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TWO-DIMENSIONAL STELLAR EVOLUTION CODE INCLUDING ARBITRARY MAGNETIC FIELDS. I. MATHEMATICAL TECHNIQUES AND TEST CASES

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ABSTRACT

A high-precision two-dimensional stellar evolution code has been developed for studying solar variability due to structural changes produced by varying internal magnetic fields of arbitrary configurations. Specifically, we are interested in modeling the effects of a dynamo-type field on the detailed internal structure and on the global parameters of the Sun. The high precision is required to model both very small solar changes (of the order of 10^{-4}) and short timescales (of the order of 1 yr). It is accomplished by using the mass coordinate to replace the radial coordinate, by using fixed and adjustable time steps, a realistic stellar atmosphere, and element diffusion, and by adjusting the grid points. We have also built into the code the potential to subsequently include rotation and turbulence. The current code has been tested for several cases, including its ability to reproduce the one-dimensional results.

Subject headings: stars: evolution — stars: variables: other — Sun: interior — Sun: oscillations

1. INTRODUCTION

Modern standard solar models are known to yield the solar structure to an amazing degree of precision (see e.g., Guenther & Demarque 1997; Basu et al. 2000; Winnick et al. 2002). These models, however, cannot explain the solar cycle, and other solar-cycle-related variability. The reason for this shortcoming is that these models do not include the dynamo magnetic fields and relevant temporal variability.

Following the suggestion by Sofia et al. (1979) that any change in the solar luminosity L must be accompanied by a change in the radius R, a number of theoretical investigations have attempted to establish the relationship between these changes (denoted as $W = \Delta \ln R / \Delta \ln L$), by including internal processes designed to mimic the effects of dynamo fields. We classify them into three broad categories:

1. perturbation calculation (see Endal et al. 1985 for a review of the early work; Balmforth et al. 1996 for subsequent work),
2. approximation analysis (see Spruit 1991, 2000 for reference), and
3. stellar evolution with magnetic fields (this method was initiated by Lydon & Sofia [1995], updated by Li & Sofia [2001], generalized to include turbulence by Li et al. [2002], and further generalized to include the interaction between turbulence and magnetic fields by Li et al. [2003]).

The first two are illustrative, but not conclusive. The third can model the effects of arbitrary magnetic field configurations. Li et al. (2003) attempted to produce the observed cycle variations of all global solar parameters and the p-mode oscillation frequencies. The result is promising (e.g., Sofia et al. 2005), but it is not final both because the one-dimensional approximation is used and because not all global parameter data exist for the same time span. The one-dimensional approximation only allows us to use a shell-like magnetic field configuration. This approximation is relatively limiting. For example, in one-dimensional codes the energy flux can only advance to the surface by penetrating the magnetic field shell. If the magnetic field were toroidal, as most dynamo models require, energy flow could circumvent the field. The aim of this paper is to describe a mathematical technique that can model arbitrary magnetic field configurations by generalizing our one-dimensional technique into the two-dimensional case.

In order to match the observed variations of solar global parameters and helioseismic frequencies, two-dimensional solar models should fulfill at least the following precision requirements:

1. a luminosity resolution equal or better than 10^{-2}% per year, because the observed cyclic variation of total solar irradiance is about 0.1% per cycle;
2. a radius resolution equal or better than 10^{-5}% per year, because the observed cyclic variation of solar radius may be as small as 10^{-4}% per cycle;
3. a realistic atmosphere model, because the helioseismic frequencies are sensitive to it;
4. suitable boundary conditions, because the model is sensitive to them;
5. element diffusion, because the helioseismic frequencies are sensitive to composition;
6. a magnetic field, because there is no cyclic variation without magnetic field;
7. turbulence, because helioseismic observations require it; and
8. the interaction between turbulence and magnetic fields, because helioseismic observations require it.

Our one-dimensional code, which is based on the Yale Stellar Evolution Code YREC (Guenther et al. 1992), meets all these requirements, which is a nontrivial accomplishment. It is difficult to modify the other existing two- or three-dimensional codes (e.g., Deupree 1990; Turcotte 2001), since each of them was developed with specific objectives not requiring this degree of precision.

We attempted to include magnetic fields in Deupree's two-dimensional stellar evolution code (Deupree 1990), but we were unable to compare the model results with solar observations and our one-dimensional results, probably because

1. the two-dimensional model has different center and surface boundary conditions than the one-dimensional model,
2. the two-dimensional model does not include an atmosphere model, and

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3. the numerical accuracy is not high enough to match the solar observations.

This experience convinced us that it would be easier to develop a high-precision two-dimensional stellar structure and evolution code by straightforwardly generalizing our one-dimensional code rather than modifying an existing two-dimensional code. Our experience shows that this conviction was well founded.

The highest precision requirement is that the cyclic variation of solar radius should be better than 10^{-5}% per year, because the observed cyclic variation of solar radius may be as small as 10^{-4}% per cycle. There are various uncertainties in the input physics (e.g., Boothroyd & Sackmann 2003; Sackmann & Boothroyd 2003). Although these uncertainties affect the interior structure of the Sun, they have little influence on the cyclic variations of solar global parameters such as solar radius, solar luminosity, and solar effective temperature because of calibration and subtraction of the same parameter at two different times, which remove various possible uncertainties in the cyclic variations of global solar parameters. Such a high precision for the cyclic variations of global solar parameters is thus achievable.

We outline here the basic schematic of the method in order to prevent the readers from getting lost in the detailed derivations.

As is common practice, the starting points are the conservation laws of mass, momentum, energy, and composition, as well as the Newtonian universal gravitational law. Both momentum conservation equations and the Poisson equation are second-order differential equations. We use the radiation transport equation to relate the temperature gradient to the energy flux in the radiative zone and use the mixing-length theory to calculate the temperature gradient in the convective zone. We include magnetic fields in this paper and include in the code the potential to subsequently include turbulence and rotation.

The main relation is the coordinate transformation from the radial coordinate r to the mass coordinate m. Regarding mass, we should specify the spatial range that the mass occupies. We use the equipotential surface S_Φ on which

$$\Phi(r,\theta;t) = \Phi_c \tag{1}$$

to indicate the spatial range, where we have assumed that the system is azimuthally symmetric or axisymmetric and that Φ_c may vary with time. The time coordinate t is taken as a parameter. Solving equation (1) for r, we obtain the equipotential surface

$$r = R(\Phi_c,\theta;t). \tag{2}$$

This equipotential surface encloses volume V_Φ, which is defined by

$$V_\Phi \equiv \begin{cases} \phi\in[0,2\pi], \\ \theta\in[0,\pi], \\ r\in[0,R(\Phi_c,\theta;t)]. \end{cases} \tag{3}$$

The mass contained in V_Φ is defined by

$$\begin{aligned} m = m(\Phi_c;t) &\equiv \int^{V_\Phi} \rho\, dV \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \int_0^{R(\Phi_c,\theta;t)} ds\, \rho(s,\theta;t)s^2 \\ &= 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{R(\Phi_c,\theta;t)} ds\, \rho(s,\theta;t)s^2, \end{aligned} \tag{4}$$

where $\rho = \rho(r,\theta;t)$ is the density. Solving equation (4) for Φ_c, we obtain

$$\Phi_c = \Phi_c(m;t). \tag{5}$$

Substituting equation (5) into equation (2), we obtain the coordinate transformation relation from $(r,\theta;t)$ to $(m,\theta;t)$:

$$r = R(\Phi_c(m;t),\theta;t) = r(m,\theta;t), \quad \theta=\theta, \quad t=t. \tag{6}$$

For any dependent variable X, for example, P, T, F_r, or ρ, we have

$$\left(\frac{\partial X}{\partial\theta}\right)_m = \left(\frac{\partial X}{\partial\theta}\right)_r + \left(\frac{\partial X}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial\theta}\right)_m. \tag{7}$$

In order to achieve a high precision that is comparable to the one-dimensional solar model in the two-dimensional case, using limited computational resources, we cannot directly numerically solve those conservation equations and the Poisson equation. For example, even in the hydrostatic case, we have five dependent variables such as pressure (P), temperature (T), radius (r), gravitational potential (Φ), and flux (F_r or $L = 4\pi r^2 F_r$). The coefficient matrix of the linearized difference equations with grids $M\times N$ has $\mathcal{N} = 5MN \times 5MN$ elements, where M (N) is the number of grid points for the mass (colatitude) coordinate. The one-dimensional solar model has $M \geq 2000$. If we take $N = 20$, we obtain $\mathcal{N} \geq 4\times 10^{10}$. Since $2^{32} = 4\times 1024^3$, a 32 bit computer can handle only $2\times 1024^3 \sim 2\times 10^9$ elements, noting that 1 bit is used to represent the sign of a number. Of course, a 64 bit computer does not impose such constraint, but the computation speed will become an obstacle.

Analytical solutions are accurate, but such solutions are hard to obtain in the general case. The one-dimensional case is accurate because we do not need to numerically solve the second-order Poisson equation for the gravitational potential Φ_0. It is well known that the gravitational acceleration in the spherically symmetric case is

$$g = d\Phi_0/dr = Gm/r^2. \tag{8}$$

In order to take a similar advantage in the two-dimensional case, we show in this paper that equation (8) can be generalized as

$$\frac{\partial\Phi}{\partial r} = \frac{Gm}{r^2} + 2\pi G r(\rho - \rho_m) - \frac{\cot\theta}{2r}\frac{\partial\Phi}{\partial\theta} + O(2), \tag{9}$$

where $O(2)$ represents a much smaller correction than the retained terms and ρ_m is defined by

$$\rho_m(m,\theta;t) \equiv \frac{1}{2r^2}\int_0^{\pi} d\theta\, R^2(\Phi_c,\theta;t)\rho(R(\Phi_c,\theta;t),\theta;t)\sin\theta. \tag{10}$$

Like equation (8) in the one-dimensional case, equation (9) substantially simplifies the two-dimensional stellar structure equations.

In the two-dimensional case, the radial component of the energy flux vector $\boldsymbol{F}$, F_r, and the θ-dependent luminosity, $L \equiv 4\pi r^2 F_r(r,\theta;t)$, are equivalent to each other, but the actual luminosity L^* is different from the θ-dependent luminosity L because

$$L^* \equiv 2\pi \int_0^{\pi} r^2 F_r(r,\theta;t)\sin\theta\, d\theta. \tag{11}$$

The basic equations are described in § 2, and then the coordinate transformation from the radial coordinate to the mass coordinate is performed in § 3. Various possible magnetic field configurations are converted into suitable expressions that appear in the stellar structure equations in § 4. Boundary conditions are equally important, so we use a whole section (§ 5) to elaborate them. The method of solution is detailed in § 6. The coefficient matrix and input physics used in § 6 are presented in Appendices A and B, respectively. The evolution sequences without any magnetic field and with a shell-like magnetic field are presented in §§ 7 and 8 to test the method.

2. BASIC EQUATIONS

The basic equations consist of the time-dependent conservation laws of mass, momentum, energy, and composition and the Poisson equation (Deupree 1990), as well as the radiative transfer equation (Unno & Spiegel 1966):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{12a}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla P - \rho \nabla \Phi + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{12b}$$

$$\rho T \frac{dS_T}{dt} = \rho \epsilon - \nabla \cdot \boldsymbol{F}_{\rm rad}, \tag{12c}$$

$$\frac{d\rho_i}{dt} = Q_i, \tag{12d}$$

$$\nabla^2 \Phi = 4\pi G \rho, \tag{12e}$$

$$\nabla \cdot \boldsymbol{F}_{\rm rad} = -4\kappa\rho(J - B), \tag{12f}$$

where $\boldsymbol{v}$ is the velocity of a fluid element, $\boldsymbol{B}$ is the magnetic field, ϵ is the nuclear energy generation rate per unit mass, $\boldsymbol{F}_{\rm rad}$ is the radiative energy flux, ρ_i is the density of species i, Q_i is the creation rate of species i, G is the universal gravitational constant, J is the mean radiative intensity, κ is the absorption coefficient, and B is the Kirchhoff-Planck function. The total derivative is defined by $d/dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla$.

The specific entropy S_T includes both nonmagnetic and magnetic components, as shown in the first law of thermodynamics (Callen 1966, p. 242; Lydon & Sofia 1995),

$$T\,dS_T = dU + P\,dV - d\chi, \tag{13}$$

where U is the nonmagnetic specific internal energy, $V = 1/\rho$ is the specific volume, $\chi = |\boldsymbol{B}|^2/8\pi\rho$ is the specific magnetic energy, and P is the nonmagnetic pressure. Since the magnetic work $d\chi$ is taken from the nonmagnetic internal energy, the total internal U_T energy decreases:

$$U_T = U - \chi. \tag{14}$$

The isotropic magnetic pressure component P_m can be expressed by χ and ρ:

$$P_m = \chi\rho. \tag{15}$$

The total isotropic pressure component P_T can thus be defined as

$$P_T = P + P_m. \tag{16}$$

Using P_T, T, and χ as independent thermodynamic variables, the equation of state and the first law of thermodynamics read (Lydon & Sofia 1995)

$$d\rho/\rho = \alpha\,dP_T/P_T - \delta\,dT/T - \nu\,d\chi/\chi, \tag{17a}$$

$$T\,dS_T = C_P\,dT - (\delta/\rho)\,dP_T + (P_T\delta\nu/P_m\alpha)\,d\chi, \tag{17b}$$

where

$$\alpha \equiv (\partial \ln \rho/\partial \ln P_T)_{T,\chi;\,t}, \quad \delta \equiv -(\partial \ln \rho/\partial \ln T)_{P_T,\chi;\,t}, \tag{18a}$$

$$\nu \equiv -(\partial \ln \rho/\partial \ln \chi)_{P_T,T;\,t}, \quad C_P \equiv (\partial U_T/\partial T)_{P_T,\chi;\,t}. \tag{18b}$$

From the first law of thermodynamics (eq. [17b]), we can define two adiabatic gradients. One fixes the specific magnetic energy,

$$\nabla_{\rm ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P_T}\right)_{S_T,\chi} = \frac{P_T\delta}{\rho C_P T}, \tag{19}$$

and another does not fix the specific magnetic energy,

$$\nabla'_{\rm ad} \equiv \left(\frac{\partial \ln T}{\partial \ln P_T}\right)_{S_T} = \nabla_{\rm ad}\left(1 - \frac{\nu\nabla_\chi}{\alpha}\right), \tag{20}$$

where the magnetic energy gradient ∇_χ is defined as

$$\nabla_\chi \equiv \frac{\partial \ln \chi}{\partial \ln P_T}. \tag{21}$$

In order to close the radiative transfer equation (eq. [12f]), we use the Eddington approximation (Unno & Spiegel 1966),

$$\boldsymbol{F}_{\rm rad} = -\frac{4\pi}{3\kappa\rho}\nabla J. \tag{22}$$

Unlike Deupree (1990), we do not directly solve these equations. We first perform some analytic work to make some approximations in advance.

2.1. *Mass Conservation Equation*

Deupree (1990) uses the constancy of the total mass during the model evolution to determine the radius at the equator. In contrast, we want to determine the equipotential surface S_Φ, $r = R(\Phi_c, \theta; t) = r(m, \theta; t)$, as in the one-dimensional case.

Mass conservation can be expressed by either equation (4) or its differential form,

$$\frac{\partial m}{\partial r} = \frac{\partial m}{\partial R} = 4\pi r^2(m, \theta; t)\rho_m(m, \theta; t), \tag{23}$$

where

$$r^2(m, \theta; t)\rho_m(m, \theta; t) \equiv \frac{1}{2}\int_0^\pi d\theta R^2(\Phi_c, \theta; t)\rho(R(\Phi_c, \theta; t), \theta; t)\sin\theta$$
$$= f(m; t). \tag{24}$$

It should be pointed out that in general,

$$\rho_m(m, \theta; t) \neq \rho(R(\Phi_c, \theta; t), \theta; t). \tag{25}$$

Nevertheless, in the spherically symmetric case, $\rho_m(m; t)$ is indeed equal to $\rho(R(\Phi_c); t)$. Since $f(m; t)$ is an integral, the two-dimensional case is much more complicated (i.e., nonlocal) than its one-dimensional counterpart (local). This complexity may be the price we have to pay to go from one dimension to two dimensions.

2.2. *Gravitational Acceleration*

We want to show here that the last two terms [excluding $O(2)$] on the right-hand side of equation (9) are due to the two-dimensional corrections to the gravitational acceleration. To this end, we should start from the Poisson equation, equation (12e), which can be expanded as follows in the spherical polar coordinate system:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Phi}{\partial \theta}\right)=4\pi G\rho, \tag{26}$$

where we have assumed that $\Phi = \Phi(r, \theta; t)$ does not vary with the ϕ coordinate. We expand Φ around its spherically symmetric state:

$$\Phi(r, \theta; t) = \Phi_0(r; t) + \delta\Phi(r, \theta; t), \tag{27}$$

where $\delta\Phi$ is a small correction and

$$\frac{\partial \Phi_0}{\partial r} = \frac{Gm}{r^2}. \tag{28}$$

Substituting equations (27) and (28) into equation (26), we obtain

$$\frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2} + 2\pi G r(\rho - \rho_m) - \frac{\cot\theta}{2r}\frac{\partial \Phi}{\partial \theta} + O(2), \tag{29}$$

where

$$O(2) = -\frac{r}{2}\left(\frac{\partial^2 \delta\Phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\right). \tag{30}$$

2.3. *Momentum Conservation Equation*

Generally, we can decompose the total velocity $\boldsymbol{v}$ in the basic equations into three components:

$$\boldsymbol{v} = \boldsymbol{V}_0 + \boldsymbol{V}_{\rm rot} + \boldsymbol{v}', \tag{31}$$

where $\boldsymbol{V}_0$ is a secular evolution velocity, $\boldsymbol{V}_{\rm rot}$ is the rotation velocity, and $\boldsymbol{v}'$ is the turbulent convection velocity. We neglect the secular expansion and rotation velocity components in the momentum conservation, i.e., we assume

$$\boldsymbol{v} = \boldsymbol{v}' \tag{32}$$

in equation (12b). We checked in the one-dimensional case that the term dV_0/dt in the momentum equation is negligible. Substituting equation (32) into equation (12b) and averaging the resulting equation over the time t and azimuthal angle ϕ, we obtain

$$\rho\nabla\overline{v'^2} = -\nabla P - \rho\nabla\Phi + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{33}$$

where $\overline{v'^2} = \overline{v_x'^2} + \overline{v_y'^2} + \overline{v_z'^2}$ is computed by solving the basic equations in the three-dimensional convection simulations of the outer layers of the Sun (Robinson et al. 2003), in which the average is taken over the time t and the horizontal coordinates x and y in a sample box. We have shown how to include turbulence in the one-dimensional case (Li et al. 2002). We neglect the turbulent contribution to the momentum equation here so as to stress the two-dimensional effects due to magnetic fields, i.e., we simply set

$$\overline{v'^2} = 0 \tag{34}$$

in this paper.

We assume that the system is azimuthally symmetric. Under this assumption, the vector equation (33) is equivalent to the two scalar equations

$$\frac{\partial P_T}{\partial r} = -\rho\frac{\partial \Phi}{\partial r} + \mathcal{H}_r, \tag{35a}$$

$$\frac{1}{r}\frac{\partial P_T}{\partial \theta} = -\frac{\rho}{r}\frac{\partial \Phi}{\partial \theta} + \mathcal{H}_\theta, \tag{35b}$$

where $P_T = P + P_m$ is the total pressure, including the magnetic pressure $P_m = B^2/8\pi$, and

$$\mathcal{H} \equiv \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \tag{36}$$

noticing that

$$\frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}. \tag{37}$$

In the one-dimensional case, we have only a single scalar equation to describe the momentum conservation, i.e., equation (35a). In contrast, we need three scalar equations for the momentum conservation in the two-dimensional case, i.e., equations (29), (35a), and (35b). It would be much better if we could combine these three equations into a single scalar equation. Fortunately, we can. To this end, solving equation (35b) for $\partial\Phi/\partial\theta$, we obtain

$$\frac{\partial \Phi}{\partial \theta} = -\frac{1}{\rho}\frac{\partial P_T}{\partial \theta} + \frac{r}{\rho}\mathcal{H}_\theta. \tag{38}$$

Then substituting this into equation (29), we obtain

$$\frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2} + 2\pi G r(\rho - \rho_m) + \frac{\cot\theta}{2r\rho}\frac{\partial P_T}{\partial \theta} - \frac{\cot\theta}{2\rho}\mathcal{H}_\theta + O(2). \tag{39}$$

Substituting equation (39) into equation (35a), we finally obtain

$$\frac{\partial P_T}{\partial r} = -\frac{Gm\rho}{r^2} + \mathcal{H}_r - 2\pi G r\rho(\rho - \rho_m) - \frac{\cot\theta}{2r}\frac{\partial P_T}{\partial \theta} + \frac{1}{2}\mathcal{H}_\theta \cot\theta + O(2). \tag{40}$$

This is our momentum conservation equation. The last three right-hand-side terms represent the two-dimensional effects.

2.4. *Energy Conservation Equation*

The energy conservation equation (eq. [12c]) depends on the velocity in the total derivative:

$$\frac{dS_T}{dt} = \frac{\partial S_T}{\partial t} + (\boldsymbol{V}_0 + \boldsymbol{v}')\cdot\nabla S_T. \tag{41}$$

The secular expansion velocity $\boldsymbol{V}_0$ cannot be neglected, and from now on we define

$$\frac{dS_T}{dt} \equiv \frac{\partial S_T}{\partial t} + \boldsymbol{V}_0\cdot\nabla S_T. \tag{42}$$

The statistical average of $\rho T \boldsymbol{v}' \cdot \nabla S_T$, namely, $\langle \rho T \boldsymbol{v}' \cdot \nabla S_T \rangle$, will determine the divergence of the convective flux $\boldsymbol{F}_{\rm conv}$:

$$\nabla \cdot \boldsymbol{F}_{\rm conv} \equiv \langle \rho T \boldsymbol{v}' \cdot \nabla S_T \rangle. \quad (43)$$

By defining the total energy flux to be the sum of both the convective and radiative flux, $\boldsymbol{F} = \boldsymbol{F}_{\rm rad} + \boldsymbol{F}_{\rm conv}$, equation (12c) becomes

$$\nabla \cdot \boldsymbol{F} = \rho \left(\epsilon - T \frac{dS_T}{dt} \right), \quad (44)$$

where

$$T \frac{dS_T}{dt} = C_P T \left[\frac{d \ln T}{dt} - \nabla_{\rm ad} \left(1 - \frac{\nu \nabla_\chi}{\alpha} \right) \frac{d \ln P_T}{dt} \right]. \quad (45)$$

In the azimuthal case, equation (44) is equivalent to the equation

$$\frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r} = \rho \left(\epsilon - T \frac{dS}{dt} \right) - \frac{1}{r \sin \theta} \frac{\partial (\sin \theta F_\theta)}{\partial \theta}. \quad (46)$$

We work out both the radial flux component F_r and polar flux component F_θ in the next subsections.

2.5. *Energy Transport by Radiation*

The radiative flux is given by equation (22), in which the mean radiative intensity J is governed by the radiative transfer equation (eq. [12f]). The Planck function B is known:

$$B = \frac{ac}{4\pi} T^4, \quad (47)$$

where a is the radiative constant and c is the speed of light in vacuum. In stellar interior, local thermodynamic equilibrium is a good approximation, which leads to

$$J \approx B = \frac{ac}{4\pi} T^4. \quad (48)$$

The more accurate solution of equations (12f) and (22) is (see Unno & Spiegel 1966)

$$J = B + \frac{l_p^2}{3} \nabla^2 B + \frac{l_p^4}{5} \nabla^4 B + \ldots, \quad (49)$$

where $l_p = 1/\kappa\rho$ is the mean free path of photons. Since

$$\nabla^2 B = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial B}{\partial r} \right), \quad (50)$$

using equation (47) in equation (50), we obtain

$$\nabla^2 B = 4 \nabla_s \left(4 \nabla_s - 1 + \alpha - \delta \nabla_s + \frac{\partial \ln \nabla_s}{\partial \ln P_T} - \frac{4\pi r^2 H_P \rho_m}{m} \right) \frac{B}{H_P^2}, \quad (51)$$

where $H_P \equiv -dr/d \ln P_T = P/\rho g$ is the pressure scale height and ∇_s is the actual temperature gradient. Substituting equation (51) into equation (49), we obtain the mean radiative intensity that goes beyond the local thermodynamic equilibrium approximation with one more term correction:

$$J = \left(1 + \lambda_0 \frac{l_p^2}{H_P^2} \right) B, \quad (52)$$

where

$$\lambda_0 = \frac{4}{3} \nabla_s \left(4 \nabla_s - 1 + \alpha - \delta \nabla_s + \frac{\partial \ln \nabla_s}{\partial \ln P_T} \right). \quad (53)$$

We want to note that the term $(4\pi r^2 H_P \rho_m / m)(l_p^2 / H_P^2) \ll 1$ is negligible in the whole star. Using this solution in equation (22), we obtain

$$\boldsymbol{F}_{\rm rad} = -\frac{4acT^3}{3\kappa\rho} (1 + \lambda) \nabla T, \quad (54)$$

where

$$\lambda \equiv \lambda_0 \left\{ 1 - \frac{1}{2} \left(\frac{\partial \ln \kappa}{\partial \ln T} \right)_{P_T} - \frac{1}{2} \frac{1}{\nabla_s} \left[1 + \left(\frac{\partial \ln \kappa}{\partial \ln P_T} \right)_T - \frac{2H_P}{r} \right] + \frac{1}{4} \frac{\partial \ln \lambda_0}{\partial \ln T} \right\} \frac{l_p^2}{H_P^2}. \quad (55)$$

Since l_p is much smaller than H_P in the optically thick region, we know $\lambda \approx 0$, so that equation (54) reduces to the widely used approximation expression without λ. However, l_p can be comparable to or larger than H_P near the surface, and the correction factor λ cannot be neglected.

2.6. *Energy Transport by Convection*

Without solving the turbulent convection problem, equation (43) only tells us that the convective flux may depend on the convective velocity $v_{\rm conv}$ and the entropy S_T, where the convective velocity $v_{\rm conv}$ has only the statistical meaning. We use the mixing-length theory to obtain an analytic expression for $\boldsymbol{F}_{\rm conv}$ in terms of $v_{\rm conv}$ and S_T (e.g., Stix 1989; Lydon & Sofia 1995). Since the convective velocity has only the statistical meaning, we assume that the turbulent convection is isotropic, so that $\boldsymbol{F}_{\rm conv}$ depends on the amplitude of the convective velocity $v_{\rm conv}$:

$$\boldsymbol{F}_{\rm conv} = -\tfrac{1}{2} \rho T l_m f(v_{\rm conv}) \nabla S_T, \quad (56)$$

where $f(v)$ will be determined by the mixing-length theory and l_m is the mixing length. It is well known that $f(v) = v$ when the radiative loss of the convective element and the magnetic fields are neglected (e.g., Stix 1989).

The starting point of the mixing-length theory (MLT) is to calculate the excess heat flux in the radial direction:

$$\begin{aligned} F^r_{\rm conv} &= \rho v_{\rm conv} DQ = \rho v_{\rm conv} (Q_e - Q_s) \\ &= \rho v_{\rm conv} \left[C_P (T_e - T_s) - (\delta/\rho)(P_{T_e} - P_{T_s}) \right. \\ &\quad \left. + (P_T \delta \nu / P_m \alpha)(\chi_e - \chi_s) \right], \end{aligned} \quad (57)$$

where we have used the first law of thermodynamics $DQ = TDS_T$. The subscripts e and s stand for a convective eddy and its surroundings. If the eddy is always assumed to be in pressure equilibrium ($DP_T = P_{Te} = P_{Ts} = 0$) and magnetic equilibrium ($D\chi = \chi_e - \chi_s = 0$) with its surroundings, we have

$$F^r_{\rm conv} = \rho v_{\rm conv} C_P (T_e - T_s) = \frac{l_m}{2H_P} \rho v_{\rm conv} C_P T (\nabla_s - \nabla_e), \quad (58)$$

where the mixing-length approximation in MLT is used to calculate the temperature (or density) difference:

$$T_e - T_s = \frac{l_m}{2} \left(\frac{\partial T_e}{\partial r} - \frac{\partial T_s}{\partial r} \right) = \frac{l_m T}{2H_P} (\nabla_s - \nabla_e). \quad (59)$$

We have also defined the eddy and surrounding temperature gradients and the pressure scale height as

$$\nabla_e \equiv \left(\frac{\partial \ln T}{\partial \ln P_T}\right)_e, \quad \nabla_s \equiv \left(\frac{\partial \ln T}{\partial \ln P_T}\right)_s, \quad H_P \equiv -\frac{\partial r}{\partial \ln P_T}. \tag{60}$$

The convective velocity $v_{\rm conv}$ is generated by the radial buoyancy. The radial buoyancy acceleration is related to the density difference by

$$\frac{d^2 r}{dt^2} = -g\left(\frac{D\rho}{\rho}\right), \tag{61}$$

where g is the gravitational acceleration. For standard MLT, the density difference is related to the temperature difference via the equation of state with $DP_T = 0$ and $D\chi = 0$ (see eq. [17a]):

$$\frac{D\rho}{\rho} = -\left(\frac{DT}{T}\right)\delta = \frac{l_m \delta}{2H_P}(\nabla_e - \nabla_s). \tag{62}$$

We also use the mixing-length approximation to calculate buoyancy acceleration

$$\frac{d^2 r}{dt^2} = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{dr}{dt}\right)^2 = \frac{1}{2}\left(\frac{dr}{dt}\right)^2_{\rm max}\frac{2}{l_m} = \frac{4v_{\rm conv}^2}{l_m}, \tag{63}$$

where we have assumed that the convective velocity $v_{\rm conv}$ equals half of the maximum velocity $(dr/dt)_{\rm max}$. Substituting equations (62) and (63) into equation (61), we obtain

$$v_{\rm conv}^2 = g\delta(\nabla_s - \nabla_e)(l_m^2/8H_P). \tag{64}$$

This gives

$$\nabla_s - \nabla_e = \frac{8H_P}{g l_m^2 \delta} v_{\rm conv}^2. \tag{65}$$

Substituting this into equation (58), we obtain

$$F_{\rm conv}^r = (4\rho C_P T / g l_m \delta) v_{\rm conv}^3. \tag{66}$$

Equation (54) yields

$$F_{\rm rad}^r = \frac{4acT^4}{3\kappa\rho H_P}(1+\lambda)\nabla_s. \tag{67}$$

Defining a "radiative" gradient

$$\nabla_{\rm rad} = \frac{3\kappa\rho H_P F_r}{4acT^4}, \tag{68}$$

we obtain

$$F_r = \frac{4acT^4}{3\kappa\rho H_P}(1+\lambda)\nabla_{\rm rad}. \tag{69}$$

We use the energy flux conservation law $F_{\rm conv}^r + F_{\rm rad}^r = F_r$ to constrain the convective velocity by

$$\frac{1}{1+\lambda}\frac{4\rho C_P T}{g l_m \delta}\frac{3\kappa\rho H_P}{4acT^4} v_{\rm conv}^3 + \nabla_s = \nabla_{\rm rad}. \tag{70}$$

2.6.1. *Nonmagnetic Adiabatic Approximation*

When the convective eddy is adiabatic, its temperature gradient equals the adiabatic gradient. The nonmagnetic approximation implies $\chi = 0$. Therefore, the temperature gradient in a nonmagnetic adiabatic eddy is determined by

$$\nabla_e = \nabla'_{\rm ad} = \nabla_{\rm ad}. \tag{71}$$

Equation (58) thus becomes

$$F_{\rm conv}^r = -\frac{1}{2}\rho T l_m v_{\rm conv}\left(\frac{\partial S}{\partial r}\right)_s, \tag{72}$$

where we have used the equality

$$\left(\frac{\partial S}{\partial r}\right)_s = -\frac{C_P}{H_P}(\nabla_s - \nabla_{\rm ad}). \tag{73}$$

Comparing equation (72) with the radial component of equation (56), we find

$$f(v) = v, \tag{74}$$

as stated above.

Using equations (65) and (71) in equation (70), we obtain the cubic equation of the convective velocity,

$$\frac{1}{1+\lambda}\frac{4\rho C_P T}{g l_m \delta}\frac{3\kappa\rho H_P}{4acT^4} v_{\rm conv}^3 + \frac{8H_P}{g l_m^2 \delta} v_{\rm conv}^2 = \nabla_{\rm rad} - \nabla_{\rm ad}. \tag{75}$$

The convective instability condition in the adiabatic approximation is

$$\nabla_{\rm rad} \geq \nabla_s > \nabla_e = \nabla_{\rm ad}, \tag{76}$$

according to equation (64).

2.6.2. *Nonmagnetic Nonadiabatic Approximation*

During its rise the eddy radiates energy into its environment. For this reason the eddy gradient ∇_e differs from the adiabatic gradient $\nabla_{\rm ad}$. We decompose the convective flux (eq. [58]) into the adiabatic (the first right-hand-side term) and nonadiabatic (the second right-hand-side term) fluxes:

$$\begin{aligned} F_{\rm conv}^r &= \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla_s - \nabla_e) \\ &= \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla_s - \nabla_{\rm ad}) \\ &\quad + \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla_{\rm ad} - \nabla_e). \end{aligned} \tag{77}$$

If the effective cross section of the convective eddy is q, the heat energy-loss rate of the eddy due to radiation can be expressed by

$$\frac{dQ_r}{dt} = \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla_{\rm ad} - \nabla_e) q. \tag{78}$$

We can also use equation (54) to calculate the radiative loss by

$$\frac{dQ_r}{dt} = -\frac{4acT^3}{3\kappa\rho}\frac{T_e - T_s}{d}\Sigma = -\frac{1}{2}\frac{l_m \Sigma}{H_P d}\frac{4acT^4}{3\kappa\rho}(\nabla_s - \nabla_e), \tag{79}$$

where d is the effective radius of the eddy and Σ is the eddy surface. Comparing equation (79) with equation (78), we obtain

$$\nabla_e - \nabla_{\rm ad} = (v_0/v_{\rm conv})(\nabla_s - \nabla_e), \tag{80}$$

where

$$v_0 = \frac{l_m \Sigma}{qd}\frac{4acT^3}{3\rho C_P}\frac{1}{l_m \kappa \rho}. \tag{81}$$

Substituting equation (80) into equation (77), we can express $\nabla_s - \nabla_e$ by $\nabla_s - \nabla_{\rm ad}$:

$$\begin{aligned}\nabla_s - \nabla_e &= \frac{1}{1 + v_0/v_{\rm conv}}(\nabla_s - \nabla_{\rm ad}) \\ &= -\frac{1}{1 + v_0/v_{\rm conv}}\frac{H_P}{C_P}\left(\frac{\partial S_T}{\partial r}\right)_s,\end{aligned} \tag{82}$$

where we have used equation (73). Finally, using equation (82) in equation (77) we obtain

$$F^r_{\rm conv} = -\frac{1}{2}\rho T l_m \frac{v_{\rm conv}}{1 + v_0/v_{\rm conv}}\left(\frac{\partial S}{\partial r}\right)_s. \tag{83}$$

This shows that

$$f(v) = \frac{v}{1 + v_0/v}. \tag{84}$$

Using equation (65) in equation (82), we obtain

$$\nabla_s - \nabla_{\rm ad} = \frac{8H_P}{g l_m^2 \delta} v_{\rm conv}^2 \left(1 + \frac{v_0}{v_{\rm conv}}\right). \tag{85}$$

Substituting this into equation (70), we obtain the cubic equation of the convective velocity,

$$\frac{4\rho C_P T}{g l_m \delta}\frac{3\kappa\rho H_P}{4acT^4} v_{\rm conv}^3 + \frac{8H_P}{g l_m^2 \delta} v_{\rm conv}^2 \left(1 + \frac{v_0}{v_{\rm conv}}\right) = \nabla_{\rm rad} - \nabla_{\rm ad}. \tag{86}$$

The convective instability condition in the nonmagnetic nonadiabatic approximation is

$$\nabla_{\rm rad} \geq \nabla_s > \nabla_e > \nabla_{\rm ad}, \tag{87}$$

according to equation (64).

2.6.3. *General Case*

When magnetic fields are present, we have

$$\left(\frac{\partial S_T}{\partial r}\right)_s = -\frac{C_P}{H_P}(\nabla_s - \nabla'_{\rm ad}). \tag{88}$$

We decompose the convective flux (eq. [58]) into the adiabatic (the first right-hand-side term) and nonadiabatic (the second right-hand-side term) fluxes:

$$\begin{aligned}F^r_{\rm conv} &= \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla_s - \nabla_e) \\ &= \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla_s - \nabla'_{\rm ad}) \\ &\quad + \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla'_{\rm ad} - \nabla_e).\end{aligned} \tag{89}$$

The heat energy-loss rate of the eddy due to radiation now can be expressed by

$$\frac{dQ_r}{dt} = \frac{1}{2}\frac{l_m v_{\rm conv}}{H_P}\rho T C_P(\nabla'_{\rm ad} - \nabla_e) q. \tag{90}$$

The radiation loss rate calculated by equation (54) is the same as that given in equation (79). Comparing equation (79) with equation (90), we obtain

$$\nabla_e - \nabla'_{\rm ad} = (v_0/v_{\rm conv})(\nabla_s - \nabla_e). \tag{91}$$

Substituting equation (91) into equation (89), we can express $\nabla_s - \nabla_e$ by $\nabla_s - \nabla'_{\rm ad}$:

$$\begin{aligned}\nabla_s - \nabla_e &= \frac{1}{1 + v_0/v_{\rm conv}}(\nabla_s - \nabla'_{\rm ad}) \\ &= -\frac{1}{1 + v_0/v_{\rm conv}}\frac{H_P}{C_P}\left(\frac{\partial S_T}{\partial r}\right)_s,\end{aligned} \tag{92}$$

where we have used equation (88). Finally, substituting equation (92) into equation (89), we obtain

$$F^r_{\rm conv} = -\frac{1}{2}\rho T l_m \frac{v_{\rm conv}}{1 + v_0/v_{\rm conv}}\left(\frac{\partial S_T}{\partial r}\right)_s, \tag{93}$$

which leads up to equation (84).

Using equation (65) in equation (92), we obtain

$$\nabla_s - \nabla'_{\rm ad} = \frac{8H_P}{g l_m^2 \delta} v_{\rm conv}^2 \left(1 + \frac{v_0}{v_{\rm conv}}\right). \tag{94}$$

Substituting this into equation (70), we obtain the cubic equation of the convective velocity in a magnetic system,

$$\frac{4\rho C_P T}{g l_m \delta}\frac{3\kappa\rho H_P}{4acT^4} v_{\rm conv}^3 + \frac{8H_P}{g l_m^2 \delta} v_{\rm conv}^2 \left(1 + \frac{v_0}{v_{\rm conv}}\right) = \nabla_{\rm rad} - \nabla'_{\rm ad}. \tag{95}$$

The convective instability condition in the magnetic nonadiabatic case is

$$\nabla_{\rm rad} \geq \nabla_s > \nabla_e > \nabla_{\rm ad}, \tag{96}$$

according to equation (64).

Equation (95) can be rewritten as

$$2A_0 y^3 + V y^2 + V^2 y - V = 0, \tag{97}$$

where we have defined the dimensionless variable

$$y = V v_{\rm conv}/v_0 \tag{98}$$

and the dimensionless parameters

$$\begin{aligned}v_0 &= 6acT^3/\rho C_P l_m \kappa \rho, \\ C &= \frac{g l_m^2 \delta}{8H_p}, \\ V &= v_0/\left[C^{1/2}(\nabla_{\rm rad} - \nabla'_{\rm ad})^{1/2}\right], \\ A_0 &= \frac{9}{8}\frac{1}{1+\lambda}.\end{aligned}$$

We choose $l_m\Sigma/qd = 9/2$ for spherical eddies and $d/l_m = 8/9$. The convective gradient can be expressed by y as

$$\nabla_{\rm conv} = \nabla_s = \nabla'_{\rm ad} + (\nabla_{\rm rad} - \nabla'_{\rm ad})y(y+V), \qquad (99)$$

according to equation (94).

When magnetic fields are neglected, $\nabla'_{\rm ad} = \nabla_{\rm ad}$, all formulae automatically reduce to their counterparts in § 2.6.2.

2.7. *Energy Flux Vector*

In the radiative zone, the total energy flux vector equals the radiative flux (eq. [54]),

$$\boldsymbol{F} = -\frac{4acT^3}{3\kappa\rho}(1+\lambda)\nabla T. \qquad (100)$$

In the convective zone, the total energy flux vector equals the sum of the radiative (eq. [54]) and convective (eq. [56]) fluxes,

$$\begin{aligned}\boldsymbol{F} &= -\frac{4acT^3}{3\kappa\rho}(1+\lambda)\nabla T - \frac{1}{2}\frac{\rho T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\nabla S_T \\ &= -\left[\frac{4acT^3}{3\kappa\rho}(1+\lambda) + \frac{1}{2}\frac{\rho C_P l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\right]\nabla T \\ &\quad + \frac{1}{2}\frac{\rho C_P T \nabla'_{\rm ad} l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\frac{1}{P_T}\nabla P_T, \end{aligned} \qquad (101)$$

where we have used the formula

$$\nabla S_T = (C_P/T)\nabla T - (C_P\nabla'_{\rm ad}/P_T)\nabla P_T. \qquad (102)$$

2.8. *Composition Conservation*

Equation (12d) describes the composition conservation law, which can be rewritten as

$$\rho\frac{\partial X_i}{\partial t} + X_i\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho X_i \boldsymbol{v}') = Q_i, \qquad (103)$$

where we have used equation (32) and have used the mass fraction $X_i \equiv \rho_i/\rho$ to replace density ρ_i. We have also assumed $\nabla\cdot\boldsymbol{v}' = 0$. Equation (103) involves two timescales: one is the thermonuclear reaction timescale $\tau_{\rm nucl}$, which determines Q_i and is quite long, and another is the convection timescale $\tau_{\rm conv}$, which determines the convection mixing and is much shorter than the former.

As before, taking the statistical average over equation (103), we obtain

$$\rho\frac{\partial X_i}{\partial t} + \frac{1}{\rho}\nabla\cdot\langle\rho X_i\boldsymbol{v}'\rangle = q_i, \qquad (104)$$

where we have used the assumption $\langle\partial\rho/\partial t\rangle = 0$ and defined $q_i \equiv Q_i/\rho$. Using the mixing-length theory, we can express the mass flux $\boldsymbol{F}_i \equiv \langle\rho X_i\boldsymbol{v}'\rangle$ as

$$\boldsymbol{F}_i = -\tfrac{1}{2}\rho v_{\rm conv} l_m \nabla X_i. \qquad (105)$$

Substituting equation (105) into equation (104), we obtain

$$\frac{\partial X_i}{\partial t} = q_i + \frac{1}{2\rho}\nabla\cdot(\rho v_{\rm conv} l_m \nabla X_i). \qquad (106)$$

In the radiative zone, the element diffusion velocity w_i (e.g., Thoul et al. 1994) changes the local composition in addition to the thermonuclear reactions. Element diffusion in stars is driven by pressure gradients (or gravity), temperature gradients, composition gradients, and radiation pressure. Gravity tends to concentrate the heavier elements toward the center of the star. Temperature gradients lead to thermal diffusion, which tends to concentrate more highly charged and more massive species toward the hottest region of the star, its center. Concentration gradients oppose the above two processes. Radiation pressure causes negligible diffusion in the solar core. Element diffusion affects the element abundances, the mean molecular weight, and the radiative opacity in the radiative zone, and therefore affects the calculated neutrino fluxes and oscillation frequencies, on which observations impose strict constraints on the solar model.

The characteristic time for elements to diffuse a solar radius under solar conditions is of the order of 6×10^{13} yr, much longer than the age of the Sun. Element diffusion therefore introduces only a small correction. Many authors have studied this topic carefully (see Thoul et al. 1994 and references therein), and both portable subroutine and analytic formulae for element diffusion calculations are available. In particular, the formulae for the element diffusion velocity fit our theoretical framework developed in this paper. We use the formula given by Thoul et al. (1994) with q_i included,

$$\frac{\partial X_i}{\partial t} = q_i - \frac{1}{r^2\rho}\frac{\partial}{\partial r}(r^2\rho X_i w_i), \qquad (107)$$

where

$$w_i(r) = \frac{T^{5/2}}{\rho}\left(A_p^i\frac{\partial\ln P_T}{\partial r} + A_T^i\frac{\partial\ln T}{\partial r} + A_H^i\frac{\partial\ln C_H}{\partial r}\right). \qquad (108)$$

See Thoul et al. (1994) for the expansion coefficients, which are actually computed by numerically solving the multifluid equations for all species. These formulae just give readers the main idea. We use the portable subroutine provided by the authors to compute the element diffusion correction. Diffusion in the polar direction is negligible.

3. COORDINATE TRANSFORMATION FROM r TO m

So far, all derivatives with respect to θ assume r to be constant. What we need is to obtain the corresponding derivatives at the constant m. This can be done by using the so-called implicit-function rule, that is,

$$\left(\frac{\partial}{\partial\theta}\right)_m = \left(\frac{\partial}{\partial\theta}\right)_r + \left(\frac{\partial r}{\partial\theta}\right)_m\frac{\partial}{\partial r} = \left(\frac{\partial}{\partial\theta}\right)_r + \left(\frac{\partial\ln r}{\partial\theta}\right)_m\frac{\partial}{\partial\ln r}. \qquad (109)$$

From now on, we use the following shortcuts to save writing:

$$r' = \ln r, \quad \rho' = \ln\rho, \quad P' = \ln P_T, \quad T' = \ln T, \quad s = \ln m. \qquad (110)$$

We note that ln is the natural logarithm.

Another formula we need for this purpose is the mass conservation equation, equation (23), which can be rewritten as

$$\frac{\partial r'}{\partial s} = \frac{m}{4\pi r^3\rho_m}. \qquad (111)$$

3.1. *Momentum Conservation Equation*

We perform the necessary coordinate transformation from r to m in equation (40). The only term that needs to be transformed is the term that contains $\partial P_T/\partial\theta$, which is equivalent to $(\partial P_T/\partial\theta)_r$. Using equation (109), we obtain

$$\frac{\partial P_T}{\partial\theta} = \left(\frac{\partial P_T}{\partial\theta}\right)_m - \frac{\partial P_T}{\partial r'}\left(\frac{\partial r'}{\partial\theta}\right)_m. \tag{112}$$

Consequently, equation (40) becomes

$$\frac{\partial P_T}{\partial r} = \left[1 - \frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial\theta}\right)_m\right]^{-1}\left[-\frac{Gm\rho}{r^2} - 2\pi Gr\rho(\rho-\rho_m) - \frac{\cot\theta}{2r}\left(\frac{\partial P_T}{\partial\theta}\right)_m + \mathcal{H}_r + \frac{1}{2}\mathcal{H}_\theta\cot\theta\right] + O(2). \tag{113}$$

The first factor on the right-hand side is caused by the coordinate transformation from r to m.

Since $\partial P_T/\partial r = (P_T/r)(\partial s/\partial r')(\partial P'/\partial s)$, using equation (111), we can rewrite equation (113) as

$$\frac{\partial P'}{\partial s} = -\frac{Gm^2}{4\pi r^4 P_T}\frac{\rho}{\rho_m} + \Theta + \mathcal{M} + O(2), \tag{114}$$

where

$$\Theta = -\frac{Gm(\rho-\rho_m)}{2rP_T}\frac{\rho}{\rho_m}\left[1-\frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial\cot\theta}\right)_m\right]^{-1} - \frac{m}{4\pi r^3\rho_m}\frac{\cot\theta}{2}\left(\frac{\partial P'}{\partial\theta}\right)_m\left[1-\frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial\theta}\right)_m\right]^{-1} - \frac{Gm^2}{4\pi r^4 P_T}\frac{\rho}{\rho_m}\frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial\theta}\right)_m\left[1-\frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial\theta}\right)_m\right]^{-1},$$

$$\mathcal{M} = \frac{m}{4\pi r^2\rho_m P_T}\left[1-\frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial\theta}\right)_m\right]^{-1}\left(\mathcal{H}_r + \frac{1}{2}\mathcal{H}_\theta\cot\theta\right).$$

3.2. *Energy Conservation Equation*

The starting equation is equation (46). The only term that needs to be transformed is the term that contains the derivative of $(\sin\theta F_\theta)$ with respect to θ. This term is a small two-dimensional correction to the energy conservation equation, since F_θ, which is given in equation (101), is already a combination of the first-order derivatives of T and P_T,

$$F_\theta = -\left[\frac{4acT^4}{3\kappa\rho}(1+\lambda) + \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\right]\frac{1}{r}\frac{\partial T'}{\partial\theta} + \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\frac{\nabla'_{\rm ad}}{r}\frac{\partial P'}{\partial\theta}. \tag{116}$$

Therefore, after neglecting the higher-order corrections as we did above, the energy conservation equation becomes

$$\frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} = \rho\left(\epsilon - T\frac{dS_T}{dt}\right) - \frac{F_\theta\cot\theta}{r} + O(2). \tag{117}$$

This shows that we only need to transform F_θ from r to m. Applying equation (109) to $(\partial T'/\partial\theta)_r$ and $(\partial P'/\partial\theta)_r$ in equation (116), we obtain

$$\begin{aligned} F_\theta = &-\left[\frac{4acT^4}{3\kappa\rho}(1+\lambda) + \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\right]\frac{1}{r} \times\left[\left(\frac{\partial T'}{\partial\theta}\right)_m - \frac{\partial T'}{\partial r'}\left(\frac{\partial r'}{\partial\theta}\right)_m\right] \\ &+ \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\frac{\nabla'_{\rm ad}}{r}\left[\left(\frac{\partial P'}{\partial\theta}\right)_m - \frac{\partial P'}{\partial r'}\left(\frac{\partial r'}{\partial\theta}\right)_m\right] \\ = &-\left[\frac{4acT^4}{3\kappa\rho}(1+\lambda) + \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\right]\frac{1}{r} \times\left[\left(\frac{\partial T'}{\partial\theta}\right)_m + \frac{Gm\rho}{rP_T}\nabla\left(\frac{\partial r'}{\partial\theta}\right)_m\right] \\ &+ \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1+v_{\rm conv}/v_0}\frac{\nabla'_{\rm ad}}{r}\left[\left(\frac{\partial P'}{\partial\theta}\right)_m + \frac{Gm\rho}{rP_T}\left(\frac{\partial r'}{\partial\theta}\right)_m\right], \end{aligned} \tag{118}$$

where ∇ is the temperature gradient.

The second step is to use equation (111) to replace ∂r by ∂s in equation (46). Unlike r', P', and T', which are the natural logarithms, we define

$$L' \equiv 4\pi r^2 F_r/L_\odot, \tag{119}$$

which is not a logarithm at all. The resulting equation is

$$\frac{\partial L'}{\partial s} = \frac{1}{L_\odot}m\left(\epsilon - T\frac{dS_T}{dt}\right)\frac{\rho}{\rho_m} - \frac{1}{L_\odot}\frac{mF_\theta\cot\theta}{r\rho_m} + O(2). \tag{120}$$

3.3. *Composition Conservation*

Equation (106) involves the derivatives with respect to θ at constant r. Since what we need are the corresponding derivatives at constant m, this equation needs a coordinate transformation from r to m. To this end, we first expand it as

$$\frac{\partial X_i}{\partial t} = q_i + \frac{1}{2r^2\rho}\frac{\partial}{\partial r}\left(r^2\rho v_{\rm conv} l_m\frac{\partial X_i}{\partial r}\right) + \frac{1}{2r^2\rho\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\rho v_{\rm conv} l_m\frac{\partial X_i}{\partial\theta}\right). \tag{121}$$

The last right-hand-side term needs the transformation. We retain its most important part. The resulting equation is

$$\frac{\partial X_i}{\partial t} = q_i + \left(1 + \frac{1}{2}\frac{\partial\rho'}{\partial r'}\right)v_{\rm conv} l_m\frac{\partial X_i}{\partial r} + \frac{\cot\theta v_{\rm conv} l_m}{2r^2}\left(\frac{\partial X_i}{\partial\theta}\right)_m + O(2), \tag{122}$$

where

$$O(2) = \frac{1}{2}\frac{\partial^2 X_i}{\partial r^2} + \frac{v_{\rm conv} l_m}{2r^2}\frac{\partial\rho'}{\partial\theta}\frac{\partial X_i}{\partial\theta} + \frac{v_{\rm conv} l_m}{2r^2}\frac{\partial^2 X_i}{\partial\theta^2} - \frac{\cot\theta v_{\rm conv} l_m}{2r^2}\frac{\partial X_i}{\partial r'}\left(\frac{\partial r'}{\partial\theta}\right)_m. \tag{123}$$

We have taken advantage of the fact that $v_{\rm conv} l_m \approx$ constant in stars.

Since the convection timescale is much shorter than the evolution timescale, the convection zone is well mixed on the evolution timescale. As a result, the detailed expression for the composition conservation equation in the convection zone does not matter much. We do it here just for the sake of completeness.

3.4. *Two-dimensional Stellar Structure Equations*

In summary, we obtain the two-dimensional stellar structure equations with (m, θ) as independent variables:

$$\frac{\partial r'}{\partial s} = \frac{m}{4\pi r^3 \rho}\left\{\frac{\rho}{\rho_m}\right\}, \tag{124a}$$

$$\frac{\partial P'}{\partial s} = -\frac{Gm^2}{4\pi r^4 P_T}\left\{\frac{\rho}{\rho_m}\right\} + \{\Theta\} + \{\mathcal{M}\} + O(2), \tag{124b}$$

$$\frac{\partial T'}{\partial s} = \frac{\partial P'}{\partial s}\begin{cases}\nabla_{\rm rad}, & \text{radiative},\\ \nabla_c, & \text{convective},\end{cases} \tag{124c}$$

$$\frac{\partial L'}{\partial s} = \frac{1}{L_\odot} m\left(\epsilon - T\frac{dS_T}{dt}\right)\left\{\frac{\rho}{\rho_m}\right\} - \left\{\frac{1}{L_\odot}\frac{mF_\theta \cot\theta}{r\rho_m}\right\} + O(2), \tag{124d}$$

$$\frac{\partial X_i}{\partial t} = q_i + \begin{cases} -\dfrac{1}{r^2\rho}\dfrac{\partial}{\partial r}(r^2\rho X_i w_i), & \text{radiative},\\ \left(1 + \dfrac{1}{2}\dfrac{\partial \rho'}{\partial r'}\right) v_{\rm conv} l_m \dfrac{\partial X_i}{\partial r} & \\ + \left\{\dfrac{\cot\theta}{2r^2} v_{\rm conv} l_m \left(\dfrac{\partial X_i}{\partial \theta}\right)_m\right\} + O(2), & \text{convective}. \end{cases} \tag{124e}$$

Those terms or factors associated with two-dimensional effects are indicated by curly braces in the equations. The symbols used above are defined as

$$\begin{aligned}\Theta = &-\frac{m}{4\pi r^3 \rho_m}\frac{\cot\theta}{2}\left(\frac{\partial P'}{\partial \theta}\right)_m\left[1 - \frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial \theta}\right)_m\right]^{-1}\\ &- \frac{Gm(\rho - \rho_m)}{2rP_T}\frac{\rho}{\rho_m}\left[1 - \frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial \theta}\right)_m\right]^{-1}\\ &- \frac{Gm^2}{4\pi r^4 P_T}\frac{\rho}{\rho_m}\frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial \theta}\right)_m\left[1 - \frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial \theta}\right)_m\right]^{-1},\end{aligned} \tag{125a}$$

$$\mathcal{M} = \frac{m}{4\pi r^2 \rho_m P_T}\left[1 - \frac{\cot\theta}{2}\left(\frac{\partial r'}{\partial \theta}\right)_m\right]^{-1}\left(\mathcal{H}_r + \frac{1}{2}\mathcal{H}_\theta \cot\theta\right), \tag{125b}$$

$$\begin{aligned}F_\theta = &-\left[\frac{4acT^4}{3\kappa\rho}(1 + \lambda) + \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1 + v_{\rm conv}/v_0}\right]\frac{1}{r}\\ &\times\left[\left(\frac{\partial T'}{\partial \theta}\right)_m + \frac{Gm\rho}{rP_T}\nabla\left(\frac{\partial r'}{\partial \theta}\right)_m\right]\\ &+ \frac{1}{2}\frac{\rho C_P T l_m v_{\rm conv}}{1 + v_{\rm conv}/v_0}\frac{\nabla'_{\rm ad}}{r}\left[\left(\frac{\partial P'}{\partial \theta}\right)_m + \frac{Gm\rho}{rP_T}\left(\frac{\partial r'}{\partial \theta}\right)_m\right].\end{aligned} \tag{125c}$$

From now on, we drop the subscript m in the derivatives such as $(\partial r'/\partial\theta)_m$,

$$\left(\frac{\partial r'}{\partial \theta}\right)_m \Longrightarrow \frac{\partial r'}{\partial \theta}. \tag{126}$$

Wherever needed, we specify the subscript m or r to avoid confusion.

4. MAGNETIC FIELDS

Our strategy is to take advantage of analytical results as much as possible. For this purpose, in this section we work out the explicit expressions for the terms associated with magnetic fields.

Generally, a magnetic field has three components. Using the spherical coordinate system, it can be expressed by

$$\boldsymbol{B} = (B_r, B_\theta, B_\phi). \tag{127}$$

All three components are functions of m and θ in the azimuthally symmetric case treated in this paper. The $\boldsymbol{B}$-related terms are expressed by $\mathcal{H}$ (see eq. [36]), which can be expanded as

$$\begin{aligned}4\pi r\mathcal{H} = &\,\boldsymbol{e}_r\left(r\boldsymbol{B}\cdot\nabla B_r - B_\theta B_\theta - B_\phi B_\phi\right)\\ &+ \boldsymbol{e}_\theta\left(r\boldsymbol{B}\cdot\nabla B_\theta - B_\phi B_\phi \cot\theta + B_\theta B_r\right)\\ &+ \boldsymbol{e}_\phi\left(r\boldsymbol{B}\cdot\nabla B_\phi + B_\phi B_r - B_\phi B_\theta \cot\theta\right).\end{aligned} \tag{128}$$

Consequently, we see

$$4\pi r\mathcal{H}_r = r\boldsymbol{B}\cdot\nabla B_r - B_\theta B_\theta - B_\phi B_\phi, \tag{129a}$$

$$4\pi r\mathcal{H}_\theta = r\boldsymbol{B}\cdot\nabla B_\theta - B_\phi B_\phi \cot\theta + B_\theta B_r, \tag{129b}$$

$$4\pi r\mathcal{H}_\phi = r\boldsymbol{B}\cdot\nabla B_\phi + B_\phi B_r - B_\phi B_\theta \cot\theta. \tag{129c}$$

We use $\boldsymbol{B}$ to define three stellar magnetic parameters, in addition to the conventional stellar parameters such as pressure, temperature, radius, and luminosity. The first magnetic parameter is the magnetic kinetic energy per unit mass, χ,

$$\chi = B^2/(8\pi\rho). \tag{130}$$

The second is the heat index due to the magnetic field, or the ratio of the magnetic pressure in the radial direction to the magnetic energy density, $\gamma - 1$,

$$\gamma = 1 + (B_\theta^2 + B_\phi^2)/B^2. \tag{131}$$

Lydon & Sofia (1995) introduced the first two in the one-dimensional case. Here we introduce the third one, the ratio of the magnetic pressure in the colatitude direction to the magnetic energy density, $\vartheta - 1$,

$$\vartheta = 1 + (B_\phi^2 + B_r^2)/B^2. \tag{132}$$

We can use these three magnetic parameters to express three components of a magnetic field as

$$B_r = [8\pi(2 - \gamma)\chi\rho]^{1/2}, \tag{133a}$$

$$B_\theta = [8\pi(2 - \vartheta)\chi\rho]^{1/2}, \tag{133b}$$

$$B_\phi = [8\pi(\gamma + \vartheta - 3)\chi\rho]^{1/2}. \tag{133c}$$

We discuss below various possible cases. Note that any case should satisfy the restriction

$$\nabla\cdot\boldsymbol{B} = 0. \tag{134}$$

4.1. $\boldsymbol{B} = (0, 0, 0)$

In this case,

$$\chi = 0, \quad \vartheta = 1, \quad \gamma = 1, \quad \mathcal{H} = 0. \tag{135}$$

Consequently, the term associated with magnetic fields vanishes, namely,

$$\mathcal{M} = 0. \tag{136}$$

Defining

$$\mathcal{B}^1 = -\frac{Gm^2}{4\pi r^4 P_T}\frac{\rho}{\rho_m}, \tag{137a}$$

$$\mathcal{B}^2 = -\frac{Gm(\rho-\rho_m)}{2rP_T}\frac{\rho}{\rho_m}, \tag{137b}$$

$$\mathcal{B}^3 = -\frac{m}{4\pi r^3\rho_m}, \tag{137c}$$

we can rewrite Θ as

$$\Theta = \mathcal{B}^1\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} + \mathcal{B}^2\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} + \mathcal{B}^3\frac{\cot\theta}{2}\frac{\partial P'}{\partial\theta}\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}. \tag{138}$$

Solving this case will provide us with a standard two-dimensional stellar model.

4.2. $\boldsymbol{B} = (0, 0, B_\phi)$

Since B_ϕ is assumed to depend on only r and θ, equation (134) is satisfied for any arbitrary function $B_\phi = B_\phi(r,\theta)$. In this case, since

$$\chi = B_\phi^2/(8\pi\rho), \quad \vartheta = 2, \quad \gamma = 2,$$

we have

$$B_r = 0, \tag{139a}$$

$$B_\theta = 0, \tag{139b}$$

$$B_\phi = (8\pi\chi\rho)^{1/2}. \tag{139c}$$

Substituting these into equations (129a) and (129b), we obtain

$$H_r = -2\chi\rho/r, \tag{140a}$$

$$H_\theta = H_r\cot\theta. \tag{140b}$$

Substituting them into equation (125b), we obtain

$$\mathcal{M} = -\mathcal{B}^4\left(1+\frac{1}{2}\cot^2\theta\right)\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \tag{141}$$

where

$$\mathcal{B}^4 = \frac{m}{2\pi r^3\rho_m}\frac{\chi\rho}{P_T}. \tag{142}$$

4.3. $\boldsymbol{B} = (0, B_\theta, 0)$

Equation (134) requires

$$\frac{\partial(\sin\theta B_\theta)}{\partial\theta} = 0. \tag{143}$$

This leads to

$$B_\theta = B(r)/\sin\theta, \tag{144}$$

where $B(r)$ is an arbitrary function of r.

In this case, since

$$\chi = B_\theta^2/(8\pi\rho), \quad \vartheta = 1, \quad \gamma = 2,$$

we have

$$B_r = 0, \tag{145a}$$

$$B_\theta = (8\pi\chi\rho)^{1/2}, \tag{145b}$$

$$B_\phi = 0. \tag{145c}$$

Equation (144) requires that $B = (8\pi\chi\rho)^{1/2}\sin\theta$ does not depend on θ.

In order to calculate $\mathcal{M}$, we have to calculate $\mathcal{H}_r$ and $\mathcal{H}_\theta$ first. Substituting equations (145a)–(145c) into equations (129a) and (129b), we obtain

$$H_r = -2\chi\rho/r, \tag{146a}$$

$$H_\theta = \frac{1}{4\pi r}\frac{1}{2}\left(\frac{\partial B_\theta^2}{\partial\theta}\right)_r \tag{146b}$$

$$= \frac{\chi\rho}{r}\left[\left(\frac{\partial\rho'}{\partial\theta}+\frac{\partial\chi'}{\partial\theta}\right)_m - \frac{4\pi r^3\rho_m}{m}\left(\frac{\partial\rho'}{\partial s}+\frac{\partial\chi'}{\partial s}\right)\left(\frac{\partial r'}{\partial\theta}\right)_m\right]. \tag{146c}$$

Substituting them into equation (125b), we obtain $\mathcal{M}$,

$$\mathcal{M} = -\mathcal{B}^4\left[1-\frac{\cot\theta}{4}\left(\frac{\partial\rho'}{\partial\theta}+\frac{\partial\chi'}{\partial\theta}\right)\right]\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} - \mathcal{B}^5\left(\frac{\partial\rho'}{\partial s}+\frac{\partial\chi'}{\partial s}\right)\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \tag{147}$$

where

$$\mathcal{B}^5 = \chi\rho/P_T. \tag{148}$$

4.4. $\boldsymbol{B} = (B_r, 0, 0)$

In this case, since

$$\chi = B_r^2/(8\pi\rho), \quad \vartheta = 2, \quad \gamma = 1,$$

we have

$$B_r = (8\pi\chi\rho)^{1/2}, \tag{149a}$$

$$B_\theta = 0, \tag{149b}$$

$$B_\phi = 0. \tag{149c}$$

Equation (134) requires

$$\frac{\partial(r^2B_r)}{\partial r} = 0. \tag{150}$$

This leads to

$$B_r = B(\theta)/r^2, \tag{151}$$

where $B(\theta)$ is an arbitrary function of θ. Therefore, we know

$$B = (8\pi\chi\rho)^{1/2}r^2 \tag{152}$$

varies with only θ.

Substituting equations (149a)–(149c) into equations (129a) and (129b), we obtain

$$H_r = \frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\left(\frac{\partial\rho'}{\partial s}+\frac{\partial\chi'}{\partial s}\right),$$

$$H_\theta = 0.$$

Substituting them into equation (125b), we obtain $\mathcal{M}$,

$$\mathcal{M} = \mathcal{B}^5\left(\frac{\partial\rho'}{\partial s}+\frac{\partial\chi'}{\partial s}\right)\left(1-\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}. \tag{153}$$

4.5. $\boldsymbol{B} = (0, B_\theta, B_\phi)$

In this case, since

$$\chi = \frac{B_\theta^2 + B_\phi^2}{8\pi\rho}, \quad \vartheta = 1 + \frac{B_\phi^2}{B_\theta^2 + B_\phi^2}, \quad \gamma = 2,$$

we have

$$B_r = 0, \tag{154a}$$
$$B_\theta = [8\pi(2-\vartheta)\chi\rho]^{1/2}, \tag{154b}$$
$$B_\phi = [8\pi(\vartheta - 1)\chi\rho]^{1/2}. \tag{154c}$$

Equation (134) requires

$$\frac{\partial(\sin\theta B_\theta)}{\partial\theta} = 0. \tag{155}$$

This leads to

$$B_\theta = B(r)/\sin\theta, \tag{156}$$

where $B(r)$ is an arbitrary function of r. Therefore, we have the constraint that $[8\pi(2-\vartheta)\chi\rho]^{1/2}\sin\theta$ depends only on r.

Substituting equations (154a)–(154c) into equations (129a) and (129b), we obtain

$$H_r = -2\chi\rho/r, \tag{157a}$$

$$\begin{aligned} H_\theta &= -\frac{2\chi\rho}{r}(\vartheta - 1)\cot\theta + \frac{1}{r}\left\{\frac{\partial}{\partial\theta}[(2-\vartheta)\chi\rho]\right\}_r \\ &= -\frac{2\chi\rho}{r}(\vartheta - 1)\cot\theta + \frac{1}{r}\left\{\frac{\partial}{\partial\theta}[(2-\vartheta)\chi\rho]\right\}_m \\ &\quad - \frac{1}{r}\frac{4\pi r^3\rho_m}{m}\left(\frac{\partial r'}{\partial\theta}\right)_m \frac{\partial}{\partial s}[(2-\vartheta)\chi\rho]. \end{aligned} \tag{157b}$$

Substituting these expressions into equation (125b), we obtain

$$\begin{aligned} \mathcal{M} = &-\mathcal{B}^6\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &+ \mathcal{B}^7\frac{\cot\theta}{2}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\vartheta''}{\partial\theta}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &- \mathcal{B}^8\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right)\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \end{aligned} \tag{158}$$

where

$$\begin{aligned} \mathcal{B}^6 &= \mathcal{B}^4\left[1 + \tfrac{1}{2}(\vartheta - 1)\cot^2\theta\right], \\ \mathcal{B}^7 &= t\tfrac{1}{2}\mathcal{B}^4(2-\vartheta), \\ \mathcal{B}^8 &= \mathcal{B}^5(2-\vartheta), \\ \frac{\partial\vartheta''}{\partial s} &= \frac{\partial}{\partial s}\log(2-\vartheta), \\ \frac{\partial\vartheta''}{\partial\theta} &= \frac{\partial}{\partial\theta}\log(2-\vartheta). \end{aligned}$$

4.6. $\boldsymbol{B} = (B_r, B_\theta, 0)$

In this case, since

$$\chi = \frac{B_r^2 + B_\theta^2}{8\pi\rho}, \quad \vartheta = 1 + \frac{B_r^2}{B_r^2 + B_\theta^2}, \quad \gamma = 1 + \frac{B_\theta^2}{B_r^2 + B_\theta^2},$$

we have

$$B_r = [8\pi(2-\gamma)\chi\rho]^{1/2}, \tag{159a}$$
$$B_\theta = [8\pi(2-\vartheta)\chi\rho]^{1/2}, \tag{159b}$$
$$B_\phi = 0. \tag{159c}$$

We have used the fact that

$$\gamma + \vartheta = 3. \tag{160}$$

A meaningful magnetic field should satisfy equation (134). For example, $r^2 B_r$ does not vary with r, and $\sin\theta B_\theta$ does not vary with θ.

Substituting equations (159a)–(159c) into equations (129a) and (129b), we obtain

$$\begin{aligned} H_r = &-(2-\vartheta)\frac{2\chi\rho}{r} + (2-\gamma)\frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right) \\ &+ [(2-\gamma)(2-\vartheta)]^{1/2}\frac{\chi\rho}{r}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\gamma''}{\partial\theta}\right) \\ &- [(2-\gamma)(2-\vartheta)]^{1/2}\frac{\chi\rho}{r}\frac{4\pi r^3\rho_m}{m}\frac{\partial r'}{\partial\theta}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right), \end{aligned}$$

$$\begin{aligned} H_\theta = &[(2-\gamma)(2-\vartheta)]^{1/2}\frac{2\chi\rho}{r} \\ &+ [(2-\gamma)(2-\vartheta)]^{1/2}\frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right) \\ &+ (2-\vartheta)\frac{\chi\rho}{r}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\vartheta''}{\partial\theta}\right) \\ &- (2-\vartheta)\frac{\chi\rho}{r}\frac{4\pi r^3\rho_m}{m}\frac{\partial r'}{\partial\theta}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right). \end{aligned}$$

Substituting these expressions into equation (125b), we obtain

$$\begin{aligned} \mathcal{M} = &-\mathcal{B}^9\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &+ \mathcal{B}^{10}\left(\frac{\partial\rho'}{\partial s} + \frac{\partial\chi'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &- \mathcal{B}^{11}\frac{\partial r'}{\partial\theta}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &+ \mathcal{B}^{12}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\gamma''}{\partial\theta}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &+ \mathcal{B}^{13}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &- \mathcal{B}^{14}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right)\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1} \\ &+ \mathcal{B}^{15}\frac{\cot\theta}{2}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\vartheta''}{\partial\theta}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \end{aligned} \tag{162}$$

where

$$\begin{aligned}
\mathcal{B}^9 &= \mathcal{B}^4\left\{(2-\vartheta) - \frac{\cot\theta}{2}[(2-\gamma)(2-\vartheta)]^{1/2}\right\},\\
\mathcal{B}^{10} &= \mathcal{B}^5(2-\gamma),\\
\mathcal{B}^{11} &= \mathcal{B}^5[(2-\vartheta)(2-\gamma)]^{1/2},\\
\mathcal{B}^{12} &= \tfrac{1}{2}\mathcal{B}^4[(2-\vartheta)(2-\gamma)]^{1/2},\\
\mathcal{B}^{13} &= \tfrac{1}{2}\mathcal{B}^5\cot\theta[(2-\vartheta)(2-\gamma)]^{1/2},\\
\mathcal{B}^{14} &= \mathcal{B}^8,\\
\mathcal{B}^{15} &= \mathcal{B}^7,\\
\frac{\partial\gamma''}{\partial s} &= \frac{\partial}{\partial s}\log(2-\gamma),\\
\frac{\partial\gamma''}{\partial\theta} &= \frac{\partial}{\partial\theta}\log(2-\gamma).
\end{aligned}$$

4.7. $\boldsymbol{B} = (B_r, 0, B_\phi)$

In this case, since

$$\chi = \frac{B_r^2 + B_\theta^2}{8\pi\rho}, \quad \vartheta = 2, \quad \gamma = 1 + \frac{B_\phi^2}{B_r^2 + B_\phi^2},$$

we have

$$B_r = [8\pi(2-\gamma)\chi\rho]^{1/2}, \tag{163a}$$
$$B_\theta = 0, \tag{163b}$$
$$B_\phi = [8\pi(\gamma-1)\chi\rho]^{1/2}. \tag{163c}$$

A meaningful magnetic field should satisfy equation (134), which requires $r^2 B_r$ not to vary with r.

Substituting equations (163a)–(163c) into equations (129a) and (129b), we obtain

$$\begin{aligned}
H_r &= -(\gamma-1)\frac{2\chi\rho}{r} + \frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}(2-\gamma)\left(\frac{\partial\rho'}{\partial s} + \frac{\partial\chi'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right),\\
H_\theta &= -(\gamma-1)\frac{2\chi\rho}{r}\cot\theta.
\end{aligned}$$

Substituting these expressions into equation (125b), we obtain

$$\begin{aligned}
\mathcal{M} = &-\mathcal{B}^{16}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&+\mathcal{B}^{10}\left(\frac{\partial\rho'}{\partial s} + \frac{\partial\chi'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1},
\end{aligned} \tag{164}$$

where

$$\mathcal{B}^{16} = \mathcal{B}^4(\gamma-1)\left(1 + \tfrac{1}{2}\cot^2\theta\right).$$

4.8. $\boldsymbol{B} = (B_r, B_\theta, B_\phi)$

This is the general case, in which all magnetic field parameters χ, ϑ, and γ are variables. Therefore, we use the general expressions for B_r, B_θ, and B_ϕ given at the beginning of this section. Substituting equations (133a)–(133c) into equations (129a) and (129b) to calculate H_r and H_θ, we obtain

$$\begin{aligned}
H_r = &-(\gamma-1)\frac{2\chi\rho}{r} + (2-\gamma)\frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\left(\frac{\partial\rho'}{\partial s} + \frac{\partial\chi'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right)\\
&+ [(2-\gamma)(2-\vartheta)]^{1/2}\frac{\chi\rho}{r}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\gamma''}{\partial\theta}\right)\\
&- [(2-\gamma)(2-\vartheta)]^{1/2}\frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\frac{\partial r'}{\partial\theta}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right),\\
H_\theta = &[(2-\gamma)(2-\vartheta)]^{1/2}\frac{2\chi\rho}{r} - (\gamma+\vartheta-3)\frac{2\chi\rho}{r}\cot\theta\\
&+ [(2-\gamma)(2-\vartheta)]^{1/2}\frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right)\\
&+ (2-\vartheta)\frac{\chi\rho}{r}\left(\frac{\partial\rho'}{\partial\theta} + \frac{\partial\chi'}{\partial\theta} + \frac{\partial\vartheta''}{\partial\theta}\right)\\
&- (2-\vartheta)\frac{4\pi r^3\rho_m}{m}\frac{\chi\rho}{r}\frac{\partial r'}{\partial\theta}\left(\frac{\partial\rho'}{\partial s} + \frac{\partial\chi'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right).
\end{aligned}$$

Substituting these expressions into equation (125b), we obtain

$$\begin{aligned}
\mathcal{M} = &-\mathcal{B}^{17}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&+\mathcal{B}^{10}\left(\frac{\partial\rho'}{\partial s} + \frac{\partial\chi'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&-\mathcal{B}^{11}\frac{\partial r'}{\partial\theta}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&+\mathcal{B}^{12}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\gamma''}{\partial\theta}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&+\mathcal{B}^{13}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&-\mathcal{B}^{14}\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right)\frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}\\
&+\mathcal{B}^{15}\frac{\cot\theta}{2}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\vartheta''}{\partial\theta}\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1},
\end{aligned} \tag{165}$$

where

$$\begin{aligned}
\mathcal{B}^{17} = \mathcal{B}^4\Big\{&\gamma - 1 - \tfrac{1}{2}\cot\theta[(2-\gamma)(2-\vartheta)]^{1/2}\\
&+ (\gamma+\vartheta-3)\tfrac{1}{2}\cot^2\theta\Big\}.
\end{aligned} \tag{166}$$

Realistic magnetic fields in the stellar interior should satisfy the Maxwell equations. One of them is the divergence-free condition specified by equation (134). Using the coordinate (m, θ), this equation reads

$$\frac{4\pi r\rho_m\sin\theta}{m}\frac{\partial(r^2B_r)}{\partial s} + \frac{\partial(\sin\theta B_\theta)}{\partial\theta} = O(2). \tag{167}$$

Assuming $B_r = C(m)\cos\theta/r^2$, by solving this equation for $B_\theta(r,\theta)$ we obtain

$$B_\theta(r,\theta) = -\frac{2\pi r\rho_m}{m}\frac{dC(s)}{ds}\sin\theta. \tag{168}$$

So far, we have finished the coordinate transformation from (r, θ) to (m, θ). This allows us to use the analytical formulae, for instance, Θ and $\mathcal{M}$, to describe the two-dimensional effects. This effort has at least the following rewards:

1. We can control the approximations by neglecting certain terms.
2. We can understand whether a certain factor or factors play an important role by including or excluding the corresponding term or terms in the numerical calculations.
3. We can use the existing technique to numerically solve the two-dimensional stellar structure equations.
4. We can use the analytical expressions to calculate the matrix element coefficients for the linearization correction equations.

We make use of these advantages below.

5. BOUNDARY CONDITIONS

As usual in mathematical physics, the boundary conditions constitute a serious part of the whole problem, and their influence on the solutions is not easy to foresee. In the one-dimensional stellar model calculations, the boundary conditions cannot be specified at one end of the interval $[0, M_{\rm tot}]$ only, but rather are split into some that are given at the center and some near the surface of the star. The central conditions are simple, whereas the surface conditions involve observable quantities. The boundaries in the angular direction are located at $\theta = 0$ and either $\theta = \pi/2$ or $\theta = \pi$. We follow Deupree (1990) in using symmetry conditions to determine them. Otherwise, the treatment of the boundary conditions is as described in Prather (1976, his Appendix A) and as implemented in YREC (Pinsonneault 1988).

5.1. *Central Conditions*

Two boundary conditions can be specified for the center, defined by

$$m = 0: \quad r = 0, \quad L = 0. \tag{169}$$

Rewriting equation (124a) as

$$dr^3 = \frac{3}{4\pi\rho_m}\, dm, \tag{170}$$

we can integrate it over a small mass interval $[0, m]$ in which $\rho_m = \rho_{mc}$ can be considered to be constant. The result

$$r = \left(\frac{3}{4\pi\rho_{mc}}\right)^{1/3} m^{1/3} \tag{171}$$

can be considered to be the first term in a series expansion of r around $m = 0$. Taking the logarithm, we obtain

$$r' = \tfrac{1}{3}[s - \log(4\pi\rho_m/3)]. \tag{172}$$

A corresponding integration of equation (124d) yields

$$L' = \frac{m}{L_\odot}\left(\epsilon - T\frac{dS_T}{dt}\right)\frac{\rho}{\rho_m} - \frac{m}{L_\odot}\frac{F_\theta}{r\rho_m}\cot\theta. \tag{173}$$

In both cases we have used the proper boundary conditions (eq. [169]) by taking the lower limit of integration to be zero.

Equations (172) and (173) are two central boundary conditions that are equivalent to equation (169).

5.2. *Surface Boundary Conditions*

5.2.1. *One-dimensional Surface Boundary Conditions*

Nothing is a priori known about the central values of pressure P_c and temperature T_c, so we need to define the surface and specify the surface values of pressure and temperature.

In principle, we can use a definition for the surface such as

$$m = M_{\rm tot}. \tag{174}$$

However, since near the surface m does not change much, this definition is not accurate enough. The theory of stellar atmospheres suggests the use of the photosphere, from which the bulk of the radiation is emitted into space:

$$T = T_{\rm eff}, \tag{175}$$

where $T_{\rm eff}$ is the effective temperature. The optical depth τ_s of the overlying layers,

$$\tau = \int_R^\infty \kappa\rho\, dr,$$

is equal to 2/3 for the Eddington approximation,

$$T^4 = \tfrac{3}{4}T_{\rm eff}^4\left(\tau + \tfrac{2}{3}\right), \tag{176}$$

where R is the total stellar radius. In contrast, the optical depth $\tau_s = 0.312155$ of the overlying layers is different from 2/3 if the atmosphere is assumed to obey a scaled solar $T(\tau)$ relation given by Krishna Swamy (1966),

$$T^4(\tau) = \tfrac{3}{4}T_{\rm eff}^4[\tau + 1.39 - 0.815\exp(-2.54\tau) - 0.025\exp(-30.0\tau)]. \tag{177}$$

Since $T_{\rm eff}$ is the temperature of that blackbody that yields the same surface flux of energy as the star, then

$$m = M_{\rm tot}: \quad L_s = 4\pi R^2\sigma T_{\rm eff}^4, \tag{178}$$

where $\sigma = ac/4$ is the Stefan-Boltzmann constant of radiation and L_s is the total luminosity. This is one of two surface boundary conditions.

The second surface boundary condition is the hydrostatic equilibrium condition: the pressure at the surface is given by the weight of the matter above. We can well approximate the gravitational acceleration by the constant value $g_0 = GM_{\rm tot}/R^2$, since the bulk of the matter above the surface is very close to the photosphere anyway. We hence have

$$m = M_{\rm tot}: \quad P_s = \int_R^\infty g\rho\, dr = \frac{GM_{\rm tot}}{R^2}I, \tag{179}$$

where the integration

$$I = \int_0^{\tau_s}\frac{1}{\kappa}\, d\tau$$

is calculated in the following way: The starting values of (P_0, τ_0) are chosen by selecting a small density ρ_0 and then computing

$$P_0 = (a/3)T_0^4 + \rho_0\mathcal{R}T_0,$$

where $T_0 \equiv T(\tau = 0)$. Then (P_0, τ_0) gives ρ_1, which gives $\kappa_0(\rho_1, T_0)$, which gives $\tau_1 = \kappa_0 P_0/g$ or $\delta\tau = \tau_1 - \tau_0$. Thus, we have $I_0 = \delta\tau/\kappa_0$. Then we redefine $\tau_0 = \tau_1$ and $\rho_0 = \rho_1$. This method could be iterated upon by redefining $T_0 = T(\tau_0)$ and so forth:

$$I = I_0 + \frac{1}{2}\left(\frac{1}{\kappa_0} + \frac{1}{\kappa_1}\right)\delta\tau + \ldots.$$

Sufficient accuracy was achieved in the atmosphere integration by choosing a small enough ρ_0 (e.g., $\rho_0 = 10^{-10}$) such that $\tau < 10^{-4}$.

From the calculation description, we can see that $I = I(P_s, T_{\rm eff})$. However, we do not know the explicit expression. Therefore, we cannot directly use equations (178) and (179) as our surface boundary conditions. Instead, we solve the system (Kippenhahn 1967)

$$\begin{pmatrix} P'_1 & T'_1 & 1 \\ P'_2 & T'_2 & 1 \\ P'_3 & T'_3 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{pmatrix} = \begin{pmatrix} R'_1 & \ln L_1 & T'_{\rm eff_1} \\ R'_2 & \ln L_2 & T'_{\rm eff_2} \\ R'_3 & \ln L_3 & T'_{\rm eff_3} \end{pmatrix} \tag{180}$$

for the a_i that are used for the surface boundary conditions,

$$R' = a_1 P' + a_2 T' + a_3, \tag{181a}$$
$$\ln L = a_4 P' + a_5 T' + a_6, \tag{181b}$$

and for the calculation of the effective temperature,

$$T'_{\rm eff} = a_7 P' + a_8 T' + a_9. \tag{182}$$

Here, the (P', T') refer to the values at the outermost mass point in the model. The last three equations can be considered to be the first term in the series expansions of equations (178) and (179).

The initial model with an estimated $(\ln L^*, T'^*_{\rm eff})$ is triangulated in the $(\ln L, T'_{\rm eff})$-plane by constructing three atmospheres of the form

$$\begin{aligned} &\text{A1}, \quad \left(\ln L^* - \tfrac{1}{2}\Delta_L, T'^*_{\rm eff} + \tfrac{1}{2}\Delta_T\right); \\ &\text{A2}, \quad \left(\ln L^* - \tfrac{1}{2}\Delta_L, T'^*_{\rm eff} - \tfrac{1}{2}\Delta_T\right); \\ &\text{A3}, \quad \left(\ln L^* + \tfrac{1}{2}\Delta_L, T'^*_{\rm eff}\right). \end{aligned}$$

If subsequent models or the model itself during convergence move significantly out of the triangle, the triangle is flipped until it once again constrains the model. The decision as to which point of the triangle should be flipped (if any) can be made by testing

$$\begin{aligned} c_i = f\Big[&(\ln L_{i+1} - \ln L_{i+2})\left(T'_{\rm eff} - T'_{\rm eff_{i+1}}\right) \\ &+ \left(T'_{\rm eff_{i+2}} - T'_{\rm eff_{i+1}}\right)(\ln L - \ln L_{i+1})\Big], \end{aligned}$$

where $f = \pm 1$ is the orientation of the triangle (e.g., in the example given, $f = +1$) and $\{i, i+1, i+2\}$ is $\{123\}$, $\{231\}$, or $\{312\}$. The value of c_i is tested against $\epsilon \Delta_L \Delta_T$, where setting $\epsilon = 0$ gives exact triangulation and $\epsilon > 0$ allows the point $(\ln L, T'_{\rm eff})$ to be at most ϵ outside of a triangle. We begin testing with $i = 1$–3; if $c_i < -\epsilon \Delta_L \Delta_T$, then we flip point i,

$$\begin{aligned} \ln L_i &\Leftarrow \ln L_{i+1} + \ln L_{i+2} - \ln L_i, \\ T'_{\rm eff_i} &\Leftarrow T'_{\rm eff_{i+1}} + T'_{\rm eff_{i+2}} - T'_{\rm eff_i}, \\ f &\Leftarrow -f, \end{aligned}$$

and repeat the testing again starting with $i = 1$ until c_i passes for $i = 1$–3. The atmospheres that have been flipped are then recomputed, as are all the coefficients a_i.

This treatment of the surface boundary conditions is the same as that in one-dimensional model calculation, except that we move the fitting point to the surface where $T = T_{\rm eff}$. Therefore, we do not need an envelope integration. This has been tested for the one-dimensional model calculations, and it turns out to be acceptable. This saves much computation time in the two-dimensional case.

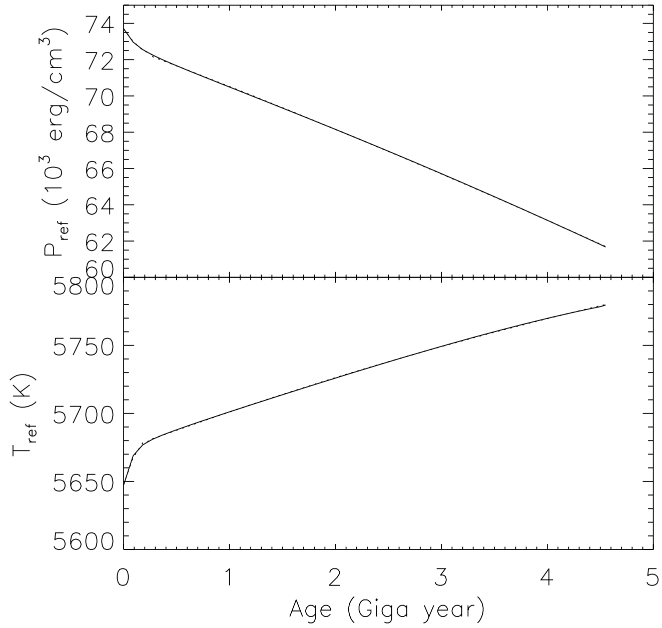

Fig. 1.—Reference values of pressure and temperature at the surface as functions of age. The dotted lines are polynomial fits to the calculated model (*solid lines*) using the standard surface boundary condition.

Our surface boundary conditions are much more complicated than Deupree's (1990), because our applications to the Sun are very sensitive to the surface conditions.

5.2.2. *Deupree's Two-dimensional Surface Boundary Conditions*

In his two-dimensional rotational models, Deupree (1990) uses the surface boundary conditions

$$\rho = \rho_{\rm ref}, \quad T = T_{\rm ref},$$

where $\rho_{\rm ref}$ and $T_{\rm ref}$ are the reference density and temperature, respectively. The most difficult part of using these surface boundary conditions is how to select the reference density and temperature at the surface.

Unlike Deupree (1990), we use P_T and T as independent thermodynamical variables. Since $\rho = \rho(P_T, T)$, the equivalent surface boundary condition is

$$P_T = P_{\rm ref}, \quad T = T_{\rm ref}. \tag{183}$$

In order to compare this with the standard surface boundary condition given above, we use the surface values of P_T and T obtained by using the standard surface boundary condition for the current Sun as the reference. Figure 1 shows the reference values as functions of age and their polynomial fits. The fitting formulae are

$$P_{\rm ref} = \begin{cases} 73695.514 - 9004.5498t & \\ \quad +13898.511t^2, & 0 \le t \le 0.27, \\ 72777.060 - 2211.7088t & \\ \quad -49.075155t^2, & 0.27 \le t \le 4.55, \end{cases} \tag{184a}$$

$$T_{\rm ref} = \begin{cases} 5647.8836 + 266.07365t & \\ \quad -539.35360t^2, & 0 \le t \le 0.27, \\ 5673.6126 + 28.625469t & \\ \quad -1.1516435t^2, & 0.27 \le t \le 4.55, \end{cases} \tag{184b}$$

The age t is in gigayears.

5.3. *Polar Boundary Conditions*

Equations (124b)–(124e) are singular at the poles ($\theta = 0$ and π) because of Θ, $\mathcal{M}$, $\mathcal{F}_\theta$, and u_θ. However, if $\partial r'/\partial\theta = 0$, the singularity due to Θ will disappear. In order to guarantee $\partial r'/\partial\theta = 0$, we also need to zero the other derivatives. Therefore, we require

$$\frac{\partial r'}{\partial\theta} = \frac{\partial L'}{\partial\theta} = \frac{\partial P'}{\partial\theta} = \frac{\partial T'}{\partial\theta} = 0 \tag{185}$$

at the poles. In order to remove the singularity due to $\mathcal{M}$, we have to zero χ at the poles, namely,

$$\chi = 0 \tag{186}$$

at the poles. Equations (185) and (186) are the polar boundary conditions. Equation (185) is similar to Deupree's (1990) polar boundary conditions, which are the symmetry conditions.

5.4. *Equatorial Conditions*

Equations (124a)–(124d) show that the two-dimensional stellar structure equations are not singular at the equator. Therefore, there are no special constraints there. If we neglect $O(2)$ in equations (124b) and (124c), the two-dimensional stellar structure equations are a set of first-order differential equations. Since we have already specified four boundary conditions at the north pole ($\theta = 0$), we do not need extra boundary conditions at the equator. If we want to include those terms that contain the second-order derivatives in $O(2)$, we have to specify four equatorial boundary conditions or five polar boundary conditions at the south pole ($\theta = \pi$). We do not include those second-order derivatives in $O(2)$ in this paper for the following reasons:

1. They are much smaller corrections than the retained terms.
2. They may cause a much bigger numerical error than the actual corrections.
3. They require a totally different method of solution (e.g., Deupree 1990).

6. METHOD OF SOLUTION

6.1. *Linearization of the Two-dimensional Stellar Structure Equations*

The dependent variables to be solved for are pressure P_T, temperature T, radius r, and luminosity L (hereafter we use L to replace L', but remember that L is in solar units); the independent variables are chosen to be mass m (or $s = \ln m$) and angular coordinate θ. The magnetic field variables χ, ϑ, and γ are also dependent variables. However, since we do not introduce their governing equations (such as the dynamo equations), we consider them to be given. All units are in cgs, except for the luminosity, which is in solar units.

The construction of a two-dimensional stellar model begins by dividing the star into M mass shells and N angular zones. The mass shells are assigned a value $s_i = \log m_i$, where m_i is the interior mass at the midpoint of shell i. The angular zones are assigned a value θ_j. A starting (or previous in evolutionary time) model is supplied with a run of (P'_{ij}, T'_{ij}, r'_{ij}, L_{ij}, χ'_{ij}, ϑ_{ij}, γ_{ij}) for $i = 1$–M and $j = 1$–N.

Here we take the general case $\boldsymbol{B} = (B_r, B_\theta, B_\phi)$ as the example to show how to solve the two-dimensional stellar structure equations. In order to write down the linearization equations, we introduce the notations

$$\mathcal{P} \equiv -\frac{Gm^2}{4\pi r^4 P_T}\frac{\rho}{\rho_m}, \tag{187a}$$

$$\mathcal{R} \equiv \frac{m}{4\pi r^3\rho}\frac{\rho}{\rho_m}, \tag{187b}$$

$$\mathcal{T} \equiv \mathcal{P}\nabla, \tag{187c}$$

$$\mathcal{L} \equiv \frac{1}{L_\odot} m\left(\epsilon - T\frac{dS_T}{dt}\right)\frac{\rho}{\rho_m}, \tag{187d}$$

$$\mathcal{T}^\ell \equiv \mathcal{B}^i\nabla, \quad \ell = 1, 2, 3, 10, \ldots, 15, 17, \tag{187e}$$

$$\mathcal{D}^1 \equiv \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \tag{187f}$$

$$\mathcal{D}^2 \equiv \left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \tag{187g}$$

$$\mathcal{D}^3 \equiv \frac{\cot\theta}{2}\frac{\partial P'}{\partial\theta}\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1}, \tag{187h}$$

$$\mathcal{D}^{10} \equiv \mathcal{D}^2\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\gamma''}{\partial s}\right), \tag{187i}$$

$$\mathcal{D}^{11} \equiv -\mathcal{D}^{10}\frac{\partial r'}{\partial\theta}, \tag{187j}$$

$$\mathcal{D}^{12} \equiv \mathcal{D}^2\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\gamma''}{\partial\theta}\right), \tag{187k}$$

$$\mathcal{D}^{13} \equiv \mathcal{D}^2\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right), \tag{187l}$$

$$\mathcal{D}^{14} \equiv -\mathcal{D}^1\left(\frac{\partial\chi'}{\partial s} + \frac{\partial\rho'}{\partial s} + \frac{\partial\vartheta''}{\partial s}\right), \tag{187m}$$

$$\mathcal{D}^{15} \equiv \mathcal{D}^2\frac{\cot\theta}{2}\left(\frac{\partial\chi'}{\partial\theta} + \frac{\partial\rho'}{\partial\theta} + \frac{\partial\vartheta''}{\partial\theta}\right), \tag{187n}$$

$$\mathcal{D}^{17} \equiv -\mathcal{D}^2, \tag{187o}$$

$$\mathcal{F}^1 \equiv \frac{4ac}{3L_\odot}\frac{mT^4}{r^2\kappa\rho\rho_m}(1+\lambda), \tag{187p}$$

$$\mathcal{F}^2 \equiv \frac{1}{2}\frac{1}{L_\odot}\frac{m}{r^2\rho_m}\frac{\rho C_p T l_m v_{\rm conv}}{1 + v_{\rm conv}/v_0}, \tag{187q}$$

$$\mathcal{F}^3 \equiv -\mathcal{F}^2\nabla'_{\rm ad}, \tag{187r}$$

$$\mathcal{F}^4 \equiv \mathcal{F}^1\frac{Gm\rho\nabla}{rP_T}, \tag{187s}$$

$$\mathcal{F}^5 \equiv \mathcal{F}^2\frac{Gm\rho\nabla}{rP_T}, \tag{187t}$$

$$\mathcal{F}^6 \equiv \mathcal{F}^3\frac{Gm\rho}{rP_T}. \tag{187u}$$

Consequently, the stellar structure equations in the general case can be rewritten as

$$\frac{\partial P'}{\partial s} = \mathcal{P} + \sum_{i=1,2,3,10}^{15,17}\mathcal{B}^i\mathcal{D}^i + O(2), \tag{188a}$$

$$\frac{\partial T'}{\partial s} = \mathcal{T} + \sum_{i=1,2,3,10}^{15,17}\mathcal{T}^i\mathcal{D}^i + O(2), \tag{188b}$$

$$\frac{\partial r'}{\partial s} = \mathcal{R}, \tag{188c}$$

$$\frac{\partial L}{\partial s} = \mathcal{L} + \left(\sum_{\ell=1}^{2}\mathcal{F}^\ell\frac{\partial T'}{\partial\theta} + \mathcal{F}^3\frac{\partial P'}{\partial\theta} + \sum_{\ell=4}^{6}\mathcal{F}^\ell\frac{\partial r'}{\partial\theta}\right)\cot\theta + O(2), \tag{188d}$$

where $\nabla = \nabla_{\rm rad}$ in the radiative zone and $\nabla = \nabla_c$ in the convective zone.

We calculate the derivatives of the dependent variables with respect to s by the central difference scheme, e.g.,

$$\frac{P'_{ij} - P'_{i-1j}}{s_i - s_{i-1}} = \frac{1}{2}\left[\left(\frac{\partial P'}{\partial s}\right)_{ij} + \left(\frac{\partial P'}{\partial s}\right)_{i-1j}\right], \quad (189)$$

but we simply use the difference scheme

$$\frac{r'_{ij} - r'_{ij-1}}{\theta_j - \theta_{j-1}} = \left(\frac{\partial r'}{\partial \theta}\right)_{ij} \quad (190)$$

to calculate the derivatives with respect to θ, because the first two of the two-dimensional stellar structure equations are singular at the poles. Thus, we can define a set of functions that should vanish at the solution of the stellar structure equations,

$$F_P^{ij} \equiv (P'_{ij} - P'_{i-1j}) - \frac{1}{2}\Delta s_i\left[(\mathcal{P}_{ij} + \mathcal{P}_{i-1j}) + \sum_{\ell=1,2,3,10}^{15,17}(\mathcal{B}^\ell_{ij} + \mathcal{B}^\ell_{i-1j})\mathcal{D}^\ell\right], \quad (191a)$$

$$F_T^{ij} \equiv (T'_{ij} - T'_{i-1j}) - \frac{1}{2}\Delta s_i\left[(\mathcal{T}_{ij} + \mathcal{T}_{i-1j}) + \sum_{\ell=1,2,3,10}^{15,17}(\mathcal{T}^\ell_{ij} + \mathcal{T}^\ell_{i-1j})\mathcal{D}^\ell\right], \quad (191b)$$

$$F_R^{ij} \equiv (r'_{ij} - r'_{i-1j}) - \frac{1}{2}\Delta s_i(\mathcal{R}_{ij} + \mathcal{R}_{i-1j}), \quad (191c)$$

$$F_L^{ij} \equiv (L_{ij} - L_{i-1j}) - \frac{1}{2}\Delta s_i\left[(\mathcal{L}_{ij} + \mathcal{L}_{i-1j}) + \sum_{\ell=1}^{2}(\mathcal{F}^\ell_{ij} + \mathcal{F}^\ell_{i-1j})\frac{\cot\theta_j}{\Delta\theta_j}(T'_{ij} - T'_{ij-1}) + (\mathcal{F}^3_{ij} + \mathcal{F}^3_{i-1j})\frac{\cot\theta_j}{\Delta\theta_j}(P'_{ij} - P'_{ij-1}) + \sum_{\ell=4}^{6}(\mathcal{F}^\ell_{ij} + \mathcal{F}^\ell_{i-1j})\frac{\cot\theta_j}{\Delta\theta_j}(r'_{ij} - r'_{ij-1})\right], \quad (191d)$$

where $\Delta s_i \equiv (s_i - s_{i-1})$ and $i = 2$–M, $j = 2$–N. The $\mathcal{D}^1_{ij}$, $\mathcal{D}^{10}_{ij}, \ldots, \mathcal{D}^{14}_{ij}$ are defined as

$$\mathcal{D}^1 = \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-1}, \quad (192a)$$

$$\mathcal{D}^2 = \left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-1}, \quad (192b)$$

$$\mathcal{D}^3 = \frac{\cot\theta_j}{2\Delta\theta_j}\left(P'_{ij} - P'_{ij-1}\right)\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-1}, \quad (192c)$$

$$\mathcal{D}^{10} = \frac{\mathcal{D}^2}{\Delta s_i}\left[\left(\chi'_{ij} - \chi'_{i-1j}\right) + \left(\rho'_{ij} - \rho'_{i-1j}\right) + \left(\gamma''_{ij} - \gamma''_{i-1j}\right)\right], \quad (192d)$$

$$\mathcal{D}^{11} = -\frac{\mathcal{D}^{10}}{\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right), \quad (192e)$$

$$\mathcal{D}^{12} = \frac{\mathcal{D}^2}{\Delta\theta_j}\left[\left(\chi'_{ij} - \chi'_{ij-1}\right) + \left(\rho'_{ij} - \rho'_{ij-1}\right) + \left(\gamma''_{ij} - \gamma''_{ij-1}\right)\right], \quad (192f)$$

$$\mathcal{D}^{13} = \frac{\mathcal{D}^2}{\Delta s_i}\left[\left(\chi'_{ij} - \chi'_{i-1j}\right) + \left(\rho'_{ij} - \rho'_{i-1j}\right) + \left(\vartheta''_{ij} - \vartheta''_{i-1j}\right)\right], \quad (192g)$$

$$\mathcal{D}^{14} = -\frac{\mathcal{D}^1}{\Delta s_i}\left[\left(\chi'_{ij} - \chi'_{i-1j}\right) + \left(\rho'_{ij} - \rho'_{i-1j}\right) + \left(\vartheta''_{ij} - \vartheta''_{i-1j}\right)\right], \quad (192h)$$

$$\mathcal{D}^{15} = \mathcal{D}^2\frac{\cot\theta_j}{2\Delta\theta_j}\left[\left(\chi'_{ij} - \chi'_{ij-1}\right) + \left(\rho'_{ij} - \rho'_{ij-1}\right) + \left(\vartheta''_{ij} - \vartheta''_{ij-1}\right)\right], \quad (192i)$$

$$\mathcal{D}^{17} = -\mathcal{D}^2. \quad (192j)$$

We want then to solve for the set of (P'_{ij}, T'_{ij}, r'_{ij}, L_{ij}) such that $F_P^{ij} = F_T^{ij} = F_R^{ij} = F_L^{ij} = 0$ with χ', ϑ, and γ specified.

The linearization of equations (191a)–(191d) with respect to ($\delta P'_{ij}$, $\delta T'_{ij}$, $\delta r'_{ij}$, δL_{ij}) yields $4MN - 4(N-1) - 4M$ equations for the $4MN$ unknowns. The $2(N-1)$ additional equations are supplied by the boundary conditions at the center. From equations (172) and (173), we can define

$$F_R^{1j} \equiv r'_{1j} - \tfrac{1}{3}[s_1 - \log(4\pi\rho_{m1j}/3)], \quad (193a)$$

$$F_L^{1j} \equiv L_{1j} - \mathcal{L}_{1j} - \frac{\cot\theta_j}{\Delta\theta_j}\left[\sum_{\ell=1}^{2}\mathcal{F}^\ell_{1j}(T'_{1,j} - T'_{1j-1}) + \mathcal{F}^3_{1j}(P'_{1j} - P'_{1j-1}) + \sum_{\ell=4}^{6}\mathcal{F}^\ell_{1,j}(r'_{1j} - r'_{1j-1})\right], \quad (193b)$$

where $j = 2$–N. Another $2(N-1)$ additional equations are supplied by the boundary conditions at the surface. From equations (181a) and (181b), we can define

$$F_R^{M+1j} \equiv R'_{Mj} - a_1P'_{Mj} - a_2T'_{Mj} - a_3, \quad (194a)$$

$$F_L^{M+1j} \equiv L'_{Mj}\left(\ln L_{Mj} - a_4P'_{Mj} - a_5T'_{Mj} - a_6\right), \quad (194b)$$

where $j = 2$–N. The $4M$ additional equations are supplied by the polar boundary conditions,

$$F_P^{i1} \equiv P'_{i1} - P'_{i2}, \quad (195a)$$

$$F_T^{i1} \equiv T'_{i1} - T'_{i2}, \quad (195b)$$

$$F_R^{i1} \equiv R'_{i1} - R'_{i2}, \quad (195c)$$

$$F_L^{i1} \equiv L_{i1} - L_{i2}, \quad (195d)$$

where $i = 1$–M. The F equations are linearized,

$$\sum_{l=1}^{M}\sum_{k=1}^{N}\left(\frac{\partial F_w^{ij}}{\partial R'_{lk}}\delta R'_{lk} + \frac{\partial F_w^{ij}}{\partial L_{lk}}\delta L_{lk} + \frac{\partial F_w^{ij}}{\partial P'_{lk}}\delta P'_{lk} + \frac{\partial F_w^{ij}}{\partial T'_{lk}}\delta T'_{lk}\right) = -F_w^{ij}, \quad (196)$$

where $w = P$, T, R, and L; $i = 1$–M; and $j = 1$–N. The summation over l has nonzero terms only for $l = i-1$, i; the summation over k has nonzero terms only for $k = j-1$, j. See Appendix A for the matrix coefficients.

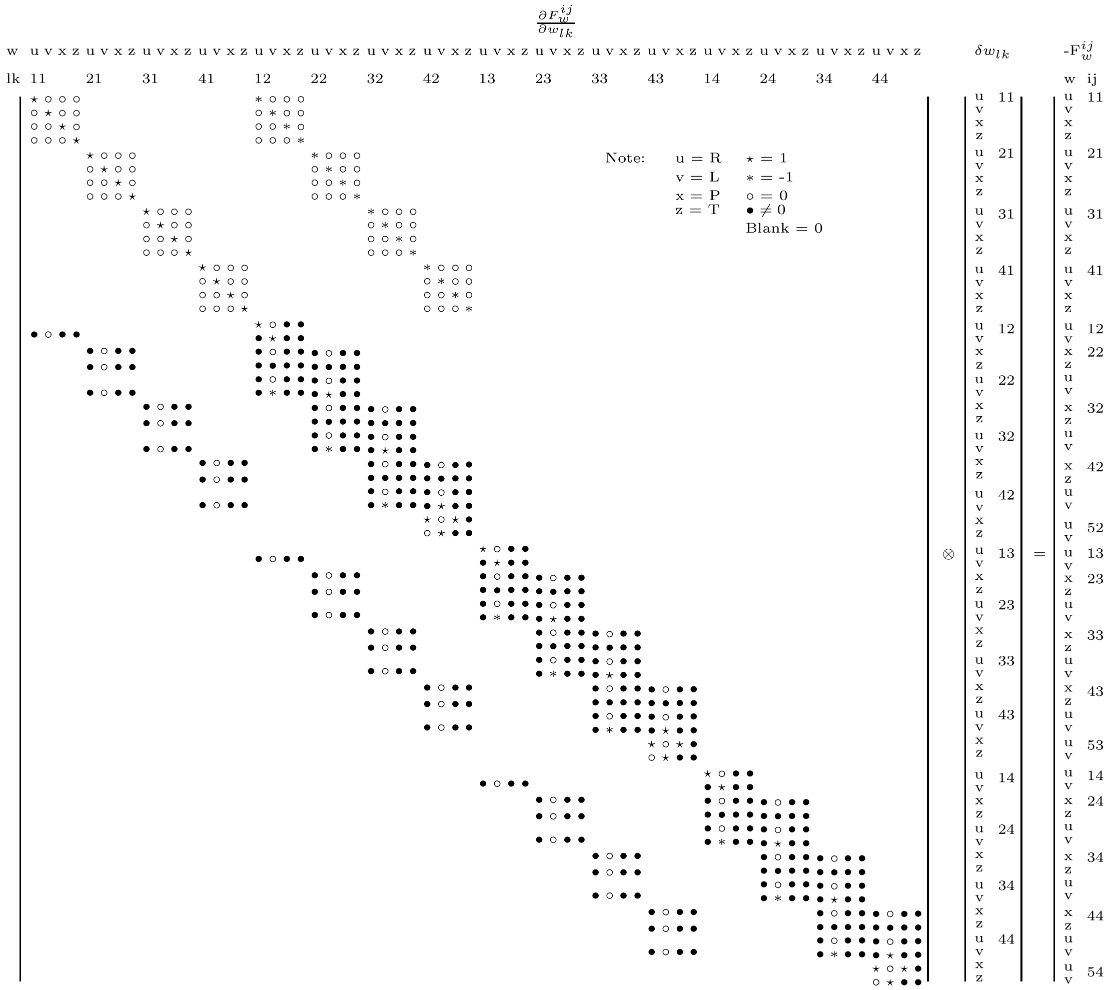

FIG. 2.—Linearization equation for a 4 × 4 point star.

6.2. *Solution of the Linearized Equations*

Rather than solving the $(4MN)^2$ system of equations directly, we take advantage of the specific form of the equations and especially of the large number of zero elements in the matrix. From Figure 2 we can see that only 12 by $4MN$ elements are nonzero at most.

The matrix is reduced in a forward direction ($i = 2$–M) as the coefficients are defined and is then solved in the backward direction ($i = M$–1) for the corrections ($\delta P'_{ij}$, $\delta T'_{ij}$, $\delta R'_{ij}$, δL_{ij}) for $j =$ 2–N. The reduction procedure begins as follows: (1) For $j = 2$, we use the polar conditions at $\theta = 0$ to eliminate those elements with subscripts $l = i$, $k = j - 1$ (i.e., block III defined in Appendix A), which can be done by simply adding block III to block II. At the end of this step, the matrix equation for a specified j looks like Figure 3*a* for a four-point star in the mass coordinate (including the center and surface boundaries). (2) We use the central boundary conditions to eliminate the first two columns in block I for $i = 2$. (3) We continue diagonalizing the four bottom rows for $i = 2$. (4) We store the right-hand side and the elements in the rightmost columns (see Fig. 3*b*). After this reduction is completed, the bottom two rows of the first part of the coefficient matrix become the "central boundary equations" for the F equations of the next pair of mass points. The method is repeatedly applied until the surface is reached, whereupon the surface boundary conditions complete the set of $4M$ equations (see Figs. 3*b* and 3*c*). For the back solution (1) the values of ($\delta P'_{M2}$, $\delta T'_{M2}$) are first calculated, then (2) the values of ($\delta R'_{i2}$, δL_{i2}, $\delta P'_{i-12}$, $\delta T'_{i-12}$) for $i = M$–2 are calculated using the stored elements of the array and ($\delta P'_{i2}$, $\delta T'_{i2}$), and finally (3) the values of ($\delta R'_{12}$, δL_{12}) are computed from the central boundary conditions and the values of ($\delta P'_{12}$, $\delta T'_{12}$) (see Figs. 3*d*–3*f*). Since the submatrix with $j = 2$ has been diagonalized, we can use it to diagonalize the submatrix with $j = 1$ and 3. For $j = 3$, we use $j = 2$ as the "polar boundary conditions," and so forth. Finally, we solve the matrix equation, and the results are stored in the right column.

6.3. *Advancing the Model*

These routines are based on the work of Prather (1976, his Appendix A) and their revised implementations in YREC (Pinsonneault 1988; Guenther et al. 1992; Guenther & Demarque 1997).

6.4. *Time Steps*

In this section we use the cgs units for luminosity L (ergs s^{-1}) and use $X(Y)$ to represent the mass fraction of hydrogen (helium). The angular zone index (i.e., j) is 2.

The timing routine calculates the time steps based on a hydrogen- or helium-burning source. Let $L_{\rm H}$ (ergs s^{-1}) be the total hydrogen-burning luminosity, and $L_{\rm He}$, the helium luminosity. There are two time steps,

$\Delta t_{\rm H}$, the hydrogen-burning time step, and
Δt_t, the total time step (i.e., for entropy and helium),

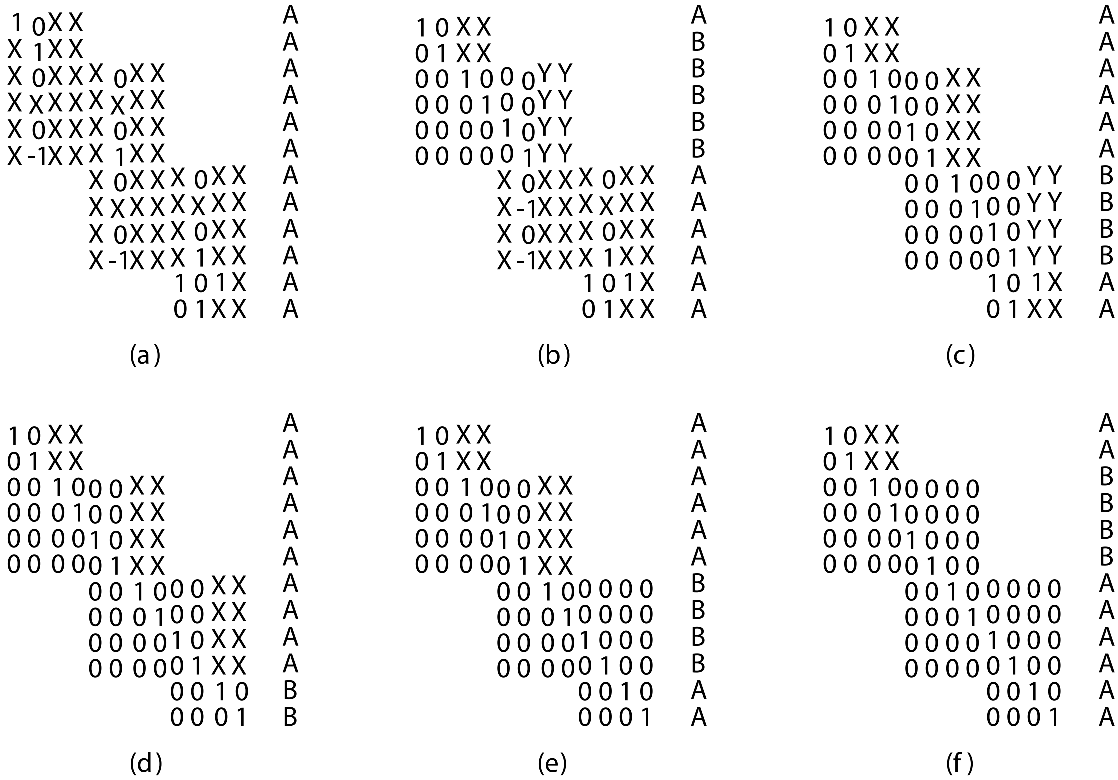

FIG. 3.—Schematic Henyey solution for a four-point star. The matrix block is denoted by 0s, 1s, and X's, which are nonzero. The right-hand side is denoted by A, and the elements changed through pivoting, by Y and B. The final reduction to the identity matrix is not shown.

where $\Delta t_t \neq \Delta t_{\rm H}$ only if the hydrogen shell is being shifted outward. If $L_{\rm H} = 0$, the following section for hydrogen burning is skipped.

For hydrogen-core burning ($X_{\rm core} > X_c^{\rm min}$), a time step corresponding to a set of reduction in $X_{\rm core}$ is calculated. Let i be the innermost point if the core is radiative ($i = 1$) or the outermost convective point if the core is convective. Then, the change in $X_{\rm core}$ is computed,

$$\Delta X_{\rm core} = \min\{\Delta X_{sc}^{\rm max}, \Delta f_X^{\rm max} X^i\},$$

and the time step is

$$\Delta t_i = \Delta t_{\rm H} = 6 \times 10^{18} \Delta X_{\rm core} m^i / L^i,$$

where m^i is the mass of the core (g) and L^i is the luminosity of the core (ergs s^{-1}, assumed to be mainly hydrogen burning).

When the core-burning criterion no longer applies ($X_{\rm core} < X_c^{\rm min}$), a limit is placed on the total amount of mass that may be burned,

$$\Delta m = \Delta f_m M_\odot X_{\rm env},$$
$$\Delta t_{\rm H}^m = 6 \times 10^{18} \Delta m / L_{\rm H}.$$

If there is a hydrogen-burning shell ($X_{\rm core} = 0$), the timing routine locates it. Let the subscript 0 denote the inner edge of the shell (first point where $X > 0$), let subscript 1/2 denote the midpoint of the shell ($X = \frac{1}{2} X_{\rm env}$); and let subscript 1 denote the end of the shell ($L^i - L^{i-1} < 10^{-4} L$ or $X = X_{\rm env}$ or $\epsilon_{\rm H} = 0$). There is a limitation set on the maximum depletion at the midpoint of the shell,

$$\Delta t_{\rm H}^{1/2} = \Delta X_{1/2}.$$

With the exception of the core-burning phase, the new hydrogen burning time step is limited by the previous total time step,

$$\Delta t_{\rm H}({\rm new}) = \min\left\{1.5 \Delta t_t({\rm old}), \Delta t_{\rm H}^m, \Delta t_{\rm H}^{1/2}\right\}.$$

If there is to be no shell shifting, then one sets $\Delta t_t = \Delta t_{\rm H}$. If the hydrogen shell is to be shifted outward through Δm_s in mass, then the shift time step is computed as

$$\Delta t_{\rm shift} = 6 \times 10^{18} X_1 \Delta m_s / L_{\rm H},$$

and the total time step is

$$\Delta t_t = \Delta t_{\rm H} + \Delta t_{\rm shift}.$$

If there is a hydrogen shell ($X_{\rm core} = 0$), the helium burning is examined. For helium-core burning ($Y_{\rm core} > X_c^{\rm min}$ and $L_{\rm core} > 0.1\, L_\odot$), the maximum helium depletion is

$$\Delta Y_{\rm core} = \min\{\Delta Y_c^{\rm min}, \Delta f_Y^{\rm max} Y_{\rm core}\},$$

and the helium time step is

$$\Delta t_{\rm He} = 5.85 \times 10^{17} \Delta Y_{\rm core} M_\odot / L_{\rm core}.$$

For helium-shell burning ($Y_{\rm core} < X_c^{\rm min}$), the amount of mass burned through by the helium shell is limited,

$$\Delta t_{\rm He} = 5.85 \times 10^{17} \Delta f_{sm} M_\odot / L_{\rm He}.$$

The helium time step places an upper limit on the previously computed hydrogen time step,

$$\Delta t_t = \min\{\Delta t_t, \Delta t_{\rm He}\},$$
$$\Delta t_{\rm H} = \min\{\Delta t_t, \Delta t_{\rm H}\}.$$

The following parameters used in the determination of the time step are read in at the beginning of each model run. Their typical values are given as

$$X_c^{\min} = 0.001, \quad \Delta X_c^{\max} = 0.04, \quad \Delta Y_c^{\max} = 0.02,$$
$$\Delta f_X^{\max} = 0.5, \quad \Delta f_Y^{\max} = 0.3, \quad \Delta f_m = 0.0015\ M_\odot,$$
$$\Delta X_{1/2}^{\max} = 0.10, \quad \Delta m_s = 5 \times 10^{-4}\ M_\odot.$$

Of course, we can also use a fixed time step to advance the model.

6.5. *Composition Advance*

The mixing routine performs all the operations on the model that are needed by the application of the time step to increase the age of the model. The routine first checks that there is no mixing within the hydrogen shell if the shell is supposed to be shifted. If there is such mixing, the shifting is suppressed (i.e., set $\Delta t_t = \Delta t_{\rm H}$).

Each mass element (the mass contained in the zone defined by $m \in [m_{i-1}, m_i]$ and $\theta \in [\theta_{j-1}, \theta_j]$) is burned individually by computing the energy generation rates for the physical conditions existing in that mass element from the previously converged model. Since the program stores only the values of hydrogen, total metal, and oxygen abundance, the change in these quantities is computed as

$$X(\text{new}) = X(\text{old}) - (dX/dt)\Delta t,$$
$$Z(\text{new}) = Z(\text{new}) + (dY/dt)\Delta t,$$
$$X_{16}(\text{new}) = X_{16}(\text{old}) - (dX_{\rm O}/dt)\Delta t,$$

where $\Delta t = \Delta t_t$ inside the hydrogen shell ($X = 0$) and $\Delta t = \Delta t_{\rm H}$ elsewhere.

The routine then mixes those zones that it is instructed to by being given a set of indices ($i = i_1 – i_2$ and $j = j_1 – j_2$),

$$X_{ij} = \left(\sum_{k=i_1}^{k=i_2} \sum_{l=j_1}^{l=j_2} a_{kl} X_{kl} \right) \left(\sum_{k=i_1}^{k=i_2} \sum_{l=j_1}^{l=j_2} a_{kl} \right)^{-1}.$$

The weights a_{kl} are proportional to the amount of mass associated with zone kl and are set up in the point readjustment routine.

If the hydrogen shell is to be shifted, the routine calculates

$$\Delta s_{\rm shift} = (\delta - \delta^2/2 + \delta^3/3 - \delta^4/4)/\ln 10,$$

where $\delta \equiv \Delta m_s / m_{1/2} \ll 1$. The points in the hydrogen shell are shifted by $\Delta s_{\rm shift}$,

$$s_0 \le s_i \le s_1 \rightarrow s_i(\text{new}) = s_i(\text{old}) + \Delta s_{\rm shift},$$

where $s_i = \log m_i$. The points up to a distance $f_s \Delta s_{\rm shift}$ in front of the shell are squeezed together,

$$s_1 < s_i < s_{\rm end} \rightarrow s_i(\text{new}) = [s_i(\text{old}) + s_{\rm end} - s_i(\text{old})]/f_s,$$

where $s_{\rm end} \equiv s_i + f_s \Delta s_{\rm shift}$.

For all of these shifted and squeezed points, the changes in P' and T' must be preserved for the calculation of the entropy energy term in the subsequent model. Thus, for every s_i(new), one must locate s_l(old) such that $s_l(\text{old}) \le s_i(\text{new}) < s_{l+1}(\text{old})$ and then interpolate linearly in s to get the old values of P' and T' that correspond to the new value of s. Then the effective changes are stored,

$$\Delta P'_{ij} = P'_{ij}(\text{new } s) - P'_{ij}(\text{preshift } s),$$
$$\Delta T'_{ij} = T'_{ij}(\text{new } s) - T'_{ij}(\text{preshift } s).$$

For the region in front of the shell that is squeezed, it is desirable to preserve the original composition gradient if such a gradient exists. The values of X, Z, and X_{16} are interpolated linearly in s, as are P' and T'. Note that the shifting process affects only the value of s and not the values of (P, T, R, L, X, Z, X_{16}) with the exception of (X, Z, X_{16}) in the squeezed region.

The mixing routine finally checks on the physical sense of the new composition at all of the points,

$$X = \max\{X, 0\},$$
$$Z = \min\{Z, 1 - X\},$$
$$X_{16} = \max\{X_{16}, 0.99 \times 10^{-3} Z_{\rm CNO}\}.$$

The first two requirements are obvious; the third requirement brings the value of X_{16} up to the approximate equilibrium value while turning off the X_{16} burning rate that is calculated if $X_{16} > 10^{-3} Z_{\rm CNO}$. The value of $Z_{\rm CNO} = Z - Z_m^0$, where Z_m^0 is the original weight abundance of all non-CNO metals. This method allows for the enrichment of CNO elements from the helium burning.

6.6. *Mixing Zones*

Consecutive mass shells, which are determined to be convective ($\nabla_{\rm rad} > \nabla_{\rm ad}$) in the previously converged model, are mixed together.

If there is a helium-burning convective zone, the semiconvective instability is treated as an overshooting (Castellani et al. 1971). The composition is first burned and mixed according to the standard convection zones. At the first radiative point outside a helium convective zone, the quantity $f \equiv \nabla_{\rm rad}^{\rm int} / \nabla_{\rm rad}^{\rm ext}$ is defined, where the radiative gradient is computed with the (s, P, T, r, L) values of the radiative point and with the composition of both the radiative point (superscript ext) and the interior convective zone (superscript int). The original convective zone is extended outward through the radiative region for all the points at which $f \nabla_{\rm rad} > \nabla_{\rm ad}$.

This overshooting region is restricted to the helium core ($X = 0$) and is limited by the condition of Castellani et al. (1971) that defines a maximum radius $R_{\max}$ of the overshooting mixing,

$$\int_{R_c}^{R_{\max}} \left[1 - \frac{\mu(r)}{\mu_c^{\rm int}} \right] dr < \left(1 - \frac{\nabla_{{\rm ad}_c}^{\rm int}}{\nabla_{{\rm rad}_c}^{\rm int}} \right) \frac{L_c \Delta t}{40 \pi P_c R_c^2},$$

where the subscript c refers to the (s, P, T, r, L) values at the edge of the original convective zone. Here μ is the mean molecular weight. The composition is then remixed from the beginning of the convective zone to the maximum extent of the overshoot region.

6.7. *Point Readjustment*

The point readjustment routine reflects all of the points between successive models. This routine starts with the central point and places each subsequent new point i so that all of the following criteria are met:

$$s_i - s_{i-1} \le \Delta s_{\max},$$
$$P'_{i2} - P'_{i-12} \le \Delta P'_{\max},$$
$$L_{i2} - L_{i-12} \le \Delta f_L L_{M2}.$$

All of the new values are interpolated linearly in s by locating the old point l such that $s_l(\text{old}) \le s_i(\text{new}) < s_{l+1}(\text{old})$. The fundamental variables (s, P_T, T, R, L), the composition (X, Z, X_{16}), and the density and entropy terms ($\Delta P'$, $\Delta T'$) are relocated between the center and outermost points for all angular zones. These variables are stored in temporary arrays and are transferred to the original arrays once the process is completed. In addition to the first and Mth points remaining fixed, other points may be retained:

1. the first radiative point (outer edge of convective zone),
2. the innermost point of the convective envelope,
3. the edge of the helium core ($X = 0$), and
4. composition discontinuities, $X_{l2} - X_{l-12} > \Delta X_{\rm disk}$ or $Z_{l2} - Z_{l-12} > \Delta Z_{\rm disk}$.

The point routine then recalculates the weights assigned to each mass shell based on the mass values at the preceding and following midpoints,

$$
\begin{aligned}
m_i &= 10^{s_i},\\
a_1 &= \tfrac{1}{2}(m_1 + m_2),\\
a_i &= \tfrac{1}{2}(m_{i+1} - m_{i-1}), \quad \text{for } i = 2 \text{ to } M - 1,\\
a_M &= M_{\rm tot} - \tfrac{1}{2}(m_M + m_{M-1}).
\end{aligned}
$$

The value m_i defines the location of the ith shell, and a_i is the number of grams contained in the shell.

In addition, the point routine adjusts the temperature of the outermost Mth point by adding a new point or deleting some old points. Given the desired temperature range $T_{\rm min}$ to $T_{\rm max}$, if $T_{\underline{M}} < T_{\rm min}$, then the outermost point $j < M$ such that $T_l > \overline{T} \equiv \frac{1}{2}(T_{\rm min} + T_{\rm max})$ is selected as the new surface point. The points $l + 1$ to M are deleted. If $T_{M2} > T_{\rm max}$ the process is more complicated. The last atmosphere that was integrated will have stored the values of ($s_{\rm atm}$, $P_{{\rm atm}j}$, $\underline{T}_{{\rm atm}j}$, $r_{{\rm atm}j}$) for the first inward integration step in which $T_{{\rm atm}j} > \overline{T}$. The new point $M + 1$ is added with the values

$$
\begin{aligned}
s_{M+1} &= s_{\rm atm}, \quad P'_{M+1j} = P'_{{\rm atm}j},\\
T'_{M+1j} &= T'_{{\rm atm}j}, \quad r'_{M+1j} = r'_{{\rm atm}j},\\
L_{M+1j} &= L_{{\rm atm}j}, \quad X_{M+1j} = X_{Mj},\\
Z_{M+1j} &= Z_{Mj}, \quad X_{16_{M+1j}} = X_{16_{Mj}}.
\end{aligned}
$$

6.8. *Model Calculation Sequence*

The following list describes the sequence of calculations that is used in computing a series of stellar models. This sequence is the same for both one- and two-dimensional model calculations.

0. Input a model and compute a time step.
1. Locate the mixing zones and advance the composition and hydrogen shell for the given time step.
2. Calculate element diffusion for the given time step.
3. Readjust the points in the mass coordinate in the model. This step is the main source of numerical errors and should be switched off for high-precision calculations such as solar variability applications.
4. Calculate the entropy terms ($\Delta P'$ and $\Delta T'$). Just zero them at the beginning, and give an estimate using their temporal change rate times the given time step.
5. Add the predictable corrections to (P', T', r', L) if their temporal change rates are available (after advancing one time step). This allows us to use a much larger time step and save a lot of computation time.
6. Specify the magnetic field configuration by selecting the functions $\chi(m, \theta)$, $\vartheta(m, \theta)$, and $\gamma(m, \theta)$.
7. Retaining the old surface (or envelope) triangle and surface boundary conditions, do two iterations for corrections to the dependent variables (P', T', r', L) and apply a given fraction ($\leq$100%) of the corrections.
8. If necessary, relocate the surface triangle for the partially converged model and compute new atmospheres and surface boundary conditions.
9. Iterate until the model converges.
10. Refine the composition and iterate until the model converges for solar applications that need a high precision.
11. Repeat step 9 once for solar applications.

12*a*. If the corrections are excessively large at any time or if the model does not converge after many iterations (say, 20), then retain the previous model that has been stored on the disk and stop.

12*b*. If the model has converged,

(*i*) compute a new time step,
(*ii*) perform the requested printing,
(*iii*) store the model temporarily on the disk, overwrite the previous model, and
(*iv*) return to step 1.

7. TEST 1: TWO-DIMENSIONAL STANDARD SOLAR MODEL

In this test, we investigate how different resolutions and different boundary conditions affect the two-dimensional solar models in the standard case (zero magnetic field).

Starting from a one-dimensional ZAMS (zero-age main sequence) model, we move the fit point to the surface where the mass coordinate $s = 1 \times 10^{-14}$ from the usual location $s = 1 \times 10^{-5}$ in a stair-stepping way. The (ZAMS and the advanced) models are determined by the following parameters: the minimum and maximum change in s between Henyey grid points, $1 \times 10^{-12} \leq \Delta s \leq 8 \times 10^{-2}$, and the maximum change in w' ($= P'$, T', r', and $L/L_\odot$) between Henyey grid points, $|\delta w'| \leq 5.2834 \times 10^{-3}$. The convergence criteria for the stellar parameters are $|\delta P'| \leq 6 \times 10^{-7}$, $|\delta T'| \leq 4.5 \times 10^{-7}$, $|\delta r'| \leq 3 \times 10^{-7}$, and $|\delta(L/L_\odot)| \leq 9 \times 10^{-7}$. The convergence tolerance on the right-hand sides of the P and r equations is 3×10^{-7}, and the convergence tolerance on the right-hand sides of the L and T equations is 2.5×10^{-7}. We also require $|\delta P'/P'| \leq 9$, $|\delta T'/T'| \leq 5$, $|\delta r'/r'| \leq 5$, and $|\delta L/L| \leq 90$. All these requirements must be satisfied simultaneously when we apply the correction to the model. This is why we have to move the fit point in a stair-stepping way. Otherwise, the correction is too large, and the solution will diverge. The model has about 2401 grid points in the mass coordinate s, i.e., $M = 2401$. We also test the cases with $M = 1201$, 601, and 301.

When this one-dimensional convergence has been obtained, the angular part of the two-dimensional grid is selected. Unlike the mass coordinate s, which is not uniform, we simply equally divide the angular coordinate θ in the range $\theta \in [0, \pi/2]$, $\theta_j = (\pi/2)(j - 1)/(N - 1)$, where $j = 1$–N. We use the converged one-dimensional model for every angular zone. We use $N = 10$, 19, and 37 in this test.

The solar mass is $M_\odot = 1.9891 \times 10^{33}$ g. The initial metal mass fraction is assumed to be $Z = 0.022$ at ZAMS. The model will evolve from ZAMS to the current age of the Sun (4.55 Gyr). The hydrogen mass fraction and mixing-length parameter (ratio of the mixing length over the pressure scale height) are determined by the requirement that the solar model at present reproduce the observed radius ($R_\odot = 6.9598 \times 10^{10}$ cm) and luminosity ($L_\odot = 3.8515 \times 10^{33}$ ergs s^{-1}). We first use the one-dimensional code to generate a one-dimensional standard solar model as the reference. We then use the two-dimensional code to generate the two-dimensional zero-magnetic-field solar models with different M and N and different surface boundary conditions and compare them with the reference. Our aim is to investigate whether we can get a two-dimensional high-precision solar model.

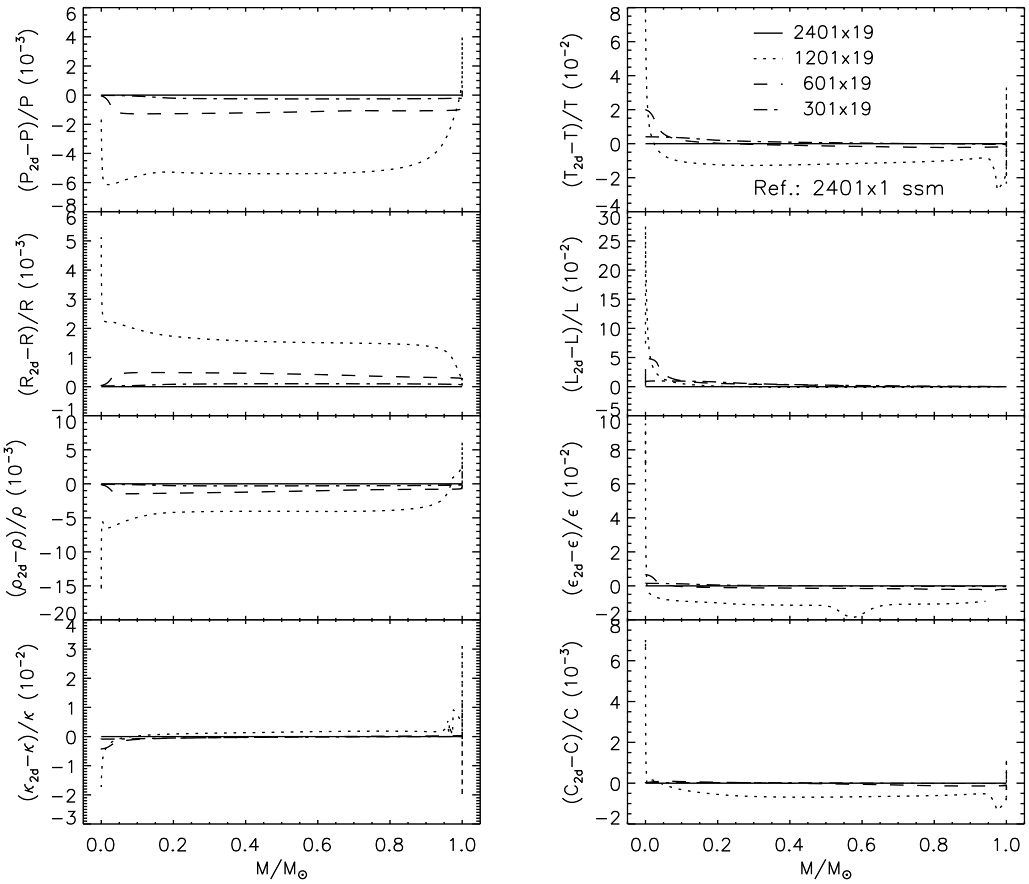

FIG. 4.—Relative changes of pressure P, temperature T, radius R, luminosity L, density ρ, nuclear energy generation rate ϵ, opacity κ, and sound speed C in two-dimensional solar models with different mass-coordinate resolutions ($M = 2401$, 1201, 601, and 301) with respect to a one-dimensional standard solar model as functions of mass coordinate.

7.1. *Convergence*

First of all, convergence is the most important requirement in model calculations. There is an intrinsic divergence at the poles in equations (188a), (188b), and (188d), which results from the terms that contain $\cot\theta$. In order to solve this intrinsic divergence problem, we require both equations (185) and (186) at and near the poles. In practice, we zero equations (195a)–(195d), where subscript 1 indicates the pole ($\theta = 0$) and subscript 2 means the point adjacent to the pole. The denser the grid in the second dimension, the more severe the intrinsic divergence problem. Therefore, it is desirable to use fewer grid points in the second dimension for the sake of convergence.

Since we have neglected the second-order derivatives with respect to θ that are believed to be smaller corrections to equations (188a), (188b), and (188d) than the first-order derivatives with respect to θ, we neglect those second-order derivatives to remove the divergence due to the numerical errors caused by them.

There is a numerical divergence problem due to the possible equality between r_{ij}, T_{ij}, and P_{ij} and r_{ij-1}, T_{ij-1}, and P_{ij-1}, respectively. When, say, r_{ij} equals r_{ij-1}, the difference between them, $\mathcal{R}' \equiv r_{ij} - r_{ij-1}$, vanishes. In this case, the derivative of the difference with respect to r_{ij} ($\partial\mathcal{R}'/\partial r_{ij}$) or r_{ij-1} ($\partial\mathcal{R}'/\partial r_{ij-1}$) should also vanish (i.e., $\partial\mathcal{R}'/\partial r_{ij} = 0$, or $\partial\mathcal{R}'/\partial r_{ij-1} = 0$, when $r_{ij} = r_{ij-1}$). If one sets $\partial\mathcal{R}'/\partial r_{ij} = 1$ and $\partial\mathcal{R}'/\partial r_{ij-1} = -1$ no matter whether r_{ij} equals r_{ij-1} or not, one will run into a numerical divergence problem. We introduce the δ_R, δ_P, and δ_T functions in Appendix A to solve this divergence problem.

The fourth divergence problem is due to the numerical error caused by numerical integration of ρ_m that affects the ratio ρ/ρ_m, which is a two-dimensional correction factor that appears in all the stellar structure equations, equations (188a)–(188d), noticing that the intrinsic singularity requires that the fewer the grid points at θ the better. The numerical integral is usually made in terms of the trapezoidal rule, which is of the second order in accuracy. Deupree (1990) adds more grid points to increase the integration precision when the numerical integral is performed. We find that it is more efficient to introduce a normalization factor in the integral, as shown in § A2.

When the radiative diffusion approximation (i.e., $\lambda = 0$) is used, the code converges very well. This approximation is not valid near the surface. If we use the temperature gradient at the surface to replace the actual gradient ∇_s, the code also converges well. However, if we use the exact expression given in equation (55), we cannot get a converged model. The main cause is due to the numerical errors in the numerical derivatives associated with λ_0.

7.2. *Resolution*

If the convergence solves the internal- or self-consistency problem, then model resolution will address the external-consistency issue. Our reference model, i.e., the one-dimensional standard solar model, is almost the same as the best model described by Winnick et al. (2002), who emphasize its comparison with various observations.

From numerical experiments using different resolutions in both dimensions, we find that the model is not sensitive to the resolution in the angular coordinate, but very sensitive to the mass coordinate (see Fig. 4). This figure compares four mass resolutions, in which the lower resolution is obtained by taking out one mass point for every two points from the adjacent higher resolution

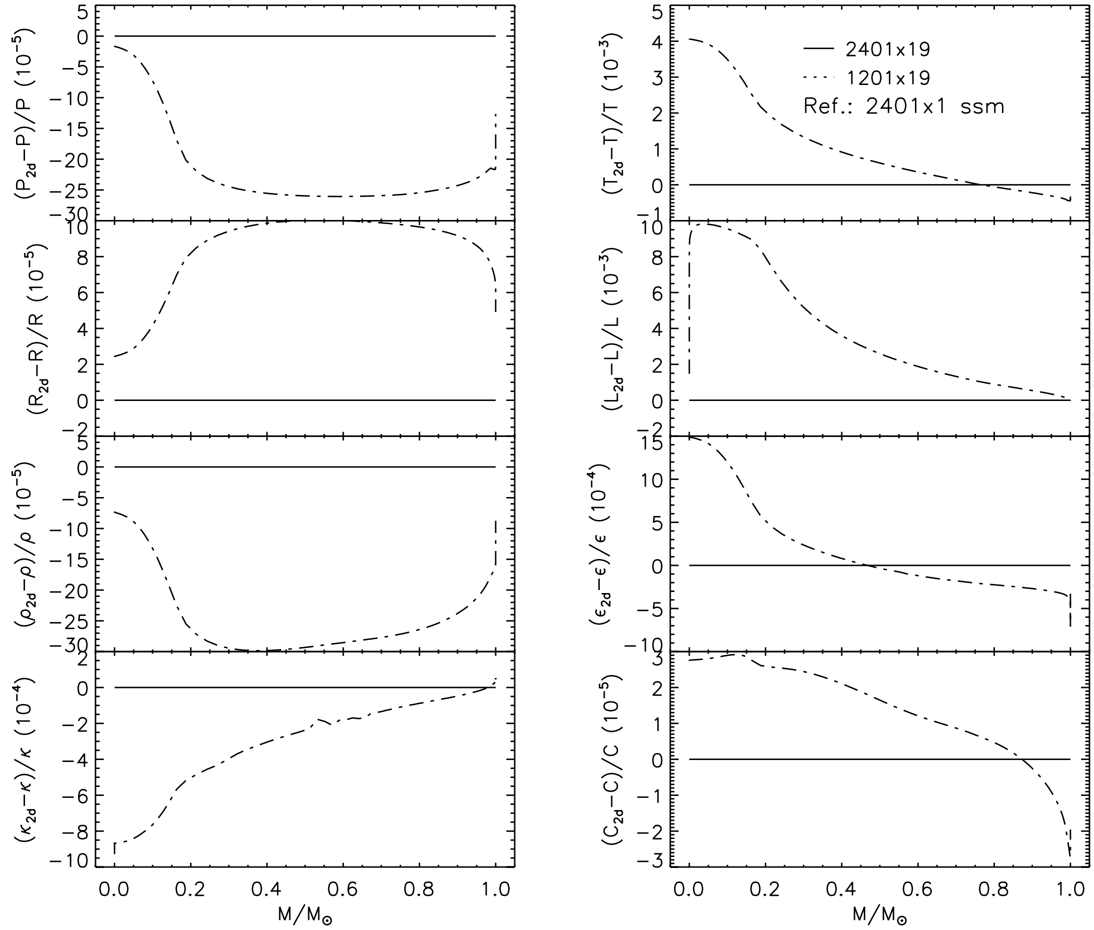

FIG. 5.—Same as Fig. 4, but only for $M = 2401$ and 1201.

model. Figure 5 zooms in to compare the models with the highest and second-highest resolutions.

We compare different angular zones in Figure 6 to make sure that the two-dimensional model is self-consistent in the angular direction. Figure 7 shows that the two- and one-dimensional solar models with the same mass resolution are in very good agreement.

7.3. *Surface Boundary Conditions*

Until now we have used only the standard surface boundary condition used in YREC (Pinsonneault 1988; Guenther et al. 1992). If we use these standard model surface values of pressure and temperature as Deupree's (1990) reference values, as indicated in § 5.2.2, we obtain the same results, as seen in Figure 8. The solid lines use equations (184a) and (184b). In order to investigate how errors in the reference pressure and temperature affect the result, we add 1% to $P_{\rm ref}$ given in equation (184a) and 0.1% to $T_{\rm ref}$ given in equation (184b). The result is shown by the dotted lines in Figure 8. From the dotted lines we can see that errors in the surface boundary condition have a larger influence on the outer layer than on the deep part of the model.

It is inevitable that some errors are introduced when $P_{\rm ref}$ and $T_{\rm ref}$ are selected in model calculations. Nevertheless, Deupree (1990) did not need to worry much about it, since his interest focused on the core convection. In contrast, we should be cautious about using Deupree's surface boundary condition, because we want to apply our model to solar variability that takes place in the convective envelope.

The model is less sensitive to the error in the reference pressure than to that in the reference temperature.

8. TEST 2: SHELL-LIKE MAGNETIC FIELDS

Shell-like magnetic fields depend on only the radial coordinate r. Any physical magnetic field should be free of divergence. The following magnetic fields are both radius dependent and divergence-free:

$$\boldsymbol{B} = (0, 0, f(r)),$$
$$\boldsymbol{B} = (C/r^2, 0, 0),$$
$$\boldsymbol{B} = (C/r^2, 0, f(r)),$$

where $f(r)$ is an arbitrary function of r and C is an arbitrary constant. If we assume that there is no magnetic field in the radiative zone of the Sun, then we have $C = 0$. Consequently, the unique physical shell-like magnetic field is

$$\boldsymbol{B} = (0, 0, f(r)). \tag{197}$$

This is the case described in § 4.2, in which

$$\mathcal{M} = -\mathcal{B}^4\left(1 + \frac{1}{2}\cot^2\theta\right)\left(1 - \frac{\cot\theta}{2}\frac{\partial r'}{\partial\theta}\right)^{-1},$$
$$\mathcal{B}^4 = \frac{m}{4\pi r^3 \rho_m}\frac{\chi\rho}{P_T}.$$

Comparing the two-dimensional stellar structure equations (eqs. [124a]–[124d]) with their one-dimensional counterparts (e.g., Li et al. 2003), we can see that the terms and/or factors in the curly braces are due to two-dimensional effects.

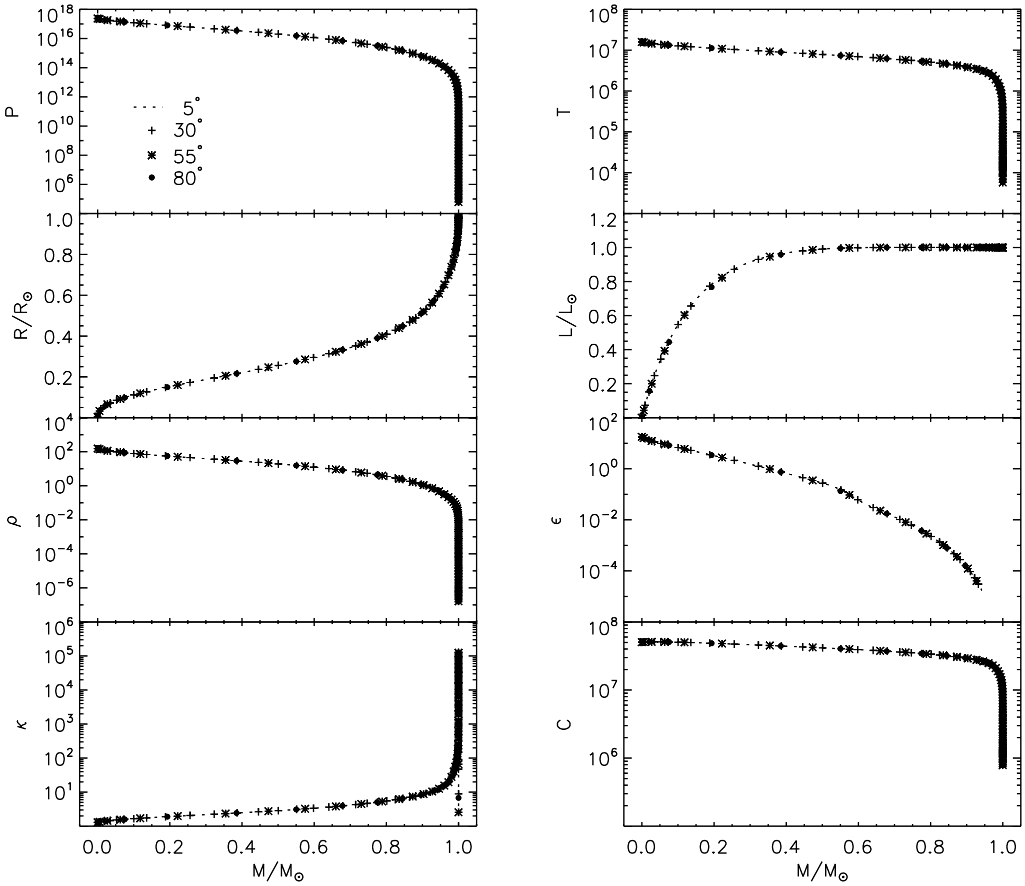

FIG. 6.—Pressure P, temperature T, radius R, luminosity L, density ρ, nuclear energy generation rate ϵ, opacity κ, and sound speed C at different angular coordinates as functions of mass coordinate.

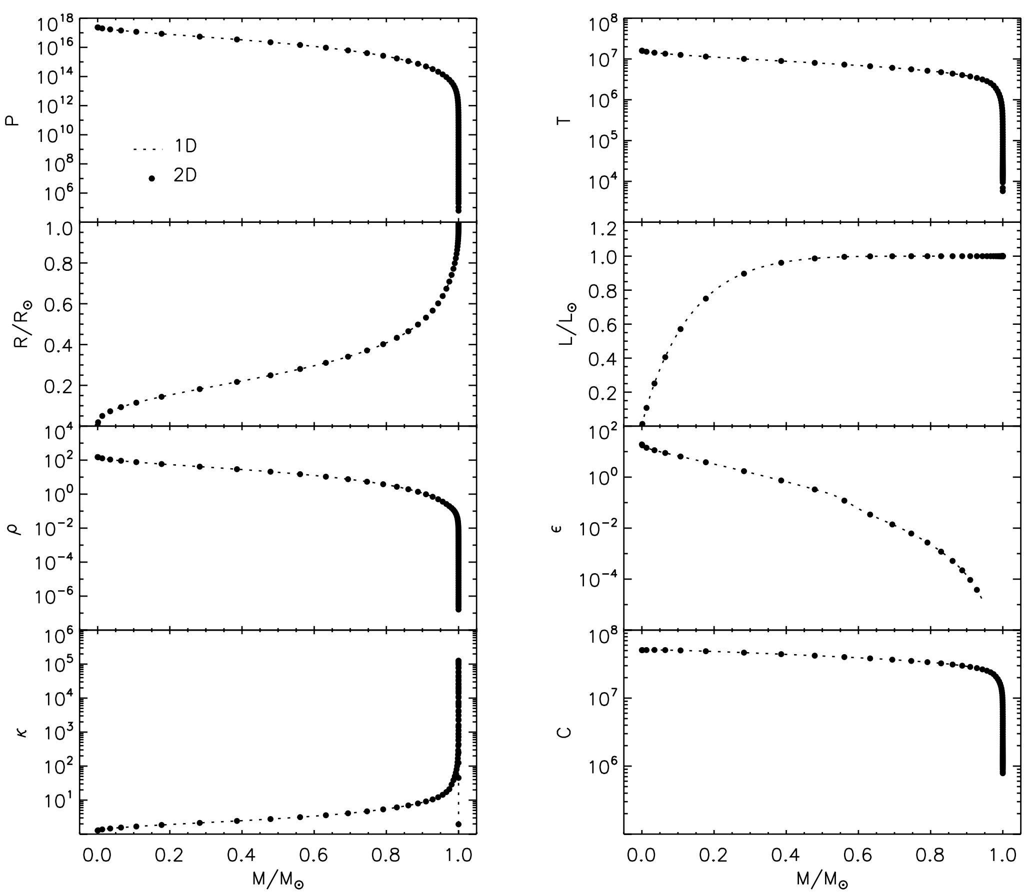

FIG. 7.—Pressure P, temperature T, radius R, luminosity L, density ρ, nuclear energy generation rate ϵ, opacity κ, and sound speed C in both one- and two-dimensional solar models that have the same mass resolution as functions of mass coordinate.

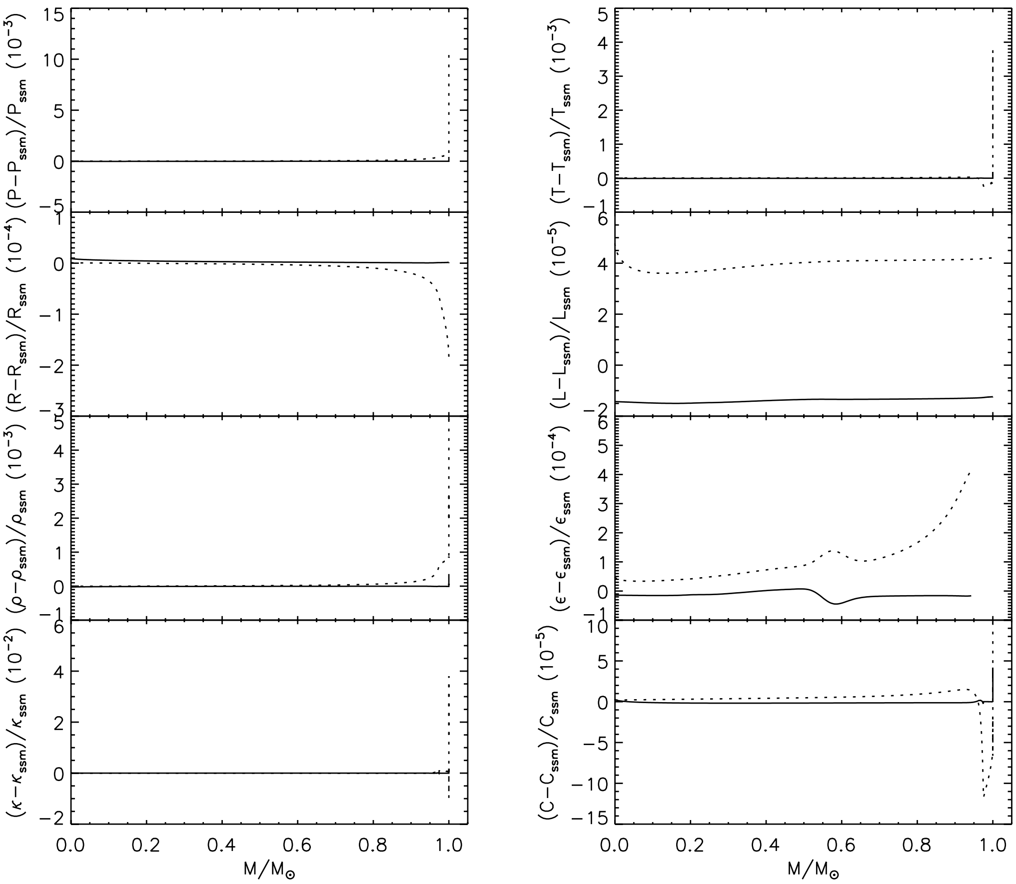

FIG. 8.—Relative changes of pressure P, temperature T, radius R, luminosity L, density ρ, nuclear energy generation rate ϵ, opacity κ, and sound speed C in the two-dimensional solar model with Deupree's surface boundary conditions (*solid line,* eqs. [184a] and [184b]; *dotted line*, 1.01 times eq. [184a] and 1.001 times eq. [184b]) with respect to a one-dimensional standard solar model as functions of mass coordinate.

In the solar variability applications, we use a standard solar model at the current age ($t = 4.55$ Gyr) as the initial model. We apply a cyclic magnetic field to the model and use 1 yr as the time step to advance the model.

As in the one-dimensional case, we specify χ as a function of time t (or sunspot number R_Z) and the mass depth $m_D = \log(1 - m/M_\odot)$ as

$$\chi(m_D, R_Z) = \chi_0(R_Z)\exp\left[-\tfrac{1}{2}(m_D - m_{Dc})^2/\sigma^2\right], \qquad (198)$$

where m_{Dc} specifies the location and σ specifies its width. Here χ_0 is determined by

$$\chi_0(R_Z) = \frac{B_0^2}{8\pi\rho_c}\left\{140 + [1 + \log(1 + R_Z)]^5\right\}^2, \qquad (199)$$

where B_0 is an adjustable parameter (in gauss) and ρ_c is the density at the mass depth of m_{Dc}. In this case the magnetic variable–related derivatives reduce to

$$\begin{aligned}\nu &= \chi\rho/P_T,\\ \nabla_\chi &= \frac{\partial\ln\chi}{\partial\ln m_D}\frac{\partial\ln m_D}{\partial\ln m}\frac{\partial\ln m}{\partial\ln P_T}\\ &= -\frac{m_D(m_D - m_{Dc})}{\sigma^2\ln 10}\frac{1 - 10^{m_D}}{10^{m_D}}\frac{4\pi P_T r^4}{GM_\odot^2(1 - 10^{m_D})^2}.\end{aligned}$$

In this test, χ does not depend on the angular coordinate θ, as required by a shell-like field. The resulting models should be the same as we obtained in the one-dimensional counterparts (Li et al. 2003). The method of solution used in this study guarantees this test, as confirmed by actual model calculations.

9. CONCLUSIONS

A high-precision two-dimensional framework for treating stellar evolution with magnetic fields has been developed and successfully tested. The required high precision is achieved by

1. using the mass coordinate to replace the radial coordinate,
2. including the convection instability,
3. including a stellar atmosphere,
4. allowing element diffusion,
5. using fixed and adjustable time steps, and
6. adjusting grid points.

The code has the potential to include rotation and turbulence, but does not have the potential to generate them like a fully hydrodynamic code.

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APPENDIX A

COEFFICIENT MATRIX

Equation (196) consists of a set of nonhomogeneous linear algebraic equations. If we use $\mathcal{A}$ to represent the coefficient matrix and use $\mathcal{B}$ to represent the nonhomogeneous term, this equation can be written down as

$$\mathcal{A}\cdot\delta w = \mathcal{B}, \tag{A1}$$

where

$$\delta w = \begin{pmatrix} \delta r'_{11} \\ \delta L'_{11} \\ \delta P'_{11} \\ \delta T'_{11} \\ \vdots \\ \delta r'_{M1} \\ \delta L'_{M1} \\ \delta P'_{M1} \\ \delta T'_{M1} \\ \vdots \\ \delta r'_{1N} \\ \delta L'_{1N} \\ \delta P'_{1N} \\ \delta T'_{1N} \\ \vdots \\ \delta r'_{MN} \\ \delta L'_{MN} \\ \delta P'_{MN} \\ \delta T'_{MN} \end{pmatrix} \tag{A2}$$

is a column matrix. The $\mathcal{B}$ term is also a column matrix,

$$\mathcal{B} = -\begin{pmatrix} F_R^{11} \\ F_L^{11} \\ F_P^{11} \\ F_T^{11} \\ \vdots \\ F_R^{M1} \\ F_L^{M1} \\ F_P^{M1} \\ F_T^{M1} \\ F_R^{12} \\ F_L^{12} \\ F_P^{22} \\ F_T^{22} \\ \vdots \\ F_R^{M2} \\ F_L^{M2} \\ F_P^{M+12} \\ F_T^{M+12} \\ \vdots \\ F_R^{1N} \\ F_L^{1N} \\ F_P^{2N} \\ F_T^{2N} \\ \vdots \\ F_R^{MN} \\ F_L^{MN} \\ F_P^{M+1N} \\ F_T^{M+1N} \end{pmatrix}. \tag{A3}$$

The coefficient matrix $\mathcal{A}$ has elements $\partial F_w^{ij}/\partial w_{lk}$. Only those elements with $l = i-1, i$ and $k = j-1, j$ may be nonzero, as shown in Figure 2. We work out these nonzero elements in this appendix.

A1. USEFUL PARTIAL DERIVATIVES

The partial derivatives of the differential equations are required for the linearization. By defining the shorthand notation $\partial_X Y = \partial Y/\partial \log X$, we can calculate the useful derivatives as follows.

The following derivatives are almost the same as in the one-dimensional case (see Prather 1976, his Appendix A), except for those terms due to ρ/ρ_m. These derivatives are nonzero for $l = i-1$, i and $k = j$. If $k = j-1$, they vanish:

$$\begin{aligned}
\partial_R\mathcal{P} &= -4\mathcal{P}, \\
\partial_L\mathcal{P} &= \partial_T\mathcal{P} = 0, \\
\partial_P\mathcal{P} &= -\mathcal{P}, \\
\partial_R\mathcal{R} &= -3\mathcal{R}, \\
\partial_L\mathcal{R} &= 0, \\
\partial_P\mathcal{R} &= -\alpha_m\mathcal{R}, \\
\partial_T\mathcal{R} &= \delta_m\mathcal{R},
\end{aligned}$$

$$\partial_P \mathcal{L} = \frac{M_r}{L_\odot}\left[\left(\frac{\partial \epsilon}{\partial \ln P}\right)_T + \left(\frac{\partial \tilde{S}}{\partial \ln P}\right)_T / \Delta t\right]\frac{\rho}{\rho_m},$$
$$\partial_R \mathcal{L} = \partial_L \mathcal{L} = 0,$$
$$\partial_T \mathcal{L} = \frac{M_r}{L_\odot}\left[\left(\frac{\partial \epsilon}{\partial \ln T}\right)_P + \left(\frac{\partial \tilde{S}}{\partial \ln T}\right)_P / \Delta t\right]\frac{\rho}{\rho_m}.$$

In the convective zone, we have

$$\partial_R \mathcal{T}_c = (\partial \ln \nabla_c / \partial \ln r - 4)\mathcal{T}_c,$$
$$\partial_L \mathcal{T}_c = 0,$$
$$\partial_P \mathcal{T}_c = (\partial \ln \nabla_c / \partial \ln P_T - 1)\mathcal{T}_c,$$
$$\partial_T \mathcal{T}_c = (\partial \nabla_c / \partial \ln T)\mathcal{T}_c.$$

In the radiative zone, we have

$$\partial_R \mathcal{T}_r = -4\mathcal{T}_r,$$
$$\partial_L \mathcal{T}_r = \mathcal{T}_r / L,$$
$$\partial_P \mathcal{T}_r = (\partial \ln \kappa / \partial \ln P_T)_T \mathcal{T}_r,$$
$$\partial_T \mathcal{T}_r = [(\partial \ln \kappa / \partial \ln T)_P - 4]\mathcal{T}_r.$$

The formulae for the various partial derivatives of the physical quantities are presented in the following sections. The equation of state calculates ρ, α, δ, c_p, $\nabla_{\rm ad}$, and the pressure and temperature derivatives of these quantities (see § B1). The energy generation rate ϵ is a function of ρ and T, too. Thus, $(\partial\epsilon/\partial T)_P$ and $(\partial\epsilon/\partial P)_T$ can be expressed by $(\partial\epsilon/\partial \ln T)_\rho$ and $(\partial\epsilon/\partial \ln \rho)_T$ (see § B2) as

$$(\partial\epsilon/\partial \ln T)_P = (\partial\epsilon/\partial \ln T)_\rho + (\partial\epsilon/\partial \ln \rho)_T (\partial \ln \rho / \partial \ln T)_P,$$
$$(\partial\epsilon/\partial \ln P_T)_T = (\partial\epsilon/\partial \ln \rho)_T (\partial \ln \rho / \partial \ln T)_P.$$

The derivatives of the convective gradient ∇_c are presented in § B4.

The entropy term contains the only explicit reference to any time dependence in the stellar structure equations. It can be reformulated as

$$\tilde{S} = -(P_T\delta/\rho)(\Delta T'/\nabla_{\rm ad} - \Delta P'),$$
$$\left(\partial\tilde{S}/\partial \ln T\right)_P = \tilde{S}\left[\delta + (\partial \ln \delta/\partial \ln T)_P\right] - (P\delta/\rho\nabla_{\rm ad})\left[1 - (\partial \ln \nabla_{\rm ad}/\partial \ln T)_P \Delta T'\right],$$
$$\left(\partial\tilde{S}/\partial \ln P_T\right)_T = \tilde{S}\left[1 - \alpha + (\partial \ln \delta/\partial \ln P_T)_T\right] + (P\delta/\rho)\left[1 + (\partial \ln \nabla_{\rm ad}/\partial \ln P_T)_T \Delta T'/\nabla_{\rm ad}\right],$$

where $(\Delta P', \Delta T')$ are the changes between successive models.

The following derivatives are new. Similarly, these derivatives are nonzero for $l = i - 1$, i and $k = j$. When $\ell = 1$, we have

$$\partial_R \mathcal{B}^1 = -4\mathcal{B}^1,$$
$$\partial_L \mathcal{B}^1 = \partial_T \mathcal{B}^1 = 0,$$
$$\partial_P \mathcal{B}^1 = -\mathcal{B}^1,$$
$$\partial_R \mathcal{T}_c^1 = \mathcal{T}^1[\partial \ln \nabla_c / \partial \ln r - 4],$$
$$\partial_L \mathcal{T}_c^1 = 0,$$
$$\partial_P \mathcal{T}_c^1 = \mathcal{T}_c^1[\partial \ln \nabla_c / \partial \ln P_T - 1],$$
$$\partial_T \mathcal{T}_c^1 = \mathcal{T}_c^1(\partial \ln \nabla_c / \partial \ln T),$$
$$\partial_R \mathcal{T}_r^1 = -4\mathcal{T}_r^1,$$
$$\partial_L \mathcal{T}_r^1 = \mathcal{T}_r^1 / L,$$
$$\partial_P \mathcal{T}_r^1 = \mathcal{T}_r^1(\partial \ln \kappa / \partial \ln P_T)_T,$$
$$\partial_T \mathcal{T}_r^1 = \mathcal{T}_r^1\left[(\partial \ln \kappa / \partial \ln T)_{P_T} - 4\right].$$

When $\ell = 2$, we have

$$\begin{aligned}
\partial_R \mathcal{B}^2 &= -\mathcal{B}^2, \\
\partial_L \mathcal{B}^2 &= \partial_T \mathcal{B}^2 = 0, \\
\partial_P \mathcal{B}^2 &= -\mathcal{B}^2, \\
\partial_R \mathcal{T}_c^2 &= \mathcal{T}^2(\partial \ln \nabla_c / \partial \ln R - 1), \\
\partial_L \mathcal{T}_c^2 &= 0, \\
\partial_P \mathcal{T}_c^2 &= \mathcal{T}^2(\partial \ln \nabla_c / \partial \ln P - 1), \\
\partial_T \mathcal{T}_c^2 &= \mathcal{T}^2(\partial \ln \nabla_c / \partial \ln T), \\
\partial_R \mathcal{T}_r^2 &= -\mathcal{T}_r^2, \\
\partial_L \mathcal{T}_r^2 &= \mathcal{T}_r^2 / L, \\
\partial_P \mathcal{T}_r^2 &= \mathcal{T}_r^2(\partial \ln \kappa / \partial \ln P_T)_T, \\
\partial_T \mathcal{T}_r^2 &= \mathcal{T}_r^2\Big[(\partial \ln \kappa / \partial \ln T)_{P_T} - 4\Big].
\end{aligned}$$

When $\ell = 3$, we have

$$\begin{aligned}
\partial_R \mathcal{B}^3 &= -3\mathcal{B}^3, \\
\partial_L \mathcal{B}^3 &= 0, \\
\partial_P \mathcal{B}^3 &= -\mathcal{B}^3 \alpha_m, \\
\partial_T \mathcal{B}^3 &= \mathcal{B}^3 \delta_m, \\
\partial_R \mathcal{T}_c^3 &= \mathcal{T}^3(\partial \ln \nabla_c / \partial \ln R - 3), \\
\partial_L \mathcal{T}_c^3 &= 0, \\
\partial_P \mathcal{T}_c^3 &= \mathcal{T}^3(\partial \ln \nabla_c / \partial \ln P - \alpha_m), \\
\partial_T \mathcal{T}_c^3 &= \mathcal{T}^3(\partial \ln \nabla_c / \partial \ln T + \delta_m), \\
\partial_R \mathcal{T}_r^3 &= -3\mathcal{T}_r^3, \\
\partial_L \mathcal{T}_r^3 &= \mathcal{T}_r^3 / L, \\
\partial_P \mathcal{T}_r^3 &= \mathcal{T}_r^3\big[\partial(\ln \kappa / \partial \ln P_T)_T - \alpha_m + 1\big], \\
\partial_T \mathcal{T}_r^3 &= \mathcal{T}_r^3\Big[(\partial \ln \kappa / \partial \ln T)_{P_T} + \delta_m - 4\Big].
\end{aligned}$$

When $\ell = 10$, 11, 13, and 14, we have

$$\begin{aligned}
\partial_R \mathcal{B}^\ell &= \partial_L \mathcal{B}^\ell = 0, \\
\partial_P \mathcal{B}^\ell &= \mathcal{B}^\ell(\alpha - 1), \\
\partial_T \mathcal{B}^\ell &= -\mathcal{B}^\ell \delta, \\
\partial_R \mathcal{T}_c^\ell &= \mathcal{T}_c^\ell(\partial \ln \nabla_c / \partial \ln r), \\
\partial_L \mathcal{T}_c^\ell &= 0, \\
\partial_P \mathcal{T}_c^\ell &= \mathcal{T}_c^\ell(\partial \ln \nabla_c / \partial \ln P + \alpha - 1), \\
\partial_T \mathcal{T}_c^\ell &= \mathcal{T}_c^\ell(\partial \ln \nabla_c / \partial \ln T - \delta), \\
\partial_R \mathcal{T}_r^\ell &= 0, \\
\partial_L \mathcal{T}_r^\ell &= \mathcal{T}_r^\ell / L, \\
\partial_P \mathcal{T}_r^\ell &= \mathcal{T}_r^\ell\big[(\partial \ln \kappa / \partial \ln P_T)_T + \alpha\big], \\
\partial_T \mathcal{T}_r^\ell &= \mathcal{T}_r^\ell\Big[(\partial \ln \kappa / \partial \ln T)_{P_T} - \delta - 4\Big].
\end{aligned}$$

When $\ell = 12$, 15, and 17, we have

$$\begin{aligned}
\partial_R \mathcal{B}^\ell &= -3\mathcal{B}^\ell, \\
\partial_L \mathcal{B}^\ell &= \partial_T \mathcal{B}^\ell = 0, \\
\partial_P \mathcal{B}^\ell &= -\mathcal{B}^\ell, \\
\partial_R \mathcal{T}_c^\ell &= \mathcal{T}_c^\ell(\partial \ln \nabla_c / \partial \ln r - 3), \\
\partial_L \mathcal{T}_c^\ell &= 0, \\
\partial_P \mathcal{T}_c^\ell &= \mathcal{T}_c^\ell(\partial \ln \nabla_c / \partial \ln P_T - 1), \\
\partial_T \mathcal{T}_c^\ell &= \mathcal{T}_c^\ell(\partial \ln \nabla_c / \partial \ln T), \\
\partial_R \mathcal{T}_r^\ell &= -3\mathcal{T}_r^\ell, \\
\partial_L \mathcal{T}_r^\ell &= \mathcal{T}_r^\ell / L, \\
\partial_P \mathcal{T}_r^\ell &= \mathcal{T}_r^\ell(\partial \ln \kappa / \partial \ln P_T)_T, \\
\partial_T \mathcal{T}_r^\ell &= \mathcal{T}_r^\ell\left[(\partial \ln \kappa / \partial \ln T)_{P_T} - 4\right].
\end{aligned}$$

When $k = j - 1$, all derivatives of $\mathcal{B}$ and $\mathcal{T}$ parameters vanish.

We also need similar derivatives of $\mathcal{D}$ parameters. When $l = i - 1$, $k = j$, all derivatives of $\mathcal{D}^1$, $\mathcal{D}^2$, $\mathcal{D}^3$, $\mathcal{D}^{12}$, $\mathcal{D}^{15}$, and $\mathcal{D}^{17}$ are zero. The nonzero derivatives are

$$\begin{aligned}
\partial_{P_{i-1j}} \mathcal{D}^{10} &= \partial_{P_{i-1j}} \mathcal{D}^{13} = -\mathcal{D}^2 \alpha_{i-1j} / \Delta s_i, \\
\partial_{P_{i-1j}} \mathcal{D}^{11} &= -\partial_{P_{i-1j}} \mathcal{D}^{10}\left(r'_{ij} - r'_{ij-1}\right)\Delta\theta_j, \\
\partial_{P_{i-1j}} \mathcal{D}^{14} &= \mathcal{D}^1 \alpha_{i-1j} / \Delta s_i, \\
\partial_{T_{i-1j}} \mathcal{D}^{10} &= \partial_{T_{i-1j}} \mathcal{D}^{13} = \mathcal{D}^2 \delta_{i-1j} / \Delta s_i, \\
\partial_{T_{i-1j}} \mathcal{D}^{11} &= -\partial_{T_{i-1j}} \mathcal{D}^{10}\left(r'_{ij} - r'_{ij-1}\right)/\Delta\theta_j, \\
\partial_{T_{i-1j}} \mathcal{D}^{14} &= -\mathcal{D}^1 \delta_{i-1j} / \Delta s_i.
\end{aligned}$$

When $l = i$, $k = j$, the nonzero derivatives are

$$\begin{aligned}
\partial_{R_{ij}} \mathcal{D}^1 &= \left(\frac{\cot\theta_j}{2\Delta\theta_j}\right)^2 \left(r'_{ij} - r'_{ij-1}\right)\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-2}\delta_R + \frac{\cot\theta_j}{2\Delta\theta_j}\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-1}\delta_R, \\
\partial_{R_{ij}} \mathcal{D}^2 &= \frac{\cot\theta_j}{2\Delta\theta_j}\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-2}\delta_R, \\
\partial_{R_{ij}} \mathcal{D}^3 &= \left(\frac{\cot\theta_j}{2\Delta\theta_j}\right)^2 \left(P'_{ij} - P'_{ij-1}\right)\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-2}\delta_R, \\
\partial_{R_{ij}} \mathcal{D}^{10} &= \frac{1}{\Delta s_i}\partial_{R_{ij}} \mathcal{D}^2\left[\left(\chi'_{ij} - \chi'_{i-1j}\right) + \left(\rho'_{ij} - \rho'_{i-1j}\right) + \left(\gamma''_{ij} - \gamma''_{i-1j}\right)\right], \\
\partial_{R_{ij}} \mathcal{D}^{11} &= -\frac{\delta_R}{\Delta\theta_j}\mathcal{D}^{10} - \frac{1}{\Delta\theta_j}\partial_{R_{ij}} \mathcal{D}^{10}\left(r'_{ij} - r'_{ij-1}\right), \\
\partial_{R_{ij}} \mathcal{D}^{12} &= \frac{1}{\Delta\theta_j}\partial_{R_{ij}} \mathcal{D}^2\left[\left(\chi'_{ij} - \chi'_{ij-1}\right) + \left(\rho'_{ij} - \rho'_{ij-1}\right) + \left(\gamma''_{ij} - \gamma''_{ij-1}\right)\right], \\
\partial_{R_{ij}} \mathcal{D}^{13} &= \frac{1}{\Delta s_i}\partial_{R_{ij}} \mathcal{D}^2\left[\left(\chi'_{ij} - \chi'_{i-1j}\right) + \left(\rho'_{ij} - \rho'_{i-1j}\right) + \left(\vartheta''_{ij} - \vartheta''_{i-1j}\right)\right], \\
\partial_{R_{ij}} \mathcal{D}^{14} &= -\frac{1}{\Delta s_i}\partial_{R_{ij}} \mathcal{D}^1\left[\left(\chi'_{ij} - \chi'_{i-1j}\right) + \left(\rho'_{ij} - \rho'_{i-1j}\right) + \left(\vartheta''_{ij} - \vartheta''_{i-1j}\right)\right], \\
\partial_{R_{ij}} \mathcal{D}^{15} &= \frac{\cot\theta_j}{2\Delta\theta_j}\partial_{R_{ij}} \mathcal{D}^2\left[\left(\chi'_{ij} - \chi'_{ij-1}\right) + \left(\rho'_{ij} - \rho'_{ij-1}\right) + \left(\vartheta''_{ij} - \vartheta''_{ij-1}\right)\right], \\
\partial_{R_{ij}} \mathcal{D}^{17} &= -\partial_{R_{ij}} \mathcal{D}^2, \\
\partial_{P_{ij}} \mathcal{D}^3 &= \frac{\cot\theta_j}{2\Delta\theta_j}\left[1 - \frac{\cot\theta_j}{2\Delta\theta_j}\left(r'_{ij} - r'_{ij-1}\right)\right]^{-1}\delta_P, \\
\partial_{P_{ij}} \mathcal{D}^{10} &= \mathcal{D}^2 \alpha_{ij} / \Delta s_i,
\end{aligned}$$

$$
\begin{aligned}
\partial_{P_{ij}}\mathcal{D}^{11} &= -\,\partial_{P_{ij}}\mathcal{D}^{10}\Big(r'_{ij} - r'_{ij-1}\Big)/\Delta\theta_j,\\
\partial_{P_{ij}}\mathcal{D}^{12} &= \mathcal{D}^2\alpha_{ij}\delta_\rho/\Delta\theta_j,\\
\partial_{P_{ij}}\mathcal{D}^{13} &= \mathcal{D}^2\alpha_{ij}/\Delta s_i,\\
\partial_{P_{ij}}\mathcal{D}^{14} &= -\,\mathcal{D}^1\alpha_{ij}/\Delta s_i,\\
\partial_{P_{ij}}\mathcal{D}^{15} &= \mathcal{D}^2\frac{\cot\theta_j}{2\Delta\theta_j}\alpha_{ij}\delta_\rho,\\
\partial_{T_{ij}}\mathcal{D}^{10} &= -\,\mathcal{D}^2\delta_{ij}/\Delta s_i,\\
\partial_{T_{ij}}\mathcal{D}^{11} &= -\,\partial_{T_{ij}}\mathcal{D}^{10}\Big(r'_{ij} - r'_{ij-1}\Big)/\Delta\theta_j,\\
\partial_{T_{ij}}\mathcal{D}^{12} &= -\,\mathcal{D}^2\delta_{ij}\delta_\rho/\Delta\theta_j,\\
\partial_{T_{ij}}\mathcal{D}^{13} &= -\,\mathcal{D}^2\delta_{ij}/\Delta s_i,\\
\partial_{T_{ij}}\mathcal{D}^{14} &= \mathcal{D}^1\delta_{ij}/\Delta s_i,\\
\partial_{T_{ij}}\mathcal{D}^{15} &= -\,\mathcal{D}^2\frac{\cot\theta_j}{2\Delta\theta_j}\delta_{ij}\delta_\rho,
\end{aligned}
$$

where $\delta_R = 1$ when $r_{ij} - r_{ij-1} \neq 0$ and $\delta_R = 0$ when $r_{ij} - r_{ij-1} = 0$, and δ_P and δ_ρ have similar meanings. When $l = i$, $k = j - 1$, the nonzero derivatives are

$$\partial_{R_{ij-1}}\mathcal{D}^\ell = -\partial_{R_{ij}}\mathcal{D}^\ell$$

for $\ell = 1, 2, 3, 10, \ldots, 15$, and 17,

$$\partial_{P_{ij-1}}\mathcal{D}^\ell = -\partial_{P_{ij}}\mathcal{D}^\ell$$

for $\ell = 3$, 12, and 15, and

$$\partial_{T_{ij-1}}\mathcal{D}^\ell = -\partial_{T_{ij}}\mathcal{D}^\ell$$

for $\ell = 12$ and 15.

We calculate the derivatives of $\mathcal{F}^2$ and $\mathcal{F}^3$ by taking the advantage of $l_m v_{\rm conv} \sim$ const. The nonzero derivatives are listed as follows for $l = i - 1$, i and $k = j$:

$$
\begin{aligned}
\partial_R\mathcal{F}^1 &= -\,2\mathcal{F}^1,\\
\partial_L\mathcal{F}^1 &= 0,\\
\partial_P\mathcal{F}^1 &= -\,\mathcal{F}^1\big[(\partial\ln\kappa/\partial\ln P_T)_T + \alpha + \alpha_m\big],\\
\partial_T\mathcal{F}^1 &= \mathcal{F}^1\Big[4 - (\partial\ln\kappa/\partial\ln T)_{P_T} + \delta + \delta_m\Big],\\
\partial_R\mathcal{F}^2 &= -\,2\mathcal{F}^2,\\
\partial_L\mathcal{F}^2 &= 0,\\
\partial_P\mathcal{F}^2 &= \mathcal{F}^2\big\{\big(\partial\ln C_p/\partial\ln P_T\big)_T - \beta\big[2\alpha + (\partial\ln\kappa/\partial\ln P_T)_T + \big(\partial\ln C_p/\partial\ln P_T\big)_T\big]\big\},\\
\partial_T\mathcal{F}^2 &= \mathcal{F}^2\Big\{1 + \big(\partial\ln C_p/\partial\ln T\big)_{P_T} + \beta\Big[3 + 2\delta - (\partial\ln\kappa/\partial\ln T)_{P_T} - \big(\partial\ln C_p/\partial\ln T\big)_{P_T}\Big]\Big\},\\
\partial_R\mathcal{F}^3 &= -\,2\mathcal{F}^3,\\
\partial_L\mathcal{F}^3 &= 0,\\
\partial_P\mathcal{F}^3 &= -\,\partial_P\mathcal{F}^2\nabla'_{\rm ad} + \mathcal{F}^3(\partial\ln\nabla_{\rm ad}/\partial\ln P_T)_T,\\
\partial_T\mathcal{F}^3 &= -\,\partial_T\mathcal{F}^2\nabla'_{\rm ad} + \mathcal{F}^3(\partial\ln\nabla_{\rm ad}/\partial\ln T)_{P_T},\\
\partial_R\mathcal{F}^4 &= \partial_R\mathcal{F}^1\begin{cases}\dfrac{Gm\rho\nabla_c}{rP_T} + \left(\dfrac{\partial\ln\nabla_c}{\partial\ln r} - 1\right)\mathcal{F}^4, & \text{convective},\\[2ex] \dfrac{Gm\rho\nabla_{\rm rad}}{rP_T} - \mathcal{F}^4, & \text{radiative},\end{cases}\\
\partial_L\mathcal{F}^4 &= \begin{cases}0, & \text{convective},\\ \mathcal{F}^4/L, & \text{radiative},\end{cases}
\end{aligned}
$$

$$\partial_P\mathcal{F}^4 = \partial_P\mathcal{F}^1\begin{cases}\dfrac{Gm\rho\nabla_c}{rP_T} + \left(\dfrac{\partial\ln\nabla_c}{\partial\ln P_T} + \alpha - 1\right)\mathcal{F}^4, & \text{convective,}\\ \dfrac{Gm\rho\nabla_{\rm rad}}{rP_T} + \left[\left(\dfrac{\partial\ln\kappa}{\partial\ln P_T}\right)_T + \alpha\right]\mathcal{F}^4, & \text{radiative,}\end{cases}$$

$$\partial_T\mathcal{F}^4 = \partial_T\mathcal{F}^1\begin{cases}\dfrac{Gm\rho\nabla_c}{rP_T} + \left(\dfrac{\partial\ln\nabla_c}{\partial\ln T} - \delta\right)\mathcal{F}^4, & \text{convective,}\\ \dfrac{Gm\rho\nabla_{\rm rad}}{rP_T} + \left[\left(\dfrac{\partial\ln\kappa}{\partial\ln T}\right)_P - \delta - 4\right]\mathcal{F}^4, & \text{radiative,}\end{cases}$$

$$\partial_R\mathcal{F}^5 = \partial_R\mathcal{F}^2\frac{Gm\rho\nabla_c}{rP_T} + \left(\frac{\partial\ln\nabla_c}{\partial\ln r} - 1\right)\mathcal{F}^5,$$

$$\partial_L\mathcal{F}^5 = 0,$$

$$\partial_P\mathcal{F}^5 = \partial_P\mathcal{F}^2\frac{Gm\rho\nabla_c}{rP_T} + \left(\frac{\partial\ln\nabla_c}{\partial\ln P_T} + \alpha - 1\right)\mathcal{F}^5,$$

$$\partial_T\mathcal{F}^5 = \partial_T\mathcal{F}^2\frac{Gm\rho\nabla_c}{rP_T} + \left(\frac{\partial\ln\nabla_c}{\partial\ln T} - \delta\right)\mathcal{F}^5,$$

$$\partial_R\mathcal{F}^6 = \partial_R\mathcal{F}^3\frac{Gm\rho}{rP_T} - \mathcal{F}^6,$$

$$\partial_L\mathcal{F}^6 = 0,$$

$$\partial_P\mathcal{F}^6 = \partial_P\mathcal{F}^3\frac{Gm\rho}{rP_T} + (\alpha - 1)\mathcal{F}^6,$$

$$\partial_T\mathcal{F}^6 = \partial_T\mathcal{F}^3\frac{Gm\rho}{rP_T} - \delta\mathcal{F}^6,$$

where $\beta = (v_{\rm conv}/v_0)/(1 + v_{\rm conv}/v_0)$.

A2. NUMERICAL INTEGRALS

The quantities ρ_m, α_m, and δ_m are integrals over θ:

$$\rho_m(m,\theta) \equiv \frac{1}{r^2}\frac{1}{2}\int_0^\pi d\theta\, r^2(m,\theta)\rho(m,\theta)\sin\theta,$$

$$\alpha_m(m) \equiv \left(\frac{\partial\ln\rho_m}{\partial\ln P_T}\right)_T = \frac{\int_0^\pi d\theta\, r^2(m,\theta)\rho(m,\theta)\alpha(m,\theta)\sin\theta}{\int_0^\pi d\theta\, r^2(m,\theta)\rho(m,\theta)\sin\theta},$$

$$\delta_m(m) \equiv -\left(\frac{\partial\ln\rho_m}{\partial\ln T}\right)_{P_T} = \frac{\int_0^\pi d\theta\, r^2(m,\theta)\rho(m,\theta)\delta(m,\theta)\sin\theta}{\int_0^\pi d\theta\, r^2(m,\theta)\rho(m,\theta)\sin\theta}.$$

Of course, the luminosity L is an integral, too:

$$L(m) \equiv 2\pi\int_0^\pi d\theta\, r^2(m,\theta)F_r(m,\theta)\sin\theta = \frac{1}{2}\int_0^\pi d\theta\, L'(m,\theta)L_\odot\sin\theta,$$

where $L' = 4\pi r^2 F_r/L_\odot$. In the one-dimensional case, we know the relationship on the solar surface,

$$L = 4\pi R^2\sigma T_{\rm eff}^4. \tag{A4}$$

If we define

$$R^2 \equiv \frac{1}{2}\int_0^\pi d\theta\, r^2(M_{\rm tot},\theta)\sin\theta,$$

$$T_{\rm eff}^4 \equiv \frac{1}{2R^2}\int_0^\pi d\theta\, r^2 T^4(M_{\rm tot},\theta)\sin\theta,$$

equation (A4) holds well in the two-dimensional case, where $M_{\rm tot}$ is the total mass of the star.

We use the trapezoidal rule to compute these integrals. For example,

$$\rho_m^{ij} = \frac{\left(1/r_{ij}^2\right) \sum_{\ell=2}^{N} (1/2)(r_{i\ell}^2 \rho_{i\ell} \sin\theta_\ell + r_{i\ell-1}^2 \rho_{i\ell-1} \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})}{\sum_{\ell=2}^{N} (1/2)(\sin\theta_\ell + \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})},$$

$$\frac{L}{L_\odot} = \frac{\sum_{\ell=2}^{N} (1/2)(L'_{M\ell} \sin\theta_\ell + L'_{M\ell-1} \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})}{\sum_{\ell=2}^{N} (1/2)(\sin\theta_\ell + \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})},$$

$$R^2 = \frac{\sum_{\ell=2}^{N} (1/2)(r_{M\ell}^2 \sin\theta_\ell + r_{M\ell-1}^2 \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})}{\sum_{\ell=2}^{N} (1/2)(\sin\theta_\ell + \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})},$$

where N is the total grid number in the second dimension θ. We have introduced the normalization factor $[\sum_{\ell=2}^{N} \frac{1}{2}(\sin\theta_\ell + \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})]^{-1}$ to remove the discrete error. The other three integrals do not need the normalization factor,

$$\alpha_m^i = \frac{\sum_{\ell=2}^{N} (1/2)(r_{i\ell}^2 \rho_{i\ell} \alpha_{i\ell} \sin\theta_\ell + r_{i\ell-1}^2 \rho_{i\ell-1} \alpha_{i\ell-1} \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})}{\sum_{\ell=2}^{N} (1/2)(r_{i\ell}^2 \rho_{i\ell} \sin\theta_\ell + r_{i\ell-1}^2 \rho_{i\ell-1} \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})},$$

$$\delta_m^i = \frac{\sum_{\ell=2}^{N} (1/2)(r_{i\ell}^2 \rho_{i\ell} \delta_{i\ell} \sin\theta_\ell + r_{i\ell-1}^2 \rho_{i\ell-1} \delta_{i\ell-1} \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})}{\sum_{\ell=2}^{N} (1/2)(r_{i\ell}^2 \rho_{i\ell} \sin\theta_\ell + r_{i\ell-1}^2 \rho_{i\ell-1} \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})},$$

$$T_{\rm eff}^4 = \frac{\sum_{\ell=2}^{N} (1/2)(r_{M\ell}^2 T_{M\ell}^4 \sin\theta_\ell + r_{M\ell-1}^2 T_{M\ell-1}^4 \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})}{\sum_{\ell=2}^{N} (1/2)(r_{M\ell}^2 \sin\theta_\ell + r_{M\ell-1}^2 \sin\theta_{\ell-1})(\theta_\ell - \theta_{\ell-1})},$$

because they have already had their own normalization factors.

A3. INTERIOR POINTS

A3.1. $w = P$

There are three blocks in this group. They are

block I, $l = i - 1$ and $k = j$;
block II, $l = i$ and $k = j$; and
block III, $l = i$ and $k = j - 1$.

We present the results block by block using the derivatives given above.

For block I,

$$\frac{\partial F_P^{ij}}{\partial R'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_R \mathcal{P}_{i-1j} - \frac{1}{2}\Delta s_i \sum_{\ell=1}^{3,12,15,17} \partial_R \mathcal{B}_{i-1j}^\ell \mathcal{D}^\ell,$$

$$\frac{\partial F_P^{ij}}{\partial L_{i-1j}} = 0,$$

$$\frac{\partial F_P^{ij}}{\partial P'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{P}_{i-1j} - 1 - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_P \mathcal{B}_{i-1j}^\ell \mathcal{D}^\ell + \sum_{\ell=10,11,13,14} \left(\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell\right) \partial_{P_{i-1j}} \mathcal{D}^\ell\right],$$

$$\frac{\partial F_P^{ij}}{\partial T'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{P}_{i-1j} - \frac{1}{2}\Delta s_i \left[\sum_{3,10,11,13,14} \partial_T \mathcal{B}_{i-1j}^\ell \mathcal{D}^\ell + \sum_{\ell=10,11,13,14} \left(\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell\right) \partial_{T_{i-1j}} \mathcal{D}^\ell\right].$$

For block II,

$$\frac{\partial F_P^{ij}}{\partial R'_{ij}} = -\frac{1}{2}\Delta s_i \partial_R \mathcal{P}_{ij} - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1}^{3,12,15,17} \partial_R \mathcal{B}_{ij}^\ell \mathcal{D}^\ell + \sum_{\ell=1,2,3,10}^{15,17} (\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell) \partial_{R_{ij}} \mathcal{D}^\ell\right],$$

$$\frac{\partial F_P^{ij}}{\partial L_{ij}} = 0,$$

$$\frac{\partial F_P^{ij}}{\partial P'_{ij}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{P}_{ij} + 1 - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_P \mathcal{B}_{ij}^\ell \mathcal{D}^\ell + \sum_{\ell=3,10}^{15} (\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell) \partial_{P_{ij}} \mathcal{D}^\ell\right],$$

$$\frac{\partial F_P^{ij}}{\partial T'_{ij}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{P}_{ij} - \frac{1}{2}\Delta s_i \left[\sum_{\ell=3,10,11,13,14} \partial_T \mathcal{B}_{ij}^\ell \mathcal{D}^\ell + \sum_{\ell=10}^{15} (\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell) \partial_{T_{ij}} \mathcal{D}^\ell\right].$$

For block III,

$$\frac{\partial F_P^{ij}}{\partial R'_{ij-1}} = -\frac{1}{2}\Delta s_i \sum_{\ell=1,2,3,10}^{15,17} (\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell)\partial_{R_{ij-1}}\mathcal{D}^\ell,$$

$$\frac{\partial F_P^{ij}}{\partial L_{ij-1}} = 0,$$

$$\frac{\partial F_P^{ij}}{\partial P'_{ij-1}} = -\frac{1}{2}\Delta s_i \sum_{\ell=3,12,15} (\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell)\partial_{P_{ij-1}}\mathcal{D}^\ell,$$

$$\frac{\partial F_P^{ij}}{\partial T'_{ij-1}} = -\frac{1}{2}\Delta s_i \sum_{\ell=12,15} (\mathcal{B}_{i-1j}^\ell + \mathcal{B}_{ij}^\ell)\partial_{T_{ij-1}}\mathcal{D}^\ell.$$

A3.2. w = T

There are three blocks in this group, too.

For block I,

$$\frac{\partial F_T^{ij}}{\partial R'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_R \mathcal{T}_{i-1j} - \frac{1}{2}\Delta s_i \sum_{\ell=1,2,3,10}^{15,17} \partial_R \mathcal{T}_{i-1j}^\ell \mathcal{D}^\ell,$$

$$\frac{\partial F_T^{ij}}{\partial L_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_L \mathcal{T}_{i-1j} - \frac{1}{2}\Delta s_i \sum_{\ell=1,2,3,10}^{15,17} \partial_L \mathcal{T}_{i-1j}^\ell \mathcal{D}^\ell,$$

$$\frac{\partial F_T^{ij}}{\partial P_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{T}_{i-1j} - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_P \mathcal{T}_{i-1j}^\ell \mathcal{D}^\ell + \sum_{\ell=10,11,13,14} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{P_{i-1j}}\mathcal{D}^\ell\right],$$

$$\frac{\partial F_T^{ij}}{\partial T_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{T}_{i-1j} - 1 - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_T \mathcal{T}_{i-1j}^\ell \mathcal{D}^\ell + \sum_{\ell=10,11,13,14} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{T_{i-1j}}\mathcal{D}^\ell\right].$$

For block II,

$$\frac{\partial F_T^{ij}}{\partial R'_{ij}} = -\frac{1}{2}\Delta s_i \partial_R \mathcal{T}_{ij} - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_R \mathcal{T}_{ij}^\ell \mathcal{D}^\ell + \sum_{\ell=1,2,3,10}^{15,17} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{R_{ij}}\mathcal{D}^\ell\right],$$

$$\frac{\partial F_T^{ij}}{\partial L_{ij}} = -\frac{1}{2}\Delta s_i \partial_L \mathcal{T}_{ij} - \frac{1}{2}\Delta s_i \sum_{\ell=1,2,3,10}^{15,17} \partial_L \mathcal{T}_{ij}^\ell \mathcal{D}^\ell,$$

$$\frac{\partial F_T^{ij}}{\partial P_{ij}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{T}_{ij} - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_P \mathcal{T}_{ij}^\ell \mathcal{D}^\ell + \sum_{\ell=3,10}^{15} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{P_{ij}}\mathcal{D}^\ell\right],$$

$$\frac{\partial F_T^{ij}}{\partial T_{ij}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{T}_{ij} + 1 - \frac{1}{2}\Delta s_i \left[\sum_{\ell=1,2,3,10}^{15,17} \partial_T \mathcal{T}_{ij}^\ell \mathcal{D}^\ell + \sum_{\ell=10}^{15} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{T_{ij}}\mathcal{D}^\ell\right].$$

For block III,

$$\frac{\partial F_T^{ij}}{\partial R'_{ij-1}} = -\frac{1}{2}\Delta s_i \sum_{\ell=1,2,3,10}^{15,17} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{R_{ij-1}}\mathcal{D}^\ell,$$

$$\frac{\partial F_T^{ij}}{\partial L_{ij-1}} = 0,$$

$$\frac{\partial F_T^{ij}}{\partial P'_{ij-1}} = -\frac{1}{2}\Delta s_i \sum_{\ell=3,12,15} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{P_{ij-1}}\mathcal{D}^\ell,$$

$$\frac{\partial F_T^{ij}}{\partial T'_{ij-1}} = -\frac{1}{2}\Delta s_i \sum_{\ell=12,15} \left(\mathcal{T}_{i-1j}^\ell + \mathcal{T}_{ij}^\ell\right)\partial_{T_{ij-1}}\mathcal{D}^\ell.$$

A3.3. $w = R$

In this group only the first two blocks are nonzero.

For block I,

$$\frac{\partial F_R^{ij}}{\partial R'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_R \mathcal{R}_{i-1j} - 1,$$
$$\frac{\partial F_R^{ij}}{\partial L_{i-1j}} = 0,$$
$$\frac{\partial F_R^{ij}}{\partial P'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{R}_{i-1j},$$
$$\frac{\partial F_R^{ij}}{\partial T'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{R}_{i-1j}.$$

For block II,

$$\frac{\partial F_R^{ij}}{\partial R'_{ij}} = -\frac{1}{2}\Delta s_i \partial_R \mathcal{R}_{ij} + 1,$$
$$\frac{\partial F_R^{ij}}{\partial L_{ij}} = 0,$$
$$\frac{\partial F_R^{ij}}{\partial P'_{ij}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{R}_{ij},$$
$$\frac{\partial F_R^{ij}}{\partial T'_{ij}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{R}_{ij}.$$

A3.4. $w = L$

Similar to §§ A3.1 and A3.2, all three blocks are nonzero.

For block I,

$$\frac{\partial F_L^{ij}}{\partial R'_{i-1j}} = -\frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left[\sum_{\ell=1}^{2}\partial_R \mathcal{F}_{i-1j}^{\ell}\left(T'_{ij} - T'_{ij-1}\right) + \partial_R \mathcal{F}_{i-1j}^{3}\left(P'_{ij} - P'_{ij-1}\right) + \sum_{\ell=4}^{6}\partial_R \mathcal{F}_{i-1j}^{\ell}\left(r'_{ij} - r'_{ij-1}\right)\right],$$
$$\frac{\partial F_L^{ij}}{\partial L_{i-1j}} = -1 - \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\partial_L \mathcal{F}_{i-1j}^{4}\left(r'_{ij} - r'_{ij-1}\right),$$
$$\frac{\partial F_L^{ij}}{\partial P'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{L}_{i-1j} - \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left[\sum_{\ell=1}^{2}\partial_P \mathcal{F}_{i-1j}^{\ell}\left(T'_{ij} - T'_{ij-1}\right) + \partial_P \mathcal{F}_{i-1j}^{3}\left(P'_{ij} - P'_{ij-1}\right) + \sum_{\ell=4}^{6}\partial_P \mathcal{F}_{i-1j}^{\ell}\left(r'_{ij} - r'_{ij-1}\right)\right],$$
$$\frac{\partial F_L^{ij}}{\partial T'_{i-1j}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{L}_{i-1j} - \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left[\sum_{\ell=1}^{2}\partial_T \mathcal{F}_{i-1j}^{\ell}\left(T'_{ij} - T'_{ij-1}\right) + \partial_T \mathcal{F}_{i-1j}^{3}\left(P'_{ij} - P'_{ij-1}\right) + \sum_{\ell=4}^{6}\partial_T \mathcal{F}_{i-1j}^{\ell}\left(r'_{ij} - r'_{ij-1}\right)\right].$$

For block II,

$$\frac{\partial F_L^{ij}}{\partial R'_{ij}} = -\frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left\{\sum_{\ell=1}^{2}\partial_R \mathcal{F}_{ij}^{\ell}\left(T'_{ij} - T'_{ij-1}\right) + \partial_R \mathcal{F}_{ij}^{3}\left(P'_{ij} - P'_{ij-1}\right) + \sum_{\ell=4}^{6}\left[\partial_R \mathcal{F}_{ij}^{\ell}\left(r'_{ij} - r'_{ij-1}\right) + \left(\mathcal{F}_{ij}^{\ell} + \mathcal{F}_{i-1j}^{\ell}\right)\delta_R\right]\right\},$$
$$\frac{\partial F_L^{ij}}{\partial L_{ij}} = 1 - \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\partial_L \mathcal{F}_{ij}^{4}\left(r'_{ij} - r'_{ij-1}\right),$$
$$\frac{\partial F_L^{ij}}{\partial P'_{ij}} = -\frac{1}{2}\Delta s_i \partial_P \mathcal{L}_{ij}$$
$$- \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left[\sum_{\ell=1}^{2}\partial_P \mathcal{F}_{ij}^{\ell}\left(T'_{ij} - T'_{ij-1}\right) + \partial_P \mathcal{F}_{ij}^{3}\left(P'_{ij} - P'_{ij-1}\right) + \left(\mathcal{F}_{ij}^{3} + \mathcal{F}_{i-1j}^{3}\right)\delta_P + \sum_{\ell=4}^{6}\partial_P \mathcal{F}_{ij}^{\ell}\left(r'_{ij} - r'_{ij-1}\right)\right],$$

$$\frac{\partial F_L^{ij}}{\partial T'_{ij}} = -\frac{1}{2}\Delta s_i \partial_T \mathcal{L}_{ij} - \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left\{\sum_{\ell=1}^{2}\left[\partial_T \mathcal{F}_{ij}^{\ell}\left(T'_{ij} - T'_{ij-1}\right) + \left(\mathcal{F}_{ij}^{\ell} + \mathcal{F}_{i-1j}^{\ell}\right)\delta_T\right] + \partial_T \mathcal{F}_{ij}^{3}\left(P'_{ij} - P'_{ij-1}\right) + \sum_{\ell=4}^{6}\partial_T \mathcal{F}_{ij}^{\ell}\left(r'_{ij} - r'_{ij-1}\right)\right\},$$

where $\delta_P = 1$ when $P_{ij} \neq P_{ij-1}$ and $\delta_P = 0$ when $P_{ij} = P_{ij-1}$. The definition of δ_T is similar.

For block III,

$$\frac{\partial F_L^{ij}}{\partial R'_{ij-1}} = \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\sum_{\ell=4}^{6}\left(\mathcal{F}_{ij}^{\ell} + \mathcal{F}_{i-1j}^{\ell}\right)\delta_R,$$

$$\frac{\partial F_L^{ij}}{\partial L_{ij-1}} = 0,$$

$$\frac{\partial F_L^{ij}}{\partial P'_{ij-1}} = \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\left(\mathcal{F}_{ij}^{3} + \mathcal{F}_{i-1j}^{3}\right)\delta_P,$$

$$\frac{\partial F_L^{ij}}{\partial T'_{ij-1}} = \frac{1}{2}\frac{\Delta s_i \cot\theta_j}{\Delta\theta_j}\sum_{\ell=1}^{2}\left(\mathcal{F}_{ij}^{\ell} + \mathcal{F}_{i-1j}^{\ell}\right)\delta_T,$$

where $\delta_R = 1$ when $r_{ij} \neq r_{ij-1}$ and $\delta_R = 0$ when $r_{ij} = r_{ij-1}$.

A4. BOUNDARY POINTS

A4.1. Center: $\omega = r$

Central boundary points have only block II for $w = r$:

$$\frac{\partial F_R^{1j}}{\partial R'_{1j}} = 1,$$

$$\frac{\partial F_R^{1j}}{\partial L_{1j}} = 0,$$

$$\frac{\partial F_R^{1j}}{\partial P'_{1j}} = \frac{1}{3}\alpha_{m1j},$$

$$\frac{\partial F_R^{1j}}{\partial T'_{1j}} = -\frac{1}{3}\delta_{m1j}.$$

A4.2. Center: $\omega = L$

Central boundary points have blocks II and III for $\omega = L$.

For block II,

$$\frac{\partial F_L^{1j}}{\partial R'_{1j}} = -\frac{\cot\theta_j}{\Delta\theta_j}\left\{\sum_{\ell=1}^{2}\partial_R \mathcal{F}_{1j}^{\ell}\left(T'_{1j} - T'_{1j-1}\right) + \partial_R \mathcal{F}_{1j}^{3}\left(P'_{1j} - P'_{1j-1}\right) + \sum_{\ell=4}^{6}\left[\partial_R \mathcal{F}_{1j}^{\ell}\left(r'_{1j} - r'_{1j-1}\right) + \mathcal{F}_{1j}^{\ell}\delta_R\right]\right\},$$

$$\frac{\partial F_L^{1j}}{\partial L_{1j}} = 1,$$

$$\frac{\partial F_L^{1j}}{\partial P'_{1j}} = -\partial_P \mathcal{L}_{1j} - \frac{\cot\theta_j}{\Delta\theta_j}\left[\sum_{\ell=1}^{2}\partial_P \mathcal{F}_{1j}^{\ell}\left(T'_{1j} - T'_{1j-1}\right) + \partial_P \mathcal{F}_{1j}^{3}\left(P'_{1j} - P'_{1j-1}\right) + \mathcal{F}_{1j}^{3}\delta_P + \sum_{\ell=4}^{6}\partial_P \mathcal{F}_{1j}^{\ell}\left(r'_{1j} - r'_{1j-1}\right)\right],$$

$$\frac{\partial F_L^{1j}}{\partial T'_{1j}} = -\partial_T \mathcal{L}_{1j} - \frac{\cot\theta_j}{\Delta\theta_j}\left\{\sum_{\ell=1}^{2}\left[\partial_T \mathcal{F}_{1j}^{\ell}\left(T'_{1j} - T'_{1j-1}\right) + \mathcal{F}_{1j}^{\ell}\delta_T\right] + \partial_T \mathcal{F}_{1j}^{3}\left(P'_{1j} - P'_{1j-1}\right) + \sum_{\ell=4}^{6}\partial_T \mathcal{F}_{1j}^{\ell}\left(r'_{1j} - r'_{1j-1}\right)\right\}.$$

For block III,

$$\frac{\partial F_L^{1j}}{\partial R'_{1j-1}} = \frac{\cot\theta_j}{\Delta\theta_j}\sum_{\ell=4}^{6}\mathcal{F}_{1j}^{\ell}\delta_R,$$
$$\frac{\partial F_L^{1j}}{\partial L_{1j-1}} = 0,$$
$$\frac{\partial F_L^{1j}}{\partial P'_{1j-1}} = \frac{\cot\theta_j}{\Delta\theta_j}\mathcal{F}_{1j}^{3}\delta_P,$$
$$\frac{\partial F_L^{1j}}{\partial T'_{1j-1}} = \frac{\cot\theta_j}{\Delta\theta_j}\sum_{\ell=1}^{2}\mathcal{F}_{1j}^{\ell}\delta_T.$$

A4.3. Surface

A4.3.1. Standard

The surface boundary conditions are linearized as

$$\delta r'_{Mj} + 0(\delta L_{Mj}) - a_1\delta P'_{Mj} - a_2\delta T'_{Mj} = -F_R^{M+1j},$$
$$0\left(\delta r'_{Mj}\right) + \delta L_{Mj} - L_{Mj}\,a_4\delta P'_{Mj} - L_{Mj}\,a_5\delta T'_{Mj} = -F_L^{M+1j}.$$

A4.3.2. Deupree's

His surface boundary equations are simpler:

$$0\left(\delta r'_{Mj}\right) + 0(\delta L_{Mj}) + 1\left(\delta P'_{Mj}\right) + 0\left(\delta T'_{Mj}\right) = -F_R^{M+1j},$$
$$0\left(\delta r'_{Mj}\right) + 0(\delta L_{Mj}) + 0\left(\delta P'_{Mj}\right) + 1\left(\delta T'_{Mj}\right) = -F_L^{M+1j},$$

where

$$F_R^{M+1j} = P'_{M+1j} - P'_{\rm ref},$$
$$F_L^{M+1j} = T'_{M+1j} - T'_{\rm ref}.$$

A4.4. Pole

The polar boundary equations are extremely simple:

$$\delta P'_{i1} - \delta P'_{i2} = 0,$$
$$\delta T'_{i1} - \delta T'_{i2} = 0,$$
$$\delta r'_{i1} - \delta r'_{i2} = 0,$$
$$\delta L_{i1} - \delta L_{i2} = 0.$$

APPENDIX B

INPUT PHYSICS

B1. THE EQUATIONS OF STATE

When a magnetic field is present, the equation of state relates the density ρ to the pressure P, temperature T, magnetic energy per unit mass χ, the ratio of specific heats γ, and the chemical composition:

$$\rho = \rho(P_T, T, \chi; X, Z),$$

where $P = P_0 + P_r + P_m$ is the total pressure, P_0 is the gas pressure, $P_r = aT^4/3$ is the radiative pressure, $P_m = \chi\rho$ is the magnetic pressure, X is the mass fraction of hydrogen, and Z is the mass fraction of elements heavier than helium (the so-called metal mass fraction). Its differential form is

$$\frac{d\rho}{\rho} = \alpha\frac{dP_T}{P} - \delta\frac{dT}{T} - \nu\frac{d\chi}{\chi},$$

where

$$\alpha = (\partial \ln \rho / \partial \ln P) \text{ at constant } T, \chi,$$
$$\delta = -(\partial \ln \rho / \partial \ln T) \text{ at constant } P, \chi,$$
$$\nu = -(\partial \ln \rho / \partial \ln \chi) \text{ at constant } P, T.$$

Here X and Z are assumed to be constant.

Since it is tedious to accurately calculate the equation of state from first principles, the equations of state are usually provided by the numerical tables as functions of (ρ, T, X, Z) for P_0, S (entropy), U (internal energy), $(\partial U/\partial \rho)_T$, $c_v = (\partial U/\partial T)_\rho$, $\chi_\rho = (\partial \ln P_0/\partial \rho)_T$, $\chi_T = (\partial \ln P_0/\partial T)_\rho$, $\Gamma_1 = (\partial \ln P_0/\partial \ln \rho)_S$, $\Gamma_2' = \Gamma_2/(1-\Gamma_2) = 1/\nabla_{\rm ad}$, and $\Gamma_3' = (\partial \ln T/\partial \ln \rho)_{P_0} - 1$. The equation of state (EOS) for the gas is taken from Rogers et al. (1996). In order to take into account a magnetic field based on the EOS tables, one can use the following correction method:

1. Use the total pressure $P = P_0 + P_r + P_m$, the total internal energy $U = U_0 + 3P_r/\rho + \chi$, and the total entropy $S = S_0 + 4P_r/\rho T + \chi/T$ to replace the gas pressure P_0, the gas internal energy U_0, and the gas entropy S_0, respectively, when interpolating to obtain the density for the given P and T.
2. Use $(P_0 + P_m)/P$ to rescale χ_ρ.
3. Use P_0/P to rescale χ_T from the EOS tables and add $4P_r/P$.
4. Add $12P_r/T$ to c_v from the EOS tables.
5. Compute $\Gamma_3' = P\chi_T/c_v\rho T$, $\Gamma_1 = \chi_\rho + \chi_T\Gamma_3'$, and $\Gamma_2' = \Gamma_1/\Gamma_3'$.

Taking these as known, we can calculate

$$\alpha = 1/\chi_\rho, \qquad \delta = \chi_T/\chi_\rho, \qquad \nu = P_m/P, \qquad \nabla_{\rm ad} = 1/\Gamma_2', \qquad c_p = P\delta/\rho T\nabla_{\rm ad}.$$

These quantities are used in calculating the convective gradient ∇_c.

B2. ENERGY GENERATION

The calculation of the energy generation includes the individual rates for the PP chain (PPI, PPII, PPIII), the CNO cycle with a simplified NO approach to equilibrium. The coefficients of all of the reaction rates and the formulae for most of them are taken from Fowler et al. (1975).

The reaction rate for the PP chain is actually that for the $^1{\rm H}(p, e^+\nu)^2{\rm D}$ reaction and assumes that all the other reactions in the chain are relatively instantaneous. The burning rate is then

$$(dX/dt)_{\rm PP} = 4.181 \times 10^{-15} \rho X^2 T_9^{-2/3} \exp\left(-3.380/T_9^{1/3}\right) \phi(\alpha') \left(1.0 + 0.123 T_9^{1/3} + 1.09 T_9^{2/3} + 0.938 T_9\right) \text{ s}^{-1},$$

where $T_9 = T/10^9$ K, the screening factor f_s is set equal to 1,

$$\phi(\alpha') = 1 + \alpha'\left[(1 + 2/\alpha')^{1/2} - 1\right],$$
$$\alpha' = 1.93 \times 10^{17} (Y/2X)^2 \exp\left(-10.0/T_9^{1/3}\right).$$

The total energy of the PP chain (subtracting the energy of the neutrinos that are produced) is

$$\epsilon_{\rm PP} = 6.398 \times 10^{18} \psi (dX/dt)_{\rm PP} \text{ ergs g}^{-1} \text{ s}^{-1},$$

where

$$\psi = 0.979 f_{\rm I} + 0.960 f_{\rm II} + 0.721 f_{\rm III},$$
$$f_{\rm I} = \left[(1 + 2/\alpha')^{1/2} - 1\right] / \left[(1 + 2/\alpha')^{1/2} + 3\right],$$
$$f_{\rm II} = (1 - f_{\rm I})/(1 + \Gamma),$$
$$f_{\rm III} = 1 - f_{\rm I} - f_{\rm II},$$
$$\Gamma = 10^{15.6837} [X/(1 + X)] T_9^{-1/6} \exp\left(-10.262/T_9^{1/3}\right).$$

The derivatives of $\epsilon_{\rm PP}$ can be found directly by

$$(\partial \ln \epsilon_{\rm PP}/\partial \ln \rho)_T = \epsilon_{\rm PP},$$
$$(\partial \ln \epsilon_{\rm PP}/\partial \ln T)_\rho = \epsilon_{\rm PP}\Big[-2/3 + 1.1267/T_9^{1/3} + (\partial \ln \phi/\partial \ln T)_\rho + (\partial \ln \psi/\partial \ln T)_\rho$$
$$+ \left(0.041 T_9^{1/3} + 0.727 T_9^{2/3} + 0.938 T_9\right) / \left(1 + 0.123 T_9^{1/3} + 1.09 T_9^{2/3} + 0.938 T_9\right)\Big],$$

$$
\begin{aligned}
(\partial \ln \phi/\partial \ln T)_\rho &= (2/\phi - 1)(1 + 2/\alpha')^{-1/2}\, 3.333/T^{1/3},\\
(\partial \ln \psi/\partial \ln T)_\rho &= \psi^{-1}\Big\{[0.258 - 0.239/(1+\Gamma)](\partial f_{\rm I}/\partial \ln T)_\rho - 0.239 f_{\rm III}/(1+\Gamma)(\partial \ln \Gamma/\partial \ln T)_\rho\Big\},\\
(\partial \ln \Gamma/\partial \ln T)_\rho &= -1/6 + 3.4207/T_9^{1/3},\\
(\partial f_{\rm I}/\partial \ln T)_\rho &= -4\Big\{\alpha'(1+2/\alpha')^{1/2}\big[(q+2/\alpha')^{1/2}+3\big]^2\Big\}^{-1} 3.333/T_9^{1/3}.
\end{aligned}
$$

In the calculation of the CNO bi-cycle, CN equilibrium is assumed and the CN cycle is assumed to be the only source of energy. The hydrogen-burning rate due to the CN cycle is then

$$(dX/dt)_{\rm CN} = 1.202 \times 10^7 \rho X X_{\rm N} T_9^{-2/3} \exp\Big(-15.228/T_9^{1/3}\Big)\ {\rm s}^{-1},$$

and the energy produced is

$$\epsilon_{\rm CN} = 5.977 \times 10^{18} (dX/dt)_{\rm CN}\ {\rm ergs\ g^{-1}\ s^{-1}}.$$

The value of $X_{\rm N}$ (^{14}N abundance by weight) assumes that all the carbon and nitrogen is in the form of ^{14}N,

$$X_{\rm N} = Z - Z_m - X_{\rm O},$$

where Z is the total metal abundance by weight, Z_m is the weight abundance of all non-CNO metals, and $X_{\rm O}$ is the weight abundance of ^{16}O. The approach to NO equilibrium is taken as a simple burning rate of ^{16}O assuming ^{17}O equilibrium,

$$dX_{\rm O}/dt = 9.54 \times 10^7 \rho X X_{\rm O} T_9^{-17/21} \exp\Big(-16.693/T_9^{1/3}\Big) - 1.6 \times 10^{-3} (dX/dt)_{\rm CN}\ {\rm s}^{-1}.$$

Between successive models the value of $X_{\rm O}$ is decreased at a rate of $(dX_{\rm O}/dt)$ per second, and thus the value of $X_{\rm N}$ is correspondingly increased. The derivatives of the CN energy production are

$$
\begin{aligned}
(\partial \epsilon_{\rm CN}/\partial \ln \rho)_T &= \epsilon_{\rm CN},\\
(\partial \epsilon_{\rm CN}/\partial \ln T)_\rho &= \epsilon_{\rm CN}\Big(-2/3 + 5.076/T_9^{1/3}\Big).
\end{aligned}
$$

B3. RADIATIVE OPACITIES

An estimate of magnetic effects on the radiative opacities [$\kappa = \kappa(T, \rho, X, Z)$] can be found in Li & Sofia (2001). Since they are small, we use only the OPAL opacity tables (Iglesias & Rogers 1996) together with the low-temperature opacities from Alexander & Ferguson (1994). For X and Z the linear interpolation is used, but for T and ρ the cubic spline interpolation is used. The cubic spline interpolation scheme allows one to obtain the derivatives of κ with respect to T and ρ. These derivatives are needed in the linearization of the equations of energy transport.

B4. THE CONVECTIVE GRADIENT AND ITS LINEARIZATION

The calculation of the convective temperature gradient ∇_c in the envelope of the stellar models employs the mixing-length theory (Henyey et al. 1965; Lydon & Sofia 1995) when magnetic fields are taken into account.

Defining $\delta' \equiv \nabla_{\rm rad} - \nabla'_{\rm ad}$, the Schwarzschild (1906) criterion is used to determine convection: $\delta > 0$ means convective. In the deep interior convection zones ∇_c is set equal to the adiabatic gradient $\nabla'_{\rm ad}$.

In the envelope the evaluation of ∇_c is more complex, and we solve

$$F(x) \equiv a_3 x^3 + x^2 + a_1 x - 1 = 0$$

for $y > 0$ such that $F(y) = 0$, where

$$
\begin{aligned}
a_1 &\equiv V = \alpha_3 \phi\big(\kappa T^3/C_p\big)\big(H_P/g\delta\delta'\big)^{1/2},\\
a_3 &\equiv \tfrac{3}{4}\phi\omega^2/V.
\end{aligned}
$$

We have defined $\delta' \equiv (\nabla_{\rm rad} - \nabla'_{\rm ad})$, $\phi \equiv (1 + \frac{1}{3}\omega^2)^{-1}$, $\omega = \rho\kappa l_m$, and $\alpha_3 \equiv 16\sqrt{2}\sigma$. The root y is guaranteed to lie in the interval $(0, +1)$, since $F(0) = -1 < 0$ and $F(1) = a_1 + a_2 > 0$. Furthermore, this root y is unique, since the derivative of F, $F'(x) = 3a_3x^2 + 2x + a_1$, is positive definite for $x > 0$. An initial estimate of the root y is made, and a second-order Newton-Raphson correction is applied:

$$\Delta y = -F(y)/F'(y) - \tfrac{1}{2}[F(y)/F'(y)]^2 F''(y)/F'(y).$$

The initial estimate of y is $y = 1/a_1$, unless $a_3 > 10^3$, in which case $y = (1/a_3)^{1/3}$, which follows the asymptotic behavior of the solution in either limit. Given the solution y, the convective gradient is computed by

$$\nabla_c = \nabla'_{\rm ad} + (\nabla_{\rm rad} - \nabla'_{\rm ad}) y(y + a_1).$$

The linearization of the convective gradient is cumbersome but can be calculated. We consider derivatives with respect to $\ln P_T$, $\ln T$, $\ln R$ and L:

$$\frac{\partial \nabla_c}{\partial \ln x} = \frac{\partial \nabla_{\rm ad}}{\partial \ln x} + y(y + a_1)\frac{\partial \delta'}{\partial \ln x} + \delta\left[(2y + a_1)\frac{\partial y}{\partial \ln x} + a_1 y \frac{\partial \ln a_1}{\partial \ln x}\right],$$

where

$$\frac{\partial \delta'}{\partial \ln x} = \frac{\partial \nabla'_{\rm rad}}{\partial \ln x} - \frac{\partial \nabla_{\rm ad}}{\partial \ln x}.$$

The derivatives of $\nabla'_{\rm ad}$ come from the equation of state and are nonvanishing only for $x = P_T$ or T. The derivatives of

$$\nabla_{\rm rad} = (3/16\pi ac)(\kappa L L_\odot P_T)/(GMT^4)$$

are nonvanishing for $x = P_T$, T, or L:

$$\frac{\partial \nabla_{\rm rad}}{\partial \ln P_T} = \nabla_{\rm rad}\left(1 + \frac{\partial \ln \kappa}{\partial \ln P_T}\right)_T,$$
$$\frac{\partial \nabla_{\rm rad}}{\partial \ln P_T} = \nabla_{\rm rad}\left(-4 + \frac{\partial \ln \kappa}{\partial \ln T}\right)_P,$$
$$\frac{\partial \nabla_{\rm rad}}{\partial L} = \frac{\nabla_{\rm rad}}{L}.$$

In the radiative zone, the actual temperature gradient is equal to the radiative temperature gradient $\nabla = \nabla_{\rm rad}$; its derivatives are given here. The opacity tables provide $\log \kappa$ versus ($\log \rho$, $\log T$). In order to calculate

$$(\partial \ln \kappa / \partial \ln T)_P = (\partial \ln \kappa / \partial \ln T)_\rho + (\partial \ln \kappa / \partial \ln \rho)_T (\partial \ln \rho / \partial \ln T)_P,$$
$$(\partial \ln \kappa / \partial \ln P)_T = (\partial \ln \kappa / \partial \ln \rho)_T (\partial \ln \rho / \partial \ln T)_P,$$

one needs $(\partial \ln \kappa / \partial \ln T)_\rho$ and $(\partial \ln \kappa / \partial \ln \rho)_T$ (see Iglesias & Rogers 1996; Alexander & Ferguson 1994). The derivatives of y are functions of a_1 and a_3,

$$\frac{\partial y}{\partial \ln x} = -\frac{1}{F'(y)}\left(a_1 y \frac{\partial \ln a_1}{\partial \ln x} + a_3 y^3 \frac{\partial a_3}{\partial \ln x}\right),$$

which need the derivatives of a_1 and a_3,

$$\frac{\partial \ln a_1}{\partial \ln x} = \frac{\partial \ln \phi}{\partial \ln x} + \frac{\partial \ln \kappa}{\partial \ln x} + 3\frac{\partial \ln T}{\partial \ln x} - \frac{\partial \ln C_p}{\partial \ln x} + \frac{1}{2}\frac{\partial \ln H_P}{\partial \ln x} - \frac{1}{2\delta'}\frac{\partial \delta'}{\partial \ln x} - \frac{1}{2}\frac{\partial \ln g}{\partial \ln x} - \frac{1}{2}\frac{\partial \ln \delta}{\partial \ln x},$$
$$\frac{\partial \ln a_3}{\partial \ln x} = 2\frac{\partial \ln \omega}{\partial \ln x} + \frac{\partial \ln \phi}{\partial \ln x} - \frac{\partial \ln a_1}{\partial \ln x}.$$

The derivatives of $\delta \equiv -(\partial \ln \rho / \partial T)_P$ and C_p are computed by the equation of state. By calculating the derivative of ϕ,

$$\frac{\partial \ln \phi}{\partial \ln x} = -\frac{2}{3}\omega^2 \phi \frac{\partial \ln \omega}{\partial \ln x},$$

and by expressing H_P and g explicitly, one can obtain

$$\frac{\partial \ln a_1}{\partial \ln x} = -\frac{2}{3}\omega^2 \phi \frac{\partial \ln \omega}{\partial \ln x} - \frac{\partial \ln C_p}{\partial \ln x} - \frac{1}{2\delta}\frac{\partial \delta}{\partial \ln x} - \frac{1}{2}\frac{\partial \ln \delta}{\partial \ln x} - \frac{P_T}{\ln x} + 3\frac{\partial \ln T}{\partial \ln x} + 2\frac{\partial \ln r}{\partial \ln x} + \frac{\partial \ln \kappa}{\partial \ln x} - \frac{1}{2}\frac{\partial \ln \rho}{\partial \ln x},$$
$$\frac{\partial \ln a_3}{\partial \ln x} = 2\phi \frac{\partial \ln \omega}{\partial \ln x} - \frac{\partial \ln a_1}{\partial \ln x}.$$

The derivatives of ω with respect to P_T, T, and r are straightforward,

$$\frac{\partial \ln \omega}{\partial \ln P_T} = 1 + \left(\frac{\partial \ln \kappa}{\partial \ln P_T}\right)_T,$$

$$\frac{\partial \ln \omega}{\partial \ln T} = \left(\frac{\partial \ln \kappa}{\partial \ln T}\right)_P,$$

$$\frac{\partial \ln \omega}{\partial \ln r} = 2.$$

REFERENCES

Alexander, D. R., & Ferguson, J. W. 1994, ApJ, 437, 879
Balmforth, N. J., Gough, D. O., & Merryfield, W. J. 1996, MNRAS, 278, 437
Basu, S., Pinsonneault, M. H., & Bahcall, J. N. 2000, ApJ, 529, 1084
Boothroyd, A. I., & Sackmann, I. J. 2003, ApJ, 583, 1004
Callen, H. B. 1966, Thermodynamics (New York: Wiley)
Castellani, V., Giannone, P., & Renzini, A. 1971, Ap&SS, 10, 340
Deupree, R. G. 1990, ApJ, 357, 175
Endal, A. S., Sofia, S., & Twigg, L. W. 1985, ApJ, 290, 748
Fowler, W. A., Caughlan, G. R., & Zimmerman, B. A. 1975, ARA&A, 13, 69
Guenther, D. B., & Demarque, P. 1997, ApJ, 484, 937
Guenther, D. B., Demarque, P., Kim, Y.-C., & Pinsonneault, M. H. 1992, ApJ, 387, 372
Henyey, L. E., Vardya, M. S., & Bodenheimer, P. 1965, ApJ, 142, 841
Iglesias, C. A., & Rogers, F. J. 1996, ApJ, 464, 943
Kippenhahn, R. 1967, Z. Astrophys., 67, 271
Krishna Swamy, K. S. 1966, ApJ, 145, 174
Li, L. H., Basu, S., Sofia, S., Demarque, P., & Guenther, D. B. 2003, ApJ, 591, 1267
Li, L. H., Robinson, F. J., Demarque, P., Sofia, S., & Guenther, D. B. 2002, ApJ, 567, 1192
Li, L. H., & Sofia, S. 2001, ApJ, 549, 1204
Lydon, T. J., & Sofia, S. 1995, ApJS, 101, 357
Pinsonneault, M. H. 1988, Ph.D. thesis, Yale Univ.
Prather, M. J. 1976, Ph.D. thesis, Yale Univ.
Robinson, F. J., Demarque, P., Li, L. H., Sofia, S., Kim, Y.-C., Chan, K. L., & Guenther, D. B. 2003, MNRAS, 340, 923
Rogers, F. J., Swenson, F. J., & Iglesias, C. A. 1996, ApJ, 456, 902
Sackmann, I. J., & Boothroyd, A. I. 2003, ApJ, 583, 1024
Schwarzschild, K. 1906, Göttingen Nachr., 41
Sofia, S., Basu, S., Demarque, P., Li, L. H., & Thuillier, G. 2005, ApJ, 632, L147
Sofia, S., O'Keefe, J., Lesh, J. R., & Endal, A. S. 1979, Science, 204, 1306
Spruit, H. C. 1991, in The Sun in Time, ed. C. Sonett, M. Giampapa, & M. S. Mathews (Tucson: Univ. Arizona Press), 118
———. 2000, Space Sci. Rev., 94, 113
Stix, M. 1989, The Sun: An Introduction (Berlin: Springer)
Thoul, A. A., Bahcall, J. N., & Loeb, A. 1994, ApJ, 421, 828
Turcotte, S. 2001, in IAU Colloq. 185, Radial and Nonradial Pulsations as Probes of Stellar Physics, ed. C. Aerts, T. R. Bedding, & J. Christensen-Dalsgaard (ASP Conf. Ser. 259; San Francisco: ASP), 258
Unno, W., & Spiegel, E. A. 1966, PASJ, 18, 85
Winnick, R. A., Demarque, P., Basu, S., & Guenther, D. B. 2002, ApJ, 576, 1075