DE GRUYTER OPEN

GESTALT THEORY, DOI 10.1515/gth-2017-0004
© 2017 (ISSN 2519-5808); Vol. 39, No.1, 6–29

Original Contributions - Originalbeiträge

Luigi Burigana & Michele Vicovaro

"Invariants" in Koffka's Theory of Constancies in Vision: Highlighting Their Logical Structure and Lasting Value

1. Introduction

In present-day perceptual psychology, Gestalt theory is mostly referred to in terms of its structural side, that is, as a theory regarding how the visual field organises into definite perceptual units, the grouping or unit formation problem[1]. But the Gestalt theory also has a metrical side, which concerns the properties – size, distance, colour, shape, orientation, etc. – possessed by those perceptual units, the factors that are influential on such properties, and how coexisting perceptual properties may depend on one another.

This article is about a special aspect of the metrical side of the Gestalt theory of vision. Specifically, it is about the concept of "invariants", as elaborated by Kurt Koffka in his *Principles of Gestalt Psychology* (1935) – in particular, in Chapter VI of the book, concerning "constancies". Besides the saliency of the concept in Koffka's work, there are other reasons that draw our interest towards it. One is that the concept, as we interpret it, allows for a description in set-theoretic terms, so that it is open to formal developments. Another reason is that the concept has enduring significance, witnessed by the fact that tracks and implications of it may be detected, possibly under different names, in present-day theories of vision.

We develop our study in three stages. First, starting from basic examples of Koffka's invariants, we propose a formal definition of the concept (Section 2) and go into the meaning of the terms and assumptions it involves (Section 3). Then, we illustrate the use of that formal definition by commenting on the relationship between Koffka's invariants and the so-called "constancy hypothesis" (Section 4) and by discussing "conditional indeterminacy" of perceptual properties and possible ways of overcoming the indeterminacy (Section 5). Lastly, we illustrate the lasting value and theoretical power of Koffka's construct by indicating concepts in modern perceptual psychology that are intrinsically associated with it and by describing a model of perceptual transparency (Section 6).

[1] Detailed information on the status of Gestalt theory in present-day vision science may be found in the studies by Spillmann (2012) and Wagemans et al. (2012a, 2012b), as well as in a special issue of the journal *Vision Research* introduced by Jäkel, Singh, Wichmann, and Herzog (2016).

In our discourse, we shall use technical words consistent – as much as possible – with those used by Koffka. Specifically, by an "invariant", we mean an invariant in Koffka's sense exemplified in Section 2; the same word also has other meanings in vision science. By "distal stimuli", "proximal stimuli", and "perceptual scene", we mean, in this order, the set of optically relevant physical properties of the environment under view, the set of its optical effects upon the eyes of an observer, and the scene subjectively experienced by the observer in an act of vision. By a "property", we mean any identifiable characteristic possessed by units or events in the distal stimuli (distal property), proximal stimuli (proximal property), or perceptual scene (perceptual property). Under suitable conditions, a property acquires the logical form of a "variable", i.e., an entity that may be tested on several occasions and that may show different "values" on those occasions. By the "range" of a property thus conceived, we mean the set of values it may take on during the several occasions in which it can be tested. For example, the range of "perceptual size", as referring to a definite kind of visual situation, is the variety of possible degrees in perceived size, from those of small objects to those of large objects, in the visual situations of the kind considered[2].

2. Examples and Logical Structure of Koffka's Invariants

This section provides a profile of the subject of our study. First, we cite two examples of invariants discussed by Koffka and point out the main terms in them; then, taking account of those terms, we propose a formal general definition of invariants; lastly, we return to the examples by illustrating how they fit that general definition.

A simple and generally known example is the "size–distance invariant", as discussed on p. 222 in Koffka (1935). In the book, there are several references to it, among which, we quote the following from p. 229:

> "a relation of proportionality exists between perceived size and distance, so that if two equal retinal lines give rise to the perception of two behavioural lines of different length, these two lines appear at correspondingly different distances." (1)

From this description, the size–distance invariant involves three properties: the size of a component of the retinal image (e.g., the length of a "retinal line"),

[2] The expressions preferred by Koffka are "distant stimuli" (rather than "distal stimuli") and "perceptual organization" or "behavioural environment" (rather than "perceptual scene") (e.g., Koffka, 1935, pp. 33, 80, and 211). What we call "range of a property", Koffka called "field of a property" (e.g., p. 244). We prefer the term "range", as the term "field" also has other meanings (e.g., "psychophysical field", p. 67; "field of stress", p. 231; and "surrounding vs. inlying fields", p. 248). We adopt this classic and relatively simple ontological set-up – distal stimuli, proximal stimuli, etc. – because it is the one actually used by Koffka in his discussion of invariants and other perceptual issues, and because it is sufficient for the purposes of our analysis.

the size of a corresponding entity in the perceptual scene (e.g., the length of a "behavioural line"), and the apparent egocentric distance of the same entity in the scene. The first, denoted here as X, is a proximal property; the second and third, denoted Y_1 and Y_2, are perceptual properties. The invariant itself presents both a "freedom" and a "constriction" aspect. The *freedom aspect* lies in that the proximal property X *is not* determinative on the perceptual properties Y_1 and Y_2; that is, for any fixed value of X, there may be different values of Y_1 compatible with it (e.g., "behavioural lines of different lengths" may correspond to "equal retinal lines"), and the same for Y_2. The *constriction aspect* lies in that the proximal property X *is* determinative on a relation between the perceptual properties Y_1 and Y_2; that is, for any fixed value of X, a definite "relation of proportionality" holds true between Y_1 and Y_2, so that any difference in Y_1 is accompanied by a proportionate difference in Y_2, and vice versa. Note that the word "invariant" literally evokes the constriction aspect of the concept, i.e., the relation between Y_1 and Y_2 "does not vary" when the value of X is fixed; but the meaning of the concept also depends substantially on its freedom aspect, i.e., Y_1 and Y_2, individually considered, are to some extent "free to vary" when the value of X is fixed.

A peculiarity of the example now considered is that the perceptual properties it involves are properties of *different types* (size and distance) but reside on an *equal bearer* (properties of one and the same object or "field part" in the perceptual scene). This peculiarity is shared by other invariants discussed by Koffka, such as the shape–orientation invariant (pp. 228–229), the whiteness–brightness invariant (pp. 243–244), and the coloured surface–coloured illumination invariant (pp. 256–258)[3].

The second example we mention may be named the "direction–direction invariant" as discussed on pp. 218–219. Among the statements referring to it, we quote the following from p. 255:

> "the angle between two lines is an invariant, whereas the absolute orientation of the perceived lines depends upon the general field conditions." (2)

As we interpret it, this invariant involves four properties, which are the directions of the two linear components of the retinal image and the directions of the two objects in the perceptual scene that correspond to those linear components. The first two properties, denoted here as X_1 and X_2, are proximal properties, and the other two, denoted Y_1 and Y_2, are perceptual properties. The *freedom aspect* is implicit in the phrase "the absolute orientation of the perceived lines depends upon the general field conditions"; this means that, for fixed values of X_1 and X_2,

[3] From p. 247 of Koffka (1935), we infer that "whiteness" is the perceptual analogue to albedo ("lightness" is the term preferred in present-day perceptual psychology; cf. Gilchrist, 2006), and "brightness" is the perceptual analogue to illuminance (how much light a surface appears to receive as it stands in a perceptual scene).

the values of Y_1 and Y_2 may vary for varying "general field conditions", so that X_1 and X_2 fail to determine Y_1 and Y_2 uniquely. The *constriction aspect* is implicit in the phrase "the angle between two lines is an invariant", which means that, for fixed values of X_1 and X_2, a relation between Y_1 and Y_2 becomes determined; that relation lies in the constant "angle" between "perceived lines" that vary in "absolute orientation".

The perceptual properties involved in this invariant are of *equal type*, as both are directions, but reside on *different bearers*, as they are properties of two distinct objects or "field parts" in the scene. In this respect, the direction–direction invariant is representative of a kind of invariant different from that illustrated by the size–distance invariant (perceptual properties of equal bearer but different types). In Koffka's analysis, there are also other exemplars of the kind now considered. We mean, in particular, the invariants that involve the concept of a "gradient", when this is understood as the difference or ratio between values taken by a perceptual property for two distinct objects in the scene – e.g., gradient in size on p. 244; gradient in whiteness on p. 246; and gradient in apparent colour on p. 255[4].

In light of these examples, we now describe a set-theoretic framework of Koffka's invariants, composed of three parts.

Part 1. Any invariant concerns a set $Y = (Y_1,...,Y_n)$ of two or more perceptual properties, and a proximal property X – this may be a composite property, so that $X = (X_1,...,X_m)$. Each of the properties, viewed as a variable, has a range of possible values. Symbols $\mathcal{Y}_1,...,\mathcal{Y}_n$, $\mathcal{X}$ will denote their ranges.

Part 2. An invariant on the system of properties $(X,Y) = (X,Y_1,...,Y_n)$ consists in a family of *conditional dependences* as follows:

$$\mathcal{D} = (D_x : x \in \mathcal{X}).$$

Family $\mathcal{D}$ is indexed by the possible values of the proximal property X. Each member D_x of the family is a relation between the perceptual properties $Y_1,...,Y_n$, that is,

$$D_x \subseteq \mathcal{Y}_1 \times \dots \times \mathcal{Y}_n,$$

which means that D_x is a subset of the Cartesian product of the ranges of the perceptual properties involved.

Part 3. On the one hand, the fact that for any $x \in \mathcal{X}$ (i.e., any x belonging to $\mathcal{X}$), there corresponds a definite $D_x \subseteq \mathcal{Y}_1 \times ... \times \mathcal{Y}_n$ means that under condition $X = x$, which concerns the proximal property, a definite dependence D_x holds true between the perceptual properties. This association between

[4] This meaning of the word "gradient" is quite different from that prevailing in contemporary perceptual psychology. Regarding the change of the concept "gradient" from Koffka's theory to Gibson's theory, refer to the book by Cutting (1986, p. 73).

values of X and dependences on $(Y_1,...,Y_n)$ is tantamount to the *constriction aspect* of the invariant. On the other hand, each dependence D_x may be such that its projection on dimension $\mathcal{Y}_j$ (for $j = 1,...,n$) is a set of *several* possible values of property Y_j (the projection may even coincide with the whole range $\mathcal{Y}_j$), which means that condition $X = x$ is unable to fix the value of Y_j. This is evidence of residual indeterminacy of each perceptual property Y_j under the condition $X = x$ and thus points to the *freedom aspect* of the invariant.

Figure 1 schematically illustrates this definition. It represents an invariant acting on a stimulus property X with range $\mathcal{X} = \{x_1,\ldots,x_5\}$ of possible values, mapped on the vertical dimension of the diagram, in addition to two perceptual properties Y_1 and Y_2 with ranges $\mathcal{Y}_1 = \{y_{1,1},\ldots,y_{1,4}\}$ and $\mathcal{Y}_2 = \{y_{2,1},\ldots,y_{2,4}\}$, mapped on the frontal and depth dimensions of the diagram. To each value $x \in \mathcal{X}$, a conditional dependence $D_x \subseteq \mathcal{Y}_1 \times \mathcal{Y}_2$ is associated, which is represented by the set of filled small circles in the x-layer; e.g., $D_{x_3}=\{(y_{1,1},y_{2,2}), (y_{1,2},y_{2,4}), (y_{1,3},y_{2,1}), (y_{1,4},y_{2,3})\}$. The constriction aspect of the invariant lies in the fact that each conditional dependence D_x is a *proper* subset of the Cartesian product $\mathcal{Y}_1 \times \mathcal{Y}_2$, which means that under any stimulus condition $X = x$, a definite dependence holds true between the perceptual properties Y_1 and Y_2; in the example cited, these dependences are one-to-one correspondences between ranges $\mathcal{Y}_1$ and $\mathcal{Y}_2$, which means that each value of Y_1 uniquely determines a value of Y_2, and vice versa. The freedom aspect lies in the fact that the stimulus property X is unable to specify either Y_1 or Y_2 separately considered; in the example, the projection of the filled circles on the dimensions $(\mathcal{X},\mathcal{Y}_1)$ covers the whole product $\mathcal{X} \times \mathcal{Y}_1$, which means that every value of X is compatible with every value of Y_1, and the same is true of X and Y_2. Note that the association between elements x of $\mathcal{X}$ and subsets D_x of $\mathcal{Y}_1 \times \mathcal{Y}_2$ may be termed a "psychophysical" dependence, because X is a stimulus property and (Y_1,Y_2) constitutes a pair of perceptual properties. Instead, for any fixed x, the conditional relation D_x between Y_1 and Y_2 may be termed an "intra-perceptual" dependence, because Y_1 and Y_2 are both perceptual properties[5].

We also illustrate the proposed framework by applying it to the two examples cited herein. The size–distance invariant concerns a system (X,Y_1,Y_2) of one proximal and two perceptual properties. Statement (1) implies that property X (proximal size) fails to determine Y_1 (perceptual size) as well as Y_2 (perceptual distance), so that Y_1 and Y_2 have residual indeterminacy under any condition $X = x$. Statement (1), however, also includes the hypothesis of a "relation of

[5] Koffka (1935, p. 229) refers to such dependences by the phrase "aspects of the percept coupled together". The same concept has received various names by subsequent authors. For example: "response–response correlation" in Oyama (1969), "perceptual interaction" in Gogel (1973), "percept–percept coupling" in Epstein (1982), and "perceptual interdependence" in Rock (1983).

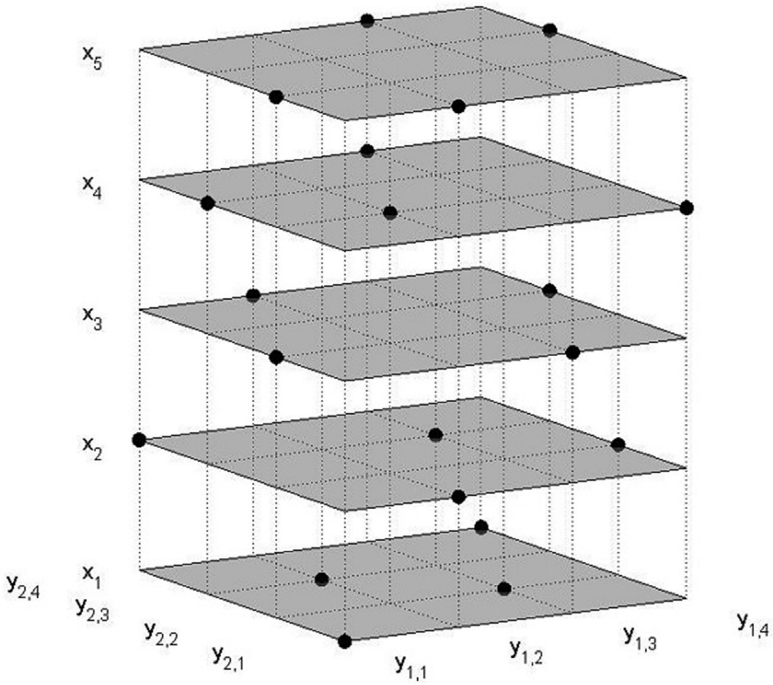

Fig. 1 Diagram of an abstractly conceived invariant involving a stimulus variable (represented on the vertical dimension) and two perceptual variables (represented on the two horizontal dimensions). The ternary dependence relation characterising the invariant is represented by the set of small filled circles in the diagram.

proportionality" between Y_1 and Y_2, which may be given this interpretation: under any condition $X = x$, the equation

$$Y_1 / Y_2 = c(x) \tag{3}$$

holds true, where $c(x)$ (the proportionality coefficient) depends on the value x and, possibly, on other characteristics of the visual context. In terms of the suggested framework, this is tantamount to hypothesising that, under any condition $X = x$, the dependence relation $D_x = \{(y_1,y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2: y_1/y_2 = c(x)\}$ holds true for the pair (Y_1,Y_2) of perceptual properties. In turn, the direction–direction invariant concerns a system (X_1,X_2,Y_1,Y_2) of two proximal and two perceptual properties. Statement (2) implies that, although X_1 does not determine Y_1, nor does X_2 determine Y_2, the pair (X_1,X_2) of proximal directions – more precisely, the angle between them – does determine the angle between the perceptual directions Y_1 and Y_2. A reasonable expression of this hypothesis is the following equation:

$$Y_1 - Y_2 = a(x_1, x_2), \tag{4}$$

in which the term $a(x_1,x_2)$ is the angle between the directions X_1 and X_2 under a condition $(X_1,X_2) = (x_1,x_2)$. This is tantamount to hypothesising that, under such a condition, a definite dependence relation $D_{(x_1,x_2)} = \{(y_1,y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2 : y_1 - y_2 = a(x_1,x_2)\}$ is in force on the pair (Y_1,Y_2) of perceptual properties.

The proposed framework is flexible enough to fit invariants more complex than the examples mentioned so far. In Section 6, we consider an invariant concerning the transparency phenomenon, which involves five properties, one of the stimulus type plus four of the perceptual type [equation (11)].

3. Salient Points in the Logical Structure

In this section, we elaborate on the suggested framework by presenting specifications of crucial terms in it. The specifications are given as responses to three questions.

Question 1. In the framework, for the sake of generality, the conditional dependences implicit in an invariant have been defined in set-theoretic terms: any dependence D_x has been meant as a subset of the Cartesian product $\mathcal{Y}_1 \times \ldots \times \mathcal{Y}_n$. But, in more practical terms, what may be the form of such dependences? What may be their algebraic expression?

One response to this question is that the dependences implicit in some exemplary invariants are suitably expressed in *transformational form*. By this, we mean that two transformations f and g may be defined, one acting on the values of the proximal property X, the other on the combined values of the perceptual properties $Y_1,\ldots,Y_n$. Then, for any $x \in \mathcal{X}$, the associated dependence D_x is expressed by the following equation:

$$g(y_1,\ldots,y_n) = f(x). \qquad (5)$$

In other words, the dependence D_x is the set of all n-tuples $(y_1,\ldots,y_n)$ in $\mathcal{Y}_1 \times \ldots \times \mathcal{Y}_n$ that satisfy this equation. In the words used by Koffka, the term $g(y_1,\ldots,y_n)$ in the equation is a "combination" of perceptual properties; for example, "combination of shape and orientation" on p. 233, "of size and distance" on p. 236, and "of whiteness and brightness" on p. 244. Thus, equation (5) signifies that, under any condition $X = x$ fixing the value of X [and consequently of $f(X)$], a suitable combination $g(Y_1,\ldots,Y_n)$ of the perceptual properties itself becomes fixed in its value, in spite of the individual freedom of those perceptual properties under that condition.

The size–distance invariant illustrates this possibility, because equation (3) is an instance of equation (5), with $f(x)$ replaced by $c(x)$ (the proportionality coefficient) and $g(y_1,y_2)$ replaced by y_1/y_2 (the ratio between perceived size and distance). The same is true of the direction–direction invariant, as seen when comparing equations (4) and (5). Other examples are offered by Koffka's invariants that involve coupled "gradients". In discussing such a case – e.g., gradient in perceived whiteness coupled with gradient in light intensity on the retina, p. 244; and gradient in apparent colour coupled with gradient in stimulus colour, p. 255 – a system of properties (X_1,X_2,Y_1,Y_2) is considered, a transformation g is defined that is applicable both to proximal and to perceptual properties [e.g., $g(x_1,x_2) = x_1–x_2$, or else $g(x_1,x_2) = x_1/x_2$], and an equation in the following form is hypothesised:

$$g(Y_1,Y_2) = g\left(X_1,X_2\right). \tag{6}$$

This equation expresses the hypothesis that the perceptual gradient (on the left) is equal to the corresponding stimulus gradient (on the right) and is obviously of the general form shown in equation (5). Such a hypothesis implies that, under any condition $(X_1,X_2) = (x_1,x_2)$, the perceptual gradient $g(Y_1,Y_2)$ has a fixed value [equal to $g(x_1,x_2)$], so that $g(Y_1,Y_2)$ constitutes an invariant in Koffka's sense.

Although suitable for several examples in Koffka, the transformational form in equation (5) is not the only form admissible. Suppose, for instance, that X_1 and X_2 are the light intensities on two regions of the retinal image, Y_1 and Y_2 are the whitenesses of the corresponding surfaces in the perceptual scene, and the following rule is hypothesised:

> Y_1 is larger or smaller than Y_2 depending on whether X_1 is larger or smaller than X_2.

In a sense, this rule expresses an invariant for properties (X_1,X_2,Y_1,Y_2); it has a constriction aspect (the order relation between X_1 and X_2 determines the order relation between Y_1 and Y_2), and a freedom aspect (Y_1 and Y_2 are not uniquely determined by X_1 and X_2, they may also depend on other "field conditions"). A natural way of expressing the conditional dependences implicit in this "ordinal invariant" is by splitting the product set $\mathcal{Y}_1 \times \mathcal{Y}_2$ into parts $P = \{(y_1, y_2) : y_1 \leq y_2\}$ and $Q = \{(y_1, y_2) : y_1 > y_2\}$ and then setting for all $(x_1,x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$

$$D_{(x_1,x_2)} = P \text{ or } = Q \text{ depending on whether } x_1 \leq \text{or} > x_2. \tag{7}$$

This observation illustrates the convenience of framing a *general* definition of invariants in *set-theoretic* terms, as we proposed. The variables involved in an invariant are not necessarily of the quantitative type – they could be variables on ordinal scales, categorical or predicative variables, etc. Furthermore, even when they are of a quantitative type, the dependences connecting them could not be expressed naturally by numerical equations.

Question 2. An invariant, as an item of a vision theory (e.g., Koffka's theory), may be viewed as a scientific hypothesis. As a hypothesis, it comes up as the result of some argument developed by vision scientists, that is, as the endpoint of some "heuristic path". We then ask: are there typical paths at the origin of hypotheses of invariants in Koffka's sense? Are there typical arguments apt to suggest the form of the conditional dependences implicit in an invariant?

As a (partial) response to this question, we suggest reference to so-called "reverse projection" (Johansson, 1970) or "inverse optics" (Poggio, Torre, & Koch, 1985), here understood as a heuristic method followed at times by vision scientists in deriving psychophysical hypotheses from known optical principles. On the optical side of the visual process – i.e., the optical transition from distal to proximal stimuli – a scientist may recognise the "compound nature" of a proximal property X; that is, there may be *several* distal properties $Z_1,...,Z_n$ that concur when determining X. For example, the length X of the retinal image of a pole in the physical environment jointly depends on the length Z_1 of the pole, its distance Z_2 from the observer, and its slant Z_3 relative to the line of sight. Thus, the proximal property in question may be regarded as a joint function $X = g(Z_1,...,Z_n)$ of the distal properties contributing to its specification, and (in principle) the algebraic form of function g may be made explicit on the basis of known optical principles. In particular, for any fixed value x of X, its "inverse image" $g^{-1}(x)$ under function g can be determined, i.e., the set of *all* combined values $(z_1,...,z_n)$ of $(Z_1,...,Z_n)$ that give rise to the *same* value x in the optical process.

The inverse optics heuristic, in simplified terms, consists of a replacement and a transfer operation. In reference to the notation cited herein, this means that the distal properties $Z_1,...,Z_n$ are *replaced* by analogous perceptual properties $Y_1,...,Y_n$ (e.g., the distal length Z_1 of a physical pole is replaced by the apparent length Y_1 of a perceptual pole, etc.), and the function g linking X to $(Z_1,...,Z_n)$ is *transferred* as a relation between X and $(Y_1,...,Y_n)$ (i.e., equation $X = g(Y_1,...,Y_n)$ is hypothesised). As a consequence, for any $x \in \mathcal{X}$, the set $g^{-1}(x)$ of the combined values of $(Z_1,...,Z_n)$ becomes reinterpreted as a set D_x of combined values of $(Y_1,...,Y_n)$. This D_x may be the conditional dependence (under condition $X = x$) of a hypothetical invariant on properties $(X,Y_1,...,Y_n)$, as represented in our set-theoretic framework. For example, the conditional dependences implicit in the size–distance–direction invariant – i.e., for a fixed retinal size, there is a definite dependence between the apparent size, distance, and slant of a *perceived* pole – could be conjectured on the basis of the optical dependence of retinal size on the size, distance, and slant of any *physical* pole in the environment.

In Koffka, we did not find explicit assertions of this approach. We did find, however, clues that the approach may have played a part in indicating possible invariants and suggesting the form of the conditional dependences in them. A clue is the following phrase (on p. 262), belonging to a discussion of perceptual transparency: "we apply the laws of colour mixture to the splitting up of the effect of neutral stimulation". It shows that reference to the "laws of colour mixture" – specifically, Talbot's law, which is an optical principle – may have served as a guide in conjecturing the dependences holding true between the perceptual properties involved in the transparency phenomenon. Another clue can be found on pp. 243–244, concerning the whiteness–brightness invariant, where the optical principle is reminded that luminance (symbol i) amounts to the product of albedo (L) and illuminance (I). We shall refer again to these invariants in Section 6, when describing a model of perceptual transparency.

Question 3. The word "invariant", in its literal sense, stands for something that does not vary, in spite of variations of other entities concomitant with it. We may ask: what is specifically invariant in Koffka's invariants? What are the reasons for using this word?

There are at least three mutually compatible responses to this question. One is that, in an invariant concerning properties $(X, Y_1, \ldots, Y_n)$, the unchanging entities are the *conditional dependences* between the perceptual properties $Y_1, \ldots, Y_n$. More precisely, for any $x \in \mathcal{X}$, under condition $X = x$, a definite dependence D_x is hypothesised to be in force between the perceptual properties, which does not change in spite of variations of the values of those properties, or of other components of the visual context. This response is implicit, e.g., in the statement "the invariants here are the relations between the different shapes" (p. 223).

A second response specifically applies to invariants whose conditional dependences are expressible in transformational form, generally represented by equation (5). The term $g(Y_1, \ldots, Y_n)$ in that equation may be interpreted as a derived perceptual property – e.g., if Y_1 and Y_2 are the directions of two oblong objects in a perceptual scene, and $g(Y_1, Y_2) = Y_1 - Y_2$, then $g(Y_1, Y_2)$ amounts to a derived perceptual property, i.e., the apparent angular separation between both objects. For such an invariant, we may then mean that the unchanging entity is the *value of the derived perceptual property* $g(Y_1, \ldots, Y_n)$. Specifically, for any $x \in \mathcal{X}$, under condition $X = x$, the value $f(x)$ becomes fixed, so that according to equation (5), the value of property $g(Y_1, \ldots, Y_n)$ also becomes fixed, in spite of possible variations of the values of properties $Y_1, \ldots, Y_n$. This alternative response to Question 3 is suggested, e.g., by the statement "the angle between two lines is an invariant", in reference to the direction–direction invariant (p. 255).

A third response applies to invariants whose conditional dependences are expressible in transformational form [equation (5)] with $f = g$, that is, the *same* transformation applies to the proximal and the perceptual properties involved, a possibility illustrated by equation (6). For such an invariant, we may mean that the unchanging entity is the *value of the transformation g*, which remains the same from $g(X_1, \ldots, X_n)$ to $g(Y_1, \ldots, Y_n)$. Note that this third meaning is structurally different from the first two:

here the change of reference (for judging invariance) is the replacement of proximal properties ($X_1,\ldots,X_n$) by perceptual properties ($Y_1,\ldots,Y_n$), whereas earlier, the change consisted of possible variations of the perceptual properties in their values.

4. "Our Invariance Principle Taking the Place of the Old Constancy Hypothesis"

The phrase set as the title of this section comes from p. 224 of Koffka (1935), wherein it concludes a couple of paragraphs entitled "Empiristic explanations of these constancies, and the reason for their popularity". The phrase suggests that, in spite of the similarity between the words "constancy" and "invariance" in their ordinary meanings, the theory of invariants is something quite different from the traditional "constancy hypothesis" and is intended to positively overcome the latter's shortcomings. In Koffka's book, there are other statements to the same effect. We quote the following (p. 97), which seems to us especially instructive:

> "All we intend to do is to replace laws of local correspondence, laws of machine effects, by laws of a much more comprehensive correspondence between the total perceptual field and the total stimulation, and we shall, in the search for these laws, find at least indications of some more specific constancies, though never one of the type expressed by the constancy hypothesis." (8)

In this section, we intend to illustrate what is peculiar in Koffka's invariants as compared with the so-called "constancy hypothesis", and we do this by taking advantage of the formal framework presented in Section 2.

In the Gestalt theory tradition, the constancy hypothesis (Konstanzannahme) is particularly known as having been the main polemic target of one of the earliest studies ascribable to the theory, a paper by Wolfgang Köhler dated 1913. The following excerpts from Koffka may be of help for specifying its meaning – or better, the meaning attributed to the hypothesis by Gestalt theorists, which is what really matters for our discussion.

> "the constancy hypothesis which derives the looks of things from a universal point-to-point relation with the proximal stimulation" (p. 87);
>
> "the constancy hypothesis maintains that the result of a local stimulation is constant, provided that the physiological condition of the stimulated receptor is constant" (p. 96);
>
> "the implicit assumption, a special case of the constancy hypothesis, that what happens under a particular set of conditions must happen under all conditions" (p. 145). (9)

In the terms we used in formulating our framework, the meaning implicit in these statements may be cast as follows: the constancy hypothesis, referring to a perceptual property Y, maintains that a proximal property X can be found that is *fully and universally determinative* upon Y; in other words, for any value x of X, a definite value y of Y does exist that regularly corresponds to x in visual experiences, irrespective of other varying "field conditions" in those experiences. Note that, in this use, the term "constancy" has no direct relationship with the "perceptual constancies" discussed in perceptual psychology (e.g., constancy in size, shape, colour, etc.). Rather, the term "constancy" in the "constancy hypothesis" is for marking the supposed *universality* of the connection between proximal property X and perceptual property Y: the association between any value of X and a corresponding value of Y is supposed to be "constant", that is, independent of other peculiar features of the visual contexts in which those values occur [phrase "under all conditions" in excerpt (9)].

Now, let us come back to Koffka's invariants. In our framework (Part 3), we noted that such an invariant has a freedom and a constriction aspect. For an invariant on a system of properties $(X, Y_1,\ldots,Y_n)$, the freedom aspect lies in that the proximal property X is unable to determine the values of perceptual properties $(Y_1,\ldots,Y_n)$ uniquely. In set-theoretic terms, this means that, for any value x of X, there may be *several* assignments of values $(y_1,\ldots,y_n)$ to $(Y_1,\ldots,Y_n)$ that are each compatible with the stimulus condition $X = x$ – for instance, several triples (y_1,y_2,y_3) of perceptual size, distance, and slant that are each compatible with a fixed length x of an item in the retinal image, in the size–distance–direction invariant. Of course, the freedom aspect of an invariant, thus understood, is in direct contrast with the "old constancy hypothesis", which (in suitable conditions) would maintain that X is fully and universally determinative upon $(Y_1,\ldots,Y_n)$, so that for any value x of the former there should be one single assignment of values $(y_1,\ldots,y_n)$ to the latter compatible with it. Thus, if the freedom aspect of invariants represents a *real* feature of perceptual phenomena, then the constancy hypothesis is untenable as a general principle.

But an invariant also has a constriction aspect. For any value x of X, a definite dependence D_x is presumed to hold true between perceptual properties $Y_1,\ldots,Y_n$ under stimulus condition $X = x$; in set-theoretic terms, D_x is nothing but the set of all assignments $(y_1,\ldots,y_n)$ compatible with that condition. In view of this, the "invariance principle" of Koffka may be regarded (as suggested, e.g., by Hochberg, 1957, p. 76) as a *higher-order constancy hypothesis*: any condition $X = x$, supposedly unable to determine the values of $(Y_1,\ldots,Y_n)$, nevertheless is presumed to be able to determine a definite dependence D_x between those values. This shift from the "old" to a higher-order constancy hypothesis goes hand-in-hand with a shift from studying the association between *values* of proximal properties and *values* of perceptual properties, to studying the association between *relations* of proximal properties and *relations* of perceptual properties (statement "relative properties of the stimulus distribution determining relative properties of the objects and events in the behavioural world", on p. 219). The latter view is tantamount to the "relational approach" in the study of vision, which is generally recognised as one salient characteristic of Gestalt theory of perception (e.g., Hatfield, 2003, p. 359; Kogo, Strecha, van Gool, & Wagemans, 2010, p. 411; Sarris, 2012, pp. 257–258).

5. Conditional Indeterminacy of Percepts in Koffka's Invariants

The freedom aspect of invariants is a sign of their theoretical flexibility, so that it is worthy of closer scrutiny. Actually, this aspect reveals that the relationship between a proximal property X and perceptual properties $Y_1,\ldots,Y_n$ may be a mix of determinacy (any value of X determines a definite dependence between $Y_1,\ldots,Y_n$) and indeterminacy ($Y_1,\ldots,Y_n$, in their values, have residual freedom relative to X). This view naturally leads a scientist to inquire about the boundary between determinacy and indeterminacy in invariants, as well as about factors by which the indeterminacy is overcome – possibly, the "laws of a much more comprehensive correspondence" alluded to in excerpt (8). In this section, we present comments on these aspects of invariants, again by taking advantage of the formal framework proposed. We present our comments as responses to three further questions.

Question 4. What are the roots of the conditional indeterminacy of perceptual properties in Koffka's invariants? Are they substantial roots, to be found in the vision processes themselves? Or methodological roots, which relate to how a scientist defines an invariant as a theoretical construct?

In preparation for a response to this question, we quote the following passage from Koffka (p. 248):

> "it is at least a *real* theory, i.e., an explanation which deduces the observed effects from the only available causes, the proximal stimulation which gives rise to perceptual organization."

This is a judgement by the author on the discussion he had developed on the preceding pages, concerning phenomena relating to "brightness constancy". The passage is significant for our purposes, as it indicates that, in Koffka's view, a "real theory" of perceptual effects is one that searches for "only available causes" in the underlying "proximal stimulation". This view presupposes the assumption that the proximal stimulation in an act of vision is indeed *sufficient* to determine the perceptual scene *uniquely*, so that if a vision scientist would take account of stimulus data *exhaustively*, then – by applying the laws of a suitable theory – the scientist should be able to predict the perceptual scene precisely, i.e., without residual indeterminacy of any property in the scene. In Koffka's book, there are also other passages that appear to attest to this view, e.g., the criticisms towards "empiristic explanations" of perceptual phenomena (on pp. 103, 209, 223–224, etc.), i.e., explanations that involve supplementary cognitive factors outside optical stimulation.

If this was the view of Koffka, then we conjecture that his response to Question 4 would correspond to the second alternative indicated, namely, the roots of the conditional indeterminacy of the perceptual properties involved in an invariant are of a methodological, not of a substantial, kind. More precisely, the composite proximal property $X = (X_1,\ldots,X_m)$ considered in an invariant regarding perceptual properties $(Y_1,\ldots,Y_n)$ may be *non-exhaustive* of the stimulus data that are influential on these, so that for any condition $X = x$, there may be several assignments of values $(y_1,\ldots,y_n)$ compatible with it [conditional indeterminacy of $(Y_1,\ldots,Y_n)$]. This argument naturally entails that if (besides X) other data implicit in the proximal stimulation were also taken into account – e.g., the "general field conditions" mentioned by Koffka in excerpt (2) and elsewhere – then the conditional indeterminacy of the perceptual properties would vanish.

For example, the perceptual size and distance of a pole appearing in a scene – i.e., the perceptual properties in the size–distance invariant – have conditional indeterminacy so far as only the size of the corresponding image on the retina is considered. But optical stimuli may also include data (e.g., the "depth factors" mentioned on p. 235) capable of determining the perceptual distance of the pole, so that (through the invariant) its perceptual size would also become determined, and the conditional indeterminacy of both perceptual properties, namely, size and distance, would vanish.

Question 5. Indeterminacy and variability are distinct but related concepts. In particular, an effective method of showing that a perceptual property is not (fully) determined by a definite stimulus condition is by describing a set of vision situations across which this condition remains fixed whereas that property undergoes variations. Is this method used by Koffka when arguing about the conditional indeterminacy implicit in his invariants?

The demonstrations on "Hering's hole method" and "Gelb's experiment" (pp. 244–245), the effect of a "reduction screen" on colour perception (pp. 254–255), etc., constitute the evidence for a positive response to this question. In the terms of our set-theoretic framework, such a demonstration amounts to showing that distinct vision situations may exist, or be constructed, across which the value of a definite proximal property X remains the same, but the combined values of perceptual properties $(Y_1,\ldots,Y_n)$ are variable, thus proving that the latter are not uniquely determined by the former (the freedom aspect of an invariant).

The conjoint variability of coexisting properties is also a standard method used in science for specifying the "direction" of a dependence relation – or, with some abuse of terms, for deciding which is the "cause" and which is the "effect" among those properties. We note, in this regard, that the conditional dependences between the perceptual properties, as represented in our set-theoretic framework, do not have a prearranged direction. They are "functional relations", as meant by Heidelberger (2010), and allow for different directions, depending on the context. Let us consider, for example, the conditional dependences implicit in the size–distance invariant. An observational context may be such that it includes cues for distance but not cues for size – e.g., what is shown is an abstract or unfamiliar figure, with no *a priori* privileged size in the perceptual representation. In such a context, the conditional dependence will be directed from distance to size: the perceptual distance, specified by the cues, will determine the perceptual size, through the conditional dependence. The orientation of the dependences will be the opposite in a hypothetical context in which there are cues for size (e.g., what is shown is a highly familiar object, which evokes a definite size in the perceptual representation), but the distance cues are weakened or removed (Gogel, 1976).

Question 6. Consider the view, apparently implicit in Koffka's theory, that the whole of the stimulus information in an act of vision is sufficient to determine the perceptual scene unambiguously, so that if two acts of vision had equal sets of optical stimuli, then the perceptual scenes experienced in them should be equal[6]. The question is: what arguments are used by Koffka to show that the conditional indeterminacy implicit in an invariant is indeed compatible with this deterministic view?

Two distinct kinds of arguments are worthy of note in this regard, one referring to "internal forces", and the other to "external forces" of perceptual organisation – a distinction on pp. 138–139. The former is signalled by the use of words such as "simple", "unique", "normal", "well-balanced", etc. (e.g., pp. 221, 224, and 231). In terms of our framework, for an invariant on properties (X, Y_1, Y_2), such an argument would mean that, within the range $\mathcal{Y}_1$ of perceptual property Y_1, there may exist a value y_1^* which is *a priori* privileged, because of its simplicity, regularity, or other peculiarities. Thus, under a stimulus condition $X = x$, property Y_1 will spontaneously settle on value y_1^*, in the absence of factors opposing this solution. In consequence, the perceptual property Y_2 will take on a definite value y_2^*, specifically, a value that is compatible with y_1^* according to the conditional dependence D_x. Conclusively, a definite combined value (y_1^*, y_2^*) becomes determined for properties (Y_1, Y_2) involved in the invariant, viz., the indeterminacy is overcome.

[6] Examples such as ambiguous figures (Rock, 1983, pp. 65–67) are not by themselves contradictory to this view, as the alternation of different perceptual organisations under stable stimulus conditions might be thought of as a (surprising) property of the resulting perceptual scene, a property which *would occur again* if the same stimuli were viewed again.

Arguments of this type may be found, e.g., in the following locations: on p. 219 concerning the direction–direction invariant, with the vertical and horizontal directions of the linear elements in the scene as privileged attributes; on pp. 231–232 concerning the shape–orientation invariant, with the frontal parallel position of a planar figure as a privileged attribute; and on p. 262 concerning the transparency phenomenon, with uniformity of colour as a preferential perceptual attribute.

The other kind of argument, in terms of "external forces", is illustrated by the following statement, which is extracted from a discussion of the shape–orientation invariant (on p. 235):

> "All factors, therefore, which determine orientation must *pari passu* influence perceived shape." (10)

It suggests that in the "totality of stimulation", there may be "factors" capable of determining the value of a perceptual property Y_1 (viz., orientation), such that the value of another perceptual property Y_2 (viz., shape) also becomes determined, due to the conditional dependence D_x (between Y_1 and Y_2) imposed by a stimulus condition $X = x$ (within a definite invariant). In Koffka, recurrent references to "total stimulation", "extended processes", "general field conditions", etc. are tantamount to references to such "factors" in the "psychophysical field", as well as their contribution to overcoming the conditional indeterminacy implicit in invariants.

6. Koffka's Insights on Invariants Are Alive in Modern Perceptual Psychology

Koffka's invariants are multifaceted constructs; in the previous sections, we have examined a sample of their aspects. Now we can see that some of those aspects occur – possibly in elaborate form and without reference to Koffka – in contemporary theories of visual perception, and we view this circumstance as proof of the substantial and productive character of the problems addressed under the title of "invariants".

One aspect is the relational approach to perception (mentioned in Section 4), which is a basic tenet of the Gestalt theory in general and is accepted and suitably developed in contemporary theories. To name a few: the "relational psychophysics" of human and animal perception (Sarris, 2006), the "anchoring theory" of lightness perception (Gilchrist et al., 1999), and the "differentiation–integration model" of anomalous surfaces and illusory contours (Kogo et al., 2010). A second aspect is the hypothesis of intra-perceptual dependences (mentioned in Section 2), which occurs under various names in authors following Koffka and is emphasised in certain present-day theories of vision, e.g., the "indirect/constructivist view" of perception (Rock, 1997) and the "experimental phenomenology" of vision, which looks at those dependences as the privileged subject of study (Savardi & Bianchi, 2012, p. 194; Sinico, 2013, p. 371).

A third aspect is the hypothesis (mentioned in Section 5) that the dependence of perceptual properties on stimulus properties is flexible or non-deterministic in character, as well as the question about what factors may intervene to absorb or compensate for the residual indeterminacy in the psychophysical relationship. Similar questions and hypotheses are considered in current theories that deal with the integration of "sources of stimulus information" (Cutting & Vishton, 1995; Trommershäuser, Körding, & Landy, 2011) and discuss possible interventions of *a priori* factors in perceptual processes (e.g., "prior constraints" in the computational approach to vision, and "prior probabilities" in Bayesian modelling; Marr, 1982, ch. 3; Kersten, Mamassian, & Yuille, 2004). As a fourth aspect, we mention the notion of "invariant" itself, which has promoted the discovery of other notable exemplars (e.g., the shape–colour–illumination and the orientation–lightness–illumination invariants; Bergström, 1977; Bloj & Hurlbert, 2002) and has been modulated into new concepts, such as "invariance" as a criterion for categorising perceptual properties (Chen, 2005; Todd, Chen, & Norman, 1998) and "invariants" as higher-order features of the optical stimulus (Gibson, 1979, ch. 14; Cutting, 1986, ch. 5).

Besides listing these correspondences, we wish to illustrate the theoretical convenience of Koffka's invariants by reasoning on a specific problem of perceptual psychology. We choose the problem of *perceptual transparency* because it is a topic explicitly discussed by Koffka in the chapter on invariants (pp. 260–264); it is a subject of substantial research in modern perceptual psychology, and it is a problem that allows us to show how Koffka's invariants may be the "gears" of articulate models of salient perceptual phenomena.

The phenomenon of perceptual transparency may be illustrated with stimuli of various complexities. For definiteness of our example, we refer to a homogeneous class of pictorial stimuli, of which the images (i)–(iv) in Figure 2 are four representative members. These images – and all members of the intended class – share the property of being composed of six *regions*, topologically distinguished into three *internal* regions $R_{1,1}$, $R_{1,2}$, and $R_{1,3}$, and three *external* regions $R_{2,1}$, $R_{2,2}$, and $R_{2,3}$. Each region is filled with a grey colour, which is uniform within the region and different from the greys in the adjacent regions. Furthermore, there is *continuity* of the peripheral margins of the internal regions, which is a favourable condition in order that their aggregate $R_{1,1}+R_{1,2}+R_{1,3}$ may give rise to a unitary entity in the perceptual rendering [a "figure" surface, denoted by A in Figure 2(vi)]. For example, thanks to continuity of margins, the aggregate $R_{1,1}+R_{1,2}+R_{1,3}$ has the general form of a disk in image (i), of a square in image (ii), and so on. Similarly, there is *continuity* between the radial margins of each internal region and the corresponding margins of the external region adjacent to it, which is a favourable condition in order that those two regions may give rise to a unitary entity (a "background" surface) in the perceived scene. The "background" surfaces are denoted B_1, B_2, and B_3 in Figure 2(vi). It is seen that the aggregates to be considered are not mutually disjoint, as each internal region belongs both to the aggregate $R_{1,1}+R_{1,2}+R_{1,3}$ (supporting the "figure" surface) and to one of the aggregates $R_{1,1}+R_{2,1}$, $R_{1,2}+R_{2,2}$, $R_{1,3}+R_{2,3}$ (supporting the "background" surfaces). This is a typical property of the transparency phenomenon: "double representation" in transparency (Koffka, 1935, p. 261).

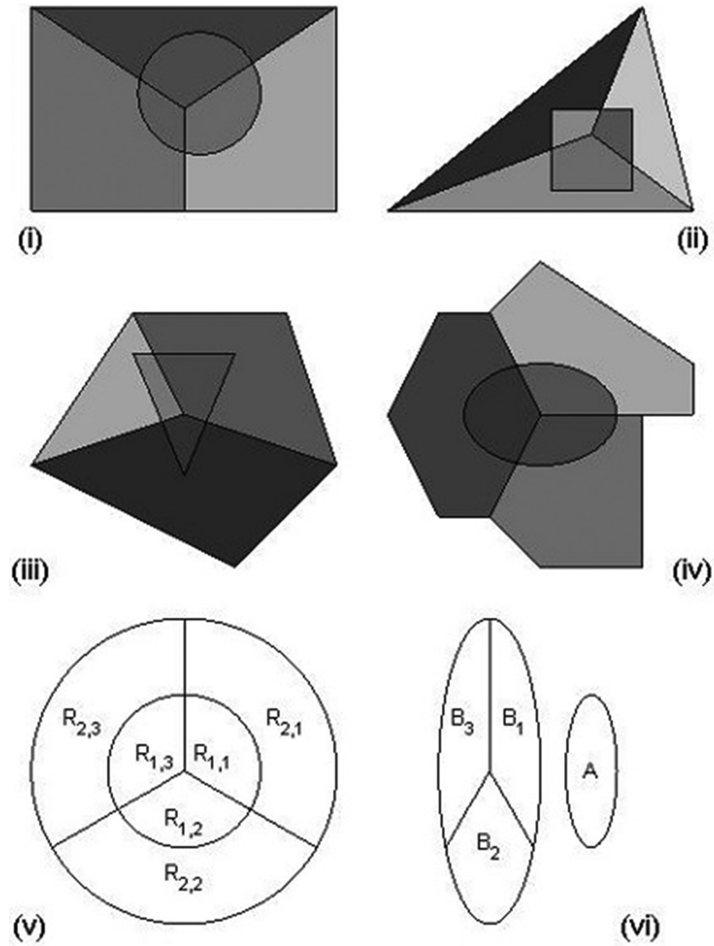

Fig. 2 A set of achromatic pictorial stimuli for testing perceptual transparency [parts (i)–(iv)], their common topology [part (v)], and the topology of a corresponding perceptual scene with transparency [part (vi)].

In an observational context of this kind, six light-related *stimulus* properties are especially important. They are the intensities (luminances) of the light sent towards the observer by the regions in the stimulus. In optical terms, each luminance $L_{i,j}$ (for i = 1,2 and j = 1,2,3) is the product of the reflectance (ability to reflect light) of region $R_{i,j}$ and the intensity (illuminance) of the light falling upon the region from the light source. Furthermore, if we presume that the effect of perceptual transparency takes place, then in suitable conditions, six light-related *perceptual* properties also should be considered. They are the apparent transparency T and whiteness W of the figure surface A; the apparent whitenesses W_1, W_2, and W_3 (of the background surfaces B_1, B_2, and B_3); and the apparent intensity I of the illumination in the scene. On the whole, 12 light-related properties are involved in the transparency problem under the stated conditions: six are properties measurable on the stimulus (the luminance values $L_{1,1}$, $L_{1,2}$, $L_{1,3}$, $L_{2,1}$, $L_{2,2}$, and $L_{2,3}$) and six are properties available in the percept (properties T, W, W_1, W_2, W_3, and I, as defined previously).

In line with modern treatments of the transparency problem (e.g., Beck, Prazdny, & Ivry, 1984; Koenderink, van Doorn, Pont, & Richards, 2008; Metelli, 1970), two kinds of psychophysical dependence may be hypothesised among those properties. One kind involves the luminance $L_{1,j}$ of any internal region $R_{1,j}$ in the stimulus, the apparent transparency T and whiteness W of the figure surface A in the percept, the apparent whiteness W_j of the background surface B_j, and the apparent intensity I of the illumination in the scene, which is expressed by the equation

$$L_{1,j} = \left((1 - T) \times W + T \times W_j\right) \times I. \tag{11}$$

This equation is tantamount to the hypothesis that the psychophysical relationship linking the optical property $L_{1,j}$ to the set (T, W, W_j, I) of perceptual properties is formally similar to the physical relationship linking the same $L_{1,j}$ to the set (T^*, W^*, W_j^*, I^*) of physical properties, in which T^* is the transmittance of a filter (a number from zero to one), W^* and W_j^* are the reflectances of the filter and of a surface lying behind it, and I^* is the illuminance upon the stimulus. The relationship defined by equation (11) has the features of a Koffka's invariant acting on five variables. Relationships of this kind are actually alluded to by Koffka on pp. 261–262, when he refers to "Talbot's law" and "laws of colour mixture" in the discussion of "transparency and constancy". The other kind of psychophysical dependence involves the luminance $L_{2,j}$ of any external region $R_{2,j}$ in the stimulus and the perceptual properties W_j and I as defined earlier, and this is expressed by the equation

$$L_{2,j} = W_j \times I. \tag{12}$$

It amounts to the hypothesis that the psychophysical relationship linking the optical property $L_{2,j}$ to the pair (W_j,I) of perceptual properties is formally similar to the physical relationship linking $L_{2,j}$ to the pair (W_j^*,I^*) of physical properties (i.e., luminance equals the product of reflectance and illuminance). In addition, the relationship expressed by equation (12) has the features of a Koffka's invariant acting on three variables. It is implicit in the discussion on pp. 243–244 about "whiteness constancy".

When these two kinds of psychophysical dependences are made explicit for all regions in the stimulus, a system of six equations for 12 variables is obtained, whose graph is represented in Figure 3. This is a descriptive model of perceptual transparency for observational contexts of the type in Figure 2. Operating analytically on such models proved helpful in addressing both the existence problem of the phenomenon – i.e., what conditions the luminance values $L_{1,1},\ldots,L_{2,3}$ should satisfy in order that the figure surface A may appear as a transparent filter – and the evaluation problem – i.e., given apparent transparency, how to predict the perceptual properties T, W, etc. based on the luminance values $L_{1,1},\ldots,L_{2,3}$ (cf. Da Pos & Burigana, 2013). The model represented in Figure 3 is a system of local dependences, each dependence acting on a part of an exhaustive set of variables. More precisely, it is a system of numerical equations [of the type represented by equation (11) or equation (12)], such that analyses or computations with that model can be carried out following rules of ordinary algebra. However, when applying this approach to other perceptual problems, systems of dependences that are of a freer kind, not expressible in terms of numerical equations, could come out [refer the "ordinal invariant" defined by expression (7)]. Mathematical tools from the combinatorial theories of "networks of constraints" and "Bayesian networks" could prove useful for dealing with such generalised systems of local psychophysical dependences (Dechter, 2003; Neapolitan, 2004).

The network of dependences underlying perceptual transparency allows us to exemplify salient aspects of invariants extracted through conceptual analysis in the preceding sections. We mention four of them. First, any single invariant is a "tolerant" or non-deterministic dependence, in which the stimulus property is able to determine not the values of the perceptual properties involved, but a relation between them. For example, in an invariant of type (12), for any fixed value of $L_{2,j}$, there corresponds not a single pair of values for (W_j,I), but an (infinite) set of such pairs, which however is a constrained set, as all pairs are supposed to satisfy equation (12). Second, any invariant (which is a psychophysical dependence) is tantamount to a family of intra-perceptual conditional dependences. For example, in an invariant of type (11), for any fixed value of $L_{1,j}$, there corresponds a quaternary relation between variables (T,W,W_j,I), which may be termed a (conditional) intra-perceptual dependence, as those variables are defined as

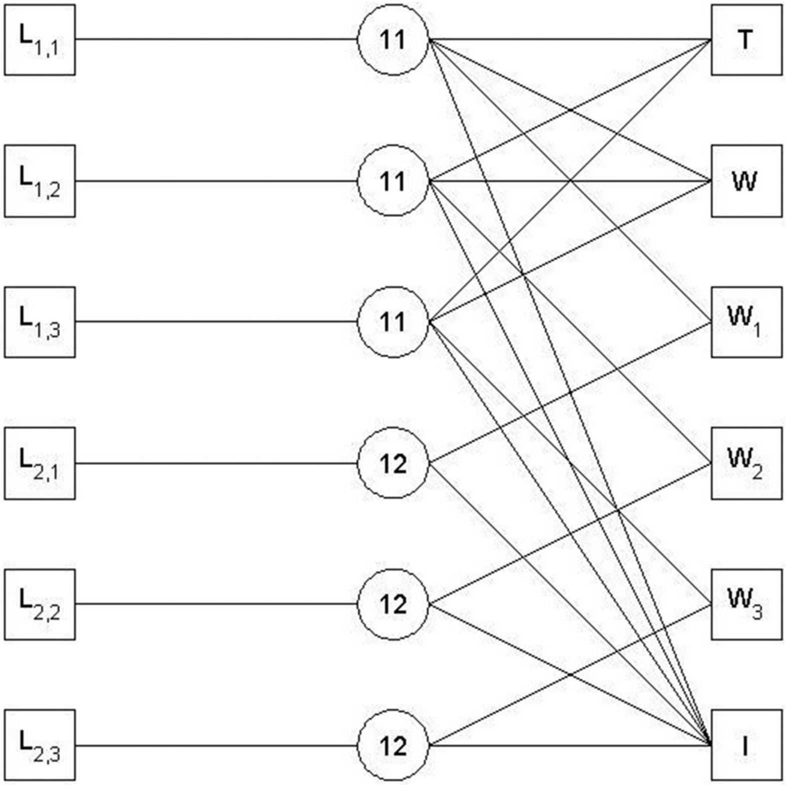

Fig. 3 Graph of a network of psychophysical dependences for stimuli similar to those in Figure 2 and the hypothesis of perceptual transparency; on the left are the stimulus variables, and on the right are the perceptual variables. Numbers in the circles indicate whether the dependences are defined by equation (11) or equation (12) in the text.

properties of the percept. Thus, in the view we are suggesting, it becomes almost unavoidable, in the discussion of a perceptual problem, to bring into play both psychophysical and intra-perceptual dependences. Third, although every single invariant is generally local and tolerant in its action, nevertheless, a whole system of invariants, like that in Figure 3, could be fully deterministic in its effect, so that once the values of the stimulus variables are assigned, one single "solution" for the set of perceptual variables becomes determined by the system. This may happen because the same perceptual variable could belong to the fields of action of two or more invariants (which would cooperate in specifying its value) and because a network of dependences allows for processes of propagation of constraining effects [an idea implicit, e.g., in excerpt (10) from Koffka]. Lastly, the invariants defined by equations (11) and (12) are examples of psychophysical dependences that (possibly) are conjectured following the "inverse optics heuristic", i.e., by resting on a known physical dependence (from projective optics) and turning it into a hypothetical psychophysical dependence, by suitable replacement of variables. The transition from the physical equation $L_{2,j} = W_j^* \times I^*$ to the psychophysical equation $L_{2,j} = W_j \times I$ illustrates this possibility.

We are far from believing that invariants in Koffka's sense may be the cornerstones of a *general* theory of visual perception. Our analyses and examples show that they are a concept of limited extension, whose applicability is subject to definite conditions. What we believe is that, when those conditions are satisfied, then invariants may prove to be a flexible and effective tool of psychophysical modelling. More precisely, we recognise their main strength in the modulation of the concept of "psychophysical dependence" they imply, namely, to state that certain perceptual properties depend upon certain stimulus properties does *not* necessarily mean that the values of the latter *uniquely* determine the values of the former. As noted above, such a modulation of psychophysical dependences may allow for representing a perceptual problem as a system of several local and partial constraints, within which there is place for intra-perceptual dependences, propagation of constraining effects, and possible contribution of *a priori* tendencies or preferences of the perceptual system. In turn, that modulation is the natural theoretical consequence of the refusal of the "constancy hypothesis" in strict form (commented on in Section 4) and the adoption of a "relational approach" to the study of vision, which are generally acknowledged as fundamental tenets of the Gestalt theory since its very beginning.

Summary

By introducing the concept of "invariants", Koffka (1935) endowed perceptual psychology with a flexible theoretical tool, which is suitable for representing vision situations in which a definite part of the stimulus pattern is relevant but not sufficient to determine a corresponding part of the perceived scene. He characterised his "invariance principle" as a principle conclusively breaking free from the "old constancy hypothesis", which rigidly surmised point-to-point relations between stimulus and perceptual properties. In this paper, we explain the basic terms and assumptions implicit in Koffka's concept, by representing them in a set-theoretic framework. Then, we highlight various aspects and implications of the concept in terms of answers to six separate questions: forms of invariants, heuristic paths to them, what is invariant in an invariant, roots of conditional indeterminacy, variability vs. indeterminacy, and overcoming of the indeterminacy. Lastly, we illustrate the lasting value and theoretical power of the concept, by showing that Koffka's insights relating to it do occur in modern perceptual psychology and by highlighting its role in a model of perceptual transparency.

Keywords: Invariance principle, Constancy hypothesis, Stimulus insufficiency, Perceptual indeterminacy, Intra-perceptual dependence.

Zusammenfassung

Mit der Einführung des Invarianzkonzepts bietet Koffka (1935) der Wahrnehmungspsychologie ein flexibles theoretisches Instrument. Mit diesem Konzept kann dargestellt

werden, dass ein bestimmtes Reizmuster für die entsprechende visuelle Wahrnehmung relevant, aber nicht ausreichend ist, um diese zu bestimmen. Er charakterisiert sein "Prinzip der Invarianz" als ein Prinzip, das die alte Konstanzannahme, die starr die Existenz einer direkten Beziehung zwischen Stimulus und perzeptiven Eigenschaften annahm, ersetzt. In diesem Artikel sollen die Grundbegriffe und die impliziten Annahmen Koffkas geklärt werden, indem sie in einem mengentheoretischen Rahmen dargestellt werden. Im Folgenden werden Aspekte und Folgerungen des Konzepts in Form von Antworten auf sechs separate Fragestellungen hervorgehoben: die Form der Invarianten, heuristische Wege zu ihnen, das Unabänderliche einer Invariante, das Wesen der bedingten Indeterminiertheit, Variabilität vs. Indeterminiertheit, Überwindung der Indeterminiertheit. Schließlich wird die dauerhafte Bedeutung des Konzepts erläutert, indem gezeigt wird, inwiefern Koffkas Erkenntnisse aktuelle Theorien der Wahrnehmungspsychologie beeinflussen.
Schlüsselwörter: Invarianzkonzept, Konstanzannahme, Stimulusinsuffizienz, perzeptive Indeterminiertheit, intra-perzeptuelle Abhängigkeit.

References

Beck, J., Prazdny, K., & Ivry, R. (1984). The perception of transparency with achromatic colors. *Perception & Psychophysics, 35*(5), 407–422.

Bergström, S. S. (1977). Common and relative components of reflected light as information about the illumination, colour, and three-dimensional form of objects. *Scandinavian Journal of Psychology, 18*(3), 180–186.

Bloj, M. G., & Hurlbert, A. C. (2002). An empirical study of the traditional Mach card effect. *Perception, 31*(2), 233–246.

Chen, L. (2005). The topological approach to perceptual organization. *Visual Cognition, 12*(4), 553–637.

Cutting, J. E. (1986). *Perception with an eye for motion*. Cambridge, MA: MIT Press.

Cutting, J. E., & Vishton, P. M. (1995). Perceiving layout and knowing distances: The integration, relative potency, and contextual use of different information about depth. In W. Epstein & S. Rogers (Eds.), *Perception of space and motion* (pp. 69–117). New York, NY: Academic Press.

Da Pos, O., & Burigana, L. (2013). Qualitative inference rules for perceptual transparency. In L. Albertazzi (Ed.), *Handbook of experimental phenomenology: Visual perception of shape, space and appearance* (pp. 343–367). New York, NY: Wiley.

Dechter, R. (2003). *Constraint processing*. San Mateo, CA: Morgan Kaufmann.

Epstein, W. (1982). Percept-percept couplings. *Perception, 11*(1), 75–83.

Gibson, J. J. (1979). *The ecological approach to visual perception*. Boston, MA: Houghton Mifflin.

Gilchrist, A. L. (2006). *Seeing black and white*. Oxford, UK: Oxford University Press.

Gilchrist, A. L., Kossyfidis, C., Bonato, F., Agostini, T., Cataliotti, J., Li, X., Spehar, B., Annan, V., & Economou, E. (1999). An anchoring theory of lightness perception. *Psychological Review, 106*(4), 795–834.

Gogel, W. C. (1973). The organization of perceived space. I. Perceptual interactions. *Psychologische Forschung, 36*(3), 195–221.

Gogel, W. C. (1976). An indirect method of measuring perceived distance from familiar size. *Perception & Psychophysics, 20*(6), 419–429.

Hatfield, G. C. (2003). Representation and constraints: The inverse problem and the structure of visual space. *Acta Psychologica, 114*(3), 355–378.

Heidelberger, M. (2010). Functional relations and causality in Fechner and Mach. *Philosophical Psychology, 23*(2), 163–172.

Hochberg, J. E. (1957). Effects of the Gestalt revolution: The Cornell symposium on perception. *Psychological Review, 64*(2), 73–84.

Jäkel, F., Singh, M., Wichmann, F. A., & Herzog, M. H. (2016). An overview of quantitative approaches in Gestalt perception. *Vision Research, 126*, 3–8.

Johansson, G. (1970). On theories for visual space perception. A letter to Gibson. *Scandinavian Journal of Psychology, 11*(2), 67–74.

Kersten, D., Mamassian, P., & Yuille, A. L. (2004). Object perception as Bayesian inference. *Annual Review of Psychology, 55*, 271–304.

Koenderink, J. J., van Doorn, A. J., Pont, S., & Richards, W. (2008). Gestalt and phenomenal transparency. *Journal of the Optical Society of America, Series A, 25*(1), 190–202.
Koffka, K. (1935). *Principles of Gestalt psychology*. New York, NY: Harcourt, Brace and Company.
Kogo, N., Strecha, C., van Gool, L., & Wagemans, J. (2010). Surface construction by a 2-D differentiation-integration process: A neurocomputational model for perceived border ownership, depth, and lightness in Kanizsa figures. *Psychological Review, 117*(2), 406–439.
Köhler, W. (1913). Über unbemerkte Empfindungen und Urteilstäuschungen. *Zeitschrift für Psychologie, 66*(Hefte 1 und 2), 51–80.
Marr, D. (1982). *Vision. A computational investigation into the human representation and processing of visual information*. San Francisco, CA: Freeman.
Metelli, F. (1970). An algebraic development of the theory of perceptual transparency. *Ergonomics, 13*(1), 59–66.
Neapolitan, R. E. (2004). *Learning Bayesian networks*. Upper Saddle River, NJ: Pearson Prentice Hall.
Oyama, T. (1969). S-S relations in psychophysics and R-R correlations in phenomenology. *Psychologia, 12*(1), 17–23.
Poggio, T., Torre, V., & Koch, C. (1985). Computational vision and regularization theory. *Nature, 317*(No. 6035), 314–319.
Rock, I. (1983). *The logic of perception*. Cambridge, MA: MIT Press.
Rock, I. (1997). *Indirect perception*. Cambridge, MA: MIT Press.
Sarris, V. (2006). *Relational psychophysics in humans and animals: A comparative developmental approach*. London, UK: Psychology Press.
Sarris, V. (2012). Epilogue: Max Wertheimer in Frankfurt and thereafter. In L. Spillmann (Ed.), *On perceived motion and figural organization* (pp. 253–265). Cambridge, MA: MIT Press.
Savardi, U., & Bianchi, I. (2012). Coupling Epstein's and Bozzi's "percept-percept coupling". *Gestalt Theory, 34*(2), 191–200.
Sinico, M. (2013). Epistemic line of explanation for experimental phenomenology. *Gestalt Theory, 35*(4), 365–376.
Spillmann, L. (2012). The current status of Gestalt rules in perceptual research: Psychophysics and neurophysiology. In L. Spillmann (Ed.), *On perceived motion and figural organization* (pp. 191–251). Cambridge, MA: MIT Press.
Todd, J. T., Chen, L., & Norman, J. F. (1998). On the relative salience of Euclidean, affine, and topological structure for 3-D form discrimination. *Perception, 27*(3), 273–282.
Trommershäuser, J., Körding, K. P., & Landy, M. S. (Eds.). (2011). *Sensory cue integration*. Oxford, UK: Oxford University Press.
Wagemans, J., Elder, J. H., Kubovy, M., Palmer, S. E., Peterson, M. A., Singh, M., & von der Heydt, R. (2012a). A century of Gestalt psychology in visual perception: I. Perceptual grouping and figure-ground organization. *Psychological Bulletin, 138*(6), 1172–1217.
Wagemans, J., Feldman, J., Gepshtein, S., Kimchi, R., Pomerantz, J. R., van der Helm, P. A., & van Leeuwen, C. (2012b). A century of Gestalt psychology in visual perception: II. Conceptual and theoretical foundations. *Psychological Bulletin, 138*(6), 1218–1252.

Luigi Burigana, born 1944, is a retired full Professor of Psychometrics, University of Padua, Italy. From 1972 to 2015, he gave courses in Psychometrics, Methods for Data Analysis, and Mathematical Psychology for students of psychology at the University of Padua. His current scientific interests are in epistemological issues in vision research, probabilistic theories of visual perception, and discrete mathematical models in psychology.
Address (corresponding author): Department of General Psychology, University of Padua, via Venezia 8, 35131 Padova, Italy.
Email: luigi.burigana@unipd.it

Michele Vicovaro, born 1985, received his PhD in Experimental Psychology from the University of Padua, Italy, in 2012. Presently, he is a research assignee at the Department of General Psychology, University of Padua. His main research interests are in experimental intuitive physics, multisensory information processing, and psychophysical models and methods. He is a lecturer in Basic Research Methods in Psychology.