ON THE BEST POSSIBLE CHARACTER
OF THE L^Q NORM IN SOME A PRIORI ESTIMATES
FOR NON-DIVERGENCE FORM EQUATIONS
IN CARNOT GROUPS

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ABSTRACT. Let $\boldsymbol{G}$ be a group of Heisenberg type with homogeneous dimension Q. For every $0 < \epsilon < Q$ we construct a non-divergence form operator L^ϵ and a non-trivial solution $u^\epsilon \in \mathcal{L}^{2,Q-\epsilon}(\Omega) \cap C(\overline{\Omega})$ to the Dirichlet problem: $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$. This non-uniqueness result shows the impossibility of controlling the maximum of u with an L^p norm of Lu when $p < Q$. Another consequence is the impossiblity of an Alexandrov-Bakelman type estimate such as

$$\sup_{\Omega} u \le C \left(\int_{\Omega} |\det(u_{,ij})| \, dg \right)^{1/m},$$

where m is the dimension of the horizontal layer of the Lie algebra and $(u_{,ij})$ is the symmetrized horizontal Hessian of u.

1. INTRODUCTION

In the recent paper [DGN2] we formulated the following conjecture. Consider a stratified, nilpotent Lie group $\boldsymbol{G}$ of step r, with Lie algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_r$. This means that $[V_1, V_j] = V_{j+1}$ for $j = 1, ..., r-1$, whereas $[V_1, V_r] = \{0\}$. If $m = dim\, V_1$, let $X_1, ..., X_m$ be a fixed orthonormal basis of the first layer V_1. Continue to denote with $X_1, ..., X_m$ the corresponding system of left-invariant vector fields on $\boldsymbol{G}$. Given an $m \times m$ matrix-valued function $g \to A(g) = (a_{ij}(g))$ on $\boldsymbol{G}$ having measurable entries, suppose in addition that $A(g) = A(g)^T$, and that there exist $\nu > 0$ such that

$$(1.1) \qquad \nu \, |\zeta|^2 \; \le \; \sum_{i,j=1}^{m} a_{ij}(g) \, \zeta_i \, \zeta_j \; \le \; \nu^{-1} \, |\zeta|^2 \, ,$$

for a.e. $g \in \boldsymbol{G}$, and for every $\zeta \in \mathbb{R}^m$. For a function $u : \boldsymbol{G} \to \mathbb{R}$ we introduce the *symmetrized horizontal Hessian* of u as the $m \times m$ matrix-valued function

$g \to (u_{,ij}(g))_{i,j=1}^m$ defined by

$$u_{,ij} \overset{def}{=} \frac{1}{2} \left\{ X_i X_j u + X_j X_i u \right\}, \qquad i,j = 1,...,m .$$

The stratification of the Lie algebra carries a natural family of non-isotropic dilations $\Delta_\lambda \xi = \lambda \xi_1 + \lambda^2 \xi_2 + ... + \lambda^r \xi_r$, for any $\xi = \xi_1 + \xi_2 + ... + \xi_r \in \mathfrak{g}$. Such dilations are transferred to $\boldsymbol{G}$ by means of the exponential map as follows: $\delta_\lambda = \exp \circ \Delta_\lambda \circ \exp^{-1}$. We denote by dg the bi-invariant Haar measure on $\boldsymbol{G}$ obtained by pushing forward Lebesgue measure on $\mathfrak{g}$ via the exponential map. One has $d(g \circ \delta_\lambda) = \lambda^Q dg$. For this reason the number $Q = \sum_{j=1}^{r} j \ dim \ V_j$ is called the homogeneous dimension of $\boldsymbol{G}$.

Conjecture. *Given a connected, bounded open set $\Omega \subset \boldsymbol{G}$, let $F \in L^Q(\Omega)$. Suppose that $u \in \mathcal{L}_{loc}^{2,Q}(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$Lu \overset{def}{=} \sum_{i,j=1}^m a_{ij} \ u_{,ij} \geq F$$

in Ω. There exists a constant $C = C(\boldsymbol{G}, \nu, \Omega) > 0$ such that

(1.2) $$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C \ ||F||_{L^Q(\Omega)} .$$

Here, $\mathcal{L}_{loc}^{2,Q}(\Omega)$ indicates the Sobolev space of functions $u \in L_{loc}^Q(\Omega)$ having weak derivatives $X_i X_j u \in L_{loc}^Q(\Omega)$. We note explicitly that, thanks to the Sobolev type embedding in [F2], we have $\mathcal{L}_{loc}^{2,Q}(\Omega) \subset C(\Omega)$. In a standard fashion, an important consequence of (1.2) would be the uniqueness for the Dirichlet problem

(1.3) $$\begin{cases} Lu = F, & u \in \mathcal{L}_{loc}^{2,Q}(\Omega) \cap C(\overline{\Omega}) , \\ u = \phi, & \phi \in C(\partial\Omega) . \end{cases}$$

In the abelian case, when $\boldsymbol{G} = \mathbb{R}^n$ with the ordinary isotropic dilations, one has $\mathfrak{g} = V_1 = \mathbb{R}^n$, so that $Q = m = n$, and $(u_{,ij}) = D^2 u$, the standard Hessian matrix of u. The above conjecture, in this situation, is in fact the celebrated geometric maximum principle of Alexandrov-Bakelman-Pucci (ABP, henceforth); see [A1], [Ba1], [Pu2], and also [Ba2], [Ba3]. In this setting it is well-known that an estimate such as (1.2) can only hold with the L^n norm of F in the right-hand side, in the sense that it is in general impossible to have it with any smaller norm; see Alexandrov [A2] and Pucci [Pu1].

In the sub-Riemannian setting, however, there is an unsettling discrepancy between the two geometric parameters involved, namely the dimension m of the first layer of the Lie algebra, and the homogeneous dimension Q of the group. The former plays a pervasive role in the intrinsic notion of convexity introduced in [DGN2], since such a notion involves a quantitative control of u only on the horizontal sub-bundle of planes generated by the basis $X_1,...,X_m$ of V_1. Furthermore, m is also the correct dimension for the *horizontal Monge-Ampère equation*

$$\det(u_{,ij}) = F$$

associated with the geometric estimate (1.2) ([DGN3]). One has, in fact, from the classical geometric-arithmetic mean inequality,

$$\det\ (u_{,ij})\ \leq\ \frac{1}{m^m}\ \frac{\left[\sum_{i,j=1}^m a_{ij}\ u_{,ij}\right]^m}{\det\ (a_{ij})}\ .$$

On the other hand, the fundamental solution of the horizontal Laplacian corresponding to the basis $\{X_1, ..., X_m\}$

$$(1.4) \qquad\qquad \mathcal{L}u\ =\ tr\ (u_{,ij})\ =\ \sum_{j=1}^m X_j^2 u$$

is homogeneous of degree $2 - Q$ with respect to the non-isotropic group dilations $\{\delta_\lambda\}_{\lambda>0}$; see [F2]. This implies (see (4.3) in Section 4) that when $a_{ij} \equiv \delta_{ij}$, $i, j = 1, ..., m$, one can considerably improve the conjectured estimate. In this case, in fact, (1.2) holds with the L^Q norm of F replaced by $||F||_{L^p(\Omega)}$, where $p > Q/2$. Such a result continues to be valid provided that the Green function $G(g, g')$ for the operator L and the domain Ω be controlled near the singularity by the fundamental solution of the horizontal Laplacian.

In view of these considerations it is not a priori clear whether in (1.2) the L^Q norm is optimal, or rather a smaller norm is permissible. The purpose of this note is to clarify this point and prove the best possible character of the L^Q norm. Precisely, we will establish the following.

Theorem 1.1. *Let $\mathbf{G}$ be a group of Heisenberg type, with homogeneous dimension Q, and denote by Ω the gauge ball $B(e, 1)$, where e is the group identity. For every $0 < \epsilon < Q$ there exists a (real) matrix-valued function $A^\epsilon(g) = (a_{ij}^\epsilon(g))$ with symmetric and bounded measurable entries, and satisfying (1.1) for some $\nu_\epsilon > 0$, such that the Dirichlet problem*

$$(1.5) \qquad \begin{cases} L^\epsilon u\ =\ \sum_{i,j=1}^m a_{ij}^\epsilon\ u_{,ij}\ =\ 0 & in \qquad \Omega\ , \\ u\ =\ 0 & on \qquad \partial\Omega\ , \end{cases}$$

admits a solution $u = u^\epsilon \in \mathcal{L}^{2,Q-\epsilon}(\Omega) \cap C(\overline{\Omega})$ different from the trivial one. As a consequence, we cannot replace the $L^Q(\Omega)$ norm in the right-hand side of (1.2) with the smaller one in $L^{Q-\epsilon}(\Omega)$.

Theorem 1.1 is analogous to the well-known results of Alexandrov [A2] and Pucci [Pu1] for elliptic equations. Our proof is inspired to Serrin's note [Se] on the existence of pathological solutions for divergence form operators. The main new fact here, and this is also the reason why in Theorem 1.1 we confine our attention to groups of Heisenberg type, is that our construction crucially exploits a deep property of such groups, namely that the non-isotropic gauge is a fundamental solution of the nonlinear operator

$$(1.6) \qquad\qquad \mathcal{L}_\infty u\ \overset{def}{=}\ \sum_{i,j=1}^m u_{,ij}\ X_i u\ X_j u\ .$$

The role of this ∞-*horizontal Laplacian* will appear clearly in the course of the proof of Theorem 1.1. We mention that the solutions of the operator $\mathcal{L}_\infty$ can be viewed as the asymptotics as $p \to \infty$ of solutions of the one-parameter family of

quasi-linear equations

$$(1.7) \qquad \mathcal{L}_p u \; = \; \sum_{j=1}^{m} X_j(|Xu|^{p-2} X_j u) \; = \; 0 \; , \qquad\qquad 1 < p < \infty \; .$$

In closing, we mention the paper [GT] for some recent progress on ABP type estimates in the Heisenberg group.

2. Preliminaries

The definition of a Carnot group has been given in the Introduction. Since Carnot groups of step 2 will play a special role in this note, we briefly recall some additional notions and definitions which will be of use in the sequel. For more details the reader is referred to the paper [F2], and also to [Be], [G]. Given a Carnot group $\boldsymbol{G}$ of step 2 with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$, we assume that a scalar product $\langle \cdot, \cdot \rangle$ is given on $\mathfrak{g}$ for which the V_i's are mutually orthogonal.

The notation $\{X_1, ..., X_m\}$, will indicate a fixed orthonormal basis of the first layer V_1. We denote by $\{Y_1, \; ... \; , \; Y_k\}$ an orthonormal basis of the second layer V_2. Elements of V_i, $i = 1, 2$, are assigned the formal degree i. We use the letters g, g' for points in $\boldsymbol{G}$, whereas the letters ξ, ξ' will indicate elements of the Lie algebra $\mathfrak{g}$. We will denote by $L_{g_o}(g) \; = \; g_o \, g$ the left-translations on $\boldsymbol{G}$ by an element $g_o \in \boldsymbol{G}$. Recall that the exponential map $exp : \mathfrak{g} \to \boldsymbol{G}$ is a global analytic diffeomorphism [V]. Denote by $\xi : \boldsymbol{G} \to \mathfrak{g}$ its inverse. We define analytic maps $\xi_i : \boldsymbol{G} \to V_i$, $i = 1, 2$, by letting $\xi_i = \pi_i \circ \xi$, where $\pi_i : \mathfrak{g} \to V_i$ denotes the projection onto the i-th layer. Clearly, $\xi(g) = \xi_1(g) + \xi_2(g)$. We will indicate with

$$(2.1) \quad x_j(g) \; = \; \langle \xi_1(g), X_j \rangle, \quad j = 1, ..., m, \qquad y_s(g) \; = \; \langle \xi_2(g), Y_s \rangle, \quad s = 1, ..., k,$$

the projections of the exponential coordinates of g onto V_1 and V_2. Letting $x(g) = (x_1(g), ..., x_m(g))$, $y(g) = (y_1(g), ..., y_k(g))$, we will often identify $g \in \boldsymbol{G}$ with its exponential coordinates $(x(g), y(g))$. Such identification is justified by the fundamental Baker-Campbell-Hausdorff formula [V]

$$\exp \; \xi \; \exp \; \xi' \; = \; \exp \; (\xi + \xi' + \frac{1}{2} \, [\xi, \xi']) \; , \qquad \xi, \xi' \in \mathfrak{g} \; .$$

We notice that

$$(2.2) \quad |\xi_1(g)|^2 \; = \; |x(g)|^2 \; = \; \sum_{j=1}^{m} x_j(g)^2 \; , \qquad |\xi_2(g)|^2 \; = \; |y(g)|^2 \; = \; \sum_{l=1}^{k} y_l(g)^2 \; .$$

We continue to denote by $X_1, ..., X_m$ and $Y_1, ..., Y_k$ the corresponding systems of left-invariant vector fields on $\boldsymbol{G}$ defined by $X_j(g) = (L_g)_*(X_j)$, $Y_s(g) = (L_g)_*(Y_s)$, where $(L_g)_*$ denotes the differential of L_g. The system $X_1, ..., X_m$ defines a basis for the so-called *horizontal subbundle* $H\boldsymbol{G}$ of the tangent bundle $T\boldsymbol{G}$. The *horizontal gradient* of u is defined by $Xu = X_1 u X_1 + ... + X_m u X_m$, and $|Xu| = (\sum_{j=1}^{m} (X_j u)^2)^{1/2}$.

The natural non-isotropic *gauge* is defined by

$$(2.3) \qquad\qquad N(g) \; = \; \left(|x(g)|^4 \; + \; 16 \, |y(g)|^2 \right)^{1/2} \; .$$

Such a gauge introduces a suitable pseudo-metric structure in $\boldsymbol{G}$ by the formula $d(g, g') = N(g^{-1} g')$. Pseudo-balls with respect to d are denoted by $B(g, r)$. We note that by left-translation invariance $B(g, r) = L_g(B(e, r))$, where $e \in \boldsymbol{G}$ indicates the identity. We note explicitly that the (otherwise immaterial) factor 16 has been

introduced in (2.3) for normalization purposes. Its role will be better explained by (3.4) in Section 3.

3. Groups of Heisenberg type, ∞-horizontal Laplacian and the gauge

In this section we analyze some basic geometric properties of Carnot groups of step 2, with special attention to those ambients which are relevant to the proof of Theorem 1.1. Let G be a group of step 2 with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We define the *Kaplan mapping* $J : V_2 \to End(V_1)$ by the equation

$$(3.1) \qquad \langle J(\eta)\xi_1, \xi_2 \rangle = \langle [\xi_1, \xi_2], \eta \rangle, \qquad \xi_1, \xi_2 \in V_1 \ , \ \eta \in V_2 \ .$$

It is clear from (3.1) that for every $\eta \in V_2$ and $\xi \in V_1$ one has $\langle J(\eta)\xi, \xi \rangle = 0$. Our first result is a basic orthogonality property noted in [GV2]; see also [DGN1].

Lemma 3.1. *In a Carnot group G of step 2, consider the function $\psi(g) \overset{def}{=} |x(g)|^2$. One has $X\psi = 2\xi_1$, and for any $s = 1, ..., k$,*

$$(3.2) \qquad \langle X\psi, Xy_s \rangle \equiv 0 \ , \qquad \langle X\psi, X(|y|^2) \rangle = 0 \ .$$

As a consequence,

$$(3.3) \qquad \langle \xi_1, X(|y|^2) \rangle = 0 \ .$$

Definition 3.2. A Carnot group G of step 2 is called of *Heisenberg type* if for every $\eta \in V_2$, such that $|\eta| = 1$, the map $J(\eta) : V_1 \to V_1$ is orthogonal.

Definition 3.2 is due to A. Kaplan [K], who introduced such a class of Lie groups in connection with questions of hypoellipticity. It was later recognized that there exists in nature a very rich supply of groups of Heisenberg type. For instance, it was proved in [CDKR] that the nilpotent component N in the Iwasawa decomposition of a simple group of rank one KAN is a group of Heisenberg type. Definition 3.2 implies

$$|J(\eta)\xi| = |\eta| \, |\xi| \ , \quad \langle J(\eta')\xi, J(\eta'')\xi \rangle = \langle \eta', \eta'' \rangle |\xi|^2, \quad \eta, \eta', \eta'' \in V_2, \ \xi \in V_1.$$

A striking property of groups of Heisenberg type is that for an appropriate constant $C = C(G) > 0$ the function

$$(3.4) \qquad \Gamma(g, g') = \frac{C}{N(g^{-1}g')^{Q-2}}$$

satisfies the equation $\mathcal{L}\Gamma(g, \cdot) = -\delta_g$ in $\mathcal{D}'(G)$. This result generalized Folland's fundamental solution for the Heisenberg group $\mathbb{H}^n$ [F1]. We emphasize here that in a group of Heisenberg type the gauge (2.3) defines a true distance [Cy], not just a pseudo-distance as for a general Carnot group.

When the dimension of the second layer is $k = 1$, then a group of Heisenberg type is isomorphic to the Heisenberg group $\mathbb{H}^n$; see [S]. In such a case there is essentially one Kaplan mapping J, given by the symplectic matrix in $\mathbb{R}^{2n}$

$$S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The next lemma expresses some basic symmetry properties of groups of Heisenberg type; see [K], [CDG], [GV2].

Lemma 3.3. *Let $\boldsymbol{G}$ be a group of Heisenberg type. One has*

$$(3.5) \qquad\qquad |XN(g)|^2 \; = \; \frac{|x(g)|^2}{N(g)^2},$$

$$(3.6) \qquad\qquad |X(y_s)(g)|^2 \; = \; \frac{1}{4} \, |x(g)|^2 \, , \qquad s = 1, ..., k,$$

$$(3.7) \qquad\qquad |X(|y|^2)(g)|^2 \; = \; |x(g)|^2 \, |y(g)|^2 \, .$$

In [CDG] it was discovered that, remarkably, an appropriate power of the gauge (2.3) also provides a fundamental solution of the quasilinear operator $\mathcal{L}_p$ in (1.7). It was proved there that for every $1 < p < \infty$, $p \neq Q$, there exists $C_p = C(\boldsymbol{G}, p) > 0$ such that the function

$$(3.8) \qquad\qquad \Gamma_p(g, g') \; = \; \frac{p-1}{p-Q} \, \sigma_p^{-1/(p-1)} \, \frac{1}{N(g^{-1}g')^{(Q-p)/(p-1)}}$$

solves the equation $\mathcal{L}_p \Gamma_p(g, \cdot) = -\delta_g$ in an appropriate weak sense. In (3.8) we have

$$(3.9) \qquad\qquad \sigma_p \; \overset{def}{=} \; Q \, \omega_p \; \overset{def}{=} \; Q \int_{B(e,1)} |XN(g)|^p \, dg \, .$$

We stress that the one-parameter family of quasilinear operators $\mathcal{L}_p$ formally approaches in the limit as $p \to \infty$ the ∞-horizontal Laplacian defined by (1.6). What we mean by this is that

$$\mathcal{L}_p u \; = \; (p-2) \, |Xu|^{p-4} \, \left\{ \frac{1}{p-2} \, |Xu|^2 \, \mathcal{L}u \; + \; \mathcal{L}_\infty u \right\} \, .$$

If u_p is a solution to $\mathcal{L}_p u_p = 0$, then the previous equation gives

$$\frac{1}{p-2} \, |Xu_p|^2 \, \mathcal{L}u_p \; + \; \mathcal{L}_\infty u_p \; = \; 0 \, .$$

If we assume that $u_p \to u$, and that $|Xu_p|^2 \mathcal{L}u_p$ is bounded independently of p large, then by letting $p \to \infty$ we find that u must be a solution to $\mathcal{L}_\infty u = 0$.

This being said, we return to (3.9) to note that

$$\lim_{p \to \infty} \sigma_p^{1/(p-1)} \; = \; ||XN||_{L^\infty(B(e,1))};$$

therefore, for every $g, g' \in \boldsymbol{G}$, $g \neq g'$, we have

$$(3.10) \qquad\qquad \lim_{p \to \infty} \Gamma_p(g, g') \; = \; \frac{N(g^{-1}g')}{||XN||_{L^\infty(B(e,1))}} \, .$$

This remarkable formula suggests that the gauge N should be a fundamental solution for the fully non-linear operator (1.6). This intuition is confirmed by the following property of groups of Heisenberg type which constitutes the essential tool in the proof of Theorem 1.1.

Theorem 3.4. *Let $\boldsymbol{G}$ be a group of Heisenberg type. Then one has in the classical sense*

$$\mathcal{L}_\infty N \; = \; 0 \quad in \quad \boldsymbol{G} \setminus \{e\} \, .$$

Proof. We observe the following alternative form of (1.6):

$$(3.11) \qquad \mathcal{L}_\infty u \ = \ \frac{1}{2} \, \langle X(|Xu|^2), Xu \rangle.$$

Using (3.11) we find

$$\mathcal{L}_\infty N \ = \ \frac{1}{2} \, \langle X(|XN|^2), XN \rangle = \ \frac{1}{2} \, \langle X(N^{-2}|x|^2), XN \rangle,$$

where in the latter equation we have used (3.5) in Lemma 3.3. We now have, again by (3.5),

$$(3.12) \qquad \langle X(N^{-2}|x|^2), XN \rangle = \ - \ 2 \, N^{-3} \, |x|^2 \langle XN, XN \rangle + \ 2 \, N^{-2} \, \langle \xi_1, XN \rangle$$
$$= \ - \ 2 \, N^{-5} \, |x|^4 \ + \ 2 \, N^{-2} \, \langle \xi_1, XN \rangle.$$

Finally, we obtain

$$XN \ = \ \frac{1}{4} \, N^{-3} \, X(\psi^2 + 16|y|^2) \ = \ N^{-3} \, \left\{ |x|^2 \xi_1 + 4X|y|^2 \right\},$$

which gives

$$\langle \xi_1, XN \rangle = \ N^{-3} \, \left\{ |x|^4 + 4\langle \xi_1, X|y|^2 \rangle \right\} \ = \ N^{-3} \, |x|^4,$$

where in the last equality we have used (3.3) in Lemma 3.1. Replacing the latter equation in (3.12) we reach the conclusion. $\qquad\square$

4. PROOF OF THEOREM 1.1

The main goal of this section is proving Theorem 1.1. We begin with some preliminary considerations which have an independent interest. In a Carnot group $\boldsymbol{G}$ consider a connected bounded open set $\Omega \subset \boldsymbol{G}$ and the horizontal Laplacian (1.4). In his pioneering paper [Bo] Bony proved the existence of a Green function for $\mathcal{L}$ and Ω, i.e., a positive distribution $G(g, g') = G(g', g)$ such that $G(g, \cdot) \in C^\infty(\Omega \setminus \{g\})$ and for which

$$\begin{cases} \mathcal{L}G(g, \cdot) \ = \ - \ \delta_g & \text{in} \quad \Omega, \\ G(g, \cdot) \ = \ 0 & \text{on} \quad \partial\Omega. \end{cases}$$

If we consider the solution to the Dirichlet problem

$$(4.1) \qquad \begin{cases} \mathcal{L}u \ = \ - \ F & \text{in} \quad \Omega, \\ u \ = \ 0 & \text{on} \quad \partial\Omega, \end{cases}$$

then the following representation formula holds:

$$(4.2) \qquad u(g) \ = \ \int_\Omega G(g, g') \, F(g') \, dg'.$$

Let $\Gamma(g, g')$ denote the positive fundamental solution of the horizontal Laplacian. Bony's maximum principle implies

$$G(g, g') \ \leq \ \Gamma(g, g'), \qquad\qquad g, g' \in \Omega, g \neq g'.$$

Using this information in (4.2), and setting $F = 0$ in Ω^c, we conclude that

$$|u(g)| \ \leq \ \int_\Omega \Gamma(g, g') \, |F(g')| \, dg' \ \leq \ \int_{B(g,d)} \Gamma(g, g') \, |F(g')| \, dg',$$

where $B(g, d)$ is a gauge ball with radius $d = diam(\Omega)$. Holder's inequality implies

$$|u(g)| \leq \left\{ \int_{B(g,d)} \Gamma(g, g')^{p'} \, dg' \right\}^{1/p'} ||F||_{L^p(\Omega)} \, ,$$

where for $p > 1$ we have $p' = p/(p-1)$. Since $\Gamma(g, g')$ is homogeneous of degree $(2 - Q)$ with respect to the non-isotropic group dilations [F2], we obtain by left-translation and rescaling that

$$\left\{ \int_{B(g,d)} \Gamma(g, g')^{p'} \, dg' \right\}^{1/p'} \leq C(\boldsymbol{G}, p) \left\{ \int_{B(e,d)} N(g)^{p'(2-Q)} \, dg' \right\}^{1/p'} \, .$$

The latter integral is finite if and only if $p' < Q/(Q-2)$, which is equivalent to $p > Q/2$. If this is the case, a simple argument shows that

$$\left\{ \int_{B(e,d)} N(g)^{p'(2-Q)} \, dg' \right\}^{1/p'} = C(\boldsymbol{G}, p) \, d^{(2p-Q)/p} \, .$$

These considerations allow us to conclude that when $F \in L^p(\Omega)$ for some $p > Q/2$, then the unique solution u to (4.1) belongs to $L^\infty(\Omega)$, and one has

(4.3) $$\sup_\Omega |u| \leq C(\boldsymbol{G}, p) \, d^{2-Q/p} \, ||F||_{L^p(\Omega)} \, .$$

If in (4.1), instead of the horizontal Laplacian, we consider an operator of the type

$$Lu = \sum_{i,j=1}^m a_{ij} \, u_{,ij} \, ,$$

with a (real) $m \times m$ symmetric matrix (a_{ij}) satisfying (1.1), then the validity of an estimate similar to (4.3) constitutes a basic open question. For the validity of (4.3) it would suffice that L and Ω admit a Green function satisfying a global estimate of the type

$$G(g, g') \leq C(\boldsymbol{G}, \Omega) \, N(g, g')^{2-Q} \, .$$

We will come back to these general questions in a forthcoming study.

Proof of Theorem 1.1. Let $\boldsymbol{G}$ be a group of Heisenberg type. We are going to construct an $m \times m$ matrix-valued function $g \to (a_{ij}(g))$ in the form

(4.4) $$a_{ij} = \delta_{ij} + \gamma \, \frac{X_i N \, X_j N}{|XN|^2} \, , \qquad i, j = 1, ..., m \, ,$$

where N is the gauge in (2.3), and the number $\gamma > 0$ will be chosen subsequently. We note that $A(g) = (a_{ij}(g))$ is symmetric. Moreover, its entries belong to $L^\infty(\boldsymbol{G})$. To see this we observe that $a_{ij}(g)$ are defined and continuous for $g \in \boldsymbol{G} \setminus E$, where $E = \{ g \in \boldsymbol{G} \mid |XN(g)| = 0 \}$. On the other hand Proposition 4.7 in [DGN1] shows that in a group of Heisenberg type the horizontal gradient of any function of the type $\phi(g) = \psi(|x(g)|^4 + |y(g)|^2)$ can only vanish in a subset of the second layer V_2. Applying this result to $\phi = N$ we conclude that

$$E \subset \{ g \in \boldsymbol{G} \mid \xi_1(g) = 0 \},$$

and therefore the dg measure of E is zero. This proves that $a_{ij} \in L^\infty(\boldsymbol{G})$. We next observe that if $\gamma > 0$, the matrix $(a_{ij}(g))$ satisfies (1.1). In fact, for any $\zeta \in V_1$ and dg-a.e. $g \in \boldsymbol{G}$ we have

$$|\zeta|^2 \ \leq \ \langle A(g)\zeta, \zeta \rangle = \ |\zeta|^2 \ + \ \gamma \ \frac{\langle XN(g), \zeta \rangle^2}{|XN|^2} \ \leq \ (1 + \gamma) \ |\zeta|^2 \ .$$

In the domain $\Omega = B(e, 1)$ we now try to construct a solution to $Lu = 0$ in the form $u = F(N)$. From (4.4) we obtain

$$(4.5) \qquad\qquad Lu \ = \ \mathcal{L}u \ + \ \gamma \ \sum_{i,j=1}^m u_{,ij} \ \frac{X_i N \ X_j N}{|XN|^2} \ ,$$

where $\mathcal{L}u$ is given by (1.4). The chain rule gives

$$(4.6) \qquad\qquad u_{,ij} \ = \ F''(N) \ X_i N \ X_j N \ + \ F'(N) \ N_{,ij} \ ,$$

and therefore

$$(4.7) \qquad\qquad \mathcal{L}u \ = \ F''(N) \ |XN|^2 \ + \ F'(N) \ \mathcal{L}N \ .$$

From the proof of (3.4) in [K] (see also [CDG]), one obtains the formula

$$(4.8) \qquad\qquad \mathcal{L}N \ = \ \frac{Q-1}{N} \ |XN|^2 \ .$$

Replacing (4.8) in (4.7) we find the following beautiful formula:

$$(4.9) \qquad\qquad \mathcal{L}u \ = \ |XN|^2 \ \left\{ F''(N) \ + \ \frac{Q-1}{N} \ F'(N) \right\} \ .$$

At this point we substitute (4.9), (4.6) into (4.5) obtaining in $G \backslash \{e\}$

$$(4.10) \quad Lu \ = \ |XN|^2 \ \left\{ F''(N) \ + \ \frac{Q-1}{N} \ F'(N) \right\} \ + \ \gamma \ F''(N) \ |XN|^2$$

$$+ \ \gamma \ \frac{F'(N)}{|XN|^2} \ \sum_{i,j=1}^m N_{,ij} \ X_i N \ X_j N$$

$$= \ (1 + \gamma) \ |XN|^2 \ \left\{ F''(N) \ + \ \frac{Q-1}{1+\gamma} \ \frac{F'(N)}{N} \right\} \ + \ \gamma \ \frac{F'(N)}{|XN|^2} \ \mathcal{L}_\infty N \ .$$

With (4.10) we have reached the crucial point of the proof. It is worth pausing at this moment to mention that, despite the appearances, up to this point we have not really used the special structure of groups of Heisenberg type yet. In fact, (4.10) can be established in every Carnot group provided that we use as "gauge" the function

$$(4.11) \qquad\qquad N(g) \ \overset{def}{=} \ \Gamma(g, e)^{1/(2-Q)} \ ,$$

where Γ is the fundamental solution of the horizontal Laplacian. It is easy to check that such N originates a pseudo-distance, not a distance as is the case for a group of Heisenberg type. This, however, would have no bearing on the questions considered here.

What really matters instead is that in order to make use of (4.10), and eventually construct a solution to $Lu = 0$, one needs to know that $\mathcal{L}_\infty N = 0$. Such deep property is unknown for a general Carnot group and for (4.11), and in fact it is

most likely false. For a group of Heisenberg type, however, we can invoke Theorem 3.4 which allows us to reduce the equation (4.10) to the following form:

$$(4.12) \qquad Lu \;=\; (1+\gamma)\,|XN|^2\,\left\{ F''(N) \;+\; \frac{Q-1}{1+\gamma}\,\frac{F'(N)}{N}\right\}\;.$$

It is now clear that $u = F(N)$ will solve $Lu = 0$ provided that F solves the ode

$$(4.13) \qquad F''(N) \;+\; \frac{Q-1}{1+\gamma}\,\frac{F'(N)}{N} \;=\; 0\;.$$

The equation (4.13) admits the solution $F(N) = N^\lambda$ with

$$(4.14) \qquad \lambda \;=\; 1 \;-\; \frac{Q-1}{1+\gamma} \qquad \text{or} \qquad \gamma \;=\; -\,1\;+\;\frac{Q-1}{1-\lambda}\;.$$

We assume that $0 < \lambda < 1$. With this constraint the number γ defined by (4.14) will be > 0 provided that $Q > 2 - \lambda$. This is guaranteed since one always has $Q = m + 2k \geq 4 > 2 - \lambda$.

We can now easily complete the proof of the theorem. Let $0 < \epsilon < Q$. The function N^λ will belong to the Sobolev space $\mathcal{L}^{2,Q-\epsilon}(\Omega)$ provided that

$$(2 - \lambda)(Q - \epsilon) < Q,$$

or equivalently,

$$(4.15) \qquad \lambda \;=\; \lambda_\epsilon \;>\; 1 \;-\; \frac{\epsilon}{Q-\epsilon}\;.$$

This can be easily verified by homogeneity considerations (or by direct calculations). It is clear that (4.15) still leaves room for choosing $\lambda < 1$. In conclusion, letting $u_\epsilon = 1 - N^{\lambda_\epsilon}$ we have produced for any $\epsilon \in (0, Q)$ a function $u_\epsilon \in \mathcal{L}^{2,Q-\epsilon}(\Omega)$ which solves the equation $L^\epsilon u_\epsilon = 0$ in Ω and continuously takes up the boundary datum $\phi = 0$. Since on the other hand $v \equiv 0$ is also a solution to the same equation, we conclude that uniqueness in the non-homogeneous Dirichlet problem fails in the class $\mathcal{L}^{2,Q-\epsilon}(\Omega)$, for any $0 < \epsilon < Q$. As a consequence, an estimate such as the ABP cannot hold in such class since, if it did, it would imply such uniqueness. $\quad\square$

In closing we recall that in his paper [Mi] C. Miranda obtained estimates of the solution of non-divergence form elliptic equations in terms of the norm $||Lu||_{L^p(\Omega)}$, when $p > n/2$, provided that the coefficients belong to the Sobolev space $W^{1,n}(\Omega)$. Subsequently, Chiarenza, Frasca and Longo considerably improved on Miranda's results by establishing the solvability of the Dirichlet problem in $W^{2,p}$ under the hypothesis that the coefficients belong to the class VMO of functions with vanishing mean oscillation (we recall that $W^{1,n}$ is contained in VMO). More recently, the interior estimates in [CFL], but not the global ones, have been obtained in [BB] for subelliptic operators of the type $Lu = \sum_{i,j=1}^m a_{ij} X_i X_j u$, with smooth vector fields of Hörmander type [H], under the assumption that the coefficients a_{ij} belong to the space VMO with respect to the Carnot-Carathéodory metric generated by the X_i's. The basic question remains open of finding geometric conditions of the ground domain that would guarantee the existence of global results, thus obtaining appropriate extensions of the theorems in [Mi] and [CFL]. In this respect we note that the existence of the operators L^ϵ in Theorem 1.1 would not prevent the possibility of such results since the coefficients (4.4) in our construction fail to belong to the spaces $\mathcal{L}^{1,Q}(\Omega)$ or VMO.

References

[A1]   A. D. Alexandrov, *Certain estimates for the Dirichlet problem*, Soviet Math. Dokl., **1** (1960), 1151-1154. MR **26:**5290

[A2]   _____, *The impossibility of general estimates for solutions and of uniqueness conditions for linear equations with norms weaker than L^n*, Vestnik Leningrad Univ., **21** (1966), (13), 5-10. MR **33:**6111

[Ba1]   I. Ja. Bakelman, *Theory of quasilinear elliptic equations*, Sibirsk. Mat. Ž. (in Russian), **2** (1961), 179-186. MR **23:**A3900

[Ba2]   _____, *Geometric methods for solving elliptic equations*, "Nauka", Moscow, 1965. MR **33:**2933

[Ba3]   _____, *Convex Analysis and Nonlinear Geometric Elliptic Equations*, Springer-Verlag, 1994. MR **95k:**35063

[Be]   A. Bellaïche and J.-J. Risler, ed., *Sub-Riemannian Geometry*, Birkhäuser, 1996. MR **97f:**53002

[Bo]   J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier, Grenoble, 1, **19** (1969), 277-304. MR **41:**7486

[BB]   M. Bramanti and L. Brandolini, *L^p estimates for nonvariational hypoelliptic operators with VMO coefficients*, Trans. Amer. Math. Soc., **352** (1999), no. 2, 781-822. MR **2000c:**35026

[CDG]   L. Capogna, D. Danielli, and N. Garofalo, *Capacitary estimates and the local behavior of solutions of nonlinear subelliptic equations*, Amer. J. Math., **118** (1996), no. 6, 1153-1196. MR **97k:**35033

[CFL]   F. Chiarenza, M. Frasca, and P. Longo, *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **336** (1993), no. 2, 841-853. MR **93f:**35232

[CDKR]   M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci, *H-type groups and Iwasawa decompositions*, Adv. Math., **87** (1991), 1-41. MR **92e:**22017

[Cy]   J. Cygan, *Subadditivity of homogeneous norms on certain nilpotent Lie groups*, Proc. Amer. Math. Soc., **83** (1981), 69-70. MR **82k:**22009

[DGN1]   D. Danielli, N. Garofalo, and D. M. Nhieu, *Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot-Carathéodory spaces*, Memoirs of the Amer. Math. Soc., to appear.

[DGN2]   _____, *Notions of convexity in Carnot groups*, Comm. Anal. Geom., **11** (2) (2003), 263–341.

[DGN3]   _____, *The Monge-Ampère equation in Carnot groups*, work in preparation.

[F1]   G. B. Folland, *A fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc., **79** (1973), 373-376. MR **47:**3816

[F2]   _____, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Math., **13** (1975), 161-207. MR **58:**13215

[G]   N. Garofalo, *Analysis and Geometry of Carnot-Carathéodory Spaces, with Applications to PDE's*, Birkhäuser, book in preparation.

[GT]   N. Garofalo and F. Tournier, *Monotonicity and comparison theorems of the Monge-Ampère equation in the Heisenberg group*, preprint, 2003.

[GV2]   _____, *Symmetry properties of positive entire solutions of Yamabe-type equations on groups of Heisenberg type*, Duke Math. J., **106** (2001), no.3, 411-448. MR **2002j:**35075

[H]   H. Hörmander, *Hypoelliptic second-order differential equations*, Acta Math., **119** (1967), 147-171. MR **36:**5526

[K]   A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc., **258** (1980), 147-153. MR **81c:**58059

[Mi]   C. Miranda, *Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui*, Ann. Mat. Pura Appl., **63** (1963). MR **30:**331

[Pu1]   C. Pucci, *Operatori ellittici estremanti*, Ann. Mat. Pura Appl., (4) **72** (1966), 141-170. MR **34:**7960

[Pu2]   _____, *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl., (4) **74** (1966), 15-30. MR **35:**5752

[Se]   J. Serrin, *Pathological solutions of elliptic differential equations*, Ann. Scuola Norm. Sup. Pisa, **18** (1964), 385-387. MR **30:**335

[S]   E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, 1993. MR **95c:**42002

[V]   V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1974.

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