DOI: 10.1515/auom-2016-0050
An. Şt. Univ. Ovidius Constanţa VERSITA **Vol. 24(3),2016, 101–121**

HX-groups and Hypergroups

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Abstract

One considers the hypergroups associated with the HX-groups $\mathbf{Z}/n\mathbf{Z}$ and with the set of square matrices of order 2, with coefficients in $\mathbf{Z}/2\mathbf{Z}$ and one calculates their fuzzy grade.

1 Introduction

One finds in the literature on HX- groups some examples, but, at least those I have seen, they are all on infinite groups. So, it has been interesting to find some on finite groups. One has determined those corresponding with $\mathbf{Z}/n\mathbf{Z}$ and one has found the associated hypergroups (after the correspondence that, in [3] it is shown to exist).

Then one has considered the Fuzzy Grade [3, 4, 6, 7, 8, 9, 12, 13, 15] of these hypergroups, finding that for every n, it is always equal to one.

Finally, one has constructed the tables of associated hypergroups in the cases: $n = 8,\ n = 9,\ n = 12,\ n = 15,\ n = 16$ and of a set of subsets of $\mathbf{Z}_2^{(2,2)}$.

2 HX- groups on finite groups

Set $n \in \mathbf{N}$. We shall see that there is in $\mathbf{Z}/n\mathbf{Z}$ an HX-group, for every divisor of n.

Set $n = aq$ and let us consider the following subsets:

$$A_0 = [0, a, 2a, ..., (q-1)a]$$

Key Words: HX-group, Fuzzy grade.
2010 Mathematics Subject Classification: Primary 20N20; Secondary 20N25.
Received: 11.05.2015
Accepted: 30.06.2015

$$A_1 = [1, a+1, 2a+1, ..., (q-1)a+1]$$
$$A_2 = [2, a+2, 2a+2, ..., (q-1)a+2]$$
$$\cdots$$
$$A_s = [s, a+s, 2a+s, ..., (q-1)a+s]$$

where $s = a - 1$.

Let us denote $\mathcal{G}_a$ the set

$$\mathcal{G}_a \subseteq \mathcal{P}^*(\mathbf{Z}/n\mathbf{Z}), \quad \mathcal{G}_a = \{A_i \mid 0 \leq i \leq a-1\}$$

For all (i, j),
if $i + j < q$, set $A_i \oslash A_j = A_{i+j}$;
if $i + j \geq q$, set $A_i \oslash A_j = A_{(i+j)-q}$.

So, the structure $(\mathcal{G}_a, \oslash)$ is an abelian group and it is an HX-group in $\mathbf{Z}/n\mathbf{Z}$. We have the following structure described in the table here below:

$\oslash$	A_0	A_1	A_2	$\cdots$	A_{a-1}
A_0	A_0	A_1	A_2		A_{a-1}
A_1	A_1	A_2	A_3		A_0
A_2	A_2	A_3			A_1
$\cdots$					
A_{a-1}	A_{a-1}				A_{a-2}

So, setting $K = \{A_i \mid 0 \leq i \leq a-1\}$, we have that $(K, \oslash) \simeq \mathbf{Z}/a\mathbf{Z}$.

Let q be a divisor of n, that is $n = qa$.

Then there is an HX-group $\mathcal{G}_a^q$ in $\mathcal{P}^*(\mathbf{Z}/n\mathbf{Z})$ of q elements $A_0, A_1, \cdots, A_{q-1}$, such that for all j, $|A_j| = a$, whence a Chinese hypergroupoid ${}_nH^a$ is constructed where for all x, y elements of ${}_nH^a$, we have $|x \oplus y| = a$.

We shall see now that $(H, \oplus)$ is reproductible.

Let us consider the hypergroup associated with $\mathcal{G}_a$.

We know that $\forall (x, y) \in H^2 = (\mathbf{Z}/n\mathbf{Z})^2$, we have

$$x \oplus y = \bigcup_{x \in A,\ y \in B} A \cdot B.$$

We have $\bigcup_{i=0}^{a-1} A_i = H$. Indeed, $(q-1)a + a - 1 = qa - 1$.

For t, such that $0 \leq t \leq qa - 1 = n - 1$, there exists (u, v), such that $u \leq q-1,\ v \leq a-1,\ t = ua + v$.

Let us remark that the following implications hold:

(i_1) $$t = 0 \Rightarrow t \in A_0$$

$$t = a \Rightarrow t \in A_0$$

(i'_1) $$t = a + 1 \Rightarrow t \in A_1$$

(i_2) $$t = a + 2 \Rightarrow t \in A_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

(i'_2) $$t = a + a - 1 \Rightarrow t \in A_{a-1}$$

(i_3) $$t = 2a \Rightarrow t \in A_0$$

(i'_3) $$t = 2a + 1 \Rightarrow t \in A_1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

(i''_3) $$t = 2a + a - 1 \Rightarrow t \in A_{a-1}.$$

One can continue in the same way. So, for all k, we have

$$t = ka \Rightarrow t \in A_0$$

$$t = ka + 1 \Rightarrow t \in A_1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$t = ka + a - 1 \Rightarrow t \in A_{a-1}$$

and finally

$$t = (q-1)a \Rightarrow t \in A_0$$

$$t = (q-1)a + 1 \Rightarrow t \in A_1$$

$$t = (q-1)a + 2 \Rightarrow t \in A_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$t = (q-1)a + a - 1 = qa - 1 \Rightarrow t \in A_{a-1}$$

$$t = qa = n \Rightarrow t \in A_0$$

So, $\bigcup_{j=0}^{a-1} A_j = H$.

We shall prove now that $(H, \oplus)$ is a hypergroup, and then we shall see the table of $(H, \oplus)$.

We find that the hypergroupoid Q constructed on $\mathcal{G} = \{A_i\}_{i<q}, \quad Q = (\mathcal{G}, \oslash)$ is a group isomorphic with $\mathbf{Z}/q\mathbf{Z}$.

Therefore, $(\mathbf{Z}/n\mathbf{Z} : \mathcal{G}, \oslash)$ is a HX-group. Let us show that $(H, \oplus)$ is a hypergroup.

For all $(w, z) \in H^2, \ \exists x : \ z \in w \oplus x$. Indeed, $\exists A_k \in \mathcal{G}_a : \ w \in A_k, \ \exists h : z \in A_h$. Since $\mathcal{G}_a$ is a group, there exists $A_r : \ A_k \oslash A_r = A_h$, therefore if $x \in A_r$ we have $z \in A_h = A_k \oslash A_r = w \oplus x$. So H is a hypergroup.

Let us consider for $\mathbf{Z}/n\mathbf{Z}, \ n = qa$. The set $\mathcal{G}_a^n$ of subsets is an HX-group, such that $\forall A \in \mathcal{G}_a^n$ we have $|A| = a$. Then for all $x \in H$, we have $\mu_1(x) = 1/a$ so H_1 is total, whence $\partial H = 1$.

H_0	0	1	2	$\cdots$	$a-1$	a	$a+1$	$a+2$	$\cdots$				$q-1$
0	A_0	A_1	A_2	$\cdots$	A_{a-1}	A_0	A_1	A_2	$\cdots$	A_{a-1}	A_0	$\cdots$	A_{a-1}
1		A_2	A_3	$\cdots$	A_0	A_1	A_2	A_3					A_{a-2}
2			A_4	$\cdots$	A_1	A_2							
$\cdots$						$\cdots$							
a						A_{2a}							
$a+1$							A_0						
$a+2$								A_1					
$\cdots$													
													A_{a-3}
													A_{a-2}

For $n = aq$, we have clearly

$$\forall s, \ A_s = [s, a+s, 2a+s, \cdots, (q-1)a+s]$$

$$\cdots\cdots\cdots\cdots$$

$$A_{a-1} = [a-1, 2a-1, \cdots, (q-1)a + a - 1].$$

3 The associated hypergroup for $n = 8$

Let us consider $\mathbf{Z}/8\mathbf{Z}$. Set $\mathcal{G}_2^8 = \{(0,4), (1,5), (2,6), (3,7)\}$. We have

K_2^8	$(0,4)$	$(1,5)$	$(2,6)$	$(3,7)$
$(0,4)$	$(0,4)$	$(1,5)$	$(2,6)$	$(3,7)$
$(1,5)$		$(2,6)$	$(3,7)$	$(0,4)$
$(2,6)$			$(0,4)$	$(1,5)$
$(3,7)$				$(2,6)$

From K_2 one obtains the Chinese hypergroup H_2^8.

H_2^8	0	1	2	3	4	5	6	7
0	0,4	1,5	2,6	3,7	0,4	1,5	2,6	3,7
1	1,5	2,6	3,7	0,4	1,5	2,6	3,7	0,4
2	2,6	3,7	0,4	1,5	2,6	3,7	0,4	1,5
3	3,7	0,4	1,5	2,6	3,7	0,4	1,5	2,6
4	0,4	1,5	2,6	3,7	0,4	1,5	2,6	3,7
5	1,5	2,6	3,7	0,4	1,5	2,6	3,7	0,4
6	2,6	3,7	0,4	1,5	2,6	3,7	0,4	1,5
7	3,7	0,4	1,5	2,6	3,7	0,4	1,5	2,6

We find $A(0) = A(4) = 16/2$, $q(0) = q(4) = 16$, whence
$\mu_1(0) = \mu_1(4) = 0.5$.

By the same way, one finds $\forall j$, $\mu_1(j) = 0.5$, so ${}_1H_2^8$ is total, whence $\partial(H_2^8) = 1$.

We can consider also $\mathcal{G}_4^8 = \{(0,2,4,6),(1,3,5,7))\}$.
We find

K_4^8	$(0,2,4,6)$	$(1,3,5,7)$
$(0,2,4,6)$	$(0,2,4,6)$	$(1,3,5,7)$
$(1,3,5,7)$	$(1,3,5,7)$	$(0,2,4,6)$

from which one obtains the Chinese hypergroup H_4^8.

H_4^8	0	1	2	3	4	5	6	7
0	0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7
1		0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7	0,2,4,6
2			0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7
3				0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7	0,2,4,6
4					0,2,4,6	1,3,5,7	0,2,4,6	1,3,5,7
5						0,2,4,6	1,3,5,7	0,2,4,6
6							0,2,4,6	1,3,5,7
7								0,2,4,6

We have clearly $A(0) = A(2) = A(4) = A(6) = 30/4$, and
$\forall i \in \{0, 2, 4, 6\},\ \mu_1(i) = 0.25$.
Analogously, one finds $\forall j \in \{1, 3, 5, 7\},\ \mu_1(j) = 0.25$.

4 The associated hypergroup for $n = 9$

Let us consider $\mathbf{Z}/9\mathbf{Z}$. Set $\mathcal{G}_3^9 = \{(0,3,6), (1,4,7), (2,5,8)\}$. We have the following

K_3^9	$(0,3,6)$	$(1,4,7)$	$(2,5,8)$
$(0,3,6)$	$(0,3,6)$	$(1,4,7)$	$(2,5,8)$
$(1,4,7)$		$(2,5,8)$	$(0,3,6)$
$(2,5,8)$			$(1,4,7)$

which is an HX-group. One obtains the hypergroup H_3^9.

H_3^9	0	1	2	3	4	5	6	7	8
0	0,3,6	1,4,7	2,5,8	0,3,6	1,4,7	2,5,8	0,3,6	1,4,7	2,5,8
1		2,5,8	0,3,6	1,4,7	2,5,8	0,3,6	1,4,7	2,5,8	0,3,6
2			1,4,7	2,5,8	0,3,6	1,4,7	2,5,8	0,3,6	1,4,7
3				0,3,6	1,4,7	2,5,8	0,3,6	1,4,7	2,5,8
4					2,5,8	0,3,6	1,4,7	2,5,8	0,3,6
5						1,4,7	2,5,8	0,3,6	1,4,7
6							0,3,6	1,4,7	2,5,8
7								2,5,8	0,3,6
8									1,4,7

We find $A(0) = A(3) = A(6) = 1/3 + 4/3 + 7/3 + 8/3 + 5/3 + 2/3 = 9$. whence $\forall j, \mu_1(j) = 0.333$, whence ${}_1H_3^9$ is total, whence $\partial(H_3^9) = 1$.

5 The associated hypergroup for $n = 12$

Let us consider $\mathbf{Z}/12\mathbf{Z}$.
Set $\mathcal{G}_3^{12} = \{(0,4,8), (1,5,9), (2,6,10), (3,7,11)\}$. We have the following

K_3^{12}	$(0,4,8)$	$(1,5,9)$	$(2,6,10)$	$(3,7,11)$
$(0,4,8)$	$(0,4,8)$	$(1,5,9)$	$(2,6,10)$	$(3,7,11)$
$(1,5,9)$	$(1,5,9)$	$(2,6,10)$	$(3,7,11)$	$(0,4,8)$
$(2,6,10)$	$(2,6,10)$	$(3,7,11)$	$(0,4,8)$	$(1,5,9)$
$(3,7,11)$	$(3,7,11)$	$(0,4,8)$	$(1,5,9)$	$(2,6,10)$

Denote $A_0 = (0,4,8),\ A_1 = (1,5,9), A_2 = (2,6,10),\ A_3 = (3,7,11)$.
K_3^{12} is an HX-group. We obtain the Chinese hypergroup H_3^{12}.

H_3^{12}	0	1	2	3	4	5	6	7	8	9	10	11
0	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3
1	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0
2	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1
3	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2
4	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3
5	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0
6	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1
7	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2
8	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3
9	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0
10	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1
11	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2	A_3	A_0	A_1	A_2

Set $\mathcal{G}_4^{12} = \{(0,3,6,9),(1,4,7,10),(2,5,8,11)\}$. We find

K_4^{12}	$(0,3,6,9)$	$(1,4,7,10)$	$(2,5,8,11)$
$(0,3,5,9)$	$(0,3,6,9)$	$(1,4,7,10)$	$(2,5,8,11)$
$(1,4,7,10)$	$(1,4,7,10)$	$(2,5,8,11)$	$(0,3,6,9)$
$(2,5,8,11)$	$(2,5,8,11)$	$(0,3,6,9)$	$(1,4,7,10)$

So, K_4^{12} is clearly an HX-group and we obtain the Chinese hypergroup H_4^{12}.

First, we set $K_0 = (0, 3, 6, 9)$, $K_1 = (1, 4, 7, 10)$, $K_2 = (2, 5, 8, 11)$.

H_4^{12}	0	3	6	9	1	4	7	10	2	5	8	11
0	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2
3	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2
6	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2
9	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2
1	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0
4	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0
7	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0
10	K_1	K_1	K_1	K_1	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0
2	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1
5	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1
8	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1
11	K_2	K_2	K_2	K_2	K_0	K_0	K_0	K_0	K_1	K_1	K_1	K_1

For $\forall j_0 \in K_0$, $A(j_0) = 16/4 + 16/4 + 16/4 = 12$, $q(j_0) = 48$, $\mu(j_0) = 0.25$;
$\forall j_1 \in K_1$, $\mu(j_1) = 0.25$;
$\forall j_2 \in K_2$, $\mu(j_2) = 0.25$.
So H_2 is total, whence $\partial H_4^{12} = 1$.

Set $\mathcal{G}_6^{12} = \{(0, 2, 4, 6, 8, 10), (1, 3, 5, 7, 9, 11)\}$.
Set $A_0 = (0, 2, 4, 6, 8, 10)$, $A_1 = (1, 3, 5, 7, 9, 11)$.

K_6^{12}	A_0	A_1
A_0	A_0	A_1
A_1	A_1	A_0

H_6^{12}	0	1	2	3	4	5	6	7	8	9	10	11
0	A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1
1		A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1	A_0
2			A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1
3				A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1	A_0
4					A_0	A_1	A_0	A_1	A_0	A_1	A_0	A_1
5						A_0	A_1	A_0	A_1	A_0	A_1	A_0
6							A_0	A_1	A_0	A_1	A_0	A_1
7								A_0	A_1	A_0	A_1	A_0
8									A_0	A_1	A_0	A_1
9										A_0	A_1	A_0
10											A_0	A_1
11												A_0

$(\mathbf{Z}/12\mathbf{Z})_2$ Let us consider now $\mathcal{G}_2^{12}$.

$\mathcal{G}_2^{12} = \{(0,6),(1,7),(2,8),(3,9),(4,10),(5,11)\}$, from which we obtain the HX-group K_2^{12}.

K_2^{12}	$(0,6)$	$(1,7)$	$(2,8)$	$(3,9)$	$(4,10)$	$(5,11)$
$(0,6)$	$(0,6)$	$(1,7)$	$(2,8)$	$(3,9)$	$(4,10)$	$(5,11)$
$(1,7)$		$(2,8)$	$(3,9)$	$(4,10)$	$(5,11)$	$(0,6)$
$(2,8)$			$(4,10)$	$(5,11)$	$(0,6)$	$(1,7)$
$(3,9)$				$(0,6)$	$(1,7)$	$(2,8)$
$(4,10)$					$(2,8)$	$(3,9)$
$(5,11)$						$(4,10)$

and finally the Chinese hypergroup H_2^{12}

H_2^{12}	0	1	2	3	4	5	6	7	8	9	10	11
0	$0,6$	$1,7$	$2,8$	$3,9$	$4,10$	$5,11$	$0,6$	$1,7$	$2,8$	$3,9$	$4,10$	$5,11$
1		$2,8$	$3,9$	$4,10$	$5,11$	$0,6$	$1,7$	$2,8$	$3,9$	$4,10$	$5,11$	$0,6$
2			$4,10$	$5,11$	$0,6$	$1,7$	$2,8$	$3,9$	$4,10$	$5,11$	$0,6$	$1,7$
3				$0,6$	$1,7$	$2,8$	$3,9$	$4,10$	$5,11$	$0,6$	$1,7$	$2,8$
4					$2,8$	$3,9$	$4,10$	$5,11$	$0,6$	$1,7$	$2,8$	$3,9$
5						$4,10$	$5,11$	$0,6$	$1,7$	$2,8$	$3,9$	$4,10$
6							$0,6$	$1,7$	$2,8$	$3,9$	$4,10$	$5,11$
7								$2,8$	$3,9$	$4,10$	$5,11$	$0,6$
8									$4,10$	$5,11$	$0,6$	$1,7$
9										$0,6$	$1,7$	$2,8$
10											$2,8$	$3,9$
11												$4,10$

In the case $\mathcal{G}_6^{12}$ we have $\forall j,\ \mu(j) = 1/6 = 0.16666$.
In the case $\mathcal{G}_4^{12}$ we have $\forall j,\ \mu(j) = 1/4 = 0.25$.
In the case $\mathcal{G}_3^{12}$ we have $\forall j,\ \mu(j) = 1/3 = 0.333$.
In the case $\mathcal{G}_2^{12}$ we have $\forall j,\ \mu(j) = 1/2 = 0.5$.

6 The associated hypergroup for $n = 15$

Let us consider $\mathbf{Z}/15\mathbf{Z}$. Set $\mathcal{G}_5^{15} = \{(0,3,6,9,12),(1,4,7,10,13),(2,5,8,11,14)\}$.
Set $A_0 = (0,3,6,9,12),\ A_1 = (1,4,7,10,13),\ A_2 = (2,5,8,11,14)$.

K_5^{15}	A_0	A_1	A_2
A_0	A_0	A_1	A_2
A_1	A_1	A_2	A_0
A_2	A_2	A_0	A_1

H_1^{15}	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2
1	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0
2	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1
3	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2
4	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0
5	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1
6	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2
7	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0
8	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1
9	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2
10	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0
11	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1
12	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2
13	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0
14	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1	A_2	A_0	A_1

So, we have $\forall j,\quad \mu(j) = 1/5 = 0.2$.
Set $\mathcal{G}_3^{15} = \{A_0^3, A_1^3, A_2^3, A_3^3, A_4^3\}$, where
$A_0^3 = (0,5,10),\ A_1^3 = (1,6,11),\ A_2^3 = (2,7,12),\quad A_3^3 = (3,8,13)$,
$A_4^3 = (4,9,14)$.
Notice that $K_3^{15} = \{A_i^2 \mid 0 \leq i \leq 4\}$ is an HX-group. Indeed,

K_3^{15}	A_0^3	A_1^3	A_2^3	A_3^3	A_4^3
A_0^3	A_0^3	A_1^3	A_2^3	A_3^3	A_4^3
A_1^3	A_1^3	A_2^3	A_3^3	A_4^3	A_0^3
A_2^3	A_2^3	A_3^3	A_4^3	A_0^3	A_1^3
A_3^3	A_3^3	A_4^3	A_0^3	A_1^3	A_2^3
A_4^3	A_4^3	A_0^3	A_1^3	A_2^3	A_3^3

H_3^{15}	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4
1	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0
2	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1
3	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2
4	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3
5	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4
6	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0
7	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1
8	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2
9	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3
10	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4
11	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0
12	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1
13	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2
14	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3	A_4	A_0	A_1	A_2	A_3

So, we have $\forall j, \quad \mu_1(j) = 1/3 = 0.333.$

7 The associated hypergroup for $n = 16$

Let us consider $\mathbf{Z}/16\mathbf{Z}$. Denote $A_0^{16,4} = A_0 = (0,4,8,12)$, $A_1^{16,4} = A_1 = (1,5,9,13)$, $A_2^{16,4} = A_2 = (2,6,10,14)$, $A_3^{16,4} = A_3 = (3,7,11,15)$.

We obtain the HX-group K_4^{16}, whence Chinese hypergroup H_4^{16}.

K_4^{16}	A_0^{16}	A_1^{16}	A_2^{16}	A_3^{16}
A_0^{16}	A_0	A_1	A_2	A_3
A_1^{16}	A_1	A_2	A_3	A_0
A_2^{16}	A_2	A_3	A_0	A_1
A_3^{16}	A_3	A_0	A_1	A_2

H_4^{16}	0	4	8	12	1	5	9	13	2	6	10	14	3	7	11	15
0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3
4	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3
8	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3
12	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3
1	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0
5	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0
9	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0
13	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0
2	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1
6	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1
10	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1
14	A_2	A_2	A_2	A_2	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1
3	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2
7	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2
11	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2
15	A_3	A_3	A_3	A_3	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_2	A_2	A_2	A_2

We have $\forall j, \quad \mu_1(j) = 1/4 = 0.25$.
Set $A_0^{16,8} = (0, 2, 4, 6, 8, 10, 12, 14)$, $A_1^{16,8} = (1, 3, 5, 7, 9, 11, 13, 15)$.
We obtain the HX-group K_8^{16}, whence the Chinese hypergroup H_8^{16}.

K_8^{16}	$A_0^{16,8}$	$A_1^{16,8}$
$A_0^{16,8}$	$A_0^{16,8}$	$A_1^{16,8}$
$A_1^{16,8}$	$A_1^{16,8}$	$A_0^{16,8}$

H_8^{16}	0	2	4	6	8	10	12	14	1	3	5	7	9	11	13	15
0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
2	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
4	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
6	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
8	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
10	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
12	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
14	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1
1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
3	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
5	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
7	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
9	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
11	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
13	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0
15	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_1	A_0	A_0	A_0	A_0	A_0	A_0	A_0	A_0

For all j, we have $\mu_1(j) = 0.125$.
Denote
$B_0^{16,2} = B_0 = (0,8)$, $B_1^{16,2} = B_1 = (1,9)$, $B_2^{16,2} = B_2 = (2,10)$,
$B_4^{16,2} = B_4 = (4,12)$, $B_5^{16,2} = B_5 = (5,13)$, $B_3^{16,2} = B_3 = (3,11)$,
$B_6^{16,2} = B_6 = (6,14)$, $B_7^{16,2} = B_7 = (7,15)$.
We obtain the HX-group K_2^{16}, whence the Chinese hypergroup H_2^{12}.

K_2^{16}	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_0	B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_1	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_0
B_2			B_4	B_5	B_6	B_7	B_0	B_1
B_3				B_6	B_7	B_0	B_1	B_2
B_4					B_0	B_1	B_2	B_3
B_5						B_2	B_3	B_4
B_6							B_5	B_5
B_7								B_6

H_2^{16}	0	8	1	9	2	10	3	11	4	12	5	13	6	14	7	15
0	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7
8	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7
1	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0
9	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0
2	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1
10	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1
3	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2
11	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2
4	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3
12	B_4	B_4	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3
5	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4
13	B_5	B_5	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4
6	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5
14	B_6	B_6	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5
7	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6
15	B_7	B_7	B_0	B_0	B_1	B_1	B_2	B_2	B_3	B_3	B_4	B_4	B_5	B_5	B_6	B_6

We have $\forall j, \quad \mu_1(j) = 0.5$.

8 HX-groupoids

The absence in the class of HX-groups associated with $\mathbf{Z}/n\mathbf{Z}$, $\forall n \in \mathbf{N}$ of elements $H^{(n)}$, such that the fuzzy grade $\partial H^{(n)}$ is greather than 1, could let think to be true for every finite hypergroupoid. I hit upon the idea of considering the multiplicative group (with respect with the product row$\times$column) of all square matrices of $\mathbf{Z}_2^{(2,2)}$ of order 2 with coefficients in $\mathbf{Z}/2\mathbf{Z}$.

$\mathbf{Z}_2^{(2,2)}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$
$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$
$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$
$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$
$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$
$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$
$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}0&1\\1&1\end{matrix}$	$\begin{matrix}1&0\\1&1\end{matrix}$	$\begin{matrix}1&0\\0&1\end{matrix}$	$\begin{matrix}0&1\\1&0\end{matrix}$	$\begin{matrix}1&1\\0&1\end{matrix}$	$\begin{matrix}1&1\\1&0\end{matrix}$

Then one has considered the hypergroupoid H_0, as follows. First, we denote by B_i where $0 \le i \le 8$.

$B_0 = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\0&1\end{pmatrix} \right\}$, $B_1 = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix} \right\}$

$B_2 = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix} \right\}$, $B_3 = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix} \right\}$

$B_4 = \left\{ \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix} \right\}$, $B_5 = \left\{ \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix} \right\}$

$B_6 = \left\{ \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix} \right\}$, $B_7 = \left\{ \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix} \right\}$

$B_8 = \left\{ \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix} \right\}$.

We construct H_0, where for every pair (i, j), we shall denote the hyperproduct $B_i \circ_0 B_j = \{B_h, B_k\}$ as $i \circ_0 j$. Then we shall calculate the membership function μ_1 associated with the hypergroupoid $(H_0; \circ_0)$. One obtains $(H_1; \circ_1)$ where $\circ_1$ is defined by

$$x \circ_1 y = \{z \mid min\{\mu(x), \mu(y)\} \le \mu(z) \le max\{\mu(x), \mu(y)\}\},$$

see [2], continuing in the same way,

H_0	0	1	2	3	4	5	6	7	8
0	0	$0,5$	$2,4$	$0,5$	$2,4$	5	$6,7$	$6,7$	$2,4$
1	$0,1,4,6$	1	$1,2,5,7$	$1,2,5,7$	4	$1,2,5,7$	$0,1,4,6$	7	$0,1,4,6$
2	$0,7$	$2,6$	2	$0,7$	$4,5$	$4,5$	6	$0,7$	$2,6$
3	$0,7$	$0,5$	$1,2,5,7$	$3,8$	$2,4$	$4,5$	$0,1,4,6$	$6,7$	$3,8$
4	$0,1,4,6$	$4,7$	4	$0,1,4,6$	$1,2,5,7$	$1,2,5,7$	7	$0,1,4,6$	$4,7$
5	$6,7$	$0,5$	5	$6,7$	$2,4$	$2,4$	0	$6,7$	$0,5$
6	6	$2,6$	$4,5$	$2,6$	$4,5$	2	$0,7$	$0,7$	$4,5$
7	7	$4,7$	$1,2,5,7$	$4,7$	$1,2,5,7$	4	$0,1,4,6$	$0,1,4,6$	$1,2,5,7$
8	$0,1,4,6$	$2,6$	$2,4$	$3,8$	$4,5$	$1,2,5,7$	$6,7$	$0,7$	$3,8$

We have clearly
$A(0) = 2 \cdot 1 + 12 \cdot 1/2 + 10 \cdot 1/4 = 42/4$
$A(1) = 1 \cdot 1 + 19 \cdot 1/4 = 23/4$
$A(3) = A(8) = 4 \cdot 1/2 = 2$
$A(7) = 3 \cdot 1 + 17 \cdot 1/2 + 9 \cdot 1/4 = 55/4$
$A(2) = 2 \cdot 1 + 1 \cdot 1/2 + 10 \cdot 1/4 = 42/4$
$A(4) = 3 \cdot 1 + 17 \cdot 1/2 + 9 \cdot 1/4 = 55/4$
$A(6) = 2 \cdot 1 + 12 \cdot 1/2 + 10 \cdot 1/4 = 42/4$
$A(5) = 2 \cdot 1 + 12 \cdot 1/2 + 10 \cdot 1/4 = 42/4.$
By consequence we have
$q_1(0) = 24,\ q_1(1) = 20,\ q_1(3) = q_1(8) = 4$
$q_1(7) = 29,\ q_1(2) = 23,\ q_1(4) = 29,\ q_1(6) = 24$
$q_1(5) = 24$, whence $\mu_1(0) = \mu_1(2) = \mu_1(5) = \mu_1(6) = 0.4375$
$\mu_1(4) = \mu_1(7) = 0.47138,\ \mu_1(3) = \mu_1(8) = 0.5$
$\mu_1(1) = 0.2875.$

One finds

H_1	1	0	2	5	6	4	7	3	8
1	1	$0,1,2,5,6$	$0,1,2,5,6$	$0,1,2,5,6$	$0,1,2,5,6$	$0,1,2,4,5,6,7$	$0,1,2,4,5,6,7$	H	H
0		$0,2,5,6$	$0,2,5,6$	$0,2,5,6$	$0,2,5,6$	$0,2,4,5,6,7$	$0,2,4,5,6,7$	$0,2,3,4,5,6,7,8$	$0,2,3,4,5,6,7,8$
2			$0,2,5,6$	$0,2,5,6$	$0,2,5,6$	$0,2,4,5,6,7$	$0,2,4,5,6,7$	$0,2,3,4,5,6,7,8$	$0,2,3,4,5,6,7,8$
5				$0,2,5,6$	$0,2,5,6$	$0,2,4,5,6,7$	$0,2,4,5,6,7$	$0,2,3,4,5,6,7,8$	$0,2,3,4,5,6,7,8$
6					$0,2,5,6$	$0,2,4,5,6,7$	$0,2,4,5,6,7$	$0,2,3,4,5,6,7,8$	$0,2,3,4,5,6,7,8$
4						$4,7$	$4,7$	$3,4,7,8$	$3,4,7,8$
7							$4,7$	$3,4,7,8$	$3,4,7,8$
3								$3,8$	$3,8$
8									$3,8$

whence $A_2(1) = 1 + 8/5 + 4/7 + 4/9 = 2278/630$,

$q_2(1) = 17,\ \mu_2(1) = 0.2127$.

$A_2(0) = A_2(2) = A_2(5) = A_2(6) = 16/4 + 8/5 + 4/7 + 4/9 + 16/6 + 16/8 = 7108/630$,

$q_2(0) = 48,\ \mu_2(0) = \mu_2(2) = \mu_2(5) = \mu_2(6) = 0.2350$.

$A_2(4) = A_2(7) = 4/2 + 16/6 + 4/7 + 4/9 + 8/4 + 16/8 = 6100/360$,

$q_2(4) = q_2(7) = 52,\ \mu_2(4) = \mu_2(7) = 0.18620$.

$A_2(3) = A_2(8) = 4/2 + 8/4 + 16/8 + 4/9 = 464/72$,

$q_2(3) = 32,\ \mu_2(3) = \mu_2(8) = 0.20139$.

H_2	0	2	5	6	1	3	8	4	7
0	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
2	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
5	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
6	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
1	$0,1,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,$ $5,6$	1	$1,$ $3,8$	$1,$ $3,8$	$1,3,4,$ $7,8$	$1,3,4,$ $7,8$
3	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	$1,$ $3,8$	$3,8$	$3,8$	$3,4,$ $7,8$	$3,4,$ $7,8$
8	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	$1,$ $3,8$	$3,8$	$3,8$	$3,4,$ $7,8$	$3,4,$ $7,8$
4	H	H	H	H	$1,3,4$ $7,8$	$3,4,$ $7,8$	$3,4,$ $7,8$	$4,7$	$4,7$
7	H	H	H	H	$1,3,4,$ $7,8$	$3,4,$ $7,8$	$3,4,$ $7,8$	$4,7$	$4,7$

$A_3(1) = 1 + 4/3 + 4/5 + 8/5 + 16/7 + 16/9 = 5542/630,$
$q_3(1) = 49,\ \mu_3(1) = 0.179527.$
$A_3(0) = A_3(2) = A_3(5) = A_3(6) = 16/4 + 8/5 + 16/7 + 16/9 = 6088/630,$
$q_3(0) = q_3(2) = q_3(5) = q_3(6) = 56,$
$\mu_3(0) = \mu_3(2) = \mu_3(5) = \mu_3(6) = 0.17256.$
$A_3(3) = A_3(8) = 4/2 + 8/4 + 4/5 + 16/7 + 16/9 = 5584/630,$
$q_3(3) = q_3(8) = 48,\ \mu_3(3) = \mu_3(8) = 0.184656.$
$A_3(4) = A_3(7) = 4/2 + 8/4 + 4/5 + 16/9 = 4144/630,$
$q_3(4) = q_3(7) = 32,\ \mu_3(4) = \mu_3(7) = 0.20555.$
$\mu_3(0) = \mu_3(2) = \mu_3(5) = \mu_3(6) = 0.17256\ <\ \mu_3(1) = 0.179527\ <$
$\mu_3(3) = \mu_3(8) = 0.184656\ <\ \mu_3(4) = \mu_3(7) = 0.20555,$
whence one finds H_3

H_3	0	2	5	6	1	3	8	4	7
0	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
2		$0,2,$ $5,6$	$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
5			$0,2,$ $5,6$	$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
6				$0,2,$ $5,6$	$0,1,2,$ $5,6$	$0,1,2,3,$ $5,6,8$	$0,1,2,3,$ $5,6,8$	H	H
1					1	$1,$ $3,8$	$1,$ $3,8$	$1,3,4,$ $7,8$	$1,3,4,$ $7,8$
3						$3,8$	$3,8$	$3,4,$ $7,8$	$3,4,$ $7,8$
8							$3,8$	$3,4,$ $7,8$	$3,4,$ $7,8$
4								$4,7$	$4,7$
7									$4,7$

One sees that $(H_3;\circ_3)$ coincides with $(H_2;\circ_2)$.
Therefore, one can conclude that the fuzzy grade of H_0 is 2.

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