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STRATIFIED SOLUTIONS FOR SYSTEMS OF CONSERVATION LAWS

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Abstract. We study a class of weak solutions to hyperbolic systems of conservation (balance) laws in one space dimension, called stratified solutions. These solutions are bounded and "regular" in the direction of a linearly degenerate characteristic field of the system, but not in other directions. In particular, they are not required to have finite total variation. We prove some results of local existence and uniqueness.

Introduction

In this paper we are interested in weak solutions of hyperbolic systems of conservation laws (or of balance laws) in one space dimension when the system admits a *linearly degenerate* eigenvalue. We introduce a class of weak solutions which are *bounded* and "regular" along the integral curves of the characteristic field corresponding to the linearly degenerate eigenvalue. However, no more regularity is required in other directions. In several space dimensions, for semilinear hyperbolic systems, this type of regularity was already introduced by J. Rauch and M. Reed in [17], and the solutions are called *stratified solutions*. Stratified solutions of quasilinear hyperbolic systems in several dimensions were considered by G. Métivier in [14], for continuous solutions.

Here we prove local (in time) existence and stability for a class of bounded and stratified solutions. In particular, we obtain a result of existence and propagation of solutions $u(t,x)$ that can have unbounded total variation: $TV_{[a,b]}\big(u(t,\cdot)\big)$ (= the total variation of $u(t,\cdot)$ on $[a,b]$) $= +\infty$. Such solutions are not provided by the classical results on general systems of conservation laws by P. D. Lax, J. Glimm, A. Bressan, ([13], [9], [2]). Our results must be compared, on one hand, with those of W. E and A. Heibig on the propagation of high frequency oscillations with $O(1)$ amplitude ([5], [12]), and on the other hand with that of Y.-J. Peng ([16]), who constructed solutions with large total variation of the entropy for the Euler system of gas-dynamics.

The main assumption is about the existence of a special symmetrizer, which has been introduced by A. Heibig in [12]. Examples of stratified waves are given

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by the *entropy waves* for the system of Euler equations of gas-dynamics, or of magnetohydrodynamics. Other detailed examples are given in Section 1.

1. Main notations and results

We denote $\Omega = \mathbf{R}\times]-T_0, T_0[$ for some positive T_0 and let $\mathcal{O}$ be an open subset of $\mathbf{R}^N$ containing a reference constant state $\underline{u}$; these sets are supposed sufficiently small in order that all operations below are well defined. We consider the following $N \times N$ system of balance laws in one space dimension:

$$\partial_t u + \partial_x F(u) = f(t, x, u) \tag{1.1}$$

where F and f are smooth functions which are defined respectively in $\mathcal{O}$ and in $\Omega \times \mathcal{O}$; we denote $A(u) = DF(u)$. Here and in the whole paper "smooth" means simply "infinitely differentiable". For the sake of simplicity we assume that F depends only on u, but the case where F depends also on (t, x) could be treated as well; for the same reasons we assume that $f(t, x, \underline{u}) \in C_0^\infty(\overline{\Omega})$. As far as the hyperbolicity of system (1.1) is concerned, we shall make some assumptions which imply that it is symmetrizable hyperbolic in the sense of Friedrichs.

Assumption I. *The matrix A has a simple, linearly degenerate eigenvalue λ.*

This assumption means that the eigenspace $\ker\big(A(u)-\lambda(u)\big)$ has dimension 1 and that $r(u) \cdot D\lambda(u) = 0$ for every $u \in \mathcal{O}$, where $r(u)$ is any eigenvector corresponding to $\lambda(u)$. Finally, let us fix a smooth eigenvector and call it $r(u)$.

We give now some motivations to our results by a simple example. Let γ be a smooth integral curve of the eigenvector r, i.e.,

$$\begin{cases} \dot\gamma(s) = r\big(\gamma(s)\big), \\ \gamma(0) = u_0 \end{cases}$$

which is defined in some interval containing 0, for some $u_0 \in \mathcal{O}$. Because of Assumption I, the eigenvalue λ is constant along the curve γ, and so let us write $\lambda\big(\gamma(s)\big) = \lambda(u_0) = \omega$. Then the function

$$u(t, x) = \gamma\big(\alpha(x - \omega t)\big) \tag{1.2}$$

is a smooth local solution to (1.1) for any smooth scalar function α. In fact, much more general functions α are allowed. Let α be smooth outside 0, discontinuous at 0, with a jump sufficiently small in order that $\gamma\big(\alpha(0\pm)\big)$ are defined; then (1.2) defines a weak solution to (1.1), a contact discontinuity. The main point, however, is that we still have weak solutions of the form (1.2) for any L^∞ function α. Actually, let $\{\alpha_n\}$ be a uniformly bounded sequence of smooth functions converging to α a.e.; then the smooth solutions $u_n = \gamma\big(\alpha_n(x - \omega t)\big)$ converge a.e. to $\gamma\big(\alpha(x - \omega t)\big)$. Moreover, this sequence is still uniformly bounded, and the sequence $\{F(u_n)\}$ has the same properties; therefore we can pass to the limit in the equations, in the weak sense.

Let us point out that these last solutions can have, for any given t, an infinite variation $TV(u(t, .))$. They are examples of *stratified* solutions to (1.1), in a sense that we are going to prove below.

Since the eigenvalue λ is simple, we can write system (1.1) in a somewhat nicer form as follows. We assume that the open set $\mathcal{O}$ is sufficiently small in order that there exist in $\mathcal{O}$ a set $R_1(u), \ldots, R_{N-1}(u)$ of λ-Riemann invariants with linearly independent gradients, so that $DR_j \cdot r = 0$ for $j = 1, \ldots, N-1$. In order to get a basis of $\mathbf{R}^N$ we complete this set by choosing a function $\chi = \chi(u)$ satisfying $D\chi \cdot r = 1$ in $\mathcal{O}$ (see, for instance, [10]). We then make the change of dependent variables $\mathcal{R} : u \mapsto (v, w) = \big(R(u), \chi(u)\big)$ and for

$$\begin{aligned} \tilde{A}(v,w) &= \big(D\mathcal{R}(\mathcal{R}^{-1})A(\mathcal{R}^{-1})D\mathcal{R}^{-1}\big)(v,w) = \begin{pmatrix} B(v,w) & 0 \\ \eta(v,w) & \mu(v) \end{pmatrix}, \\ \tilde{f}(t,x,v,w) &= D\mathcal{R}\big(\mathcal{R}^{-1}(v,w)\big) f\big(t,x,\mathcal{R}^{-1}(v,w)\big) = \begin{pmatrix} b(t,x,v,w) \\ d(t,x,v,w) \end{pmatrix}, \end{aligned} \tag{1.3}$$

we obtain

$$\begin{cases} \partial_t v + B(v,w)\partial_x v = b(t,x,v,w), \\ \partial_t w + \eta(v,w)\partial_x v + \mu(v)\partial_x w = d(t,x,v,w). \end{cases} \tag{1.4}$$

Above we denote by η an $N-1$ line vector, and wrote $\mu(v)$ for $\lambda\big(\mathcal{R}^{-1}(v,w)\big)$. We remark that μ does not depend on w and is not an eigenvalue of the matrix B due to Assumption I. Let $\tilde{\mathcal{O}}$ be the image of the set $\mathcal{O}$ by this change of variables; we may assume that $\mathcal{R}(\underline{u}) = 0$, otherwise we make a translation of the new dependent coordinates (v, w) which does not affect in any way the structure of the system. Then let us fix an arbitrary compact neighborhood $\mathcal{K} \subset \mathcal{O}$ of $\underline{u}$ and a compact neighborhood $\tilde{\mathcal{K}}_1 \subset \tilde{\mathcal{O}}$ of $\tilde{\mathcal{K}} = \mathcal{R}(\mathcal{K})$. Let us point out that, in general, weak solutions are *not* conserved under such nonlinear change of variables; this is, however, the case for the class of solutions under consideration, as we shall prove below.

Assumption II. *There exists a smooth $(N-1) \times (N-1)$ symmetric positive definite matrix $S(v,w)$ such that the matrix $S \cdot (B - \mu I)$ is symmetric and independent of w.*

This condition was introduced by Heibig in [12] and a thorough study is contained in [22]; in particular, Sévennec provides in [22] a sufficient condition in terms of an entropy of system (1.1) for the existence of such a symmetrizer. He also shows that if Assumptions I and II hold, then system (1.1) is globally hyperbolic in the sense of Serre (see [19], [20]). On the other hand, there exist globally hyperbolic systems which do not satisfy Assumption II for some eigenvalue (the magnetohydrodynamics equations, for instance; see [22], remark 20, page 87). We refer again to [22] for an intrinsic statement of the condition above, which is expressed directly on system (1.1).

We point out that Assumption II contains an hyperbolicity assumption on the complete system (1.1). Indeed, it implies that the system $\partial_t + A(u)\partial_x$ is symmetrizable in the sense of Friedrichs (see the Appendix). However, we will make no explicit use of this fact in the paper: this remark is just for the sake of completeness, and to insist on the fact that Assumption II must be viewed as a reinforced hyperbolicity assumption.

Let us also remark that neither the change of variable $\mathcal{R}$ made above nor the formulation of Assumption II need that the eigenvalue λ, on which the whole construction relies, is linearly degenerate, in spite of the fact that we already took advantage of this assumption in writing $\mu = \mu(v)$. The assumption of being linearly degenerate is needed instead in the very definition of the function spaces that we introduce now.

For $T \in \,]-T_0, T_0[$ we write $\Omega_T = \mathbf{R} \times]-T_0, T[$ and denote by

$$Lip(\Omega_T) = \big\{u \in L^\infty(\Omega_T);\ (\partial_t u, \partial_x u) \in L^\infty(\Omega_T)\big\}$$

the class of Lipschitz-continuous functions in Ω_T. We introduce then the set

$$P(\Omega_T) = \big\{u \in L^\infty(\Omega_T;\mathcal{O});\ R(u) \in Lip(\Omega_T) \text{ for every smooth } \lambda\text{-Riemann invariant } R\big\}.$$

Since every Riemann invariant in $\mathcal{O}$ can be expressed by means of the previous $R_1, \dots, R_{N-1}$ (see [10]), it is sufficient in the definition of the set $P(\Omega_T)$ to ask that $R_j(u) \in Lip(\Omega_T)$ for $j = 1, \dots, N-1$.

Since λ is linearly degenerate, it is a λ-Riemann invariant. A consequence is the following: if $u = u(t,x)$ belongs to $P(\Omega_T)$, then the field $X_u = \partial_t + \lambda(u)\partial_x$ acts on $L^2(\Omega_T)$ functions (since $\lambda(u)$ is Lipschitz), and, in particular, $X_u v$ has a (distributional) sense in $H^{-1}(\Omega_T)$ for any $v \in L^2(\Omega_T)$.

We then define inductively for any positive integer m and any $u \in P(\Omega_T)$ the following spaces of stratified functions:

$$\begin{aligned} \Sigma^0(u;\Omega_T) &= L^2(\Omega_T), \\ \Sigma^m(u;\Omega_T) &= \big\{v \in \Sigma^{m-1}(u;\Omega_T);\ X_u v \in \Sigma^{m-1}(u;\Omega_T)\big\},\ m \geq 1. \end{aligned}$$

An induction is actually needed to define $\Sigma^m(u;\Omega_T)$, since a direct expression like $(X_u)^k v$ makes no sense in general for $k \geq 2$ and $u \in P(\Omega_T)$ and $v \in L^2(\Omega_T)$. This kind of stratified regularity was introduced by J. Rauch and M. Reed ([17]) for the study of semilinear hyperbolic problems.

Let us now introduce the following subset of $P(\Omega_T)$:

$$\mathcal{S}^m(\Omega_T) := \{u \in P(\Omega_T); u \in \Sigma^m(u;\Omega_T)\}. \tag{1.5}$$

A function u in $\mathcal{S}^m(\Omega_T)$ is stratified with respect to the foliation induced by the field X_u. We remark that the sets $\mathcal{S}^m(\Omega_T)$ are not linear spaces, in general; nevertheless, after a change of variables straightening the λ-characteristic field (see Proposition 3.1) they turn into linear spaces, for every $m \geq 0$.

Our main results follow.

Theorem 1.1. *We make the Assumptions* I *and* II. *Let $m \geq 4$ be an integer and let $u_0 \in \mathcal{S}^m(\Omega_0)$ be a solution of* (1.1) *in Ω_0. Then there exists a time $T \in \,]0, T_0[$ such that the system* (1.1) *has a unique solution $u \in \mathcal{S}^m(\Omega_T)$ with $u_{|\Omega_0} = u_0$.*

The time T in the result above depends on the quantities $\sup_{0\leq j\leq m} \|u^{(j)}\|_{L^2(\Omega_0)}$ and $\sup_{0\leq j\leq 1} \|u^{(j)}\|_{L^\infty(\Omega_0)}$, where we defined $u^{(0)} = u$, $u^{(1)} = X_u u$, $\dots$, $u^{(m)} = X_{u^{(m-1)}} u^{(m-1)}$. Theorem 1.1 implies the existence of a unique $T^* \in \,]0, T_0]$ and a unique $u \in L^\infty_{loc}(\Omega_{T^*})$ such that $u_{|\Omega_T} \in \mathcal{S}^m(\Omega_T)$ is the solution of (1.1) for every $T < T^*$. As usual we say that (Ω_{T^*}, u) is the unique maximal $\mathcal{S}^m$-solution of (1.1). In general, we have $T^* < T_0$.

Theorem 1.2. *Under the same assumptions of Theorem* 1.1*, let (Ω_{T^*}, u) be the maximal $\mathcal{S}^m$-solution of* (1.1)*. Let us suppose $T^* < T_0$ and $u \in L^\infty(\Omega_{T^*}, \mathcal{O})$. Then $u \notin P(\Omega_{T^*})$, which means that there exists a Riemann invariant R such that*

$$\lim_{T\to T^*} \|\partial_x R(u)\|_{L^\infty(\Omega_T)} = +\infty.$$

A consequence of this theorem is, for example, the following. Let us denote

$$\mathcal{S}^\infty(\Omega_T) = \bigcap_{m=0}^{\infty} \mathcal{S}^m(\Omega_T).$$

Then, if $u_0 \in \mathcal{S}^\infty(\Omega_0)$, the life span T^* does not depend on m, and the corresponding maximal solution (Ω_{T^*}, u) satisfies $u \in \mathcal{S}^\infty(\Omega_T)$ for every $T < T^*$.

Cauchy problem and compatibility conditions. Let us consider a solution $u \in \mathcal{S}^m(\Omega_T)$ of (1.1) given by Theorem 1.1. The fact that $X_u u$ is in $L^2(\Omega_T) \cap L^\infty(\mathbf{R})$ implies that u belongs to $C\big([-T_0, T] : L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})\big)$, so that the restriction $u(t_0, \cdot)$ at some given time t_0 $(-T_0 < t_0 < T)$ is well defined in the space $L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$. In general, the function $h(x) := u(t_0, x)$ has no more regularity. However, the function h satisfies some particular additional conditions (for instance $R(h) \in Lip(\mathbf{R})$), which imply that the Cauchy problem for (1.1) with the data $h(x)$ at the time $t = t_0$ can be solved in the class of stratified solution. It is then a natural question to describe a set of *sufficient compatibility conditions* for a Cauchy data in $L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$, allowing us to solve the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x F(u) = f(t, x, u), \\ u_{|t=0} = u_0 \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R}), \end{cases} \tag{1.6}$$

in the class of stratified solutions.

Let us first describe the form of the *necessary* compatibility conditions that the given solution $u \in \mathcal{S}^m(\Omega_T)$ satisfies, when $m \geq 2$. Let us call

$$\begin{array}{lcl} \Phi_0 := \chi(u), & & \Psi_0 := R(u), \\ \Phi_k := X_u \Phi_{k-1}, \quad 1 \leq k \leq m, & \text{and} & \Psi_k := X_u \Psi_{k-1}, \quad 1 \leq k \leq m. \end{array}$$

We show in Section 3.5 that $\Phi_k \in C\big([-T_0, T] : L^2(\mathbf{R}_x)\big)$ for $k \leq m-1$, and $\Psi_k \in C\big([-T_0, T] : H^1(\mathbf{R}_x)\big)$ for $k \leq m-2$. Moreover, the restrictions $\mathbf{F}_k := \Phi_{k|t=0}$ and $\mathbf{G}_\ell := \Psi_{\ell|t=0}$ (which belong respectively to $L^2(\mathbf{R})$ if $k \leq m-1$ and to $H^1(\mathbf{R})$ if $\ell \leq m-2$) satisfy the following relations:

$$\mathbf{F}_k = \mathbf{U}_k\big(x, \mathbf{F}_j, \partial_x \mathbf{G}_j;\ 0 \leq j \leq k-1\big), \qquad 1 \leq k \leq m-1, \tag{1.7}$$

$$\mathbf{G}_k = \mathbf{V}_k(x, \mathbf{F}_j, \partial_x \mathbf{G}_j;\ 0 \leq j \leq k-1), \qquad 1 \leq k \leq m-1, \tag{1.8}$$

where the functions $\mathbf{U}_k$ and $\mathbf{V}_k$ are C^∞ functions of their arguments. These functions also satisfy $\mathbf{U}_k(x, 0, \ldots, 0) \in C_0^\infty(\mathbf{R})$ and $\mathbf{V}_k(x, 0, \ldots, 0) \in C_0^\infty(\mathbf{R})$.

These relations can also be used to produce *sufficient* compatibility conditions for the Cauchy problem, as stated now in the following result.

Theorem 1.3. *Let Assumptions* I *and* II *hold and let $m \in \mathbf{N}$, $m \geq 4$. Let $u_0 \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$. Suppose that u_0 satisfies the following compatibility conditions*

$$\mathcal{G}_k \in Lip(\mathbf{R}) \cap H^1(\mathbf{R}) \quad \textit{for} \quad 0 \leq k \leq m-1, \tag{1.9}$$

where the functions $\mathcal{G}_k(x)$ and $\mathcal{F}_k(x) \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$ are defined by $\mathcal{F}_0 := \chi(u_0)$, $\mathcal{G}_0 := R(u_0)$, and by the induction

$$\mathcal{F}_k := \mathbf{U}_k\big(x, \mathcal{F}_j, \partial_x \mathcal{G}_j;\ 0 \leq j \leq k-1\big), \qquad 1 \leq k \leq m, \tag{1.10}$$

$$\mathcal{G}_k := \mathbf{V}_k\big(x, \mathcal{F}_j, \partial_x \mathcal{G}_j;\ 0 \leq j \leq k-1\big), \qquad 1 \leq k \leq m. \tag{1.11}$$

Then, there exists $T > 0$ and a unique solution $u \in \mathcal{S}^m(\mathbf{R} \times [0, T])$ of the Cauchy problem (1.6).

Let us point out that the induction in the statement above actually has a meaning: at the step k, the definition of $\mathcal{F}_k$ and $\mathcal{G}_k$ in (1.10), (1.11) makes sense since $\mathcal{F}_j \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$ for $j \leq k-1$ and thanks to the compatibility condition (1.9) of the step $k-1$ which requires that $\partial_x \mathcal{G}_j$ is in $L^\infty(\mathbf{R})$ for $j \leq k-1$.

In several space dimensions, for semilinear hyperbolic systems, the type of regularity shown in the previous theorem was already introduced by J. Rauch and M. Reed in [17], and the solutions were called *stratified solutions.* For quasilinear hyperbolic systems in several space dimensions, continuous stratified solutions were studied by G. Métivier ([14]).

Remark 1.4. For the sake of simplicity we stated our result under Assumption I. In fact, Theorem 1.1 still holds if we replace Assumption I by

Assumption I′. *The eigenvalue* λ *has constant multiplicity* $d > 1$.

This condition means that the eigenspace

$$\ker\big(A(u) - \lambda(u)\big) \tag{1.12}$$

associated to $\lambda(u)$ has dimension d for every $u \in \mathcal{O}$. This assumption implies that the eigenvalue λ is linearly degenerate and that the distribution (1.12) of d-linear subspaces is involutive (see [1], [11], [6]). As a consequence, for every $u \in \mathcal{O}$ the subspace $\ker\big(A(u) - \lambda(u)\big)$ is spanned by some smooth vectors $\{r_1(u), \dots, r_d(u)\}$, and a change of dependent variables as in (1.3) is again possible. In this case the decomposition (v, w) takes place in $\mathbf{R}^{N-d} \times \mathbf{R}^d$ and the matrix $\tilde{A}$ has the form

$$\begin{pmatrix} B'(v,w) & 0 \\ C(v,w) & \mu(v) I_d \end{pmatrix} \tag{1.13}$$

where the matrices B' and C have sizes $(N-d) \times (N-d)$, $d \times (N-d)$, respectively, and I_d stands for the $d \times d$ identity matrix. In this framework the special symmetrizer of Assumption II has size $(N-d) \times (N-d)$. We refer to the examples below for a case where Assumption I′ is needed.

Remark 1.5. The condition $u \in \mathcal{S}^m(\Omega_T)$ can be read as a condition of polarization of the microlocal singularities of the vector function u, in the sense

$$\begin{aligned} u \in \mathcal{S}^\infty(\Omega_T) \iff & WF_{\mathrm{pol}}(u) \\ & \subset \Big\{ \big((t,x), (\tau,\xi); W\big) \in T^*(\Omega_T) \setminus \{0\} \times \mathbf{R}^N; \text{condition } (*) \text{ holds} \Big\} \end{aligned}$$

where

$$(\tau, \xi) \in \mathbf{R}\Big(\lambda\big(u(t,x)\big), -1\Big), \quad W \in \mathbf{R} r\big(u(t,x)\big). \tag{$*$}$$

Here $WF_{\mathrm{pol}}(u)$ is the polarized wave-front set of u (see [4]). From this point of view, Theorem 1.1 contains a quasilinear propagation result for the polarized wave-front set of u.

Remark 1.6. In some special cases the vector η in (1.4) is identically 0, for instance in the system of gas-dynamics in Lagrangian coordinates; see below. However, this fact does not simplify in an essential way what follows. Let us also point out that Assumption II does not depend on the vanishing of η. Actually, this condition is satisfied both by the equations of gas-dynamics and the system of elasticity of wires; in the first case η vanishes and in the second it does not.

Remark 1.7. Let us consider the large-amplitude rapidly oscillating solution defined for $t < 0$ by $u(t,x) = \gamma\big(\alpha\big((x - \omega t)/\varepsilon\big)\big)$, with α *periodic* and ϵ a small parameter. A consequence of Theorem 1.1 is that this solution propagates for $t > 0$ as a stratified solution, and its life span does not depend on ε. Such oscillating solutions, with a more regular α, were studied by W. E, A. Heibig and D. Serre ([5], [12], [19]).

Examples. We now give some examples. The case $N = 2$ is particularly easy: if the system is strictly hyperbolic, then by Assumption I it can be put under the form

$$\begin{cases} \partial_t v + \alpha(v,w)\partial_x v = 0, \\ \partial_t w + \mu(v)\partial_x w = 0 \end{cases}$$

and we can take $S(v,w) = |\alpha(v,w) - \mu(v)|^{-1}$; on the other hand, if the linearly degenerate eigenvalue has multiplicity 2, then there is no need for the symmetrizer.

The gas-dynamics equations in mass-Lagrangian coordinates are

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t\big(e + \dfrac{u^2}{2}\big) + \partial_x(pu) = 0, \end{cases}$$

where v is the specific volume, u the velocity, e the specific internal energy, p the pressure; we denote by s the specific entropy. We assume that $p_v(v,s) < 0$ in the region under consideration so that the system is strictly hyperbolic with eigenvalues $-c$, 0, c, where $c = \sqrt{-p_v(v,s)}$ is the local sound speed. The central eigenvalue 0 is linearly degenerate and we can take u, p as Riemann invariants. By the second law of thermodynamics $T\,ds = de + p\,dv$ and taking u, p, s as new independent variables, we see that the system above can be written as

$$\begin{cases} \partial_t u + \partial_x p = 0, \\ \partial_t p + c^2\partial_x u = 0, \\ \partial_t s = 0. \end{cases}$$

Under the previous notations we see that

$$B = \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & (1/c)^2 \end{pmatrix}, \quad SB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and so S satisfies Assumption II.

In the next example we consider the equations of gas-dynamics in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x\big((\rho E + p)u\big) = 0, \end{cases}$$

where $\rho = 1/v$ is the density and $E = e + u^2/2$ the specific total energy. In a region where $\rho > 0$ and $p_\rho(\rho,s) > 0$ the system is strictly hyperbolic with eigenvalues $u - c$, u, $u + c$; the eigenvalue u is linearly degenerate and again a pair of Riemann invariant is u, p. In variables u, p, s we have

$$\begin{cases} \partial_t u + u\partial_x u + \dfrac{1}{\rho}\partial_x p = 0, \\ \partial_t p + \rho c^2\partial_x u + u\partial_x p = 0, \\ s_t + u s_x = 0. \end{cases}$$

In this case

$$B = \begin{pmatrix} u & 1/\rho \\ \rho c^2 & u \end{pmatrix}, \quad S = \begin{pmatrix} \rho & 0 \\ 0 & 1/(\rho c^2) \end{pmatrix}, \quad S(B - u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and again Assumption II holds.

We consider now the equations of motion of a flexible and elastic string moving in a plane or in the whole space. We denote $Y = Y(t,x)$ the position at time t of a point of the string whose position in a reference configuration is $x \in \mathbf{R}$; the function Y is then valued in $\mathbf{R}^m$, for $m = 2$ or 3. If we neglect the memory effects of the material and no external forces are present, a model for the motion of the string is given by the following system of m second-order laws:

$$\partial_t^2 Y - \partial_x \left(\frac{T(r)}{r} \partial_x Y \right) = 0.$$

Here $r = |\partial_x Y|$ is the local extension and $r = 1$ identifies the rest; the function T is the module of the tension vector, a smooth function depending on the material under consideration. We assume

$$T(1) = 0, \quad \left(\frac{T(r)}{r} \right)' > 0, \quad T''(r) \neq 0 \tag{1.14}$$

for every $r > 1$. We reduce the system above to a system of $2m$ first-order conservation laws by setting $v = \partial_t Y$, $u = \partial_x Y$; we obtain

$$\begin{cases} \partial_t v - \partial_x (c^2 u) = 0, \\ \partial_t u - \partial_x v = 0, \end{cases} \tag{1.15}$$

where $c(r) = \sqrt{T(r)/r}$; we denote also $b(r) = rc'(r)$, which is a positive quantity since of (1.14). The eigenvalues of this system are $\pm\sqrt{T'} = \pm\sqrt{c(b+2c)}$, which are simple and genuinely nonlinear because of (1.14), and $\pm\sqrt{T/r} = \pm c$, which are linearly degenerate, of multiplicity $m-1$. As eigenvectors we can take respectively ${}^t(\mp\sqrt{T'}u, u)$ and ${}^t(\mp cX, X)$, for $X \in \mathbf{R}^m$ satisfying $\langle u, X \rangle = 0$. Therefore, under assumption (1.14), in the region $r > 1$ the system (1.15) is strictly hyperbolic if $m = 2$ and hyperbolic (non-strict) if $m = 3$.

We focus at first on the case $m = 2$ and on the eigenvalue $\mu(r) = -c(r)$. A choice of Riemann invariants with linearly independent gradient is for instance r, $v_1 - c(r)u_1$, $v_2 - c(r)u_2$, and it is easy to deduce from (1.15) the equations

$$\begin{cases} \partial_t r - \partial_x (rc) - \big\langle q, \partial_x (v - cu) \big\rangle = 0, \\ \partial_t (v - cu) + b(\partial_x c) u + c\partial_x (v - cu) + c' \big\langle q, \partial_x (v - cu) \big\rangle u = 0, \end{cases}$$

where we omitted dependence on r and denoted $q = u/r$. Under these notations we have

$$B = \begin{pmatrix} -(b+c) & -{}^t q \\ b^2 q & cI + bq \otimes q \end{pmatrix}, \quad S = \frac{b}{2c} \begin{pmatrix} b + 2c & {}^t q \\ q & \frac{1}{b} I \end{pmatrix} \tag{1.16}$$

for I the identity 2×2 matrix and $q \otimes q$ the matrix $q^t q$. Clearly, the matrix S is symmetric and its eigenvalues are $1/(2c)$, $\big(\alpha \pm (\alpha^2 - 8bc)^{1/2}\big)/(4c)$, for $\alpha = b^2 + 2bc + 1$; they are strictly positive. Then one finds

$$S(B + c) = \begin{pmatrix} -b^2 & 0 \\ 0 & I \end{pmatrix}$$

which depends only on the Riemann invariant r. Therefore, also in this case Assumption II holds, and analogous calculations show that the same is true also for the eigenvalue $c(r)$.

In the case $m = 3$ we have a decomposition as in (1.13), with $d = 2$. Now B' and the related symmetrizer S are 4×4 matrices; they are given by (1.16), if we replace the 2-vector q by their 3-dimensional version and take as I the identity 3×3 matrix. We stress that in this case the system is no longer *strictly* hyperbolic, and this gives a motivation to Remark 1.1 above.

2. Background results

In this section we gather some scattered results on weighted anisotropic Sobolev spaces, Moser-type inequalities, as well as some a-priori estimates for symmetric systems. For the former topics we adhere as much as possible to the notations of [11], to which the reader is referred for some proofs and details; for the latter, see [3]. Constants are usually denoted by the letter C; a ball of center 0 and radius R is denoted by $B(0,R)$.

2.1. **Function spaces and norms.** We recall that $\Omega_T = \mathbf{R}\times]-T_0,T[$; we write

$$\|u\|_{0,T} = \|u\|_{L^\infty(\Omega_T)}, \qquad \|u\|_{1,T} = \|u\|_{0,T} + \|Du\|_{0,T}$$

and denote by $Lip(\Omega_T)$ the class of Lipschitz-continuous functions with the norm $\|\ \|_{1,T}$. Sometimes we need the notations $Lip_t(\Omega_T)$ or $Lip_x(\Omega_T)$ for spaces of bounded functions which are uniformly Lipschitz-continuous in the t variable, respectively x; their norms are denoted respectively by

$$\|u\|_{1,T,t} = \|u\|_{0,T} + \|D_t u\|_{0,T}, \qquad \|u\|_{1,T,x} = \|u\|_{0,T} + \|D_x u\|_{0,T}.$$

It is straightforward that, for $T \geq 0$,

$$\|u\|_{0,T} \leq \|u\|_{0,0} + T\|\partial_t u\|_{0,T}$$

if $u \in Lip_t(\Omega_T)$. Let $\lambda > 1$ be a real parameter, $m \geq 0$ an integer; we define

$$\Lambda^m(\Omega_T) = \left\{u \in L^2(\Omega_T);\, \partial_t^k u \in L^2(\Omega_T),\, k = 1, 2, \dots, m\right\}$$

endowed with the family of semi-norms

$$|u|_{m,\lambda,T} = \sum_{k=0}^{m} \lambda^{m-k} \|e^{-\lambda t}\partial_t^k u\|_{L^2(\Omega_T)}.$$

These functions may be valued in some $\mathbf{R}^n$; when we need to stress this fact we write $\Lambda^m(\Omega_T, \mathbf{R}^n)$. The following pair of inequalities is going to be frequently used:

$$|\partial_t u|_{m-1,\lambda,T} = |u|_{m,\lambda,T} - \lambda^m |u|_{0,\lambda,T} \leq |u|_{m,\lambda,T}, \tag{2.1}$$

$$|u|_{m-1,\lambda,T} \leq \frac{1}{\lambda}\,|u|_{m,\lambda,T}. \tag{2.2}$$

The proof is immediate. From $\Lambda^m(\Omega_T)$ we define the following spaces of functions with one normal derivative, and the related semi-norms:

$$\begin{aligned} N^m(\Omega_T) &= \left\{u \in \Lambda^m(\Omega_T);\, \partial_x u \in \Lambda^{m-2}(\Omega_T)\right\},\ m \geq 2, \\ |u|^N_{m,\lambda,T} &= |u|_{m,\lambda,T} + |\partial_x u|_{m-2,\lambda,T}; \end{aligned}$$

$$\begin{aligned} N_1^m(\Omega_T) &= \left\{u \in \Lambda^m(\Omega_T);\, \partial_x u \in \Lambda^{m-1}(\Omega_T)\right\},\ m \geq 1, \\ |u|^{N_1}_{m,\lambda,T} &= |u|_{m,\lambda,T} + |\partial_x u|_{m-1,\lambda,T}. \end{aligned}$$

2.2. **Embeddings and Moser inequalities.** This follows an embedding result for the space N^m; the proof is a simple modification of that given in [11].

Lemma 2.1. *Let us fix* $T_1 \in \,]-T_0, T_0[$ *and for some* $T \in \,]T_1, T_0[$ *consider* $u \in N^m(\Omega_T)$. *There exists a positive constant* C*, depending only on* T_1 *and* m*, such that*

$$\|u\|_{0,T} \le C e^{\lambda |T|} |u|^N_{m,\lambda,T}, \quad m \ge 2, \tag{2.3}$$

$$\|u\|_{0,T} \le \|u\|_{0,T_1} + C|T| e^{\lambda |T|} |u|^N_{m,\lambda,T}, \quad m \ge 3, \tag{2.4}$$

$$\|\partial_t u\|_{0,T} \le \|\partial_t u\|_{0,T_1} + C|T| e^{\lambda |T|} |u|^N_{m,\lambda,T}, \quad m \ge 4. \tag{2.5}$$

As a consequence of the lemma we see that $N^2(\Omega_T) \subset L^\infty(\Omega_T)$ and $N^4(\Omega_T) \subset Lip_t(\Omega_T)$. In the following lemmas we collect some Moser-type inequalities, [15]; they are consequences of Gagliardo-Nirenberg weighted inequalities. The first one deals with composition with smooth functions.

Lemma 2.2. *Let* $F \in C^\infty(\mathbf{R}^n)$ *be a scalar function satisfying* $F(0) = 0$*, let* R *be a positive real number and* $T_1 \in \,]-T_0, T_0[$. *There exists a positive constant* C *such that if* $g \in \Lambda^m(\Omega_T, \mathbf{R}^n) \cap L^\infty(\Omega_T, \mathbf{R}^n)$ *and* $\|g\|_{0,T} \le R$*, for some* $T \in \,]T_1, T_0[$*, then* $F(g) \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$ *and*

$$|F(g)|_{m,\lambda,T} \le C|g|_{m,\lambda,T}. \tag{2.6}$$

Moreover, if $g \in N^m(\Omega_T, \mathbf{R}^n) \cap Lip(\Omega_T, \mathbf{R}^n)$ *and* $\|g\|_{1,T} \le R$*, then* $F(g) \in N^m(\Omega_T) \cap Lip(\Omega_T)$ *and*

$$|F(g)|^N_{m,\lambda,T} \le C|g|^N_{m,\lambda,T}. \tag{2.7}$$

The constant C *depends on the derivatives of* F *of order less than or equal to* m *in* $B(0, R)$.

The notation $\overset{\circ}{F}(u) = F(u) - F(0)$ will be used in the following when we apply this lemma to functions which do not vanish in 0. The next lemma states that $\Lambda^m \cap L^\infty$ is an algebra.

Lemma 2.3. *For every* $T_1 \in \,]-T_0, T_0[$ *and integer* $m \ge 0$ *there exists a positive constant* C *such that if* $T \in \,]T_1, T_0[$ *and* $f, g \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$*, then also* $fg \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$ *and*

$$|fg|_{m,\lambda,T} \le C\big\{\|f\|_{0,T}|g|_{m,\lambda,T} + |f|_{m,\lambda,T}\|g\|_{0,T}\big\}. \tag{2.8}$$

More precisely, for positive integers j, h, k *satisfying* $j + h \le k \le m$ *we have*

$$\lambda^{m-k} |\partial_t^j f \partial_t^h g|_{0,\lambda,T} \le C\big\{\|f\|_{0,T}|g|_{m,\lambda,T} + |f|_{m,\lambda,T}\|g\|_{0,T}\big\}. \tag{2.9}$$

As a consequence of (2.6) and (2.8) we see that if F is a smooth function, g is as in (2.6) and in addition $h \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$, then

$$|F(g)h|_{m,\lambda,T} \le C\big\{\|g\|_{0,T}|h|_{m,\lambda,T} + |g|_{m,\lambda,T}\|h\|_{0,T} + |h|_{m,\lambda,T}\big\} \tag{2.10}$$

without assuming $F(0) = 0$.

We give now some estimates on commutators of functions.

Lemma 2.4. *Let* $m \ge 1$ *be an integer and* $T_1 \in \,]-T_0, T_0[$*; then there exists a positive constant* C *such that the following is true. For every* $T \in \,]T_1, T_0[$*, if* $A \in \Lambda^m(\Omega_T) \cap Lip_t(\Omega_T)$ *is a function valued in* $n \times n$ *matrices and* $u \in \Lambda^m(\Omega_T) \cap Lip_t(\Omega_T)$ *an* $\mathbf{R}^n$*-valued function, then*

$$\lambda^{m-k} |[A\partial_t, \partial_t^k]u|_{0,\lambda,T} \le C\big\{\|\partial_t A\|_{0,T}|\partial_t u|_{m-1,\lambda,T} + |\partial_t A|_{m-1,\lambda,T}\|\partial_t u\|_{0,T}\big\} \tag{2.11}$$

for every $k = 0, 1, \ldots, m$. *If* A *is as above but* $u \in N_1^m(\Omega_T) \cap Lip_x(\Omega_T)$, *then for the same* k

(2.12)
$$\lambda^{m-k}|[A\partial_x, \partial_t^k]u|_{0,\lambda,T} \leq C\big\{\|\partial_t A\|_{0,T}|\partial_x u|_{m-1,\lambda,T} + |\partial_t A|_{m-1,\lambda,T}\|\partial_x u\|_{0,T}\big\}.$$

Proof. Let us consider first (2.11). We remark that

$$\lambda^{m-k}[A\partial_t, \partial_t^k]u = -\lambda^{m-k} \sum_{k_1,k_2}{}' \begin{pmatrix} k \\ k_1 \end{pmatrix} \partial_t^{k_1} A\, \partial_t^{k_2} \partial_t u$$

where the prime in the sum means summation over all k_1 and k_2 such that $1 \leq k_1 \leq k$, $0 \leq k_2 \leq k-1$, $k_1 + k_2 = k$. Then we write $k_1 = k_0 + 1$ for $k_0 \geq 0$ so that $\partial_t^{k_1} A = \partial_t^{k_0}\partial_t A$, and apply (2.9) with $(m-1)-(k-1) = m-k$ to every scalar component of the summands. The result follows at once. In the same way we prove (2.12). □

2.3. L^2 **estimate.** We recall here (in the special case of one space dimension) the classical L^2 estimate for symmetric hyperbolic systems ([7], [3]). Let A_0, A_1, be two smooth $n \times n$ symmetric matrices defined in $\mathcal{O}$, with A_0 positive definite; fix some $T_1 \in\,]-T_0, T_0[$ and consider the extension problem

$$\begin{cases} A_0(b_0)\partial_t u + A_1(b_1)\partial_x u = f, \\ u_{|\Omega_{T_1}} = u_0. \end{cases} \tag{2.13}$$

Let us emphasize the condition on the functions $\partial_t b_0$ and $\partial_x b_1$ in the theorem, which is one of the key points in the proof of the main result, and motivation for Assumption II.

Theorem 2.5. *Let us fix a real number* $R > 0$ *and let* $\mathcal{K}$ *be a compact subset of* $\mathcal{O}$. *For* $T \in\,]T_1, T_0[$ *we assume that* $b_0 \in Lip(\Omega_T, \mathcal{K})$ *and* $b_1 \in Lip(\Omega_T, \mathcal{K})$ *satisfy*

$$\|\partial_t b_0\|_{L^\infty(\Omega_T)} + \|\partial_x b_1\|_{L^\infty(\Omega_T)} \leq R.$$

Let $f \in L^2(\Omega_T)$ *and let* $u_0 \in L^2(\Omega_{T_1})$ *be a solution to the above linear system in* Ω_{T_1}. *Then there exists a unique solution* $u \in L^2(\Omega_T)$ *to* (2.13), *there exist a positive real number* λ_0 *and a positive constant* C *depending only on* R *(and on the compact set* K*) such that for every* $\lambda > \lambda_0$

$$u|_{0,\lambda,T} \leq C\Big\{|u_0|_{0,\lambda,T_1} + \frac{1}{\lambda}|f|_{0,\lambda,T}\Big\}.$$

3. Proofs

3.1. **A change of variables.** Now we prepare the proof of Theorem 1.1 by an unknown change of independent variables which straightens the λ-characteristic field of system (1.4).

Without loss of generality we assume that the set $\tilde{\mathcal{O}}$ is the product $\tilde{\mathcal{O}}_{N-1} \times \tilde{\mathcal{O}}_1$, with $\tilde{\mathcal{O}}_{N-1} \subset \mathbf{R}^{N-1}$ and $\tilde{\mathcal{O}}_1 \subset \mathbf{R}$. For $T \in\,]-T_0, T_0[$ and $v \in Lip(\Omega_T)$ with values in $\tilde{\mathcal{O}}_{N-1}$ we denote the characteristic curves of the field $\partial_t + \mu(v)\partial_x$ by

$$\Gamma(s; t, x) = \big(s, \gamma(s; t, x)\big)$$

where the function γ solves

$$\begin{cases} \dfrac{d\gamma(s;t,x)}{ds} = \mu\Big(v\big(s, \gamma(s;t,x)\big)\Big), \\ \gamma(t; t, x) = x. \end{cases} \tag{3.1}$$

Since the function $\mu(v)$ is bounded and Lipschitz-continuous in Ω_T, then the maximal solutions of (3.1) are defined for $s \in\,]-T_0, T[$, for every $(t,x) \in \Omega_T$. We define

$$\Theta : (t,x) \longrightarrow \big(t, \gamma(0;t,x)\big).$$

The map Θ is then a homeomorphism of Ω_T, whose inverse function is $\Theta^{-1}(\check t, \check x) = \big(\check t, \gamma(\check t; 0, \check x)\big)$. Since $\gamma \in C^1\big(]-T_0, T[, Lip(\Omega_T)\big)$, then both Θ and Θ^{-1} are Lipschitz-continuous functions on Ω_T, and then Θ is a bi-Lipschitz change of variables. Let us point out, however, that, in general, Θ is not a C^1 diffeomorphism. We denote

$$\psi(\check t, \check x) = \gamma(\check t; 0, \check x)$$

and write a $\check{}$ to denote the functions transformed after the change of variables Θ; then

$$\partial_{\check t}\psi = \mu(\check v), \quad \psi(0,\check x) = \check x \tag{3.2}$$

and we can write (1.4) as

$$\begin{cases} \partial_{\check t}\check v + \dfrac{1}{\partial_{\check x}\psi}\big(B(\check v, \check w) - \mu(\check v)\big)\partial_{\check x}\check v = b(\check t, \check x, \check v, \check w), \\ \partial_{\check t}\check w + \dfrac{1}{\partial_{\check x}\psi}\eta(\check v, \check w)\partial_{\check x}\check v = d(\check t, \check x, \check v, \check w). \end{cases} \tag{3.3}$$

Then we define $z = \partial_{\check x}\psi$ and rewrite (3.3) as

$$\begin{cases} \partial_{\check t}\check v + \frac{1}{z}\big(B(\check v, \check w) - \mu(\check v)\big)\partial_{\check x}\check v = b(\check t, \check x, \check v, \check w), \\ \partial_{\check t}\check w + \frac{1}{z}\eta(\check v, \check w)\partial_{\check x} v = d(\check t, \check x, \check v, \check w), \\ \partial_{\check t} z - \partial_{\check x}\mu(\check v) = 0. \end{cases} \tag{3.4}$$

In these coordinates Assumption II becomes clearer. In fact, due to this assumption we can write

$$S(\check v, \check w)\big(B(\check v, \check w) - \mu(\check v)\big) = G(\check v)$$

and after multiplying each side of the first line in (3.4) by $zS(\check v, \check w)$ we get at last

$$\begin{cases} zS(\check v, \check w)\partial_{\check t}\check v + G(\check v)\partial_{\check x}\check v = S(\check v, \check w) b(\check t, \check x, \check v, \check w) z, \\ z\partial_{\check t}\check w + \eta(\check v, \check w)\partial_{\check x}\check v = d(\check t, \check x, \check v, \check w) z, \\ \partial_{\check t}\check z - \partial_{\check x}\mu(\check v) = 0. \end{cases} \tag{3.5}$$

For $m \geq 0$ we define the following function spaces which take the place of the sets $\mathcal{S}^m(\Omega_T)$:

$$\begin{aligned} \mathcal{H}^m(\Omega_T) &= \mathcal{H}^m(\Omega_T, \tilde{\mathcal{O}} \times \mathbf{R}) \\ &= \{\Lambda^m(\Omega_T, \tilde{\mathcal{O}}_{N-1}) \cap Lip(\Omega_T, \tilde{\mathcal{O}}_{N-1})\} \\ &\quad \times \{\Lambda^m(\Omega_T, \tilde{\mathcal{O}}_1) \cap Lip_t(\Omega_T, \tilde{\mathcal{O}}_1)\} \\ &\quad \times \{\mathbf{R} \oplus \Lambda^m(\Omega_T, \mathbf{R}) \cap Lip_t(\Omega_T, \mathbf{R})\}. \end{aligned}$$

Proposition 3.1. *A function $u \in P(\Omega_T)$ is a weak solution of* (1.1) *if and only if there exist $\check v \in Lip(\Omega_T)$, $(\check w, z) \in L^\infty(\Omega_T)$ and $\delta > 0$ such that $(\check v, \check w, z)$ is a weak solution of* (3.5) *and $|z(t,x)| > \delta$ for every $(t,x) \in \Omega_T$. Moreover, for any $m \geq 0$, u belongs to $\mathcal{S}^m(\Omega_T)$ if and only if $(\check v, \check w, z)$ belongs to $\mathcal{H}^m(\Omega_T)$.*

Proof. The proof of the first equivalence is very similar to the proof of the second equivalence in the case $m = 0$. So, to avoid obvious repetitions we begin directly by showing the second equivalence in the case $m = 0$, and then deal with regularity.

Let $(\check{v}, \check{w}, z) \in \mathcal{H}^0(\Omega_T)$ be a solution of (3.5) with $|z(t,x)| > \delta$, for some $\delta > 0$. We consider a sequence of functions $\check{u}_n = (\check{v}_n, \check{w}_n) \in C^\infty(\Omega_T)$ such that

$$(\check{v}_n, \partial_{\check{t}}\check{v}_n, \partial_{\check{x}}\check{v}_n) \to (\check{v}, \partial_{\check{t}}\check{v}, \partial_{\check{x}}\check{v}) \text{ a.e. in } \Omega_T , \quad \|\check{v}_n\|_{1,T} \le M,$$
$$(\check{w}_n, \partial_{\check{t}}\check{w}_n) \to (\check{w}, \partial_{\check{t}}\check{w}) \text{ a.e. in } \Omega_T , \quad \|\check{w}_n\|_{0,T} \le M$$

for some positive M. Then, in view of (3.2), we define $\psi_n \in C^\infty(\Omega_T)$ by

$$\begin{cases} \partial_{\check{t}}\psi_n = \mu(\check{v}_n), \\ \psi_n(0, \check{x}) = \check{x}; \end{cases}$$

that is,

$$\psi_n(\check{t}, \check{x}) = \check{x} + \int_0^{\check{t}} \mu\big(\check{v}_n(s, \check{x})\big)\, ds$$

and then $z_n = \partial_{\check{x}}\psi_n$. A consequence of the convergence assumptions on the sequence $\{\check{v}_n\}$ is that ψ_n converges in $L^\infty(\Omega_T)$ to the corresponding function ψ defined with the function $\check{v}$, while $z_n = \partial_{\check{x}}\psi_n$ remains bounded in $L^\infty(\Omega_T)$. Obviously

$$z_n(\check{t}, \check{x}) = 1 + \int_0^{\check{t}} \partial_{\check{x}}\mu(\check{v}_n)(s, \check{x})\, ds$$

so that z_n satisfies

$$\begin{cases} \partial_{\check{t}} z_n - \partial_{\check{x}}\mu(\check{v}_n) = 0, \\ z_{n|t=0} = 1. \end{cases}$$

Since $|z(t,x)| > \delta$, we may assume that $|z_n(t,x)| > \delta/2$. The bounded sequence $\big(\check{u}_n, z_n\big)$ converges a.e. to $\big(\check{u}, z\big)$, and from the dominated convergence theorem it follows that

$$(3.6) \quad \begin{pmatrix} z_n S(\check{v}_n, \check{w}_n)\partial_{\check{t}}\check{v}_n + G(\check{v}_n)\partial_{\check{x}}\check{v}_n - S(\check{v}_n, \check{w}_n) b(\check{t}, \check{x}, \check{v}_n, \check{w}_n) z_n \\ z_n \partial_{\check{t}}\check{w}_n + \eta(\check{v}_n, \check{w}_n)\partial_{\check{x}}\check{v}_n - d(\check{t}, \check{x}, \check{v}_n, \check{w}_n) z_n \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in $L^1_{loc}(\Omega_T)$. The left-hand side of (3.6) may be written as

$$(3.7) \quad \mathcal{M}(\check{u}_n, z_n) \\ \times \left\{ \partial_{\check{t}}\mathcal{T}(\check{u}_n) + \frac{1}{z_n}\Big(\partial_{\check{x}} F\big(\mathcal{T}(\check{u}_n)\big) - \lambda\big(\mathcal{T}(\check{u}_n)\big)\partial_{\check{x}}\big(\mathcal{T}(\check{u}_n)\big)\Big) - f\big(\check{t}, \check{x}, \mathcal{T}(\check{u}_n)\big)\right\}$$

for

$$\mathcal{M}(\check{u}_n, z_n) = z_n \begin{pmatrix} S(\check{v}_n, \check{w}_n) & 0 \\ 0 & 1 \end{pmatrix} \big(D\mathcal{T}(\check{u}_n)\big)^{-1}$$

where we denoted for brevity $\mathcal{T} = \mathcal{R}^{-1}$. Now

$$\partial_{\check{x}} F\big(\mathcal{T}(\check{u}_n)\big) - \lambda\big(\mathcal{T}(\check{u}_n)\big)\partial_{\check{x}}\big(\mathcal{T}(\check{u}_n)\big) = \Big(A\big(\mathcal{T}(\check{u}_n)\big) - \lambda\big(\mathcal{T}(\check{u}_n)\big)\Big) D\mathcal{T}(\check{u}_n)\partial_{\check{x}}\check{u}_n$$

and the important point is that the matrix $\big(A\big(\mathcal{T}(\check{u}_n)\big) - \lambda\big(\mathcal{T}(\check{u}_n)\big)\big) D\mathcal{T}(\check{u}_n)$ has the last column (the coefficients of $\partial_{\check{x}}\check{w}_n$) identically zero. This allows us to pass to the limit in the expression between braces in (3.7). We introduce then the family of changes of variables $\Theta_n \in C^\infty(\Omega_T)$ by

$$\Theta_n^{-1}(\check{t}, \check{x}) = \big(\check{t}, \psi_n(\check{t}, \check{x})\big).$$

For $\tilde{u}_n(t,x) = \check{u}_n\big(\Theta_n(t,x)\big)$, we find that

$$\partial_t \mathcal{T}(\tilde{u}_n) + \partial_x F\big(\mathcal{T}(\tilde{u}_n)\big) - f(t, x, \tilde{u}_n) \to 0$$

in $L^1_{\rm loc}(\Omega_T)$. At last $\tilde u_n = \check u_n \circ \Theta_n \to \tilde u = \check u \circ \Theta$ a.e., and since $\tilde u_n$ remains bounded in $L^\infty(\Omega_T)$, we obtain

$$\partial_t \mathcal{T}(\tilde u) + \partial_x F\big(\mathcal{T}(\tilde u)\big) = \tilde f(t,x,\tilde u).$$

Therefore $u = \mathcal{T}(\tilde u)$ solves (1.1). By a similar method one easily proves the converse; in particular, the condition $|z(t,x)| > \delta$ is implied by (3.1).

We are left to regularity. Let us assume that $(\check u, z) \in \mathcal{H}^m(\Omega_T)$ is a solution to (3.5). Then $\check v \in Lip(\Omega_T)$ implies $R(u) \in Lip(\Omega_T)$ and this fact together with $w \in L^\infty(\Omega_T)$ means that $u \in P(\Omega_T)$. At last we remark that

$$\partial_{\check t} \check u(\check t, \check x) = X_{u(t,x)} u(t,x). \tag{3.8}$$

Then $(\check u, z) \in \mathcal{H}^m(\Omega_T)$ implies $u \in \mathcal{S}^m(\Omega_T)$.

This concludes the proof. □

We write $b(\check t, \check x, \check v, \check w, z)$ for $S(\check v, \check w) b(\check t, \check x, \check v, \check w) z$ and $d(\check t, \check x, \check v, \check w, z)$ for $d(\check t, \check x, \check v, \check w) z$. If we omit for brevity all ˇ's and use the same letters for the transformed functions in the previous system, then Theorem 1.1 is a consequence of the following result.

Theorem 3.1. *Let $m \geq 4$ be an integer, $\delta > 0$ a real number, and let $U_0 = (v_0, w_0, z_0) \in \mathcal{H}^m(\Omega_0)$ be a solution of (3.5) in Ω_0 with $|z_0(t,x)| > 2\delta$ for every $(t,x) \in \Omega_0$. Then there exists a time $T \in]0, T_0[$ such that the problem*

$$\begin{cases} zS(v,w)\partial_t v + G(v)\partial_x v = b(t,x,v,w,z), \\ z\partial_t w + \eta(v,w)\partial_x v = d(t,x,v,w,z), \\ \partial_t z - \partial_x \mu(v) = 0, \\ (v,w,z)_{|\Omega_0} = (v_0, w_0, z_0) \end{cases} \tag{3.9}$$

has a unique solution $U = (v,w,z) \in \mathcal{H}^m(\Omega_T)$, with $|z(t,x)| \geq \delta$ for every $(t,x) \in \Omega_T$. More precisely, the time T depends only on δ and on the bound of the norm of U_0 in $\mathcal{H}^m(\Omega_0)$ (and of S, G, b, d).

This is the result that we are going to prove in the next paragraphs. Proposition 3.1 implies that the existence time T actually depends neither on the choice of the Riemann invariants nor on δ, but only on the data of system (1.1) and of the norms of the solution in the past specified in Theorem 1.1.

Let us point out that the function v is somewhat regular also in the x variables; in fact, from the first set of $N-1$ equations we deduce that $\partial_x v \in \Lambda^{m-1}(\Omega_T)$, since the matrix G is invertible. On the other hand, no further regularity than that provided by the theorem above can be obtained in general for the functions w and z. Some Sobolev as well as Lipschitz estimates for the solution are given in the following subsections.

Let us also mention that, as the proof will show, one could add a term $g(t,x) \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$ (with $g(t,x) = 0$ for $t < 0$) in the right-hand side of (3.9) without changing the conclusion of the theorem.

3.2. **The linearized case.** In this subsection we focus at first on the following system, which is obtained by linearizing (3.5) at (p,q,ζ):

$$\begin{cases} \zeta S(p,q)\partial_t v + G(p)\partial_x v = b, \\ \zeta \partial_t w + \eta(p,q)\partial_x v = d, \\ \partial_t z - \nabla\mu(p)\partial_x v = 0. \end{cases} \tag{3.10}$$

We denote $a = (p,q,\zeta - 1)$ and $U = (v,w,z)$ is the unknown function. This system is partially decoupled: the functions w and z are easily determined once v is found.

So we begin by giving some a-priori estimates for the first $N-1$ equations of (3.10), that is,

$$\zeta S(p,q)\partial_t v + G(p)\partial_x v = b. \tag{3.11}$$

Let us denote the associated operator

$$L(a)v = E(a)\partial_t v + G(p)\partial_x v$$

for $E(a) = \zeta S(p,q)$.

Proposition 3.2. *Let R and δ be two positive real numbers and $m \geq 1$ an integer. We can find a positive function $C = C(R,\delta)$ and a real number $\lambda_0 > 1$, such that for every $T \in]0, T_0[$ the following holds.*

Let $p \in \Lambda^m(\Omega_T) \cap Lip(\Omega_T)$, $(q, \zeta - 1) \in \Lambda^m(\Omega_T) \cap Lip_t(\Omega_T)$, with (p,q) valued in $\tilde{\mathcal{K}}$, $|\zeta(t,x)| \geq \delta$ for every $(t,x) \in \Omega_T$ and

$$\|p\|_{1,T} + \|q\|_{1,T,t} + \|\zeta - 1\|_{1,T,t} \leq R;$$

let $b \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$, and $v \in \Lambda^m(\Omega_T) \cap Lip(\Omega_T)$ be a solution to (3.11) in Ω_T; then for every $\lambda > \lambda_0$ the function v satisfies the estimates

$$\begin{aligned} |v|_{m,\lambda,T} \leq{} & C|v|_{m,\lambda,0} \\ & + \frac{C}{\lambda}\big\{|v|_{m,\lambda,T} + |a|_{m,\lambda,T}\|\partial_t v\|_{0,T} + \|b\|_{0,T}|a|_{m,\lambda,T} + |b|_{m,\lambda,T}\big\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} |\partial_x v|_{m-1,\lambda,T} \leq C\big\{ & |v|_{m,\lambda,T} + |a|_{m-1,\lambda,T}\|\partial_t v\|_{0,T} \\ & + \|b\|_{0,T}|a|_{m-1,\lambda,T} + |b|_{m-1,\lambda,T}\big\} \end{aligned} \tag{3.13}$$

and if $m \geq 2$,

$$\begin{aligned} |v|^N_{m,\lambda,T} \leq{} & C|v|_{m,\lambda,0} \\ & + \frac{C}{\lambda}\big\{|v|_{m,\lambda,T} + |a|_{m,\lambda,T}\|\partial_t v\|_{0,T} + \|b\|_{0,T}|a|_{m,\lambda,T} + |b|_{m,\lambda,T}\big\}. \end{aligned} \tag{3.14}$$

If $m \geq 4$, we have

$$\|v\|_{1,T} \leq C(\|v\|_{1,0} + \|b\|_{0,T}) + CTe^{\lambda T}|v|^N_{m,\lambda,T}. \tag{3.15}$$

Proof. In order to estimate $|v|_{m,\lambda,T}$ we write

$$L(a)\partial_t^k v = \partial_t^k b + [L(a), \partial_t^k]v \tag{3.16}$$

for $k = 0, 1, \dots, m$, since $L(a)v = b$; the idea is to check that the right-hand side in (3.16) is in $L^2(\Omega_T)$ and then apply the estimate given in the former section. We denote by C some constants depending on m and, in particular, on R and δ.

First of all $\partial_t^k b \in L^2(\Omega_T)$, because $b \in \Lambda^m(\Omega_T)$; on the other hand,

$$[L(a), \partial_t^k]v = [E(a)\partial_t, \partial_t^k]v + [G(p)\partial_x, \partial_t^k]v.$$

We begin by considering $[E(a)\partial_t, \partial_t^k]v$. By (2.11) it follows

$$\begin{aligned} & \lambda^{m-k}|[E(a)\partial_t, \partial_t^k]v|_{0,\lambda,T} \\ & \qquad \leq C\big\{\|\partial_t E(a)\|_{0,T}|\partial_t v|_{m-1,\lambda,T} + |\partial_t E(a)|_{m-1,\lambda,T}\|\partial_t v\|_{0,T}\big\}. \end{aligned}$$

By hypothesis we know that $\|\partial_t E(a)\|_{0,T} \leq C$. Since Moser inequality (2.6) implies

$$|\partial_t E(a)|_{m-1,\lambda,T} \leq C|a|_{m,\lambda,T},$$

then

$$\lambda^{m-k}|[E(a)\partial_t, \partial_t^k]v|_{0,\lambda,T} \leq C\big\{|v|_{m,\lambda,T} + |a|_{m,\lambda,T}\|\partial_t v\|_{0,T}\big\}. \tag{3.17}$$

Next we consider $[G(p)\partial_x, \partial_t^k]v$. By (2.12) and (2.6) we deduce

$$\begin{aligned} &\lambda^{m-k}|[G(p)\partial_x, \partial_t^k]v|_{0,\lambda,T} \\ &\qquad \leq C\big\{\|\partial_t G(p)\|_{0,T}|\partial_x v|_{m-1,\lambda,T} + |\partial_t G(p)|_{m-1,\lambda,T}\|\partial_x v\|_{0,T}\big\} \\ &\qquad \leq C\big\{|\partial_x v|_{m-1,\lambda,T} + |p|_{m,\lambda,T}\|\partial_x v\|_{0,T}\big\}. \end{aligned} \tag{3.18}$$

Since the matrix G is invertible, we find from (3.11)

$$\partial_x v = G^{-1}(p)\big(b - E(a)\partial_t v\big).$$

From this formula we see that

$$\|\partial_x v\|_{0,T} \leq C\big\{\|b\|_{0,T} + \|\partial_t v\|_{0,T}\big\}. \tag{3.19}$$

In order to estimate $|\partial_x v|_{m-1,\lambda,T}$ we remark that by using (2.10) we obtain

$$|G^{-1}(p)b|_{m-1,\lambda,T} \leq C\big\{|b|_{m-1,\lambda,T} + \|b\|_{0,T}|p|_{m-1,\lambda,T}\big\} \tag{3.20}$$

and analogously we estimate

$$|G^{-1}(p)E(a)\partial_t v|_{m-1,\lambda,T} \leq C\big\{|v|_{m,\lambda,T} + \|\partial_t v\|_{0,T}|a|_{m-1,\lambda,T}\big\}. \tag{3.21}$$

Then by summing (3.20) and (3.21) we find

$$|\partial_x v|_{m-1,\lambda,T} \leq C\big\{|v|_{m,\lambda,T} + |a|_{m-1,\lambda,T}\|\partial_t v\|_{0,T} + \|b\|_{0,T}|p|_{m-1,\lambda,T} + |b|_{m-1,\lambda,T}\big\}$$

which proves (3.13). Now we plug the previous estimate and (3.19) into (3.18), and obtain

$$\begin{aligned} &\lambda^{m-k}|[G(p)\partial_x, \partial_t^k]v|_{0,\lambda,T} \\ &\qquad < C\big\{|v|_{m,\lambda,T} + |a|_{m,\lambda,T}\|\partial_t v\|_{0,T} + |p|_{m,\lambda,T}\|b\|_{0,T} + |b|_{m,\lambda,T}\big\}. \end{aligned} \tag{3.22}$$

At last we sum (3.17) and (3.22) and find

$$\begin{aligned} &\lambda^{m-k}|[L(a), \partial_t^k]v|_{0,\lambda,T} \\ &\qquad \leq C\big\{|v|_{m,\lambda,T} + |a|_{m,\lambda,T}\|\partial_t v\|_{0,T} + |p|_{m,\lambda,T}\|b\|_{0,T} + |b|_{m,\lambda,T}\big\}. \end{aligned}$$

So far we have checked that the right-hand side in (3.16) is in $L^2(\Omega_T)$ and got estimates of it, with constants depending only on R and not on δ. Since we assumed also that $\|\partial_x p\|_{0,T} \leq R$, we can apply Theorem 2.5 to (3.16); therefore, for a constant depending also on δ,

$$|\partial_t^k v|_{0,\lambda,T} \leq C|\partial_t^k v|_{0,\lambda,0} + \frac{C}{\lambda}\big\{|\partial_t^k b|_{0,\lambda,T} + |[L(a), \partial_t^k]v|_{0,\lambda,T}\big\}$$

and by multiplying by λ^{m-k} and summing for $k = 0, 1, \ldots, m$ we get (3.12).

The estimate (3.14) now follows easily from (3.12), (3.13) and the inequality

$$|v|^N_{m,\lambda,T} \leq |v|_{m,\lambda,T} + \frac{1}{\lambda}|\partial_x v|_{m-1,\lambda,T}.$$

At last we prove (3.15). From the inequality $\|v\|_{0,T} \leq \|v\|_{0,0} + T\|\partial_t v\|_{0,T}$ and (3.19) we obtain

$$\|v\|_{1,T} \leq \|v\|_{0,0} + C(\|b\|_{0,T} + \|\partial_t v\|_{0,T}).$$

Since $m \geq 4$ we can apply (2.5), and this proves (3.15).

This concludes the proof of the proposition. □

Remark 3.2. The constant $C = C(R, \delta)$ in the previous proposition can be chosen to be increasing in R for δ fixed and decreasing in δ for R fixed. Moreover, we point out that in the estimate (3.15) the constant depends only on an upper bound of the L^∞ norms $\|p\|_{0,T} + \|q\|_{0,T} + \|\zeta - 1\|_{0,T}$.

The following proposition exploits the previous a priori estimates to prove the existence of solutions to (3.11). In fact, we prove much more, since we prepare suitable bounds to be used in the proof of our main result. This accounts for the smallness of the existence time T below, which is not strictly needed in linear problems.

Proposition 3.3. *Let $m \geq 4$ be an integer, $\delta > 0$, P, Q, ρ_0, be positive real numbers, and $\mu_0 = \mu_0(\tau)$ be a positive function defined for $\tau \in]0, +\infty[$; then there exist some positive constants $R > \rho_0$, λ, $M > \mu_0(\lambda)$, and a time $T \in]0, T_0[$ such that the following is true. Let $(b, d) \in \Lambda^m(\Omega_T) \cap L^\infty(\Omega_T)$ with*

$$\|(b, d)\|_{0,T} \leq P, \quad |(b, d)|_{m,\lambda,T} \leq QM,$$

let $(p, q, \zeta) \in \mathcal{H}^m(\Omega_T)$, with (p, q) valued in $\tilde{\mathcal{K}}_1$, $|\zeta(t, x)| \geq \delta$ for every $(t, x) \in \Omega_T$ and

$$\|p\|_{1,T} + \|q\|_{1,T,t} + \|\zeta - 1\|_{1,T,t} \leq R,$$
$$|p|_{m,\lambda,T} + |q|_{m,\lambda,T} + |\zeta - 1|_{m,\lambda,T} \leq M;$$

at last let $U_0 = (v_0, w_0, z_0) \in \mathcal{H}^m(\Omega_0)$ be a solution to (3.10) in Ω_0, with (v_0, w_0) valued in $\tilde{\mathcal{K}}$, $|z_0(t, x)| > 2\delta$ for every $(t, x) \in \Omega_0$, and

$$\|v_0\|_{1,0} + \|w_0\|_{1,0,t} + \|z_0 - 1\|_{1,0,t} \leq \rho_0,$$
$$|v_0|_{m,\tau,0} + |w_0|_{m,\tau,0} + |z_0 - 1|_{m,\tau,0} \leq \mu_0(\tau)$$

for every $\tau > 0$. Then the problem

$$\begin{cases} \zeta S(p, q)\partial_t v + G(p)\partial_x v = b, \\ \zeta \partial_t w + \eta(p, q)\partial_x v = d, \\ \partial_t z - \nabla\mu(p)\partial_x v = 0, \\ (v, w, z)_{|\Omega_0} = (v_0, w_0, z_0) \end{cases}$$

has a unique solution $U = (v, w, z) \in \mathcal{H}^m(\Omega_T)$, with (v, w) valued in $\tilde{\mathcal{K}}_1$, $|z(t, x)| \geq \delta$ for every $(t, x) \in \Omega_T$, which satisfies

$$\|v\|_{1,T} + \|w\|_{1,T,t} + \|z - 1\|_{1,T,t} \leq R, \tag{3.23}$$

$$|v|_{m,\lambda,T} + |w|_{m,\lambda,T} + |z - 1|_{m,\lambda,T} \leq M. \tag{3.24}$$

Proof. In order to avoid unnecessary details in the following we will be more precise about the constants R, λ, M, T. They all depend on δ, but we drop this fact since it is by no means important in Theorem 3.1. We stress instead the dependence on P, Q, ρ_0 and on the function μ_0, as well as how R, λ, M, T are chosen in turn: each one of them depends on the preceding ones, in the order we have written. So, for instance, M depends on R and λ, but not on T. More precisely, we start in step 1 below with some arbitrary R (and λ, M, T, too); while performing the proof we shall impose conditions on it (depending on P, Q, ρ_0, μ_0) and determine a constant $C = C(R)$. Then we choose λ such that

$$\lambda \geq \max\{\lambda_0, CR\} \tag{3.25}$$

for some constant λ_0 provided by Proposition 3.2. Next we fix M according to the value of R, and at last T depending on R, λ, M (say $T \le 1$ for simplicity); in particular, we need

$$T \le \frac{1}{M^2}, \tag{3.26}$$

$$Te^{\lambda T} \le \frac{R}{2}\sqrt{T}. \tag{3.27}$$

We divide the proof into many steps; at first we solve the $(N-1)\times(N-1)$ system

$$\begin{cases} \zeta S(p,q)\partial_t v + G(p)\partial_x v = b, \\ v_{|\Omega_0} = v_0. \end{cases}$$

To prove existence and uniqueness we can assume $v_0 = 0$ by a classical truncation in time. The announced estimates follow then using again Step 2 below, with a general v_0.

Step 1: *smoothing.* For some $R \ge \rho_0$, some $\lambda > 0$ and some $M \ge 0$ we construct by means of a standard mollification a sequence of functions $a^\nu = (p^\nu, q^\nu, \zeta^\nu - 1) \in C_0^\infty(\bar\Omega_T)$, $\nu = 1, 2, \ldots$, such that $a^\nu \to a$ in $H^{0,m}(\Omega_T)$ and

$$\begin{gathered} \|(p^\nu, q^\nu) - (p,q)\|_{0,T} < \operatorname{dist}\big(\tilde{\mathcal{K}}, \mathbf{R}^N \setminus \tilde{\mathcal{K}}_1\big), \\ \|p^\nu\|_{1,T} + \|q^\nu\|_{1,T,t} + \|\zeta^\nu - 1\|_{1,T,t} \le 2R, \\ |p^\nu|_{m,\lambda,T} + |q^\nu|_{m,\lambda,T} + |\zeta^\nu - 1|_{m,\lambda,T} \le 2M. \end{gathered}$$

Due to (3.2) we can then consider the problem

$$\begin{cases} L(a^\nu)v^\nu = b, \\ v^\nu_{|\Omega_0} = 0 \end{cases}$$

for $\nu = 1, 2, \ldots.$ Theorem 2.5 provides a unique solution $v^\nu \in L^2(\Omega_T)$ to this problem. Since the coefficients of the system are smooth, it follows (by induction on m) that $v^\nu \in \Lambda^m(\Omega_T)$. From the equations we see that $\partial_x v^\nu \in \Lambda^{m-1}(\Omega_T)$, and therefore $v^\nu \in N_1^m(\Omega_T)$; this implies that $v^\nu \in Lip_t(\Omega_T)$ since $m \ge 4$, and again by the equations it turns out that $v^\nu \in Lip(\Omega_T)$. In conclusion $v^\nu \in N_1^m(\Omega_T) \cap Lip(\Omega_T)$, and therefore we can apply Proposition 3.2 to these solutions.

Step 2: *boundedness.* We claim now that the sequences

$$\alpha_\nu = \|v^\nu\|_{1,T} \qquad \beta_\nu = |v^\nu|_{m,\lambda,T}$$

are bounded respectively by R/h and M/h, where $h \ge 3$ is a constant independent on R and M which is given in the final step 5. In fact, by (3.15), (3.14), and Remark 3.2 we obtain

$$\alpha_\nu \le \Gamma(\|v\|_{1,0} + P) + CTe^{\lambda T}|v_0|_{m,\lambda,0} + \frac{C}{\lambda}Te^{\lambda T}\big\{M\alpha_\nu + \beta_\nu + M(P+Q)\big\}$$

for a constant Γ depending only on the L^∞ norms $\|p\|_{0,T} + \|q\|_{0,T} + \|s\|_{0,T}$; since the pair (p,q) is valued in a compact set, and we can suppose that s does also, the constant Γ can be chosen to be independent on R. By (3.25) and (3.27) we obtain now

$$\alpha_\nu \le C_1 + \sqrt{T}\beta_\nu$$

for some constant C_1 independent on ν; we can take for instance

$$C_1 = 2\Gamma\rho_0 + \frac{CR}{M}\mu_0 + P + Q$$

where from now on μ_0 stands for some $\mu_0(\lambda)$, with λ to be fixed at the end as a function of R. Now we use the a priori estimate (3.12) combined with the previous estimate and find

$$\beta_\nu \leq \frac{1}{R}(1 + M\sqrt{T})\beta_\nu + \frac{C_2}{R}$$

with some other constant C_2 independent on ν, for instance,

$$C_2 = 2\{\Gamma M\rho_0 + CR\mu_0 + PM + QM\}$$

for the same constant C used above in this step; it follows then from (3.26) that the sequence $\{\beta_\nu\}$ is bounded by $C_2/(R-2)$. In order that $C_2/(R-2) \leq M/h$ it is sufficient to take $R \geq 2 + 2h(\Gamma\rho_0 + P + Q)$ and consequently

$$M > 2hCR\mu_0/\{R - 2 - 2h(\Gamma\rho_0 + P + Q)\}.$$

At last, we see that it is sufficient to increase R so that $R \geq 2 + 2h(2\Gamma\rho_0 + P + Q)$ and then $M \geq 2hC\mu_0$ to obtain $\alpha_\nu \leq R/h$.

Step 3: *convergence.* Let $v \in L^2(\Omega_T)$ be the solution to

$$\begin{cases} L(a^\nu)v^\nu = b, \\ v_{|\Omega_0} = 0 \end{cases}$$

by Theorem 2.5; we prove now that $v^\nu \to v$ in $L^2(\Omega_T)$. It is sufficient to remark that $v^\mu - v^\nu$ satisfy

$$\begin{cases} L(a^\mu)(v^\mu - v^\nu) = \big(L(a^\nu) - L(a^\mu)\big)v^\nu, \\ v^\mu - v^\nu_{|\Omega_0} = 0 \end{cases}$$

since by the estimates there it follows

$$|v^\mu - v^\nu|_{0,\lambda,T} \leq \frac{C}{\lambda}\|v^\nu\|_{1,T}|a^\mu - a^\nu|_{0,\lambda,T}$$

for $\lambda \geq \lambda_0$, where C is some positive constant independent on ν. The previous step shows that the sequence $\|v^\nu\|_{1,T}$ is bounded, and this proves the convergence. At last, by the boundedness of the sequences α_ν and β_ν, we deduce that the limit function $v \in \Lambda^m(\Omega_T) \cap Lip(\Omega_T)$ and

$$\|v\|_{1,T} \leq R/h, \qquad |v|_{m,\lambda,T} \leq M/h. \tag{3.28}$$

All this achieves the proof of the existence and regularity of v; uniqueness follows by a classical integration by parts.

Step 4: *the remaining equations.* Let us consider the extension problem for w, since the other is completely analogous. Since $v \in N_1^m(\Omega_T) \cap Lip(\Omega_T)$, from (2.10) and the assumptions on d it follows that $d - \eta(p,q)\partial_x v \in \Lambda^{m-1}(\Omega_T) \cap L^\infty(\Omega_T)$; estimates on this term are obtained by writing $\partial_x v = G^{-1}(p)\big(b - E(a)\partial_t v\big)$. On one hand, by (2.10) and (2.2) it follows

$$|\eta(p,q)G^{-1}(p)b|_{m-1,\lambda,T} \leq \frac{C}{\lambda}\{|b|_{m,\lambda,T} + \|b\|_{0,T}|a|_{m,\lambda,T}\}. \tag{3.29}$$

On the other hand, again by (2.10) we obtain

$$|\eta(p,q)G^{-1}(p)E(a)\partial_t v|_{m-1,\lambda,T} \le C\big\{|v|_{m,\lambda,T} + \frac{1}{\lambda}|a|_{m,\lambda,T}\|\partial_t v\|_{0,T}\big\}. \tag{3.30}$$

Now let us remark that w satisfies the initial-value problem

$$\begin{cases} \partial_t w = g, \\ w_{|\Omega_0} = w_0 \end{cases}$$

for some $g \in \Lambda^{m-1}(\Omega_T) \cap L^\infty(\Omega_T)$ and $w_0 \in \Lambda^m(\Omega_0) \cap Lip_t(\Omega_0)$ such that $\partial_t w_0 = g$ in Ω_0. Then it follows from Theorem 2.5 (or directly from the explicit formula for w) that $w \in \Lambda^m(\Omega_T) \cap Lip_t(\Omega_T)$ and satisfies the estimate

$$|w|_{m,\lambda,T} \le C\big\{|w_0|_{m,\lambda,0} + |g|_{m-1,\lambda,T}\big\} \tag{3.31}$$

for some constant C. Obviously this estimate holds for every $m \ge 1$.

Therefore, we can plug the estimate (3.12) into (3.30), sum (3.29) with (3.30) and by (3.31) we finally get an estimate for w.

We can now put together this estimate, the analogous one for z and (3.12) to obtain

$$\begin{aligned} |U|_{m,\lambda,T} &\le C|U|_{m,\lambda,0} \\ &+ \frac{C}{\lambda}\big\{|v|_{m,\lambda,T} + |a|_{m,\lambda,T}\|\partial_t v\|_{0,T} + \|b\|_{0,T}|a|_{m,\lambda,T} + |b|_{m,\lambda,T} + |d|_{m,\lambda,T}\big\} \end{aligned} \tag{3.32}$$

for some positive constant C, while the function v satisfies also (3.14). Lipschitz estimates are then quickly deduced by integrating the equations; for w and z we find

$$\|w\|_{1,T,t} \le \|w_0\|_{1,0,t} + C\big\{\|v\|_{1,T} + \|d\|_{0,T}\big\},$$
$$\|z-1\|_{1,T,t} \le \|z_0 - 1\|_{1,0,t} + C\|v\|_{1,T}$$

while v satisfies (3.15).

We now check that $|z(t,x)| \ge \delta$. From the explicit formula for z we see that it is sufficient to control $TC\|\partial_x v\|_{0,T}$ from above with δ; but this is quickly obtained by (3.28) if $T \le \delta h/(CR)$.

Step 5: *precise bounds.* In steps 2 and 3 we gave some bounds to the Lipschitz and Sobolev norms of v; now we take into account also w and z and deduce (3.23), (3.24) as well as the fact that (v,w) are valued in $\tilde{\mathcal{K}}_1$.

From the Lipschitz estimates for w we see that $\|w\|_{1,T,t} \le \rho_0 + 2P + 2\Gamma R/h$; it is now that h is determined such that $h \ge 12\Gamma$. Therefore, we have $\|w\|_{1,T,t} \le R/3$ if $R > 6(2P + \rho_0)$. Then we write the Sobolev estimate for w that we gave in the previous step, and see that if $h \ge 6$, $R > 6(Ph + Qh + 1)/(h-6)$ and $M \ge 6C\mu_0$, then $|w|_{m,\lambda,T} \le M/3$. Analogous estimates hold for z, and summing up all of them we reach (3.23), (3.24).

At last $\|(v,w)\|_{0,T} \le \|(v_0,w_0)\|_{0,0} + T\|\partial_t(v,w)\|_{0,T} \le \rho_0 + TR$ and then (v,w) are valued in $\tilde{\mathcal{K}}_1$ if $T \le \mathrm{dist}\{\tilde{\mathcal{K}}, \mathbf{R}^N \setminus \tilde{\mathcal{K}}_1\}/R$.

This concludes the proof of the proposition. ∎

3.3. **Proof of Theorem 3.1.** In order to simplify the proof we assume that the functions b and d depend only on (v, w, z) (respectively (v, w)); the general case requires only some minor changes. As a consequence of the assumption $f(t, x, \underline{u}) \in C_0^\infty(\overline{\Omega})$ we have then $b(0) = 0$, $d(0) = 0$. The same remark as at the beginning of Proposition 3.3 still holds here, since we need to impose some further conditions on λ and then on T.

Let $U_0 = (v_0, w_0, z_0) \in \mathcal{H}^m(\Omega_0)$ be a solution to (3.5) in Ω_0, with $(v_0, w_0) \in \tilde{\mathcal{K}}$ and z_0 strictly bounded away from 0 by 2δ. The proof is by an iterative scheme: the initial step 0 is defined as an extension $U^0 = (v^0, w^0, z^0) \in \mathcal{H}^m(\Omega_{T_0})$ of U_0 to Ω_{T_0}, and unless by taking T sufficiently small we can assume that (v^0, w^0) are valued in $\tilde{\mathcal{K}}_1$ and $|z^0(t, x)| \geq \delta$ for every $(t, x) \in \Omega_{T_0}$. Since $z_0 \in \mathbf{R} \oplus \Lambda^m$, and since $\delta > 0$, we can suppose that $z_0 \in 1 \oplus \Lambda^m$.

In order to apply Proposition 3.3 we define now once for all the constant ρ_0 and the function $\mu_0 = \mu_0(\tau)$ by

$$\rho_0 = \|v^0\|_{1,T_0} + \|w^0\|_{1,T_0,t} + \|z^0 - 1\|_{1,T_0,t},$$
$$\mu_0(\tau) = |v^0|_{m,\tau,T_0} + |w^0|_{m,\tau,T_0} + |z^0 - 1|_{m,\tau,T_0};$$

moreover, we denote by P and Q two positive constants (by (2.6)) such that

$$\|(b, d)\|_{L^\infty(\tilde{\mathcal{K}}_1)} \leq P, \qquad |(b, d)(U)|_{m,\tau,T} \leq Q|U|_{m,\tau,T}$$

hold for every $\tau > 0$, every $T \in]0, T_0[$ and every $U \in H^{0,m}(\Omega_T)$ with (v, w) valued in $\tilde{\mathcal{K}}_1$.

It follows at last for $\nu = 0, 1, \ldots$ our iterative scheme

$$(3.33) \qquad \begin{cases} z^\nu S(v^\nu, w^\nu)\partial_t v^{\nu+1} + G(v^\nu)\partial_x v^{\nu+1} = b(v^\nu, w^\nu, z^\nu), \\ z^\nu \partial_t w^{\nu+1} + \eta(v^\nu, w^\nu)\partial_x v^{\nu+1} = d(v^\nu, w^\nu, z^\nu), \\ \partial_t z^{\nu+1} - \nabla\mu(v^\nu)\partial_x v^{\nu+1} = 0, \\ (v^{\nu+1}, w^{\nu+1}, z^{\nu+1})_{|\Omega_0} = (v_0, w_0, z_0). \end{cases}$$

Step 1: *well definedness and boundedness.* In this first step we want to prove by induction on ν the following statement:

$(I)_\nu$ there exist positive constants R, λ, M and a time $T \in]0, T_0[$ such that the iterative scheme defines a sequence $\{U^\nu\}$ in $\mathcal{H}^m(\Omega_T)$ with $(v^\nu, w^\nu) \in \tilde{\mathcal{K}}_1$, $\|z^\nu|_{0,T} \geq \delta$, which satisfies

$$\|v^\nu\|_{1,T} + \|w^\nu\|_{1,T,t} + \|z^\nu - 1\|_{1,T,t} \leq R,$$
$$|v^\nu|_{m,\lambda,T} + |w^\nu|_{m,\lambda,T} + |z^\nu - 1|_{m,\lambda,T} \leq M.$$

Since we fixed ρ_0 above, the function μ_0, and P, Q, then Proposition 3.3 provides the constants R, λ, M, T. The proof is now easily done in a straightforward way as follows.

Since U^0 can be thought as a solution in Ω_0 to the system (3.10) where a is replaced by the same U^0, then by Proposition 3.3 there exist a solution U^1 corresponding to $\nu = 0$ in (3.33), which satisfies all the requirements of the statement.

In the same way we prove that $(I)_\nu$ implies $(I)_{\nu+1}$ by applying again directly Proposition 3.3.

Step 2: *convergence.* We prove the convergence in L^∞. Let us denote by $\{U^\nu\}$ the sequence that we have constructed in the previous step. We can apply the Ascoli-Arzelà theorem to the sequence $\{v^\nu\}$ and deduce that $v^\nu \to v$ pointwise and uniformly on compact sets of Ω_T, unless we consider subsequences. On the other hand, the boundedness of $\{\partial_t v^\nu\}$ in the $L^2(\Omega_T)$ norm implies $\partial_t v^\nu \rightharpoonup \partial_t v$ L^2-weakly, again modulo subsequences, and the same holds clearly also for $\{\partial_x v^\nu\}$.

We consider now the equation for $w^{\nu+1}$, and writing $\partial_x v^{\nu+1}$ in terms of $\partial_t v^{\nu+1}$ we obtain

$$\partial_t w^{\nu+1} = \Gamma(U^\nu) + \Delta(U^\nu)\partial_t v^{\nu+1}$$

where the definition of the functions Γ and Δ is clear. Therefore, we can write the difference as

(3.34)
$$\begin{aligned}\partial_t(w^{\nu+1} - w^\nu) = {} & \Gamma(U^\nu) - \Gamma(U^{\nu-1}) \\ & + \big(\Delta(U^\nu) - \Delta(U^{\nu-1})\big)\partial_t v^{\nu+1} + \Delta(U^{\nu-1})\big(\partial_t v^{\nu+1} - \partial_t v^\nu\big).\end{aligned}$$

An analogous expression holds for $\partial_t(z^{\nu+1} - z^\nu)$.

Let us define now $\Omega_{T,r} = \{(t,x) \in \Omega_T;\ |x| < r\}$. We claim that both $\{w^\nu\}$ and $\{z^\nu\}$ are Cauchy sequences in $L^\infty(\Omega_{T,r})$. To prove our claim let us denote

$$\begin{aligned}\alpha_\nu &= \|v^\nu - v^{\nu-1}\|_{L^\infty(\Omega_{T,r})},\\ \beta_\nu &= \|w^\nu - w^{\nu-1}\|_{L^\infty(\Omega_{T,r})},\\ \gamma_\nu &= \|z^\nu - z^{\nu-1}\|_{L^\infty(\Omega_{T,r})}.\end{aligned}$$

The first two differences in the right-hand side of (3.34) are bounded by $C(\alpha_\nu + \beta_\nu + \gamma_\nu)$ for some constant C. Integrating the inequality from 0 to t gives

$$|(w^{\nu+1} - w^\nu)(t,x)| \le 2CT(\alpha_\nu + \beta_\nu + \gamma_\nu) + \Big|\int_0^t \Delta(U^{\nu-1})\big(\partial_t v^{\nu+1} - \partial_t v^\nu\big)ds\Big|$$

and integrating by part the last integral gives

$$\beta_{\nu+1} \le CT(\alpha_\nu + \beta_\nu + \gamma_\nu) + C\alpha_{\nu+1}$$

for some new constant C. An analogous estimate holds for $\gamma_{\nu+1}$, and if we sum both of them we find for the sequence $\rho_\nu = \beta_\nu + \gamma_\nu$ the estimate

$$\rho_{\nu+1} \le CT\rho_\nu + C\alpha_{\nu+1}$$

for another constant C; the important point is that all these constants do not depend on r. Since the sequence $\{v^\nu\}$ is uniformly convergent on $\Omega_{T,r}$ we can assume $\alpha_\nu \le \alpha_0 2^{-\nu}$; then we can easily prove by induction on ν that $\rho_\nu \le c\rho_0 2^{-\nu}$ for some constant c sufficiently large (say $c \ge \max\{1, 2C\alpha_0/\rho_0\}$) and T again somewhat smaller than above (say $T \le 1/c$). This proves our claim.

Therefore, we find w and z in $L^\infty(\Omega_T)$ and by the previous estimates the existence of a function $U = (v,w,z) \in \mathcal{H}^m(\Omega_T)$ is established. In particular, let us point out that, up to a subsequence, the sequences $\{\partial_t w^\nu\}$ and $\{\partial_t z^\nu\}$ converge $L^2(\Omega_T)$-weakly, since they are bounded in $L^2(\Omega_T)$.

Step 3: *consistence.* At last we check that the function U of the former step is really a (weak) solution to (3.9).

In fact, as a byproduct of the former step we showed that the sequences $\{\partial_t v^\nu\}$, $\{\partial_x v^\nu\}$, $\{\partial_t w^\nu\}$, $\{\partial_t z^\nu\}$ converge $L^2(\Omega_T)$-weakly; moreover, the sequence $\{U^\nu\}$ is bounded in $L^\infty(\Omega_T)$ and converges in $L^\infty_{\text{loc}}(\Omega_T)$. Then any term $a(U^\nu)\partial_t U^\nu$ or $a(U^\nu)\partial_x v^\nu$ converges $L^2(\Omega_T)$-weakly.

Step 4: *uniqueness.* Let U and U' be two solutions of class $\mathcal{H}^m(\Omega_T)$ to (3.9); in particular, they agree in Ω_0 with U_0. By substraction we find that

$$\begin{aligned} &E(U)\partial_t(v-v') + G(v)\partial_x(v-v') \\ &\quad = b(U) - b(U') - \big(E(U)-E(U')\big)\partial_t v' - \big(G(v)-G(v')\big)\partial_x v' \end{aligned}$$

and by an integration by parts

$$|v-v'|_{0,\lambda,T} \le \frac{C}{\lambda}|U-U'|_{0,\lambda,T} \tag{3.35}$$

since both $\partial_t v'$ and $\partial_x v'$ are bounded in $L^\infty(\Omega_T)$. Then we write

$$\partial_t w - \Delta(U)\partial_t v = \Gamma(U)$$

with notations as above, and from the difference

$$\partial_t(w-w') - \Delta(U)\partial_t(v-v') = \Gamma(U) - \Gamma(U') - \big(\Delta(U)-\Delta(U')\big)\partial_t v'$$

we deduce the estimate

$$|w-w'|_{0,\lambda,T} \le C|v-v'|_{0,\lambda,T} + \frac{C}{\lambda}|U-U'|_{0,\lambda,T}. \tag{3.36}$$

If we now plug (3.35) into (3.36) and write an analogous estimate for $z-z'$, we finally find

$$|U-U'|_{0,\lambda,T} \le \frac{C}{\lambda}|U-U'|_{0,\lambda,T};$$

then, unless choosing λ sufficiently large, and consequently T sufficiently small, we obtain $U=U'$.

The theorem is now completely proved.

3.4. **Proof of Theorem 1.2.** The method of proof is very classical and is a consequence of the "tame" Moser estimates established on the linearized problem. First of all, we deduce from Theorem 1.1 that $u \notin \mathcal{S}^m(\Omega_{T^*})$. Now, suppose that $u \in P(\Omega_{T^*})$. We are going to prove that this implies that $u \in \mathcal{S}^m(\Omega_{T^*})$, and so we reach a contradiction. By performing the change of variables of Section 3.1, we get a function $U=(v,w,\zeta)$ solution of (3.8) on Ω_{T^*}, and such that: $U \in \mathcal{H}^m(\Omega_T)$ for any $T<T^*$, $v \in Lip(\Omega_{T^*})$, $(w,z) \in Lip_t(\Omega_{T^*})$. Let $T \in [0,T^*[$. Using the estimate for the linear problem (3.10) with $(p,q,\zeta)=(v,w,z)$, we get an estimate like (3.32):

$$\begin{aligned} |U|_{m,\lambda,T} \quad &\le \quad C|U|_{m,\lambda,0} \\ &\qquad + \frac{C}{\lambda}\Big(|v|_{m,\lambda,T} + |v|_{m,\lambda,T}\|\partial_t v\|_{0,T} + \|v\|_{0,T}|v|_{m,\lambda,T} + |b|_{m,\lambda,T} + 1\Big) \\ &\le \quad C|U|_{m,\lambda,0} + \frac{C'}{\lambda}\Big(|U|_{m,\lambda,T}+1\Big) \end{aligned}$$

where $C' = C'\big(\|v\|_{Lip(\Omega_{T^*})}, \|U\|_{Lip_t(\Omega_{T^*})}\big)$. Taking now $\lambda=\lambda_1$ large enough, we get that $|U|_{m,\lambda_1,T} \le C''|U|_{m,\lambda_1,0}$ where C'' is a constant independent of $T \le T^*$. This implies that $U \in \mathcal{H}^m(\Omega_{T^*})$, which is the contradiction expected.

3.5. **Proof of Theorem 1.3.** Let us begin with some formal calculations on the system (3.9). This system can be written

$$\partial_t U = M(U)\partial_x v + B(t,x,U) \tag{3.37}$$

where $U = (v, w, z-1) = (v, w, s)$, and with suitable smooth matrices $M(\cdot)$ and $B(\cdot)$. As is well known, it follows by induction, that all the time derivatives $\partial_t^k U$ can be expressed in terms of the space derivatives of U. Let us give a more precise description of this relations.

Let us define the function $\mathcal{U}_1 : \mathbf{R}^2 \times \mathbf{R}^{N+1} \times \mathbf{R}^{N-1} \to \mathbf{R}^{N+1}$ by the following expression (with obvious notations):

$$\mathcal{U}_1(t,x,U,\partial_x v) := M(U)\partial_x v + B(t,x,U).$$

In the following, we do not write the dependence on the variables (t,x) for simplicity. The second order derivative takes the form

$$\begin{aligned} \partial_t^2 U &= \big(M'(U)\partial_t U\big)\partial_x v + M(U)\partial_x\partial_t v + B'(U)\partial_t U \\ &=: \mathcal{U}_2\big(\partial_t^j U, \partial_x\partial_t^j v;\ 0 \le j \le 1\big). \end{aligned}$$

By induction we define the functions $\mathcal{U}_k$ such that

$$\partial_t^k U = \mathcal{U}_k\big(\partial_t^j U, \partial_x\partial_t^j v;\ 0 \le j \le k-1\big). \tag{3.38}$$

Since $U = (v, w, z-1)$, one can extract from the components of $\mathcal{U}_k$ the expression for $\partial_t^k v$ and define the function $\mathcal{V}_k$ by

$$\partial_t^k v = \mathcal{V}_k\big(\partial_t^j U, \partial_x\partial_t^j v;\ 0 \le j \le k-1\big). \tag{3.39}$$

Now, suppose that $U \in \mathcal{H}^{m+1}(\Omega_T)$ $(m \ge 0)$ is a solution of system (3.9). First, observe that, for $k \le m$, $\partial_t^k U$ and $\partial_t(\partial_t^k U)$ are both in $L^2(\Omega_T)$; this implies that

$$\partial_t^k U \in C\big([-T_0, T] : L^2(\mathbf{R}_x)\big), \quad k = 0, \dots, m. \tag{3.40}$$

It also follows from the equation (3.9) that $\partial_x v \in \Lambda^{m-1}(\Omega_T)$, because $G(v)$ is invertible. By the same argument we deduce that

$$\partial_t^\ell v \in C\big([-T_0, T] : H^1(\mathbf{R}_x)\big), \quad \ell = 0, \dots, m-1. \tag{3.41}$$

Let us define the functions $\mathcal{F}_k(x)$ and $\mathcal{G}_k(x)$ for $0 \le k \le m$, by the following induction, whose sense will be justified below:

$$\mathcal{F}_0 := U_{|t=0}\ , \quad \mathcal{G}_0 := v_{|t=0}$$

$$\mathcal{F}_k(x) := \mathcal{U}_k\Big(0, x, \mathcal{F}_j, \partial_x\mathcal{G}_j;\ 0 \le j \le k-1\Big)\ , \quad k \le m\ , \tag{3.42}$$

$$\mathcal{G}_k(x) := \mathcal{V}_k\Big(0, x, \mathcal{F}_j, \partial_x\mathcal{G}_j;\ 0 \le j \le k-1\Big)\ , \quad k \le m\ . \tag{3.43}$$

This induction makes sense because, at each step, it follows from (3.38), (3.39) that $\mathcal{F}_k(x) = (\partial_t^k U)(0,x)$ and that $\mathcal{G}_k(x) = (\partial_t^k v)(0,x)$. It follows then from (3.40) and (3.41) that

$$\mathcal{F}_k \in L^2(\mathbf{R}) \quad \text{for} \ \ 0 \le k \le m, \tag{3.44}$$

$$\mathcal{G}_k \in H^1(\mathbf{R}) \quad \text{for} \ \ 0 \le k \le m-1, \tag{3.45}$$

and this fact in turn enables us to define $\mathcal{F}_{k+1}$ and $\mathcal{G}_{k+1}$ when $k \le m-1$.

This set of *necessary conditions* can also be used to produce *sufficient conditions* on the Cauchy data U_0 in order to solve the Cauchy problem

$$\left\{\begin{array}{l}\partial_t U = M(U)\partial_x v + B(t,x,U),\\ U_{|t=0} = (v_0, w_0, s_0)\end{array}\right. \tag{3.46}$$

in the class of stratified solutions.

Theorem 3.3. *Let $m \in \mathbf{N}$, $m \geq 4$. Let $U_0 = (v_0, w_0, s_0) \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$ with $v_0 \in Lip(\mathbf{R}) \cap H^1(\mathbf{R})$ and $\inf |z_0| > 0$ for $z_0 = 1 + s_0$. Suppose that U_0 satisfies the following compatibility conditions, for $0 \leq k \leq m$, $0 \leq \ell \leq m-1$:*

$$\mathcal{F}_k \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}), \quad \mathcal{G}_\ell \in Lip(\mathbf{R}) \cap H^1(\mathbf{R}), \tag{3.47}$$

where the functions $\mathcal{F}_k$ and $\mathcal{G}_\ell$ are defined by the induction (3.42), (3.43)*, initialized with $\mathcal{F}_0 := U_0$ and $\mathcal{G}_0 := v_0$. Then, there exists $T > 0$ and a unique solution $U \in \mathcal{H}^m(\mathbf{R} \times [0,T])$ of the Cauchy problem* (3.46).

Proof. We remark first that at each step, the induction of the statement makes sense by the compatibility condition (3.47).

Consider the functions

$$U_a(t,x) := \sum_0^m \mathcal{F}_k(x) t^k/k! =: \big(v_a(t,x), w_a(t,x), z_a(t,x)\big).$$

The function U_a satisfies

$$h(t,x) := \partial_t U_a - M(U_a)\partial_x v_a - B(t,x,U_a) \in L^\infty \cap L^2$$

and

$$(\partial_t^k h)_{|t=0} = 0 \quad \text{for} \quad k = 0, \ldots, m-1. \tag{3.48}$$

Let us call $\tilde{h}$ the function defined by

$$\tilde{h}(t,x) = \left\{\begin{array}{ll} 0 & \text{if } t \geq 0,\\ h(t,x) & \text{if } t \leq 0.\end{array}\right.$$

Due to (3.48) the function $\tilde{h}$ is in $\Lambda^m(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ (we recall that $T_0 > 0$ is some fixed, positive number). Now, using Theorem 3.1 we find some $T > 0$ ($T \in]0, T_0]$) and $U = (v, w, z) \in \mathcal{H}^m(\Omega_T)$ solution of

$$\left\{\begin{array}{l}\partial_t U - M(U)\partial_x v - B(t,x,U) = \tilde{h} \textit{ in } \Omega_T,\\ U_{|\Omega_0} = U_a,\end{array}\right. \tag{3.49}$$

which proves the theorem. ∎

We deduce now the proof of Theorem 1.3. Suppose that the hypotheses of Theorem 1.3 are satisfied, $u_0(x)$ being the initial data. The change of variables in Section 3.1

$$\Theta : (t,x) \longrightarrow \big(t, \gamma(0;t,x)\big),$$

satisfies $\Theta(0,x) = x$, and this implies that the corresponding initial data for the transformed system (3.9) are given by

$$v(0,x) = \mathcal{R}\big(u_0(x)\big), \;\; w(0,x) = \chi\big(u_0(x)\big), \;\; z(0,x) = 1. \tag{3.50}$$

Now, the compatibility conditions satisfied in the hypotheses of Theorem 1.3 are exactly the compatibilities (3.47) of Theorem 3.3. One can then solve first the

Cauchy problem (3.9), which gives a solution of the original Cauchy problem by the Proposition 3.1, and so Theorem 1.3 is proved.

Let us also mention that the regularity properties of the functions Φ_k and Ψ_k in the description of the necessary compatibility conditions are a consequence of (3.40), (3.41) and (3.8).

Appendix A.

We recall that the operator $\partial_t + A(u)\partial_x$ is said to be symmetrizable hyperbolic in $\mathcal{O}$ (in the sense of Friedrichs) if there exists a smooth $N \times N$ matrix $\Sigma = \Sigma(u)$, defined in $\mathcal{O}$, which is symmetric, positive definite and such that $\Sigma(u)A(u)$ is also symmetric (see [7]). In this special situation where the space dimension is 1, the operator $\partial_t + A(u)\partial_x$ is symmetrizable if and only if the matrix $A(u)$ is (smoothly) diagonalizable. A well known sufficient condition for symmetrizability (in any dimension) is the existence of a smooth strictly convex entropy, [8]. In analogy with the previous definition we say that the operator $\partial_t + B(v,w)\partial_x$ acting on $(N-1)$-vector functions (but defined in a subset of $\mathbf{R}^N$) is symmetric hyperbolic if it has an $(N-1)\times(N-1)$ symmetrizer S sharing the same properties of the symmetrizer Σ above.

Lemma A.1. *The operator $\partial_t + A(u)\partial_x$ is symmetrizable in $\mathcal{O}$ if and only if the operator $\partial_t + B(v,w)\partial_x$ is symmetrizable in $\tilde{\mathcal{O}}$.*

Proof. The symmetrizability of the operator $\partial_t + A(u)\partial_x$ is clearly equivalent to the symmetrizability of $\partial_t + \tilde{A}(v,w)\partial_x$. It is then sufficient to prove that $\partial_t + \tilde{A}(u)\partial_x$ is symmetrizable if and only if $\partial_t + B(v,w)\partial_x$ does. In the whole proof we then omit for simplicity the dependence on the variable $u \in \tilde{\mathcal{O}}$.

Let $\Sigma_{\tilde{A}}$ be a symmetrizer for $\tilde{A}$; without any loss of generality we can assume that

$$\Sigma_{\tilde{A}} = \begin{pmatrix} S' & {}^t s \\ s & 1 \end{pmatrix}$$

where S' is a symmetric $(N-1)\times(N-1)$ matrix and s is an $N-1$ line vector. We have

$$\Sigma_{\tilde{A}}\tilde{A} = \begin{pmatrix} S'B + s\otimes\eta & \mu{}^t s \\ sB+\eta & \mu \end{pmatrix}$$

where we denoted the matrix product of a row vector (${}^t s$) with a line vector (η) with $\otimes$ for the sake of clarity. From $\Sigma_{\tilde{A}}\tilde{A} = {}^t\tilde{A}\Sigma_{\tilde{A}}$ we deduce that $s = -\eta\big(B-\mu\big)^{-1}$ and then that

$$S'B - {}^tBS' = {}^t\big(B-\mu\big)^{-1}\eta\otimes\eta - \eta\otimes\eta\big(B-\mu\big)^{-1}. \tag{A.1}$$

Define now the symmetric $(N-1)\times(N-1)$ matrix

$$\Sigma_B = S' - s\otimes s = S' - {}^t\big(B-\mu\big)^{-1}\eta\otimes\eta\big(B-\mu\big)^{-1}.$$

Then

$$\Sigma_B B - {}^tB\Sigma_B = S'B - {}^tBS' - {}^t\big(B-\mu\big)^{-1}\eta \otimes\eta\big(B-\mu\big)^{-1}B + {}^tB{}^t\big(B-\mu\big)^{-1}\eta\otimes\eta\big(B-\mu\big)^{-1}.$$

If we plug (A.1) in this last formula we see that the right-hand side vanishes, and so the matrix $\Sigma_B B$ is symmetric.

We prove now that the matrix Σ_B is positive definite. Let $p = {}^t(p', p_n) \neq 0$ be an N row vector; since the matrix $\Sigma_{\tilde{A}}$ is positive definite, we have ${}^t p \Sigma_{\tilde{A}} p > 0$, which reads

$$ {}^t p' S' p' + 2s \cdot p' \, p_N + p_N^2 > 0. $$

We denoted the scalar product by a dot. In order that this second-order inequality in p_N is satisfied a necessary and sufficient condition is that its discriminant be negative , i.e., $(s \cdot p')^2 - {}^t p' S' p' < 0$, which is just the condition of positiveness of the matrix Σ_B.

At last, since both $\Sigma_{\tilde{A}}$ and $\tilde{A}$ are smooth, then Σ_B is also, and then Σ_B is a symmetrizer for B.

We now prove the converse. Let Σ_B be a symmetrizer for B and define the symmetric matrix

$$ \Sigma_{\tilde{A}} = \begin{pmatrix} \Sigma_B + s \otimes s & {}^t s \\ s & 1 \end{pmatrix} $$

for $s = -\eta \big(B - \mu \big)^{-1}$. By the above construction it is clear that the matrix $\Sigma_{\tilde{A}} \tilde{A}$ is symmetric; its positiveness and smoothness follow from that of Σ_B as above. Then $\Sigma_{\tilde{A}}$ is a symmetrizer for $\tilde{A}$. The lemma is therefore proved. □

Another proof follows, a shorter one, but it does not give in an obvious way the smoothness of the symmetrizer. We recall first that a matrix A is symmetrizable if and only if it is diagonalizable; the proof goes as follows. Let Σ be a symmetrizer of A and then let Q be a non-singular matrix such that $\Sigma = {}^t Q Q$; moreover, let P be an orthogonal matrix such that $P^{-1} \Sigma A P = \Lambda$, where Λ is a diagonal matrix. Then the matrix $QAQ^{-1} = {}^t Q^{-1} P \Lambda P^{-1} Q^{-1}$ is symmetric, then diagonalizable, and then also A is diagonalizable. Conversely, if $V^{-1} A V$ is diagonal, then as a symmetrizer we can choose $(V {}^t V)^{-1}$.

Now let us prove the lemma. If A is symmetrizable, then $\tilde{A}$ is too, and by the previous remark $\tilde{A}$ is diagonalizable; then $V^{-1} A V$ is diagonal for some invertible matrix V. We denote by V' the $(N-1) \times (N-1)$ matrix defined by $(V')_{ij} = V_{ij}$, for $i, j = 1, \ldots, N-1$; the matrix $(V')^{-1} B V'$ is diagonal, then B is diagonalizable, then symmetrizable by the former remark.

Conversely, let B be symmetrizable and let $r'_1, \ldots, r'_{N-1}$ be a basis of $\mathbf{R}^{N-1}$ made of eigenvectors of B related to the eigenvalues $\lambda_1, \ldots, \lambda_{N-1}$. Then $\tilde{A}$ has eigenvalues $\lambda_1, \ldots, \lambda_{N-1}, \mu$ with eigenvectors

$$ \begin{pmatrix} r'_1 \\ \dfrac{\eta \cdot r'_1}{\lambda_1 - \mu} \end{pmatrix}, \cdots, \begin{pmatrix} r'_{N-1} \\ \dfrac{\eta \cdot r'_{N-1}}{\lambda_{N-1} - \mu} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} . $$

Therefore, $\tilde{A}$ is diagonalizable, and then symmetrizable.

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