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THERMODYNAMIC DESCRIPTION OF A FAMILY OF PARTIALLY RELAXED STELLAR SYSTEMS

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ABSTRACT

We examine the thermodynamic properties of a family of partially relaxed, anisotropic stellar systems, derived earlier from the Boltzmann entropy under the assumption that a third quantity Q is conserved in addition to the total energy and the total number of stars. We now show that the family of models conforms to the paradigm of the gravothermal catastrophe, which is expected to occur (in the presence of adequate energy transport mechanisms) when the one-parameter equilibrium sequence attains sufficiently high values of the concentration parameter; these are the values for which the models are well fitted by the $R^{1/4}$ law. In the intermediate concentration regime the models belonging to the sequence exhibit significant deviations from the $R^{1/4}$ law. Curiously, in the low-concentration regime, the global thermodynamic temperature associated with the models becomes negative when the models become too anisotropic so that they are unstable against the radial orbit instability; this latter behavior, while offering a new clue to the physical interpretation of the radial orbit instability, is at variance with respect to the low-concentration limit of the classical case of the isotropic, isothermal sphere investigated by Bonnor and Lynden-Bell & Wood.

Subject headings: galaxies: evolution — galaxies: formation — galaxies: kinematics and dynamics — galaxies: structure — stellar dynamics

1. INTRODUCTION

The possibility of providing a thermodynamic description of self-gravitating stellar systems has motivated a number of investigations in galactic dynamics, starting with the pioneering work of Antonov and Lynden-Bell in the 1960s. After the realization that violent relaxation is likely to lead to partially relaxed configurations in dynamic equilibrium (Lynden-Bell 1967), a reexamination of the problem of the isothermal sphere, studied earlier by Bonnor (1956) for a self-gravitating gas, led to the interesting possibility that stellar systems may undergo the process of gravothermal catastrophe (Lynden-Bell & Wood 1968; see also Antonov 1962). However, a rigorous derivation of the onset of the gravothermal catastrophe from a study of the Boltzmann entropy appeared to be available only for the case of ideal systems confined by a spherical reflecting wall. A number of convincing qualitative arguments made it clear that also unbound stellar systems with finite mass, such as those described by the King sequence (King 1966; these spherical models have a finite radius, but they do not require an external wall), should probably fall into the same physical framework and indeed the paradigm received a lot of attention, especially in the context of the dynamics of globular clusters (see Spitzer 1987), which are known to possess, at least to some extent, the desired internal collisionality (see also Lynden-Bell & Eggleton 1980). (An indirect indication that the general physical picture of the gravothermal catastrophe is likely to be robust comes also from the proof that the behavior of the classical gas case is basically independent of the assumption of spherical geometry; Lombardi & Bertin 2001.) Several investigations have aimed at producing a rigorous derivation of the gravothermal catastrophe for unbound stellar systems, focusing on the underlying argument that refers to the Poincaré stability of linear series of equilibria (Katz 1978, 1979; Padmanabhan 1989), but the proof has always been centered on an unjustified *Ansatz* in order to connect the underlying entropy S with the global temperature $T = 1/(\partial S/\partial E_{\rm tot})$ (see Appendix V in the article by Lynden-Bell & Wood 1968; Katz 1980; Magliocchetti, Pucacco, & Vesperini 1998). Other investigations have explored the possibility of setting the discussion in the context of nonstandard entropies (e.g., the Tsallis entropies; see the study of the polytropic spheres by Chavanis 2002 and Taruya & Sakagami 2002). Note that the concept of entropy for collisionless systems is quite subtle (e.g., see Stiavelli & Bertin 1987 and references therein). One might even argue whether it is actually compatible with the long-range nature of gravity, given the fact that self-gravitating systems lack additivity, a key ingredient in thermodynamics.

In the meantime, inspired by N-body simulations of collisionless collapse (van Albada 1982), which confirmed the general picture of incomplete violent relaxation and showed that it can lead to systems with realistic density profiles without ad hoc tuning of the initial conditions, some families of models were constructed able to reproduce, for quasi-spherical configurations, the characteristic feature of the anisotropy profile with an inner isotropic core and an outer radially biased envelope (Bertin & Stiavelli 1984; see Bertin & Stiavelli 1993 and references therein): these families turned out to exhibit the characteristic $R^{1/4}$ projected density profile and indeed were shown to match nicely the observed photometric and kinematical characteristics of bright ellipticals. In an attempt at providing a justification of these models (in particular, of the so-called f_∞ models, constructed initially only from dynamic arguments) from statistical mechanics, two routes were pursued (Stiavelli & Bertin 1987).

The first combines an explicit statement of partial relaxation, i.e., of a relaxation process that is expected to be inefficient in the outer regions, and the existence of a suitable weight, related to the orbital period, for the cells that make the relevant partition of phase space; it follows qualitative arguments proposed by Lynden-Bell (1967) and is physically appealing (see also Tremaine 1986). It was indeed shown to lead naturally to the f_∞ models. However, this route is not fully satisfactory from the mathematical point of view, especially since it involves an explicit approximation for the orbital period that is applicable only to the low binding energy limit of quasi-Keplerian orbits. The second route is straightforward from the mathematical point of view, being based on the classical Boltzmann entropy and on the assumed explicit conservation of a third quantity Q, in addition to the total mass M and to the total energy $E_{\rm tot}$. It was shown to lead to an analytically different family of models (that we may call the $f^{(\nu)}$ models; see the definition in § 3 below), with qualitative properties very similar to those of the f_∞ models. Those models were not studied much further and did not receive great attention, not only because the relevant distribution function is not as simple as that of the f_∞ models, but especially because the conservation of Q could only be justified approximately by inspection of a number of N-body simulations, without a clear-cut physical justification (see Stiavelli & Bertin 1987; in contrast, the conservation of the additional $\boldsymbol{A} \cdot \boldsymbol{B}$ invariant sometimes invoked in plasma physics is rather transparent; see Chandrasekhar & Woltjer 1958).

In this paper we take advantage of the simple statistical mechanics foundation of the $f^{(\nu)}$ family of unbound partially relaxed stellar systems to explore the possibility of a thermodynamic description of stellar-dynamic models that are endowed with realistic properties.

2. COMMENT ON THE EVOLUTION OF ELLIPTICAL GALAXIES

Before proceeding to illustrate the results of this paper, we make a short digression in order to bring out the connections between the present analysis and the evolution of elliptical galaxies. We start by recalling that, formally, the sequence of King (1966) models is one special family of solutions of the collisionless Boltzmann equation. Yet, it is recognized to provide a reasonable description of the current properties of globular clusters (see Djorgovski & Meylan 1994), within a framework where these stellar systems continually evolve as a result of a variety of mechanisms (among which are star evaporation and disk shocking; see Vesperini 1997 and references therein) and where the paradigm of the gravothermal catastrophe can be applied (see Spitzer 1987). Of course, it is well known that the level of internal collisionality in globular clusters is relatively high, so that the above approach is quite natural.

In contrast, one might at first think of dismissing the possibility that the paradigm of the gravothermal catastrophe should be of interest for the study of elliptical galaxies because these large stellar systems lack the desired level of collisionality, judging from the estimate of the relevant star-star relaxation times. Here, following the spirit of earlier investigations (starting with Lynden-Bell & Wood 1968), we note that real elliptical galaxies are actually complex systems the evolution of which goes well beyond the idealized framework of the collisionless Boltzmann equation. In other words, splitting their description into past (formation) processes and present (mostly collisionless equilibrium) conditions should be considered only as an idealization introduced in order to assess the properties that define their current basic state.

In practice, elliptical galaxies are expected to be in a state of continuous evolution, for which we can list several specific dynamic causes:

1. *Leftover granularity of the stellar system itself from initial collapse.*—Clumps of stars are likely to continue to form and dissolve in phase space even after the system has reached an approximate steady state. This acts as internal collisionality thus making some relaxation proceed even at current epochs. Indeed, numerical simulations of violent (partial) relaxation show that some evolution continues well after the initial collapse has taken place.
2. *Drag of a system of globular clusters or other heavier objects towards the galaxy center.*—A globular cluster system or the frequent capture of small satellites (minimergers) may provide an internal heating mechanism associated with the process of dynamic friction by the stars on the heavier objects (Bertin, Liseikina, & Pegoraro 2002).
3. *Long-term action of tidal interactions of the galaxy with external objects.*
4. *Presence of gas in various phases* (*cold, warm, and hot*).—Significant cooling flows have been observed in bright ellipticals. Traditionally, studies of processes of this kind focus on the dynamics of the cooling gas and keep the background stellar system as "frozen." In reality, energy and mass exchanges take place between the stellar system and the interstellar medium.
5. *Interaction between the galactic nucleus and the galaxy.*—A number of interesting correlations have been found between the properties of galaxy nuclei and global properties of the hosting galaxies (e.g., see Pellegrini 1999). These correlations suggest that significant energy exchanges are taking place between the galaxy and its nucleus. Eventually, if a sufficiently concentrated nucleus is generated, then star-star relaxation in the central regions may also become a significant cause of dynamic evolution.

All of the above are specific mechanisms that are expected to make elliptical galaxies evolve in spite of their very long typical star-star relaxation time. Most of these processes are hard to model and to calculate in detail. As for the evolution of other complex many-body systems, it is hoped that thermodynamic arguments may help us identify general trends characterizing such evolution. This is the basic physical scenario in which the calculations presented in this paper are expected to be of interest for real elliptical galaxies.

3. PARTIALLY RELAXED, UNBOUND, FINITE MASS SYSTEMS FROM THE BOLTZMANN ENTROPY

Let us consider the standard Boltzmann entropy $S = -\int f \ln f \, d^3x \, d^3v$ and look for functions that extremize its value under the constraint that the total energy $E_{\rm tot} = \frac{1}{3}\int Ef \, d^3x \, d^3v$, the total mass $M = \int f \, d^3x \, d^3v$, and the additional quantity

$$Q = \int J^\nu |E|^{-3\nu/4} f \, d^3x \, d^3v \tag{1}$$

are taken to be constant. Here the functions E and J^2 represent the specific energy and the specific angular momentum square of a single star subject to a spherically symmetric mean potential $\Phi(r)$. As shown elsewhere (Stiavelli & Bertin 1987), this extremization process leads to the following family of distribution functions

$$f^{(\nu)} = A \exp\left[-aE - d\left(\frac{J^2}{|E|^{3/2}}\right)^{\nu/2}\right] , \quad (2)$$

where a, A, and d are positive real constants. One may think of these constants as providing two-dimensional scales (for example, M and Q) and one dimensionless parameter; the dimensionless parameter can be taken to be $\gamma = ad^{2/\nu}/(4\pi GA)$. In principle, ν is any positive real number; in practice, we focus on values of $\nu \approx 1$. The $f^{(\nu)}$ nontruncated models are constructed by taking this form of the distribution function for $E \leq 0$, a vanishing distribution function for $E > 0$, and by integrating the relevant Poisson equation under the condition that the potential Φ be regular at the origin and that it behaves like $-GM/r$ at large radii. This integration leads to an eigenvalue problem (see the Appendix) for which a value of γ is determined by the choice of the central dimensionless potential, $\gamma = \gamma(\Psi)$, with $\Psi = -a\Phi(r = 0)$.

The main point of the following analysis is the determination of the Boltzmann entropy $S(M, Q, \Psi)$ and of the total energy $E_{\rm tot}(M, Q, \Psi)$ along the sequence of models, i.e., as a function of the concentration parameter Ψ defined above. These functions, at constant M and Q, are illustrated in Figure 1. They have been obtained by noting that, from the definitions of S and $f^{(\nu)}$,

$$S = -M \ln A + 3aE_{\rm tot} + dQ . \quad (3)$$

From the definitions $Q = Aa^{-9/4}d^{-1-3/\nu}\hat{Q}(\Psi)$ and $M = Aa^{-9/4}d^{-3/\nu}\hat{M}(\Psi)$ and the definition of γ, we can express the variables (A, a, d) in terms of the variables (M, Q, Ψ) and thus find that the entropy per unit mass can be written as $S/M = S_0(M, Q) + \sigma(\Psi)$, where S_0 is constant when the values of M and Q are fixed, with

$$\sigma = -\ln\left(\hat{M}^{(4\nu-6)/(5\nu)}\hat{Q}^{6/(5\nu)}\gamma^{-9/5}\right) + \frac{3\hat{E}}{\hat{M}} + \frac{\hat{Q}}{\hat{M}} . \quad (4)$$

Here $\hat{E} = \hat{E}(\Psi)$ is the dimensionless total energy defined from $E_{\rm tot} = Aa^{-13/4}d^{-3/\nu}\hat{E}$. From the identity $aE_{\rm tot}/M = \hat{E}/\hat{M}$ and the expression of $a = a(M, Q, \Psi)$ obtained previously, we find $E_{\rm tot}/M = H(M, Q)\epsilon(\Psi)$, with

$$\epsilon = \gamma^{4/5}\hat{M}^{-(9\nu+4)/(5\nu)}\hat{Q}^{4/(5\nu)}\hat{E} . \quad (5)$$

The factor $H(M, Q)$ is a constant when M and Q are taken to be constant. The quantities $\gamma(\Psi)$, $\hat{M}(\Psi)$, $\hat{Q}(\Psi)$, and $\hat{E}(\Psi)$ that enter the expression of σ and ϵ depend only on Ψ and are evaluated numerically on the equilibrium sequence.

This completes the derivation that allows us to draw the analogy with the classical paper of Lynden-Bell & Wood (1968). This step, straightforward for the $f^{(\nu)}$ models, is by itself interesting and new. In fact, other attempts at applying the paradigm of the gravothermal catastrophe to stellar dynamic equilibrium sequences were either based on an unjustified *Ansatz* for the identification of the relevant temperature (e.g., see Appendix V in the article by Lynden-Bell & Wood 1968; Katz 1980; Magliocchetti et al. 1998) or on the use of nonstandard entropies (for less realistic models; Chavanis 2002).

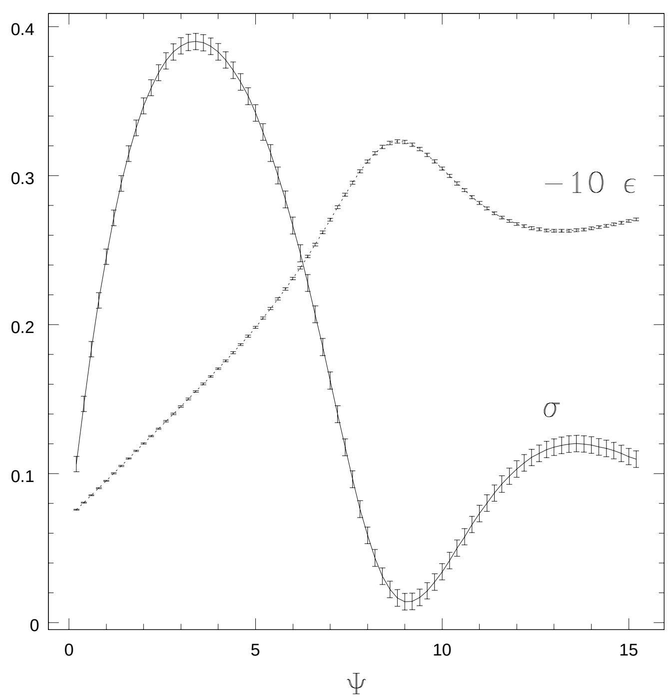

FIG. 1.—Specific entropy and total energy along the equilibrium sequence of $f^{(\nu)}$ models with $\nu = 1$ [as a function of the concentration parameter Ψ, at constant M and Q, and thus expressed by means of the functions $\sigma(\Psi)$ and $\epsilon(\Psi)$ defined in the text]. Note that for $\Psi \lesssim 3.5$ the models are characterized by a negative temperature because the derivatives of S and $E_{\rm tot}$ have opposite signs.

4. THE HIGH-CONCENTRATION REGIME: GRAVOTHERMAL CATASTROPHE

When the $f^{(\nu)}$ models were constructed (Stiavelli & Bertin 1987), it was immediately realized that they have general properties similar to those of the f_∞ models (Bertin & Stiavelli 1984); in particular, for values of $\nu \approx 1$, sufficiently concentrated models along the sequence tend to settle into a "stable" overall structure, except for the development of a more and more compact nucleus as the value of Ψ increases, and are characterized by a projected density profile very well fitted by the $R^{1/4}$ law characteristic of the surface brightness profile of bright elliptical galaxies. This property is illustrated in Figure 2.

Now, by inspection of Figure 1 and by analogy with the study of the isothermal sphere (Lynden-Bell & Wood 1968), we can identify the location at $\Psi \approx 9$ as the location for the *onset of the gravothermal catastrophe*. This sequence of models thus has the surprising result that the value of Ψ that defines the onset of the gravothermal catastrophe is precisely that around which the models appear to become realistic representations of bright elliptical galaxies. We leave to other papers (see Bertin & Stiavelli 1993) the detailed discussion of the issues that have to be addressed when comparison is made with the observations.

We note that in this regime of high concentration the general properties of the gravothermal catastrophe are reasonably well recovered by the use of the *Ansatz* that the

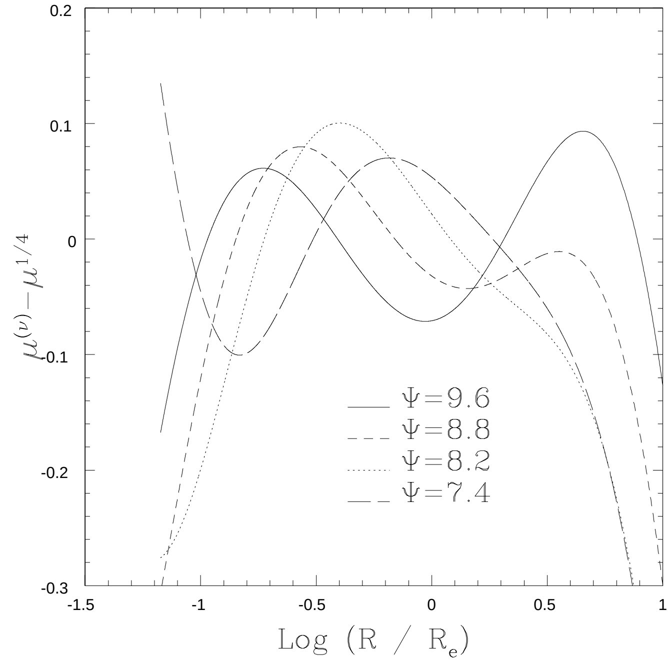

FIG. 2.—Residuals $\mu^{(\nu)}$–$\mu^{1/4}$, in magnitudes, obtained by fitting the $R^{1/4}$ law to the projected density profile of $f^{(\nu)}$ models for $\nu = 1$ and some values of Ψ.

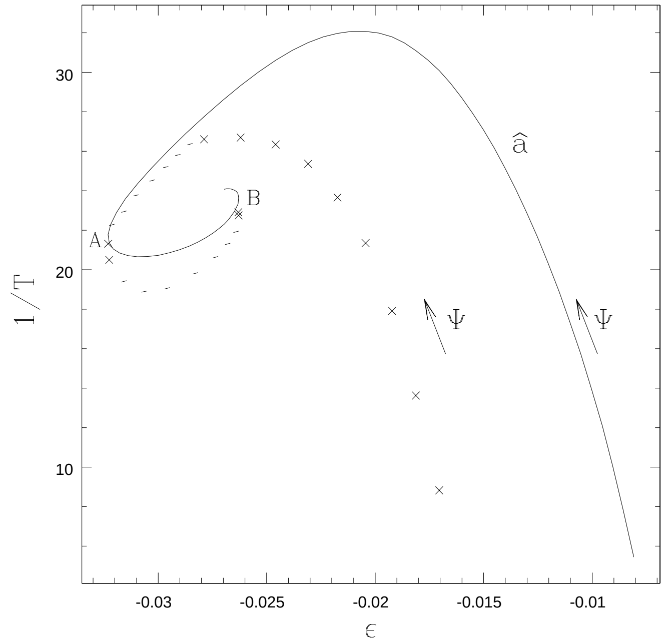

FIG. 3.—Instability spiral of $f^{(\nu)}$ models with $\nu = 1$. The solid line refers to the results obtained with the *a-Ansatz* (with $\hat{a} = \gamma^{-4/5}\hat{M}^{(\nu+1)/(5\nu)} \times \hat{Q}^{-4/(5\nu)}$). Crosses represent the global temperature from the definition $\partial S/\partial E_{\rm tot}$; other symbols indicate estimated points for which the adopted numerical differentiation is less reliable. The values of Ψ and ϵ for points A and B with a vertical tangent remain unchanged.

temperature parameter conjugate to the total energy is a, a quantity directly related to the velocity dispersion in the central regions. Basically, this was the *Ansatz* made in the discussion of the possible occurrence of the gravothermal catastrophe for the King models or for other sequences of models (e.g., see Lynden-Bell & Wood 1968; Katz 1980; Magliocchetti et al. 1998). Here we have proved that the application of a rigorous derivation, which is available in our case, gives rise to relatively modest quantitative changes in the $(E_{\rm tot}, 1/T)$ diagram for values of Ψ close to and beyond the onset of the catastrophe (see Fig. 3). However, in § 6 we draw attention to an interesting, qualitatively new phenomenon missed in the previous derivations based on the use of the *a-Ansatz*.

In passing, we note that in this regime of relatively high concentrations, the $f^{(\nu)}$ models possess one intrinsic property that makes them more appealing than the widely studied f_∞ models. This is related to the way the models compare to the phase-space properties of the products of collisionless collapse, as observed in *N*-body simulations (van Albada 1982). In fact, one noted unsatisfactory property of the concentrated f_∞ models was their excessive degree of isotropy with respect to the models produced in the simulations.[1] Here we can easily check that the anisotropy level of the concentrated $f^{(\nu)}$ models, while still within the desired (radial orbit) stability boundary and still consistent with the modest amount of radial anisotropy revealed by the observations, seems much closer to that resulting from *N*-body simulations of collisionless collapse; in particular, the anisotropy radius r_α, defined from the relation $\alpha(r_\alpha) = 1$, with $\alpha = 2 - (\langle v_\theta^2 \rangle + \langle v_\phi^2 \rangle)/\langle v_r^2 \rangle$, is close to the half-mass radius r_M (while for the f_∞ models it is about 3 times as large). This is illustrated in Figure 4. In any case, we emphasize that, while we have been clearly taking inspiration from simulations of collisionless collapse, our main interest is in comparing the structure of our models with that of observed objects rather than in providing a detailed fit to the results of *N*-body simulations.

5. THE INTERMEDIATE CONCENTRATION REGIME: THE $R^{1/4}$ LAW AND DEVIATIONS FROM IT

The intermediate concentration regime (the precise point that marks the low-concentration regime will be identified in the next section) is a regime in which the models appear to be stable, with respect not only to the gravothermal catastrophe (following the arguments provided earlier; but we should recall that the catastrophe is expected to require a sufficiently high level of effective collisionality in order to take off) but also to other instabilities (see the discussion given by Bertin et al. 1994 and references therein). The relatively wide variations, between $\Psi = 3.5$ and $\Psi = 9$, in all the representative quantities that characterize the equilibrium models suggest that this part of the sequence could be used to model the weak homology of bright elliptical galaxies (see Bertin, Ciotti, & Del Principe 2002), much like the sequence of King models is able to capture observed system-

[1] Merritt, Tremaine, & Johnstone (1989) stressed this point and thus argued that a better representation of *N*-body simulations would be obtained by considering the f_∞ family of models extended to the case of negative values of the coefficient a multiplying the energy in the exponent. Unfortunately, their proposed solution, in terms of models characterized by such a peculiar phase-space structure, is unable to reproduce both the core velocity distribution observed in numerical experiments and the modest amount of radial anisotropy revealed by observed line profiles. In addition, their proposed solution is not viable because for negative values of a the radial anisotropy level is so high that the models are violently unstable, on an extremely short timescale, with the result that their structural properties would be drastically changed by rapid evolution (Stiavelli & Sparke 1991; Bertin et al. 1994).

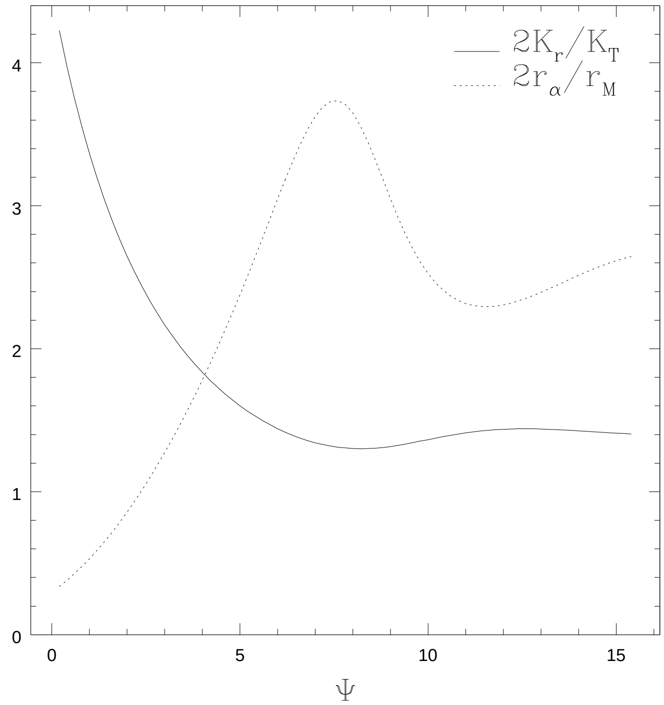

FIG. 4.—Anisotropy along the equilibrium sequence: anisotropy radius in units of the half-mass radius $2r_\alpha/r_M$ and the anisotropy parameter $2K_r/K_T$ (ratio of total kinetic energy in the radial direction to that in the tangential directions) of $f^{(\nu)}$ models with $\nu = 1$.

atic variations in the structure of globular clusters (see Djorgovski & Meylan 1994).

6. THE LOW-CONCENTRATION REGIME: NEGATIVE GLOBAL TEMPERATURE AND RADIAL ORBIT INSTABILITY

The low-concentration regime is marked by an unexpected and significant difference with respect to the low-concentration limit of the classical isothermal sphere (Bonnor 1956; Lynden-Bell & Wood 1968). In fact, while the classical case reduces to the ideal nongravitating gas, to which Boyle's law applies, for the $f^{(\nu)}$ models the system remains self-gravitating, although it develops a wide core in the density distribution. A clear-cut proof of this difference is given by inspecting the behavior of the global temperature T, identified from the thermodynamic definition $T = 1/(\partial S/\partial E_{\rm tot})$. While the temperature defined by the *a-Ansatz* remains obviously positive definite, by definition, if we look at Figure 1 we see that the global temperature T changes sign at $\Psi \approx 3.5$. This marks a drastic qualitative deviation from the classical studies.

Here we note a curious coincidence of this transition value of Ψ with the value around which the sequence is bound to change its stability properties with respect to the radial orbit instability (Polyachenko & Shukhman 1981). Indeed, the location where the sequence is expected to become unstable in this regard is precisely that defined by $\Psi \approx 3.5$, as can be judged from inspection of Figure 4; around those values of Ψ the level of radial anisotropy, as measured by $2K_r/K_T$, reaches the threshold value of 1.8–2, known to be sufficient for the excitation of the instability (the precise value of $2K_r/K_T$ corresponding to marginal stability is model dependent; for some sequences the reported value is below the range suggested by Polyachenko & Shukhman 1981).

These clues appear to be interesting and important, but more work is required before a final claim can be made that there is indeed a direct relation between the dynamic radial orbit instability and the fact that the system possesses a negative global temperature, as we found based on the simple work presented here.

7. CONCLUSIONS

A relatively straightforward and simple thermodynamic description of an equilibrium sequence constructed earlier and known to possess realistic characteristics with respect to bright elliptical galaxies shows that, on the one side, the paradigm of the gravothermal catastrophe may be adequate to explain the occurrence of realistic properties in models of collisionless stellar systems and, on the other side, a long-known dynamic instability might turn out to be interpreted in terms of a thermodynamic argument.

Probably the main open question regarding the models discussed in this paper, partially addressed in previous studies (see Stiavelli & Bertin 1987), is to what extent the quantity Q is actually reasonably well conserved during violent (partial) relaxation. It is likely that a thorough investigation of this issue may give indications that the quantity is best conserved only in certain ranges of ν. Studies of this type, combined with other dynamic and thermodynamic considerations, may turn out to lead to the identification of a family of equilibrium models with optimal behavior with respect to statistical mechanics, with respect to what we know about collisionless collapse, and with respect to the problem of providing a realistic representation of bright elliptical galaxies.

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APPENDIX

NUMERICAL INTEGRATIONS

The $f^{(\nu)}$ models are constructed by solving the Poisson equation

$$\frac{1}{\hat{r}^2}\frac{d}{d\hat{r}}\hat{r}^2\frac{d}{d\hat{r}}\hat{\Phi}(\hat{r}) = \frac{1}{\gamma}\hat{\rho}(\hat{r}, \hat{\Phi}) , \qquad (A1)$$

in which γ is considered as an eigenvalue to be determined by imposing the two natural boundary conditions $\hat{\Phi}(0) = -\Psi$ and

$\hat{\Phi}(\hat{r}) \sim -\hat{M}(\hat{r})/(4\pi\gamma\hat{r})$ as $\hat{r} \to \infty$. Here the hat over a symbol indicates that the quantity is suitably expressed in dimensionless form.

We have computed the two-dimensional integral for the density with an adaptive seven-point scheme (Berntsen, Espelid, & Genz 1991) in order to properly handle the presence of a peaked integrand for certain values of the pair $(\hat{r}, \hat{\Phi})$. The Poisson equation has then been solved with a fourth-order Runge-Kutta code by starting from $\hat{r} = 0$ with a seed value for γ and iterating the procedure until the boundary condition at large radii is matched within a certain accuracy. Finally, we have proceeded to calculate the global quantities $\hat{M}(\Psi)$, $\hat{Q}(\Psi)$, and $\hat{E}(\Psi)$ from their definitions.

In order to check the accuracy of the numerical integration we have performed the following tests: (1) the virial theorem is satisfied with accuracy of the order 10^{-6} or better; (2) the integrated mass (from its definition) and the mass derived from the asymptotic behavior of the potential at large radii are the same with accuracy of the order 10^{-4}; (3) the expression for $\hat{\Phi}(\hat{r})$ at large radii to 2 significant orders in the relevant asymptotic expansion has been checked to be correct with an accuracy from 10^{-3} to 10^{-4}; (4) the asymptotic analysis allows us to estimate the contributions to $\hat{M}$, $\hat{Q}$, and $\hat{E}$ external to a sphere of large radius R; this has been checked to help improve the numerical determination of the relevant global quantities.

We estimate that the final relative error in the quantities along the equilibrium sequence is of the order of some parts times 10^{-4} for $\hat{M}$ and $\hat{Q}$ and some parts times 10^{-5} for $\hat{E}$. The total energy is less sensitive to the finite radius truncation error, due to its $1/R^2$ convergence. The propagation of these errors leads to the error bars plotted in Figure 1, in which the entropy σ is the more difficult to determine with good accuracy.

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