at each $\mathbf{x} \in \mathbb{R}^d$ as:

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$$

***Corresponding Author: Monica Billio:** University Ca' Foscari of Venice, Department of Economics, Venice, Italy, E-mail: billio@unive.it
Lorenzo Frattarolo: European Commission, Joint Research Centre (JRC), Ispra, Italy, E-mail:lorenzo.frattarolo@ec.europa.eu
Dominique Guégan: University Paris-1 Panthéon-Sorbonne, Paris, France and University Ca' Foscari of Venice, Department of Economics, Venice, Italy, E-mail: dominique.guegan@univ-paris1.fr

in terms of the unique copula C. C is the joint CDF of $\mathbf{U}$. Analogously, let $\bar{F}_i(x_i) = 1 - F_i(x_i)$, $i = 1, \dots, d$ be the marginal survival functions of $\mathbf{X}$. The joint survival function of $\mathbf{X}$ could be expressed at each $\mathbf{x} \in \mathbb{R}^d$ as:

$$\bar{F}(x_1, \dots, x_d) = \bar{C}\left(\bar{F}_1(x_1), \dots, \bar{F}_d(x_d)\right)$$

The survival copula $\bar{C}$ is then the CDF of the random vector $\mathbf{1}_d - \mathbf{U}$.

With these definitions, equation (1.1) is equivalent to the following identity:

$$\mathcal{H}_0 : C(u_1, \dots, u_d) = \bar{C}(u_1, \dots, u_d) \tag{1.2}$$

Using equation (1.2) the test of $\mathcal{H}_0$, becomes a test of distributional identity based on a measure of their distance. Empirical distribution functions allow a non-parametric consistent estimation of their distance. Let $\left\{\{X_{ij}\}_{i=1}^{n}\right\}_{j=1}^{d} \equiv \{\mathbf{X}_i\}_{i=1}^{n}$ be an independent sample of size n from d-dimensional random vector $\mathbf{X}$. Let $\mathbb{I}(A)$ be the indicator for the set A. Moreover, let us define the pseudo-observations $\hat{U}_{ij} = \frac{1}{n+1}\sum_{k=1}^{n}\mathbb{I}\left(X_{kj} \le X_{ij}\right)$, $i = 1, \dots, n$ and $j = 1, \dots, d$. Then, the empirical copula and the empirical survival copula are:

$$C_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(\hat{\mathbf{U}}_i \le \mathbf{u}\right), \quad \bar{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(\mathbf{1}_d - \hat{\mathbf{U}}_i \le \mathbf{u}\right).$$

Any measure of the distance between C_n and $\bar{C}_n$ leads to a possible test statistic of the hypothesis $\mathcal{H}_0 : C = \bar{C}$. We focus on a Cramér–von Mises statistic under the random measure generated by the empirical copula:

$$S_n = \int_{(0,1]^d} \left(C_n - \bar{C}_n\right)^2 dC_n = \frac{1}{n}\sum_{i=1}^{n}\left(C_n\left(\hat{\mathbf{U}}_i\right) - \bar{C}_n\left(\hat{\mathbf{U}}_i\right)\right)^2. \tag{1.3}$$

In the bivariate case, this statistic was introduced in [2] and investigated in [6] and [11]. In particular, according to [11], the measure in equation (1.3) is the most powerful statistic for a random vector of dimension two. Different investigations of the same problem by other bivariate non-parametric approaches are in [26] and [19]. [23] proposes a general framework for testing the homogeneity hypothesis of a copula. In the latter paper, the asymptotic theory covers multivariate radial symmetry tests, but it lacks a simulation study of the finite sample properties of testing this symmetry. [17] and [1] are the only two papers, to our knowledge, that propose a multivariate non-parametric radial test and include an investigation of its finite sample properties.

[17] studies a statistic linear in the difference between the copula and the survival copula. The difference is weighted using the following probability density, which gives more weight to the tails:

$$h_k(\mathbf{u}) = \frac{2^{k+d}}{(-d)^d}\frac{(k+d)!}{k!}\max\left(\frac{1}{2} - \frac{\mathbf{1}_d^T\mathbf{u}}{d}, 0\right). \tag{1.4}$$

Denoting with H_k the distribution coming from the density h_k, their statistic is:

$$G_k = \int_{(0,1]^d} \left(C - \bar{C}\right) dH_k = \frac{(-d)^d\,k!}{(k+d)!}\mathbb{E}\left[\left|\frac{1}{2} - \frac{\mathbf{1}_d^T\mathbf{U}}{d}\right|^{k+d} \operatorname{sign}\left(\frac{1}{2} - \frac{\mathbf{1}_d^T\mathbf{U}}{d}\right)\right] \tag{1.5}$$

The null hypothesis

$$\mathcal{H}_{00} : G_k = 0, \tag{1.6}$$

is strictly weaker than the hypothesis of radial symmetry $\mathcal{H}_0$ for, at least, two reasons. First, as remarked by the author, due to linearity, the difference between the copula and the survival copula can change sign on the support of H_k and, in principle, G_k could be null for asymmetric models. Second, due to the weighting scheme, copulas with support in the null set of H_k have $G_k = 0$. In particular, strict d-CM Copulas (as introduced in [18]) that have support on the hyperplane with constant sum $\mathbf{1}_d^T\mathbf{u} = \frac{d}{2}$ have $G_k = 0$. An asymmetric 3-CM copula is introduced in example 7 of [22]. This copula distributes probability mass uniformly on the edges of the equilateral triangle in $[0, 1]^3$ with vertices $(0, 1/2, 1)$, $(1/2, 1, 0)$ and $(1, 0, 1/2)$. The corresponding survival copula has probability mass distributed uniformly on the edges of the reflected triangle with vertices $(1, 1/2, 0)$, $(1/2, 0, 1)$ and $(0, 1, 1/2)$.

The empirical version uses an asymptotically equivalent transformation of pseudo-observations:

$$\begin{aligned} \mathbf{V}_i &= \frac{n+1}{n}\mathbf{U}_i - \frac{\mathbf{1}_d}{2n}, \quad i = 1, \dots, n \\ G_{k,n} &= \frac{(-d)^d\, k!}{(k+d)!}\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{2} - \frac{\mathbf{1}_d^T\mathbf{V}_i}{d}\right|^{k+d} \operatorname{sign}\left(\frac{1}{2} - \frac{\mathbf{1}_d^T\mathbf{V}_i}{d}\right). \end{aligned} \tag{1.7}$$

On the other side, linearity and a fixed weighting scheme lead to the advantage of a normal asymptotic distribution for the empirical version.

The authors of [1] use an approach based on the characteristic function. They introduce the random vector $\mathbf{W} = \mathbf{U} - \frac{1}{2}\mathbf{1}_d$, marginally distributed as $U\left[-1/2, 1/2\right]$, and rephrase (1.1) as:

$$\mathcal{H}_0 : \mathbf{W} = -\mathbf{W}. \tag{1.8}$$

Using the characteristic function of $\mathbf{W}$, $\psi_C(t) = \mathbb{E}\left(\exp\left(i\mathbf{t}\mathbf{W}^T\right)\right)$, they write (1.8) in a different form :

$$\mathcal{H}_0 : \mathcal{L}_C(t) = \psi_C(t) - \psi_C(-t) = 0 \tag{1.9}$$

The empirical versions are given by

$$\begin{aligned} \hat{\mathbf{W}}_i &= \hat{\mathbf{U}}_i - \frac{1}{2}\mathbf{1}_d \qquad i = 1, \dots, n \\ \mathcal{L}_n(\mathbf{t}) &= \frac{1}{n}\sum_{i=1}^{n}\sin\left(\mathbf{t}\hat{\mathbf{W}}_i\right), \end{aligned}$$

leading to a family of test statistics indexed by the weighting function $\omega : \mathbb{R}^d \mapsto \mathbb{R}$

$$R_{n,\omega} = n\int_{\mathbb{R}^d}\mathcal{L}_n(\mathbf{t})\,\omega(\mathbf{t})\,d\mathbf{t}. \tag{1.10}$$

Both [17] and [1] propose, in the bivariate case, a simulation-based comparison of finite sample properties with the test of [11]. A comparison beyond the bivariate case is missing, even if the asymptotic theory behind a higher-dimensional generalization was in [23]. The purpose of this paper is to fill this gap by investigating the finite sample properties of the test statistic S_n in equation (1.3) using the multiplier Bootstrap under three different specifications of the multiplier empirical process, comparing its performance to the procedures proposed in [17] and [1]. Even if our methodological contribution is limited to an alternative proof of proposition 1 in [23], we hope that the breadth of our simulation study, covering also recent asymmetric copulas proposed by [24] and [10], can safely, guide the reader in most of the practical applications of the tests to i.i.d. data.

The paper is structured as follows: Section 2 studies the S_n-based test's asymptotic properties. Section 3 presents a simulation study of the finite sample properties of the test and a comparison with the procedure proposed in [17] and [1]. Section 4 summarizes our findings and proposes further developments.

2 Asymptotic Behavior of S_n Under Copula Radial Symmetry

The asymptotic null distribution of S_n relies on the limiting behavior of the empirical copula process and the empirical survival copula processes:

$$\mathbb{C}_n = \sqrt{n}\left(C_n(\mathbf{u}) - C(\mathbf{u})\right), \quad \bar{\mathbb{C}}_n = \sqrt{n}\left(\bar{C}_n(\mathbf{u}) - \bar{C}(\mathbf{u})\right). \tag{2.1}$$

[27] obtains the following weak convergence result for the empirical copula process:

$$\mathbb{C}_n \rightsquigarrow \mathbb{C} = \mathbb{B}_C(\mathbf{u}) - \sum_{d=1}^{d} \frac{\partial C(\mathbf{u})}{\partial u_d} \mathbb{B}_{d,C}(u_d), \tag{2.2}$$

where $\mathbb{B}_C$ is a d-dimensional Brownian sheet with covariance function

$$\mathbb{C}\text{ov}(\mathbb{B}_C(\mathbf{u}), \mathbb{B}_C(\mathbf{v})) = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}) \tag{2.3}$$

where $\wedge$ is the component-wise minimum.

Let us introduce an assumption on copula derivatives, analogous to the one introduced in [27]:

A 1. *For each $j \in \{1, \ldots, d\}$, the jth first-order partial derivative $\frac{\partial C}{\partial u_j}$ exists and is continuous on the set $V_{d,j} := \left\{u \in [0,1]^d : 0 < u_j < 1\right\}$.*

The weak convergence of the survival copula empirical process is not covered by proposition 3.1 in [27], contrary to what is stated in [17]. It is, instead, a corollary of proposition 1 in [23]. The latter paper develops a general class of shape hypotheses by comparing the copula C of the random vector $\mathbf{U}$ to the copula associated to a transformation of $\mathbf{U}$. The transformations considered are coordinate-wise reflections, permutation and marginalization of the random vector $\mathbf{U} \sim C$. More formally, the coupled action of a marginalization of cardinality p and a permutation of the components of $\mathbf{U}$ is equivalent to taking a subset of cardinality p, $I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, d\}$, and extracting the sub-vector $\mathbf{U}_I = \left(U_{i_1}, \ldots, U_{i_p}\right)$. Considering the subset of cardinality r, $J = \{j_1, \ldots, j_r\} \subseteq I$, the action of r reflections with respect to hyperplanes of constant j_k-th coordinate $k \in \{1, \ldots, r\}$, passing through the center of the unit hypercube is coordinate-wise equivalent to:

$$\left[R_J(\mathbf{U}_I)\right]_j = \begin{cases} U_j & \text{if} \quad j \in J \\ 1 - U_j & \text{if} \quad j \in I \setminus J \end{cases}.$$

This transformation is decreasing for $j \in J$, and increasing for coordinates in its complement, $j \in I \setminus J$. We define the copula $C_{I,J}$ of the transformed vector its empirical version, and its empirical process as:

$$\begin{aligned} C_{I,J}(\mathbf{v}) &= \mathbb{P}\left(R_J(\mathbf{U}_I) \leq \mathbf{v}\right) \\ C_{n,I,J}(\mathbf{v}) &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(R_J(\hat{\mathbf{U}}_{Ii}) \leq \mathbf{v}) \\ \mathbb{C}_{n,I,J}(\mathbf{v}) &= \sqrt{n}\left(C_{n,I,J}(\mathbf{v}) - C_{I,J}(\mathbf{v})\right) \end{aligned}$$

[23] obtains the weak convergence of the empirical process based on the copula of the transformed vector:

Proposition 1 ([23]). *Let $\left\{\{X_{ij}\}_{i=1}^{n}\right\}_{j=1}^{d} \equiv \{\mathbf{X}_i\}_{i=1}^{n}$ be an independent sample of size n from d-dimensional random vector $\mathbf{X} \sim C$. Suppose that assumption **A 1** is satisfied for $C_{I,J}$, then the empirical copula process $\mathbb{C}_{nI,J} = \sqrt{n}\left(C_{n,I,J}(\mathbf{v}) - C_{I,J}(\mathbf{v})\right)$ weakly converges towards a Gaussian field:*

$$\mathbb{C}_{n,I,J} \rightsquigarrow \mathbb{C}_{I,J} = \mathbb{B}_{C_{I,J}}(\mathbf{u}) - \textstyle\sum_{j\in I} \frac{\partial C_{I,J}(\mathbf{u})}{\partial u_j} \mathbb{B}_{j,C_{I,J}}\left(u_j\right) \quad in \quad \ell^\infty[0,1]^d$$

In the appendix, we propose an alternative proof of proposition 1, whose main line of reasoning, based on the functional delta method, was independently developed in the 2014 Ph.D. thesis of one of the authors. The weak convergence of the survival empirical process is obtained from proposition 1, considering $I = J = \{1, \ldots, d\}$.

2.1 Derivative Estimators and Multiplier Bootstraps

The dependence of the limiting processes from the unknown copula and survival copula forbids a closed-form inference. The use of multiplier central limit theorem solves this issue by obtaining the distribution of the limiting process through sampling. We will use the version of the multiplier central limit theorem introduced in [25] for i.i.d. random vectors. Given a function $f : [0,1]^d \mapsto [0,1]$, let $h = 1/\sqrt{n}$ and $\mathbf{e}_i$ be the vector corresponding to the i-th column of the d dimensional identity matrix. The partial finite difference derivative operator of step h, $\mathcal{D}_i^h$ introduced by [25] in the consistent estimation of copula derivatives is:

$$\mathcal{D}_i^h f(\mathbf{u}) = \begin{cases} \dfrac{f(\mathbf{u} + \mathbf{e}_i(2h - u_i))}{2h} & \text{if} \quad u_i \le h \\ \dfrac{f(\mathbf{u} + \mathbf{e}_i h) - f(\mathbf{u} - \mathbf{e}_i h)}{2h} & \text{if} \quad h < u_i \le 1 - h \\ \dfrac{f(\mathbf{u} - \mathbf{e}_i(u_i - 1)) - f(\mathbf{u} - \mathbf{e}_i(u_i - (1 - 2h)))}{2h} & \text{if} \quad u_i \ge 1 - h \end{cases} \tag{2.4}$$

We define a multiplier sequence $\left\{\xi_{i,n}\right\}_{i\in\mathbb{Z}}$ as an i.i.d. sequence $\left\{\xi_{i,n}\right\}_{i\in\mathbb{Z}}$ with $\mathbb{E}(\xi_{0,n}) = 0$, $\mathbb{E}\left(\xi_{0,n}^2\right) = 1$ and independent from the available sample. Given M independent copies of the multiplier sequence $\left\{\xi_{i,n}^{[1]}\right\}_{i\in\mathbb{Z}}, \dots, \left\{\xi_{i,n}^{[M]}\right\}_{i\in\mathbb{Z}}$ we can define the new processes:

$$\tilde{\mathbb{B}}_n^{[m]}(\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_{i,n}^{(m)} \left(\mathbb{I}\left(\hat{\mathbf{U}}_i \le \mathbf{u}\right) - C(\mathbf{u})\right) \tag{2.5}$$

$$\tilde{\bar{\mathbb{B}}}_n^{[m]}(\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_{i,n}^{(m)} \left(\mathbb{I}\left(\mathbf{1}_d - \hat{\mathbf{U}}_i \le \mathbf{u}\right) - \bar{C}(\mathbf{u})\right) \tag{2.6}$$

$$\tilde{\mathbb{C}}_n^{[m]}(\mathbf{u}) = \tilde{\mathbb{B}}_n^{[m]}(\mathbf{u}) - \sum_{i=1}^{d} \mathcal{D}_i^h C_n(\mathbf{u}) \tilde{\mathbb{B}}_{i,n}^{[m]}(u_i) \tag{2.7}$$

$$\tilde{\bar{\mathbb{C}}}_n^{[m],plain}(\mathbf{u}) = \tilde{\bar{\mathbb{B}}}_n^{[m]}(\mathbf{u}) - \sum_{i=1}^{d} \mathcal{D}_i^h \bar{C}_n(\mathbf{u}) \tilde{\bar{\mathbb{B}}}_{i,n}^{[m]}(u_i) \tag{2.8}$$

Results in [25] then imply:

$$\left(\mathbb{C}_n, \tilde{\mathbb{C}}_n^{[1]}, \dots, \tilde{\mathbb{C}}_n^{[M]}\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{C}^{[1]}, \dots, \mathbb{C}^{[M]}\right) \tag{2.9}$$

$$\left(\bar{\mathbb{C}}_n, \tilde{\bar{\mathbb{C}}}_n^{[1],plain}, \dots, \tilde{\bar{\mathbb{C}}}_n^{[M],plain}\right) \rightsquigarrow \left(\bar{\mathbb{C}}, \bar{\mathbb{C}}^{[1]}, \dots, \bar{\mathbb{C}}^{[M]}\right). \tag{2.10}$$

Here, $\mathbb{C}^{[1]}, \dots, \mathbb{C}^{[M]}$ are M independent copies of $\mathbb{C}$ and $\bar{\mathbb{C}}^{[1]}, \dots, \bar{\mathbb{C}}^{[M]}$ of $\bar{\mathbb{C}}$.
Furthermore, as remarked in [23], under the null we have different possibilities for consistent estimators of the derivatives, leading to the following multiplier processes:

$$\tilde{\bar{\mathbb{C}}}_n^{[m]GN}(\mathbf{u}) = \tilde{\bar{\mathbb{B}}}_n^{[m]}(\mathbf{u}) - \sum_{i=1}^{d} \mathcal{D}_i^h C_n(\mathbf{u}) \tilde{\bar{\mathbb{B}}}_{i,n}^{[m]}(u_i) \tag{2.11}$$

$$\tilde{\mathbb{C}}_n^{[m],mid}(\mathbf{u}) = \tilde{\mathbb{B}}_n^{[m]}(\mathbf{u}) - \sum_{d=1}^{d} \frac{\mathcal{D}_i^h C_n(\mathbf{u}) + \mathcal{D}_i^h \bar{C}_n(\mathbf{u})}{2} \tilde{\mathbb{B}}_{i,n}^{[m]}(u_i) \tag{2.12}$$

$$\tilde{\bar{\mathbb{C}}}_n^{[m],mid}(\mathbf{u}) = \tilde{\bar{\mathbb{B}}}_n^{[m]}(\mathbf{u}) - \sum_{i=1}^{d} \frac{\mathcal{D}_i^h C_n(\mathbf{u}) + \mathcal{D}_i^h \bar{C}_n(\mathbf{u})}{2} \tilde{\bar{\mathbb{B}}}_{i,n}^{[m]}(u_i) \tag{2.13}$$

In the multiplier process of the empirical survival copula, equation (2.11) uses the numerical derivative of the empirical copula and corresponds to the version of the multiplier used in [11]. Equations (2.12) and (2.13) use the midpoint between derivatives as the estimator of the derivatives in both processes. Under the null, all the proposed derivatives estimators are consistent, implying the convergence of the different multiplier bootstrap processes to independent copies of the empirical copula process:

$$\left(\mathbb{C}_n, \tilde{\mathbb{C}}_n^{[1]}, \ldots, \tilde{\mathbb{C}}_n^{[M]}\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{C}^{[1]}, \ldots, \mathbb{C}^{[M]}\right) \tag{2.14}$$

$$\left(\mathbb{C}_n, \tilde{\mathbb{C}}_n^{[1],mid}, \ldots, \tilde{\mathbb{C}}_n^{[M],mid}\right) \rightsquigarrow \left(\mathbb{C}, \mathbb{C}^{[1]}, \ldots, \mathbb{C}^{[M]}\right) \tag{2.15}$$

$$\left(\bar{\mathbb{C}}_n, \tilde{\bar{\mathbb{C}}}_n^{[1],plain}, \ldots, \tilde{\bar{\mathbb{C}}}_n^{[M],plain}\right) \rightsquigarrow \left(\bar{\mathbb{C}}, \bar{\mathbb{C}}^{[1]}, \ldots, \bar{\mathbb{C}}^{[M]}\right) \tag{2.16}$$

$$\left(\bar{\mathbb{C}}_n, \tilde{\bar{\mathbb{C}}}_n^{[1],GN}, \ldots, \tilde{\bar{\mathbb{C}}}_n^{[M],GN}\right) \rightsquigarrow \left(\bar{\mathbb{C}}, \bar{\mathbb{C}}^{[1]}, \ldots, \bar{\mathbb{C}}^{[M]}\right) \tag{2.17}$$

$$\left(\bar{\mathbb{C}}_n, \tilde{\bar{\mathbb{C}}}_n^{[1],mid}, \ldots, \tilde{\bar{\mathbb{C}}}_n^{[M],mid}\right) \rightsquigarrow \left(\bar{\mathbb{C}}, \bar{\mathbb{C}}^{[1]}, \ldots, \bar{\mathbb{C}}^{[M]}\right) \tag{2.18}$$

We expect that the use of both derivatives, under the null, will reduce the asymptotic variance and make the size closer to the nominal one. Under the alternative, in the plain estimator case, the survival copula derivative is estimated without bias, while in the Genest and Nešlehová approach, it has a bias equal to the difference between the derivatives of the copula and the survival copula. The mid-estimator, instead, removes part of the bias from the survival copula derivative estimator but adds it back to the copula derivative estimator. The overall benefit of the latter estimator is difficult to determine a priori. The random vectors $\mathbf{U}$ and $\mathbf{1}_d - \mathbf{U}$ are pairwise antithetical. Then, under the null, our strategy coincides with applying the antithetic variates method [12] to reduce the copula derivative estimator's variance.

2.2 Test Statistics

Following what previously said, we want to test the null hypothesis

$$\mathcal{H}_0 : \quad C(u_1, \ldots, u_d) = \bar{C}(u_1, \ldots, u_d) \tag{2.19}$$

against the alternative

$$\mathcal{H}_1 : \quad C(u_1, \ldots, u_d) \neq \bar{C}(u_1, \ldots, u_d). \tag{2.20}$$

Under the null, equation (1.3) becomes:

$$nS_n = nS_n^{GN} = nS_n^{mid} = \int_{(0,1]^d} \left(\mathbb{C}_n - \bar{\mathbb{C}}_n\right)^2 d\hat{C}_n. \tag{2.21}$$

The multiplier copies in the three different versions $ver \in \{plain, GN, mid\}$ are:

$$n\tilde{S}_n^{[m],ver} = \int_{(0,1]^d} \left(\tilde{\mathbb{C}}_n^{[m]} - \tilde{\bar{\mathbb{C}}}_n^{[m],ver}\right)^2 d\hat{C}_n, \tag{2.22}$$

In the following proposition, we obtain the weak limits under the null:

Proposition 2. *Let* $\left\{\{X_{ij}\}_{i=1}^n\right\}_{j=1}^d \equiv \{\mathbf{X}_i\}_{i=1}^n$ *an independent sample of size n from d-dimensional random vector* $\mathbf{X}$ *having copula C. If C is a copula symmetric under reflection, i.e.* $C = \bar{C}$, *under* ***A 1***, *we have, as* $n \to \infty$, *the following weak convergence results, for the three versions of the statistic,* $ver = \{plain, GN, mid\}$:

$$\left(nS_n, n\tilde{S}_n^{[1],ver}, \ldots, n\tilde{S}_n^{[M],ver}\right) \rightsquigarrow \left(\mathbb{S}, \mathbb{S}^{[1]}, \ldots, \mathbb{S}^{[M]}\right)$$
$$\mathbb{S} = \int_{(0,1]^d} \left(\mathbb{C} - \bar{\mathbb{C}}\right)^2 dC,$$

where $\mathbb{S}^{[1]}, \ldots, \mathbb{S}^{[M]}$ *are independent copies of* $\mathbb{S}$.

It follows from proposition 2, the proof of which is postponed to the appendix, that approximate P-values for the tests of H_0 based on S_n are given by:

$$\hat{P}_{ver} = \frac{1}{M} \sum_{m=1}^{M} \mathbb{I}\left(\tilde{S}_n^{[m],ver} > S_n\right),$$

where $ver = \{plain, GN, mid\}$.

3 Simulation Study

This section studies the finite sample behavior of the different tests of multivariate copula radial symmetry. Throughout the section, the number of bootstrap replicates is M = 1000, and the estimated probabilities of rejection come from 1000 Monte Carlo repetitions.

We compare the tests S_n^{plain}, S_n^{mid}, S_n^{GN}, $G_{4,n}$ and R_n^N. The test statistic $G_{4,n}$ is the statistic proposed in equation (1.7) in the case $k = 4$, as suggested in [17]. Under the assumption A1 for the derivatives of the copula and the derivatives of the survival copula, using proposition 3.1 in [27] and Proposition 1, G_{4n} is normally distributed with an asymptotic variance that can be consistently estimated with the standard bootstrap. R_n^N [1] is the statistic introduced in equation (1.10), with the kernel ω chosen as the product of standard normal densities. We set the smoothing parameter σ to 1 because, to our understanding, it is the best compromise across different dimensions and copula models. We refer to the original paper for the interesting V-statistics asymptotics and the actual implementation details of the multiplier bootstrap used in the procedure. Details on random sampling from the different copulas are in Appendix B. Before discussing the statistical performance of the different procedures, we investigate their computational performance by reporting in table 1 the running times for a large sample ($n = 500$) and high dimension ($d = 10$) under the Frank copula model, using Matlab on a Windows 10 laptop with an intel i7-6500U CPU and 8 GB of RAM. Similar results were obtained under other specifications of copula models.

Table 1: Running Times of S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$, as estimated from 1000 replicates, in the $n = 500$, $d = 10$ case, under the Frank copula model.

	Mean (sec)	Max (sec)	Min (sec)
S_n^{plain}	0.90	2.20	0.80
S_n^{mid}	0.96	2.00	0.85
S_n^{GN}	0.89	2.61	0.78
R_n^N	35.58	40.57	34.08
$G_{4,n}$	3.52	14.13	3.03

The fastest procedures are based on the Cramér–von Mises statistics, with an average time of about 1 second. S_n^{GN} slightly over-performs, the other two due to the evaluation of only one derivative. $G_{4,n}$ comes after, with an average running time of 3.5 seconds. For this test, we relied on the Matlab Statistic and Machine learning toolbox function *bootstrp*. It is possible that the running time could be lowered with a dedicated bootstrap procedure. The slowest approach is R_n^N. In our understanding, this is due to the need to evaluate a $d \times d \times n \times n$ tensor of second derivatives.

Table 2: Percentages of rejection at 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Pearson Type II copula.

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	5.0	3.7	1.1	4.8	4.9	3.2	6.1	4.9	4.2
S_n^{mid}	4.7	3.6	1.2	4.4	5.1	3.4	5.7	5.0	4.4
S_n^{GN}	2.8	2.2	0.6	3.2	3.4	1.7	4.7	3.9	3.0
R_n^N	5.9	4.5	1.9	6.2	6.1	4.3	6.0	6.3	5.6
$G_{4,n}$	4.7	4.3	2.0	5.6	5.7	2.5	5.3	4.7	3.3
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	2.2	2.9	0.3	2.4	4.1	1.8	3.9	4.9	2.6
S_n^{mid}	2.3	2.9	0.4	2.6	4.0	2.1	4.2	4.7	3.0
S_n^{GN}	0.5	0.6	0.0	1.3	2.3	0.9	2.5	3.5	0.8
R_n^N	4.1	5.0	3.0	4.8	5.4	3.1	4.6	6.1	3.3
$G_{4,n}$	4.1	6.8	2.1	5.6	6.0	2.6	5.0	5.6	3.6
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	1.4	1.7	0.1	1.2	2.4	0.5	2.2	3.2	1.1
S_n^{mid}	1.3	1.8	0.1	1.1	2.2	0.6	2.4	3.3	1.0
S_n^{GN}	0.0	0.0	0.0	0.0	0.4	0.1	0.3	0.9	0.1
R_n^N	3.5	4.0	2.7	3.3	4.2	3.0	4.5	5.0	2.8
$G_{5,n}$	2.5	9.7	4.7	4.0	7.8	4.3	4.3	7.7	4.7

3.1 Eliptical Family and Related Asymmetric Copulas

We first investigate the different procedures' success in replicating the distribution of the test statistics under the null hypothesis. We report in table 2 , as a representative example, results for the test's size under the radially symmetric elliptical family of Pearson Type II copula. Similar results for Normal, t-student with $\nu = 4$ degrees of freedom and Laplace copulas are available upon request.

The table presents different sample sizes $n \in \{125, 250, 500\}$, different dimensions of the copula $d \in \{2, 5, 10\}$ and different levels of dependence. In particular, we use as a dependence measure Kendall's τ on pairs of random variables imposing values in set $\{.25, .50, .75\}$.

Most procedures are close to their 5% nominal level in every set-up, excluding $G_{4,n}$. This statistic is often above the nominal level. Its size is almost doubled for a low number of observations, high dimension and average dependence.

S_n^{GN} is the most conservative, followed by S_n^{plain} and S_n^{mid}. R_n^N, appears slightly more liberal. Conservatism decreases with sample size and increases with dependence and dimension.

To study the power of the tests based on S_n, S_n^{mid}, S_n^{GN} and R_n^N we consider the radially asymmetric squared elliptical family, whose most notable members are the χ^2 and Fisher copula [24]. Given a random vector $\mathbf{U} \sim C$, the stochastic representation of a random vector $\mathbf{V}$, distributed according to squared copula associated to C, is $\mathbf{V} = \sqrt{(\mathbf{1}_d - 2\mathbf{U})^2}$, where the square root is taken elementwise. Squared normal copula and squared t copula are the copula of random vectors distributed as χ^2 and Fisher distributions. We use the same simulated samples used for the elliptical family experiments, obtaining results for χ^2 (table 3), Fisher (table 4), squared Laplace (table 5) and squared Pearson Type II (table 6) copulas. For this reason, Kendall's τ for this set of experiments is that of the associated elliptical random vector. We refer to [24] for the relationship among the Kendall's τ of the squared and original copula.

In the bivariate case, the most powerful test is usually $G_{4,n}$, excluding $\tau = 0.5$ in the Pearson Type II case. Instead, for a higher number of dimensions, $G_{4,n}$ is among the worst ones, with S_n^{GN}. R_n^N, S_n^{plain} and S_n^{mid} are close in performance for the χ^2 and Fisher copula. R_n^N is, usually, the most powerful test in the squared Pearson Type II case, while S_n^{plain} and S_n^{mid} are better in the squared Laplace case.

In the squared elliptic family, the monotone behavior of dependence with power suggests a non-decreasing relationship between asymmetry and dependence. To disentangle the interaction of asymmetry and dependence, we close the section by introducing another asymmetric variant of the Student t copula.

Table 3: Percentages of rejection at 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the χ^2 copula (*Kendall'τ refers to the associated normal copula).

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	5.7	42.8	73.2	9.7	78.1	97.0	17.4	98.0	100.0
S_n^{mid}	5.8	42.9	73.1	9.3	78.0	96.9	17.1	98.3	100.0
S_n^{GN}	3.5	34.8	60.5	8.4	75.2	96.5	15.6	98.1	100.0
R_n^N	9.5	58.2	83.4	15.0	88.1	98.6	27.4	99.7	100.0
$G_{4,n}$	11.5	66.0	95.4	20.7	92.3	100.0	37.9	99.8	100.0
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	7.9	94.0	94.8	25.4	100.0	100.0	70.8	100.0	100.0
S_n^{mid}	7.7	94.0	95.1	25.5	100.0	100.0	71.3	100.0	100.0
S_n^{GN}	1.6	81.1	82.0	16.3	100.0	99.9	63.8	100.0	100.0
R_n^N	19.5	98.6	95.9	50.2	100.0	100.0	88.8	100.0	100.0
$G_{4,n}$	5.5	93.4	100.0	35.2	100.0	100.0	77.0	100.0	100.0
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	1.6	95.5	93.0	13.7	100.0	100.0	74.6	100.0	100.0
S_n^{mid}	1.6	95.5	93.7	14.1	100.0	100.0	75.2	100.0	100.0
S_n^{GN}	0.0	62.6	57.6	2.1	99.9	99.8	47.5	100.0	100.0
R_n^N	29.1	100.0	94.9	84.5	100.0	100.0	99.9	100.0	100.0
$G_{4,n}$	0.1	59.6	99.9	5.7	97.1	100.0	32.9	100.0	100.0

Table 4: Percentages of rejection at 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Fisher copula (*Kendall'τ refers to the associated Student-t copula).

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	20.0	51.6	74.1	43.3	85.0	97.8	77.6	99.9	100.0
S_n^{mid}	19.6	51.7	74.7	43.8	85.5	97.8	77.6	99.9	100.0
S_n^{GN}	14.4	42.8	60.7	40.4	83.5	97.1	76.4	99.9	100.0
R_n^N	28.6	62.2	80.0	56.9	91.0	98.4	86.2	100.0	100.0
$G_{4,n}$	40.4	75.4	93.4	71.5	96.9	100.0	93.2	100.0	100.0
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	64.3	95.2	97.4	97.8	100.0	100.0	100.0	100.0	100.0
S_n^{mid}	64.6	95.6	97.5	97.8	100.0	100.0	100.0	100.0	100.0
S_n^{GN}	35.6	84.9	85.6	95.0	100.0	100.0	100.0	100.0	100.0
R_n^N	70.1	95.3	95.9	98.1	100.0	100.0	100.0	100.0	100.0
$G_{4,n}$	29.1	92.3	100.0	79.4	100.0	100.0	99.1	100.0	100.0
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	54.7	98.8	96.3	98.7	100.0	100.0	100.0	100.0	100.0
S_n^{mid}	53.2	98.8	97.1	98.7	100.0	100.0	100.0	100.0	100.0
S_n^{GN}	6.0	75.6	61.7	77.1	100.0	100.0	100.0	100.0	100.0
R_n^N	92.7	99.5	94.6	100.0	100.0	100.0	100.0	100.0	100.0
$G_{4,n}$	2.8	68.7	99.8	25.9	98.8	100.0	71.6	100.0	100.0

Table 5: Percentages of rejection at 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Squared Laplace copula (*Kendall'τ refers to the associated Laplace copula).

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	6.9	27.6	59.6	12.6	56.5	93.6	25.2	88.1	99.9
S_n^{mid}	7.1	28.0	60.8	11.9	57.6	93.9	25.2	88.4	99.9
S_n^{GN}	3.8	20.2	46.5	10.8	54.7	93.0	23.0	86.3	99.9
R_n^N	9.6	38.3	72.4	18.1	66.4	96.3	32.7	92.7	99.9
$G_{4,n}$	12.8	48.5	87.1	26.1	80.3	99.6	43.0	97.6	100.0
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	69.4	82.0	92.4	97.5	99.1	100.0	100.0	100.0	100.0
S_n^{mid}	68.8	82.0	92.6	97.2	99.1	100.0	100.0	100.0	100.0
S_n^{GN}	35.8	55.5	73.5	94.0	98.8	100.0	100.0	100.0	100.0
R_n^N	18.5	61.8	92.8	55.4	92.8	99.8	96.8	99.9	100.0
$G_{4,n}$	5.4	50.0	97.9	4.9	81.2	100.0	6.5	97.7	100.0
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	97.4	96.4	92.6	100.0	100.0	100.0	100.0	100.0	100.0
S_n^{mid}	96.4	96.4	93.8	100.0	100.0	100.0	100.0	100.0	100.0
S_n^{GN}	14.2	40.4	44.6	100.0	100.0	99.9	100.0	100.0	100.0
R_n^N	36.3	69.9	92.1	97.3	98.1	99.9	100.0	100.0	100.0
$G_{4,n}$	12.7	31.3	95.2	12.4	60.7	100.0	11.9	88.8	100.0

Table 6: Percentages of rejection at 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Squared Pearson Type II copula (*Kendall'τ refers to the associated Pearson Type II copula).

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	8.7	46.5	50.2	15.4	81.2	87.7	35.0	99.2	99.8
S_n^{mid}	8.2	45.7	51.4	16.5	82.3	87.2	37.0	99.3	99.9
S_n^{GN}	6.8	33.7	37.7	12.4	79.3	85.0	34.0	99.1	99.7
R_n^N	9.3	48.3	61.4	19.5	78.3	91.0	40.1	97.3	99.5
$G_{4,n}$	33.8	20.6	80.0	75.3	35.9	98.0	98.8	58.4	100.0
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	12.4	97.5	91.4	74.8	100.0	99.8	100.0	100.0	100.0
S_n^{mid}	11.3	97.8	91.2	74.8	100.0	99.8	99.9	100.0	100.0
S_n^{GN}	6.0	90.6	74.4	68.5	100.0	99.7	99.8	100.0	100.0
R_n^N	27.9	99.9	93.4	76.9	100.0	99.8	99.6	100.0	100.0
$G_{4,n}$	25.3	99.0	100.0	65.6	100.0	100.0	89.7	100.0	100.0
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	0.4	97.0	90.7	14.8	100.0	99.9	94.2	100.0	100.0
S_n^{mid}	0.3	97.2	91.6	14.7	100.0	99.9	94.3	100.0	100.0
S_n^{GN}	0.0	63.1	48.2	1.7	100.0	99.7	70.6	100.0	100.0
R_n^N	20.6	100.0	93.4	82.5	100.0	100.0	99.9	100.0	100.0
$G_{4,n}$	4.2	85.5	100.0	20.5	99.3	100.0	47.2	100.0	100.0

The skew-t family has a degree of asymmetry governed by the vector of parameters γ. We focus on the case $\gamma = \gamma \mathbf{1}_d$. We detail the tests' performance under a Skew-t family, with $\nu = 4$ degrees of freedom under different degrees of dependence as a function of the asymmetry parameter γ taking values in $\{-1, -0.5, 0\}$. Experiments with $\gamma = 0.5, 1$ were also conducted but not reported because they show a behavior similar to their negative counterparts. We show results for dimension 2 in table 7 and dimension 10 in table 8. As expected, the power increases with asymmetry for all the procedures. With constant asymmetry, power decreases if we increase dependence. Power decreases with dimension. In dimension 2, the test based on $G_{4,n}$ prevails, while the S_n^{GN} based test has the worst performance. The same results carry over in the $d = 10$ case, if we do not consider the case $n = 250$, $\tau = 0.25$ and $\gamma = -0.5$, in which the best performing tests appear to be the ones based on S_n^{plain} and S_n^{mid}. In general, for moderate asymmetry and high dependence, all procedures fail to deliver adequate power even with $n = 500$.

Table 7: Percentages of rejection at a 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Skew t-student copula with $\nu = 4$ degrees of freedom, in dimension 2.

n=125	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
γ	-1	0.5	0	-1	0.5	0	-1	0.5	0
S_n^{plain}	59.4	27.2	4.7	34.9	15.1	3.5	11.6	6.7	2.1
S_n^{mid}	58.3	26.5	4.5	35.7	15.0	3.2	11.9	7.0	3.0
S_n^{GN}	56.1	25.7	3.8	31.0	13.4	2.2	7.8	4.4	0.6
R_n^N	60.9	29.0	4.5	36.5	15.2	3.3	9.2	4.2	1.8
$G_{4,n}$	76.3	40.5	6.0	45.5	20.3	3.9	3.6	2.4	0.9
n=250	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
γ	-1	0.5	0	-1	0.5	0	-1	0.5	0
S_n^{plain}	87.8	47.9	4.5	63.8	28.4	3.6	31.4	12.4	3.3
S_n^{mid}	87.3	48.1	4.4	64.2	28.2	4.1	33.6	13.1	3.6
S_n^{GN}	86.9	46.4	4.1	61.0	26.6	3.3	26.3	9.0	2.3
R_n^N	89.7	52.3	4.4	67.8	30.4	3.7	41.0	14.2	2.1
$G_{4,n}$	96.4	67.6	4.8	84.0	41.8	5.3	50.1	17.7	1.0
n=500	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
γ	-1	0.5	0	-1	0.5	0	-1	0.5	0
S_n^{plain}	99.4	80.3	2.7	94.4	50.7	3.6	69.3	24.2	3.8
S_n^{mid}	99.4	79.5	3.1	94.5	50.6	3.9	69.1	25.6	4.6
S_n^{GN}	99.4	79.4	2.6	93.7	48.8	3.3	63.5	21.0	2.7
R_n^N	99.7	84.0	3.3	94.8	56.0	4.1	77.8	31.5	4.1
$G_{4,n}$	100.0	94.1	4.9	99.7	73.9	4.9	94.3	46.8	3.0

3.2 Archimedean Family

We include Archimedean copulas in the study, focusing on the non-symmetric Clayton and Gumbel families (tables 9 and 10, respectively), and on the Frank family (table 11), which is symmetric in two dimensions and mildly asymmetric beyond dimension two.

In the Clayton case, asymmetry is strong, and all tests deliver adequate power in most cases with some difficulties for $n = 125$ and low or high dependence. $G_{4,n}$ is again the champion in the bivariate case, while for higher dimensions, different procedures deliver the best performance in other cases. For this family, the overall relationship between power and dimension and between power and dependence is blurry.

In the Gumbel case, achieving sufficient power is more challenging and appears possible only for big samples in high dimensions. In this case, increasing dimension helps. The relationship with dependence is increasing for $d = 2$ and decreasing in the higher dimensional cases. In the vast majority of cases, the $G_{4,n}$ statistic outperforms the others if we neglect the case $n = 125, d = 10$ and $\tau = 0.25$, in which the best procedures are those based on S_n^{plain} and S_n^{mid}.

In table 11, we study the Frank family, which is symmetric in two dimensions and mildly asymmetric beyond dimension two. In dimension 2, we reproduce the situation discussed for the elliptic family. In the non-symmetric cases, instead, we see that Cramér von Mises based tests generally outperform $G_{4,n}$ and the

Table 8: Percentages of rejection at a 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Skew t-student copula with $\nu = 4$ degrees of freedom, in dimension 10.

n=125	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
γ	-1	0.5	0	-1	0.5	0	-1	0.5	0
S_n^{plain}	90.8	48.8	2.6	31.2	11.8	0.9	0.0	0.1	0.0
S_n^{mid}	90.9	47.3	2.4	32.7	12.5	0.9	0.0	0.1	0.0
S_n^{GN}	67.6	14.4	0.0	4.6	1.8	0.0	0.0	0.0	0.0
R_n^N	91.0	61.3	2.2	52.3	22.6	3.5	17.9	7.9	3.1
$G_{4,n}$	96.3	69.7	6.2	78.1	47.8	7.3	15.7	9.2	2.2
n=250	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
γ	-1	0.5	0	-1	0.5	0	-1	0.5	0
S_n^{plain}	100.0	96.8	3.3	93.4	50.6	2.6	7.4	3.9	0.4
S_n^{mid}	100.0	96.8	2.9	93.9	50.5	2.6	8.9	4.4	0.4
S_n^{GN}	99.9	86.8	0.1	79.9	31.1	0.2	0.1	0.3	0.0
R_n^N	100.0	93.7	2.5	86.0	46.6	3.4	45.4	17.2	3.3
$G_{4,n}$	100.0	88.5	7.4	98.8	71.6	6.9	71.6	33.0	3.4
n=500	$\tau = 0.25$			$\tau = 0.5$			$\tau = 0.75$		
γ	-1	0.5	0	-1	0.5	0	-1	0.5	0
S_n^{plain}	100.0	100.0	3.7	100.0	90.1	2.4	68.7	24.4	1.0
S_n^{mid}	100.0	100.0	3.6	100.0	90.0	2.6	71.3	26.1	1.2
S_n^{GN}	100.0	100.0	0.8	100.0	83.1	1.2	35.5	9.9	0.0
R_n^N	100.0	100.0	3.1	99.3	79.1	3.2	78.7	33.8	4.3
$G_{4,n}$	100.0	98.9	6.1	100.0	94.1	5.6	99.6	75.7	4.0

Table 9: Percentages of rejection at a 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Clayton copula.

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	43.7	89.2	97.0	72.9	99.7	100.0	95.2	100.0	100.0
S_n^{mid}	43.0	88.3	97.0	72.5	99.8	100.0	95.3	100.0	100.0
S_n^{GN}	40.9	86.8	95.8	70.6	99.7	100.0	94.9	100.0	100.0
R_n^N	45.0	90.2	97.3	77.0	99.8	100.0	96.2	100.0	100.0
$G_{4,n}$	61.6	96.5	99.3	87.9	100.0	100.0	99.1	100.0	100.0
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	90.1	99.4	93.5	100.0	100.0	100.0	100.0	100.0	100.0
S_n^{mid}	89.2	99.3	94.0	100.0	100.0	100.0	100.0	100.0	100.0
S_n^{GN}	81.6	98.4	72.9	100.0	100.0	100.0	100.0	100.0	100.0
R_n^N	85.9	98.2	98.8	99.7	100.0	100.0	100.0	100.0	100.0
$G_{4,n}$	81.8	100.0	100.0	97.0	100.0	100.0	100.0	100.0	100.0
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	88.9	99.9	50.4	100.0	100.0	100.0	100.0	100.0	100.0
S_n^{mid}	87.3	100.0	61.3	100.0	100.0	100.0	100.0	100.0	100.0
S_n^{GN}	45.0	98.2	6.1	99.8	100.0	100.0	100.0	100.0	100.0
R_n^N	97.4	99.3	98.9	100.0	100.0	100.0	100.0	100.0	100.0
$G_{4,n}$	76.4	100.0	100.0	93.2	100.0	100.0	100.0	100.0	100.0

Table 10: Percentages of rejection at a 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Gumbel copula.

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	11.4	19.8	15.1	24.8	45.7	41.0	50.3	79.6	80.2
S_n^{mid}	10.8	19.9	16.0	25.1	46.7	43.2	51.0	79.3	81.1
S_n^{GN}	7.8	13.0	6.0	22.8	42.4	35.7	48.9	76.9	76.7
R_n^N	16.7	27.9	26.7	32.7	55.6	57.6	59.8	84.0	87.7
$G_{4,n}$	23.7	41.9	29.9	46.1	73.7	75.6	78.1	95.5	97.7
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	56.9	59.4	8.9	97.3	98.8	74.2	100.0	100.0	99.7
S_n^{mid}	57.0	60.9	9.7	97.4	98.9	77.4	100.0	100.0	99.7
S_n^{GN}	38.3	33.5	0.4	96.2	97.3	48.1	100.0	100.0	99.2
R_n^N	77.5	78.9	47.7	97.8	97.9	82.9	100.0	100.0	99.2
$G_{4,n}$	79.7	93.5	74.9	98.7	99.9	98.8	100.0	100.0	100.0
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	96.9	92.6	1.0	100.0	100.0	94.0	100.0	100.0	100.0
S_n^{mid}	97.0	93.2	1.7	100.0	100.0	95.7	100.0	100.0	100.0
S_n^{GN}	73.4	46.6	0.0	100.0	100.0	43.8	100.0	100.0	100.0
R_n^N	93.3	85.7	44.5	100.0	99.8	82.0	100.0	100.0	99.3
$G_{4,n}$	65.6	96.4	85.0	98.2	100.0	99.9	100.0	100.0	100.0

characteristic function test, with differences increasing with the number of observations and dimensions. Again S_n^{plain} and S_n^{mid} have the same behavior and are comparatively better than S_n^{GN} . Power deteriorates with dependence.

Table 11: Percentages of rejection at a 5% significance level, as estimated from 1000 replicates, for the tests based on S_n^{plain}, S_n^{mid}, S_n^{GN}, R_n^N and $G_{4,n}$ under the Frank copula.

d=2	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	4.6	4.6	3.6	4.9	4.7	3.6	4.7	4.0	3.6
S_n^{mid}	4.7	4.6	3.8	5.1	4.7	4.2	5.1	4.2	3.9
S_n^{GN}	3.0	2.4	1.8	4.1	3.6	2.2	3.9	3.2	2.4
R_n^N	5.2	4.3	3.0	5.7	4.8	4.7	4.9	4.8	3.9
$G_{4,n}$	5.4	6.0	1.8	5.9	5.1	4.3	5.4	4.6	3.9
d=5	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	19.2	6.9	0.4	78.8	42.8	2.6	99.9	97.4	16.0
S_n^{mid}	18.5	7.1	0.3	79.0	44.1	2.9	99.9	97.4	17.1
S_n^{GN}	9.2	2.1	0.0	70.7	30.9	0.7	99.7	95.6	6.1
R_n^N	25.9	13.4	4.6	65.3	21.1	5.8	98.1	46.4	6.7
$G_{4,n}$	17.2	18.1	7.1	44.8	31.6	13.3	72.9	56.6	18.2
d=10	n=125			n=250			n=500		
τ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
S_n^{plain}	92.7	46.0	0.1	100.0	99.9	7.4	100.0	100.0	99.4
S_n^{mid}	91.6	48.7	0.1	100.0	99.9	8.7	100.0	100.0	99.6
S_n^{GN}	64.1	10.6	0.0	100.0	99.6	0.4	100.0	100.0	82.5
R_n^N	58.3	12.1	3.0	98.7	31.9	4.0	100.0	84.9	6.9
$G_{4,n}$	9.3	31.1	11.6	65.4	70.7	23.4	96.9	96.2	43.7

4 Conclusions

This article extends the bivariate test of copula radial symmetry introduced in [11] to more than two dimensions. Additionally, it refines the inference procedure by proposing two new estimators of copula derivatives under the null. These refinements bring the Cramér von Mises test to a performance comparable to the major competitors based on a linear statistic [17] and the characteristic function [1]. In fact, in our extensive simulation study, including the recently proposed squared elliptic copulas [10] among other models, S_n^{GN} is always worse than or equal to the two other new procedures, that are overall equivalents in terms of performance. The new methods based on the Cramér von Mises statistic and the test based on characteristic function, proposed in [1], are comparable in retaining the nominal level under the null and have overall similar power under the alternative. In particular, R_n^N shows more power under the squared Pearson Type II copula, while S_n^{plain} and S_n^{mid} perform better in the squared Laplace and Frank cases. These procedures are more or less equivalent under the symmetric models and the other asymmetric models studied. The hypothesis tested by $G_{k,n}$ is different from copula radial symmetry, and the interpretation of the test requires additional care. Nevertheless, the test based on $G_{4,n}$ is the best performer for $d = 2$ for the copula model considered in the simulation study, even if slightly too liberal under the null. In higher dimensions, the discrepancy between the empirical size and the nominal one increases, and the size becomes almost twice the nominal in the worst cases. Under the alternative, beyond dimension 2, $G_{4,n}$ is the most powerful under the skew-t and Gumbel copula. Taking into account running times (table 1), a relevant aspect in the high dimensional domain, S_n^{plain} and S_n^{mid} appear the best trade-off between statistical and computational performance.

Overall, the simulation study shows how all the inference procedures studied appear reliable in high dimensions with low and moderate levels of dependence and high asymmetry, using a sample size of 500, and often of 250, observations. High dependence and medium or low asymmetry require an increase of the sample size beyond 500 observations. The relation between power and dimension depends on the underlying copula model.

The investigation of copula radial symmetry tests in the context of time series is left for future research. However, we remark that the use of continuous mapping theorem in [23] or the functional delta method, in the present paper, in deriving the asymptotic properties of the test based on the Cramér von Mises statistic allows an easy extension to strongly mixing data along the lines of [3], covering most of the known stationary parametric models. Similar comments apply to $G_{4,k}$ if we introduce the use of circulant bootstrap or other dependent bootstraps, while the extension of R_n^N appears more challenging.

Acknowledgements: The views expressed by the second author are the author's alone and do not necessarily correspond to those of the European Commission. This project has received funding from the European Union's Seventh Framework Programme (FP7-SSH/2007-2013) for research, technological development and demonstration under grant agreement 320270 - SYRTO. Monica Billio acknowledges financial support from Italian Ministry MIUR under the PRIN project Hi-Di NET - Econometric Analysis of High Dimensional Models with Network Structures in Macroeconomics and Finance (grant 2017TA7TYC). The first version of this paper was part of the Ph.D. thesis of the second author. We wish to thank Umberto Cherubini and Olivier Scaillet for their valuable discussions and comments.

A Proofs

The pseudo-observations $\hat{U}_{ij} = \frac{1}{n+1}\sum_{k=1}^{n}\mathbb{I}\left(X_{kj} \leq X_{ij}\right)$, $i = 1, \ldots, n$ and $j = 1, \ldots, d$ are asymptotically equivalent to $\frac{1}{n}\sum_{k=1}^{n}\mathbb{I}\left(X_{kj} \leq X_{ij}\right)$, $i = 1, \ldots, n$ and $j = 1, \ldots, d$. Since the weak convergence of the empirical copula process is derived in [27] using the latter expression for pseudo-observation, in the proof we follow this convention and we call it, with a slight abuse of notation, $\hat{U}_{ij} = \frac{1}{n}\sum_{k=1}^{n}\mathbb{I}\left(X_{kj} \leq X_{ij}\right)$, $i = 1, \ldots, n$ and $j = 1, \ldots, d$. In the $n \to \infty$ limit, results obtained in this way are equivalent to the ones obtainable using the main text's convention. The use of the alternative convention in the main text is due to better finite sample behavior.

A.1 Proof of Proposition 1

The proof obtains a relationship between $C_{n,I,J}$ and the empirical distribution function of $V_{ij} = T_J\left(U_{ij}\right)$, whose empirical process is known. The relationship requires introducing an auxiliary function equivalent to a particular expected value of a product of indicators and the empirical version of this expected value. This relationship is the map studied in [4], which derives its Hadamard differentiability. The result follows from an application of the functional delta method to the map.

We define the function $F_{I,J} : \mathbb{R}^p \mapsto \mathbb{R}$,

$$F_{I,J}(\mathbf{x}) = \mathbb{E}\left[\prod_{j\in J}\mathbb{I}\left(X_j > x_j\right)\prod_{j\in I\setminus J}\mathbb{I}\left(X_j \leq x_j\right)\right],$$

For each $j \in I$, we define as $F_{I,J,j}\left(x_j\right)$ the limits of $F_{I,J}(\mathbf{x})$, letting $x_{j'} \to -\infty$ for all $j' \in J\setminus j$ and letting $x_{j'} \to \infty$ for all $j' \in I \setminus \left(J \setminus j\right)$:

$$F_{I,J,j}\left(x_j\right) = \begin{cases} \bar{F}_j\left(x_j\right) & \text{if} \quad j \in J \\ F_j\left(x_j\right) & \text{if} \quad j \in I\setminus J \end{cases},$$

For each $j \in I$, we define the generalized inverse of $F_{I,J,j}$:

$$F^{-}_{I,J,j}(u) = \begin{cases} \sup\left\{x\in\mathbb{R} : F_{I,J,j}(x) \leq u\right\}, u \in [0,1] & \text{if} \quad j\in J \\ \inf\left\{x\in\mathbb{R} : F_{I,J,j}(x) \geq u\right\}, u \in [0,1] & \text{if} \quad j\in I\setminus J \end{cases}.$$

The copula of the transformed vector $C_{I,J}$ can be expressed as a function of $F_{I,J}$ and of $F^{-}_{I,J,j}$, $j \in I$,

$$\begin{aligned} C_{I,J}(\mathbf{v}) &= \mathbb{E}\left[\prod_{j\in J}\mathbb{I}\left(\bar{F}_j\left(X_j\right) \leq v_j\right)\prod_{j\in I\setminus J}\mathbb{I}\left(F_j\left(X_j\right) \leq v_j\right)\right] \\ &= \mathbb{E}\left[\prod_{j\in J}\mathbb{I}\left(X_j > \bar{F}^{-}_j\left(v_j\right)\right)\prod_{j\in I\setminus J}\mathbb{I}\left(X_j \leq F^{-}_j\left(v_j\right)\right)\right] \\ &= F_{I,J}\left(F^{-}_{I,J,i_1}\left(v_{i_1}\right),\ldots,F^{-}_{I,J,i_p}\left(v_{i_p}\right)\right) \end{aligned} \tag{A.1}$$

The empirical versions of $F_{I,J}$ and $F_{I,J,j}$ with $j \in I$, are defined as follows:

$$F_{n,I,J}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{j\in J}\mathbb{I}\left(X_{ij} > x_j\right)\prod_{j\in I\setminus J}\mathbb{I}\left(X_{ij} \leq x_j\right)$$

$$F_{n,I,J,j}\left(x_j\right) = \begin{cases} \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(X_{ij} > x_j\right) & \text{if} \quad j\in J \\ \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(X_{ij} \leq x_j\right) & \text{if} \quad j\in I\setminus J \end{cases}$$

The empirical counterpart of equation (A.1) is:

$$\begin{aligned} C_{n,I,J}(\mathbf{v}) &= \frac{1}{n}\sum_{i=1}^{n}\prod_{j\in I}\mathbb{I}\left(R_J\left(\hat{U}_{ji}\right) \leq v_j\right) \\ &= F_{n,I,J}\left(F^{-}_{n,I,J,i_1}\left(v_{i_1}\right),\ldots,F^{-}_{n,I,J,i_p}\left(v_{i_p}\right)\right) \end{aligned} \tag{A.2}$$

We introduce the empirical distribution function of $V_{ij} = T_J\left(U_{ij}\right)$:

$$G_n(\mathbf{v}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}\left(\mathbf{V}_i \leq \mathbf{v}\right)$$

Using the results in [8], the multivariate empirical process on $[0,1]^p$, $\mathbb{G}_n = \sqrt{n}\left(G_n - C_{I,J}\right)$ has weak convergence limit:

$$\begin{aligned} \mathbb{G}_n(\mathbf{u}) &\rightsquigarrow \mathbb{B}_{C_{I,J}}(\mathbf{u}) \\ \mathbb{C}\text{ov}\left(\mathbb{B}_{C_{I,J}}(\mathbf{u}), \mathbb{B}_{C_{I,J}}(\mathbf{v})\right) &= C_{I,J}(\mathbf{u}\wedge\mathbf{v}) - C_{I,J}(\mathbf{u})\,C_{I,J}(\mathbf{v}). \end{aligned}$$

In what follows, we derive the relationship between G_n and $C_{n,I,J}$ using $F_{n,I,J}$ and equation (A.3). We have the following expression for G_n:

$$\begin{aligned} G_n(\mathbf{v}) &= \frac{1}{n}\sum_{i=1}^{n}\prod_{j_k\in J}\mathbb{I}\left(X_{ij} > F^{-}_{I,J,j}(v_j)\right)\prod_{j\in I\setminus J}\mathbb{I}\left(X_{ij} \leq F^{-}_{I,J,j}(v_j)\right) \\ &= F_{n,I,J}\left(F^{-}_{I,J,i_1}\left(v_{i_1}\right), \ldots, F^{-}_{I,J,i_p}\left(v_{i_p}\right)\right). \end{aligned} \tag{A.3}$$

For each $j \in I$, letting $v_{j'} \to 1$ for all $j' \in I\setminus j$ in equation (A.3), we obtain the following relationship between the marginals of G_n , $F^{-}_{I,J,j}$ and $F_{n,I,J,j}$ with $j \in I$:

$$G_{nj}\left(v_j\right) = F_{n,I,J,j}\left(F^{-}_{I,J,j}\left(v_j\right)\right)$$

Then, using the composition properties of the generalized inverses, it follows that:

$$G^{-}_{nj}\left(v_j\right) = F_{I,J,j}\left(F^{-}_{n,I,J,j}\left(v_j\right)\right). \tag{A.4}$$

Substituting (A.4) in equation (A.3) and using equation (A.2), we obtain the following alternative representation of $C_{n,I,J}$:

$$G_n\left(G^{-}_{ni_1}\left(v_{i_1}\right), \ldots, G^{-}_{ni_p}\left(v_{i_p}\right)\right) = C_{n,I,J}(\mathbf{v}).$$

We now introduce the map [4]

$$\Phi : \begin{cases} \mathbb{D}_{\Phi} \mapsto \ell^{\infty}[0,1]^p \\ H \mapsto H\left(H_1^{-1}, \ldots, H_p^{-1}\right) \end{cases},$$

where $\mathbb{D}_{\Phi}$ denotes the set of all distribution functions H on $[0,1]^p$, whose marginal CDFs H_j satisfy $H_j(0) = 0$, $j \in \{1,\ldots,p\}$. Using this map, the empirical copula process of the transformation can be expressed as follows:

$$\mathbb{C}_{n,I,J} = \sqrt{n}\left(\Phi\left(G_n\right) - \Phi\left(C_{I,J}\right)\right). \tag{A.5}$$

Theorem 2.4 in [4] implies that, under **A 1**, Φ is Hadamard differentiable at $C_{I,J}$, and the application of the functional delta method to (A.5) yields the result.

A.2 Proof of Proposition 2

Let $C[0,1]^d$ be the space of function $f : [0,1]^d \to \mathbb{R}$ that are continuous $D[0,1]^d$; the space of cadlag function on $[0,1]^d$; and $BV1[0,1]^d$ the subspace of $D[0,1]^d$ consisting of the functions with total variation bounded by one. For notational convenience, we consider only one multiplier replicate the generalization to M replicates being straightforward. Under the null and assumption **A1**, (2.14)-(2.18) hold. We can write:

$$\left(\left(\mathbb{C}_n - \bar{\mathbb{C}}_n\right)^2, \left(\tilde{\mathbb{C}}^{[1],ver}_n - \tilde{\bar{\mathbb{C}}}^{[1],ver}_n\right)^2, \mathbb{C}_n\right) = \sqrt{n}\left(\left(\mathbb{A}_n, \mathbb{A}^{[1],ver}_n, \hat{C}_n\right) - \left(\mathbb{A}, \mathbb{A}^{[1],ver}, C\right)\right)$$

where $\mathbb{A}_n = \sqrt{n}\left(C_n - \bar{C}_n\right)^2$, $\hat{\mathbb{A}}_n^{[1],ver} = \frac{1}{\sqrt{n}}\left(\left(\tilde{\mathbb{C}}_n^{[1],ver} - \tilde{\bar{\mathbb{C}}}_n^{[1],ver}\right)^2\right)$, $\mathbb{A} = \mathbb{A}^{[1]} = 0$ and $ver \in \{plain, GN, mid\}$.

From continuous mapping theorem, we get:

$$\left(\left(\mathbb{C}_n - \bar{\mathbb{C}}_n\right)^2, \left(\tilde{\mathbb{C}}_n^{[1],ver} - \tilde{\bar{\mathbb{C}}}_n^{[1],ver}\right)^2, \mathbb{C}_n\right) \rightsquigarrow \left(\left(\mathbb{C} - \bar{\mathbb{C}}\right)^2, \left(\mathbb{C}^{[1],ver} - \bar{\mathbb{C}}^{[1],ver}\right)^2, \mathbb{C}\right)$$

on $\left[\ell^\infty [0,1]^d\right]^4$.

Let us introduce the map $\Psi : \ell^\infty [0,1]^d \times \ell^\infty [0,1]^d \times BV1\,[0,1]^d \longrightarrow \mathbb{R}^2$, defined by

$$\Psi(\alpha, \tilde{\alpha}, \beta) = \left(\int_{(0,1]^d} \alpha d\beta, \int_{(0,1]^d} \tilde{\alpha} d\beta\right). \tag{A.6}$$

We have, then,

$$\left(n\mathbb{T}_n, n\tilde{\mathbb{T}}_n^{[1]}\right) = \sqrt{n}\left(\Psi\left(\mathbb{A}_n, \hat{\mathbb{A}}_n^{[1],ver}, C_n\right) - \Psi\left(\mathbb{A}, \mathbb{A}^{[1],ver}, C\right)\right).$$

We state the Hadamard differentiability of Ψ tangentially to $C[0,1]^d \times C[0,1]^d \times d[0,1]^d$ at each $(\alpha, \tilde{\alpha}, \beta)$ in $\ell^\infty [0,1]^d \times \ell^\infty [0,1]^d \times BV1\,[0,1]^d$, such that $\int |d\alpha| < \infty$ and $\int |d\tilde{\alpha}| < \infty$ in the lemma 1 below. Then, an application of the functional delta method gives

$$\left(n\hat{\mathbb{T}}_n, n\tilde{\mathbb{T}}_n^{[1],ver}\right) \rightsquigarrow \Psi'_{A,A^{[1],ver},C}\left(\left(\mathbb{C} - \bar{\mathbb{C}}\right)^2, \left(\mathbb{C}^{[1],ver} - \bar{\mathbb{C}}^{[1],ver}\right)^2, \mathbb{C}\right)$$

with

$$\Psi'_{A,A^{[1],ver},C}\left(\left(\mathbb{C} - \bar{\mathbb{C}}\right)^2, \left(\mathbb{C}^{[1],ver} - \bar{\mathbb{C}}^{[1],ver}\right)^2, \mathbb{C}\right)$$

$$= \left(\int_{(0,1]^d} \mathbb{A} d\mathbb{C} + \int_{(0,1]^d} \left(\mathbb{C} - \bar{\mathbb{C}}\right)^2 dC, \int_{(0,1]^d} \mathbb{A}^{[1],ver} d\mathbb{C} + \int_{(0,1]^d} \left(\mathbb{C}^{[1],ver} - \bar{\mathbb{C}}^{[1],ver}\right)^2 dC\right)$$

$$= \left(\int_{(0,1]^d} \left(\mathbb{C} - \bar{\mathbb{C}}\right)^2 dC, \int_{(0,1]^d} \left(\mathbb{C}^{[1],ver} - \bar{\mathbb{C}}^{[1],ver}\right)^2 dC\right) = \left(\mathbb{T}, \mathbb{T}^{[1],ver}\right).$$

Lemma 1. *The map Ψ defined in (A.6) is Hadamard differentiable tangentially to $C[0,1]^d \times C[0,1]^d \times d[0,1]^d$ at each $(\alpha, \tilde{\alpha}, \beta)$ in $\ell^\infty [0,1]^d \times \ell^\infty [0,1]^d \times BV1\,[0,1]^d$, such that $\int |d\alpha| < \infty$ and $\int |d\tilde{\alpha}| < \infty$ with derivative given by*

$$\Psi'_{A,\tilde{A},B}(\alpha, \tilde{\alpha}, \beta) = \left(\int_{(0,1]^d} A d\beta + \int_{(0,1]^d} \alpha dB, \int_{(0,1]^d} \tilde{A} d\beta + \int_{(0,1]^d} \tilde{\alpha} dB\right)$$

where if β is not of bounded variation, $\int \alpha d\beta$, $\int \tilde{\alpha} d\beta$ are defined via the d-dimensional integration by parts formula exemplified for 2 dimensions in Theorem 8.8 of [13].

Lemma 1 is a vectorized d-dimensional version of lemma 3.9.17 in [29] (see, also, lemma 4.3 of [5]) and, since the proof is similar, it will be omitted.

B Random Sampling

We report procedures and software used to simulate the different copula models included in the simulation study.

We start with the elliptical family. Normal copula and t-student samples come from the R *copula* package [15] and are based on the random sampling from the multivariate distribution and a component-wise probability integral transform of the obtained vector. We implemented in Matlab the same procedure for Laplace and Pearson Type II copulas. In particular, we used the general stochastic representation for elliptic multivariate distributions introduced in [9] for generation of the Pearson type II base random vector:

$$\mathbf{X} = RC\mathbf{S}$$

$\mathbf{S}$ is uniformly distributed on the d-dimensional sphere and can be easily obtained by generating a d-dimensional independent multivariate normal random vector $\mathbf{N}$ and dividing each component by its ℓ^2 norm. R is independent of $\mathbf{S}$ and can be obtained in the Pearson Type II case by simulating its square R^2 as a $\beta\left(d/2, 1/2\right)$ random variable and taking the square root. C is the Cholesky decomposition of the correlation matrix. The marginal distribution of $\mathbf{X}$ has the following expression:

$$F(x) = \frac{\Gamma\left((d-1)/2\right)}{\sqrt{\pi}\Gamma\left(d/2\right)} \int_{-1}^{x} \left(1-y^2\right)^{d/2-1} dy,$$

where $\Gamma(x)$ is the Euler gamma function, We report the expressions for $\mathbf{U}$ in the cases $d = 2, 3, 4, 5, 10$ (functions of vectors are taken component-wise):

$$\begin{aligned}
d = 2 &\quad \mathbf{U} = (\mathbf{X}+1)/2 \\
d = 3 &\quad \mathbf{U} = 1 + \frac{\mathbf{X}\sqrt{1-\mathbf{X}^2} - \arccos(\mathbf{X})}{\sqrt{\pi}} \\
d = 4 &\quad \mathbf{U} = -(\mathbf{X}+1)(\mathbf{X}-2)/4 \\
d = 5 &\quad \mathbf{U} = \frac{3\pi + 2\mathbf{X}(5-2\mathbf{X}^2)\sqrt{1-\mathbf{X}^2} + 6\arcsin(\mathbf{X})}{6\pi} \\
d = 10 &\quad \mathbf{U} = \frac{(1+\mathbf{X})^5(128 + 5\mathbf{X}(\mathbf{X}(69 + 7(\mathbf{X}-5)\mathbf{X})) - 65)}{256}
\end{aligned}$$

The stochastic representation for Laplace random vectors uses an independent normal random vector $\mathbf{N}$:

$$\begin{aligned}
\mathbf{X} &= RC\mathbf{N} \\
\mathbf{U} &= 1/2 + 1/2\,\mathrm{sign}(\mathbf{X})(1-\exp(-|\mathbf{X}|));
\end{aligned}$$

where $R^2 = 2E$ with $E \sim \mathrm{Exp}(1)$.

As already recalled in the main text, the squared elliptical copulas are obtained starting from an elliptic $\mathbf{U}$ and computing $\sqrt{(\mathbf{1}_d - 2\mathbf{U})}$.

The skew-t copula comes from the skew-t multivariate distribution, i.e. from the following stochastic representation :

$$\mathbf{X} = \gamma W + \sqrt{W}C\mathbf{N};$$

where, again, $\mathbf{N}$ is a vector of normal independent random variables, C is the Cholesky decomposition of the correlation matrix, and $W \sim Ig\left(\nu/2, \nu/2\right)$. This is a particular case of the multivariate generalized hyperbolic distribution [7]. For the sampling of $\mathbf{X}$ and the numerical integration of the CDF required for the PIT, we relied on the R package *ghyp* [30].

Finally, an exchangeable Archimedean copula with generator ψ has the following stochastic representation:

$$\mathbf{U} = \psi(\mathbf{E}/W);$$

where $\mathbf{E}$ is a random vector of independent $\text{Exp}(1)$ random variables and W is distributed as the inverse Laplace–Stieltjes transform of ψ. We refer to [14] for the distribution and random generation of W in the Frank, Clayton and Gumbel cases. The R *copula* package from which we obtained the samples uses this methodology.

References

[1] Bahraoui, T. and J.-F., Quessy (2017). Tests of radial symmetry for multivariate copulas based on the copula characteristic function. *Electron. J. Stat. 11*(1), 2066–2096.

[2] Bouzebda, S. and M., Cherfi (2012). Test of symmetry based on copula function. *J. Statist. Plann. Inference. 142*(5), 1262–1271.

[3] Bücher, A. and I., Kojadinovic (2016). A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing. *Bernoulli. 22*(2), 927–968.

[4] Bücher, A. and S., Volgushev (2013). Empirical and sequential empirical copula processes under serial dependence. *J. Multivariate Anal. 119*, 61–70.

[5] Carabarin-Aguirre, A. and B. G., Ivanoff (2010). Estimation of a distribution under generalized censoring. *J. Multivariate Anal. 101*(6), 1501–1519.

[6] Dehgani, A., Dolati, A. and M., Úbeda Flores (2013). Measures of radial asymmetry for bivariate random vectors. *Statist. Papers. 54*(2), 271–286.

[7] Demarta, S. and A. J., McNeil (2005) The t copula and related copulas. *Int. Stat. Rev. 73*(1), 111–129.

[8] Einmahl, J. H. J. (1987). *Multivariate Empirical Processes*. PhD thesis, Centrum voor Wiskunde en Informatica, Amsterdam.

[9] Fang, K. T., Kotz, S. and K. W., Ng (1990). *Symmetric Multivariate and Related Distributions*. Chapman and Hall, London.

[10] Favre, A.-C., Quessy, J.-F. and M.-H., Toupin (2018). The new family of Fisher copulas to model upper tail dependence and radial asymmetry: properties and application to high-dimensional rainfall data. *Environmetrics 29*(3), Article ID e2494, 17 pages.

[11] Genest, C. and J. G., Nešlehová (2014). On tests of radial symmetry for bivariate copulas. *Statist. Papers. 55*(4), 1107–1119.

[12] Hammersley, J. M. and K. W., Morton (1956). A new Monte Carlo technique: antithetic variates. *Proc. Cambridge Philos. Soc. 52*, 449–475.

[13] Hildebrandt, T. H. (1963) *Introduction to the Theory of Integration*. Academic Press, New York.

[14] Hofert, M. (2008). Sampling Archimedean copulas. *Comput. Statist. Data Anal. 52*(12), 5163–5174.

[15] Hofert, M., I., Kojadinovic, M., Maechler and J., Yan (2020). `copula`: *Multivariate Dependence with Copulas*. R package version 1.0-1. Available on CRAN.

[16] Joe, H. (2014). *Dependence Modeling with Copulas*. CRC Press, Boca Raton FL.

[17] Krupskii, P. (2017). Copula-based measures of reflection and permutation asymmetry and statistical tests. *Statist. Papers. 58*(4), 1165–1187.

[18] Lee, W. and J. Y., Ahn (2014). On the multidimensional extension of countermonotonicity and its applications. *Insurance Math. Econom. 56*, 68–79.

[19] Li, B. and M. G., Genton (2013). Nonparametric identification of copula structures. *J. Amer. Statist. Assoc. 108*(502), 666–675.

[20] Nelsen, R. B. (1993). Some concepts of bivariate symmetry. *J. Nonparametr. Statist. 3*(1), 95–101.

[21] Nelsen, R. B. (2006). *An Introduction to Copulas*. Second edition. Springer, New York.

[22] Nelsen, R. B. and M., Úbeda Flores (2012). Directional dependence in multivariate distributions. *Ann. Inst. Statist. Math. 64*(3), 677–685.

[23] Quessy, J.-F. (2016). A general framework for testing homogeneity hypotheses about copulas. *Electron. J. Stat. 10*(1), 1064–1097.

[24] Quessy, J.-F. and M., Durocher (2019). The class of copulas arising from squared distributions: properties and inference. *Econom. Stat. 12*, 148–166.

[25] Rémillard, B. and O., Scaillet (2009). Testing for equality between two copulas. *J. Multivariate Anal. 100*(3), 377–386.

[26] Rosco, J. F. and H., Joe, H. (2013). Measures of tail asymmetry for bivariate copulas. *Statist. Papers. 54*(3), 709–726.

[27] Segers, J. (2012). Asymptotics of empirical copula processes under non-restrictive smoothness assumptions. *Bernoulli. 18*(3), 764–782.

[28] Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris 8*, 229–231.

[29] van der Vaart, A. W. and J. A., Wellner (1996). *Weak Convergence and Empirical Processes*. Springer, New York.

[30] Weibel, M., D., Luethi and W., Breymann (2020). `ghyp`: *Generalized Hyperbolic Distribution and Its Special Cases*. R package version 1.6.1. Available on CRAN.