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TOWARD THE QUASI–STEADY STATE ELECTRODYNAMICS OF A NEUTRON STAR

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ABSTRACT

We consider the electrodynamics of a rotating magnetized neutron star with allowance for the quasi–steady state regime of charged particle ejection from the polar cap into the inner magnetosphere. We derive basic equations within the framework of a general relativistic formalism and present the analytic solutions. We demonstrate that the quasi–steady state supply of electrons into the region of open field lines reduces the total power put into the acceleration of primary particles, although the total power corresponding to a steady state solution in general relativity is increased compared to the flat-space solution. Also, we illustrate that temporarily modulated ejection of electrons from the stellar surface results in substantial modulation of an accelerating electric field and charge density along the open field lines. We briefly outline the implications of this effect for the pulsars.
Subject headings: MHD — pulsars: general — relativity — stars: neutron

1. INTRODUCTION

It is well known that observations of radio pulsars reveal a good deal of quasi-periodic behaviors (microstructure) along with the so-called moding (see, e.g., Bartel et al. 1982) and nulling (see Deich et al. 1986 and references therein) phenomena for the discrete components of an average profile. It seems very likely that at least some of these observations may reflect complexity of the physical conditions at the stellar surface and quasi–steady state character of particle supply into the magnetosphere of a neutron star. The polar cap heating, outcoming acoustic flux, and inhomogeneity in the chemical composition and physical conditions across the polar cap of a neutron star may trigger and/or modulate the nonstationary ejection of charged particles into the inner magnetosphere of a neutron star.

Most of the existing pulsar theories exploit as a fundamental assumption the possibility of a steady state supply of charged particles from a neutron star surface into the stellar magnetosphere. Also, the theories accounting for the negative feedback of the ejected space charge onto the electric potential drop above the polar cap of a neutron star unavoidably imply that the charge density of particles ejected from the stellar surface must be precisely limited by the so-called Goldreich-Julian charge density (Goldreich & Julian 1969). Within the framework of such theories it is therefore intrinsically impossible to consider any nonstationary regime of ejection of charged particles into the stellar magnetosphere and its effect on the electrodynamics of a neutron star.

In this paper we do not intend to comment on the existing pulsar models. Instead we would like to emphasize that our knowledge of fundamentals about the pulsar magnetosphere has been boosted by the pioneering works of Goldreich & Julian (1969), Sturrock (1971), Mestel (1971), Ruderman & Sutherland (1975) and Arons & Scharlemann (1979). The subsequent achievements in this subfield and some new ideas are reviewed by Arons (1991), Michel (1991), and Mestel (1992). Although a viable self-consistent model of a global pulsar magnetosphere is not available at present, it is important that the analysis of a classical problem (with simplified treatment of the effects of an electron-positron plasma) based on the "first-principles" approach (see Mestel 1992; Mestel et al. 1985; Mestel & Pryce 1992; and references therein) provides firm grounds for the construction (albeit in the next millennium) of such a model.

In this paper we attempt to extend the electrodynamics of the inner magnetosphere of a neutron star to include the effect of the quasi–steady state supply of charged particles to the domain of open field lines. We consider the situation where the number density of physical charges ejected from the polar cap of a neutron star arbitrarily varies with time, even though it does not exceed the local Goldreich-Julian number density above the stellar surface. We introduce a formalism that may adequately and consistently treat this effect. We do not discuss the problem of closure of the global magnetospheric currents. Rather we focus on a local analysis, and present the explicit solutions for the electric potential, the component of the electric field parallel to the magnetic field, and the density of charges in the domain of open field lines in the inner magnetosphere of a rotating magnetized neutron star. The approach we discuss in this paper potentially enables us to search for the effects in pulsar emission that might be associated with the temporal variation of the physical properties at the surface of a neutron star. For example, one of the most interesting implications of our study would be the manifestation of neutron star oscillations (see, e.g., Boriakoff 1976; Van Horn 1980) in the electrodynamic properties of a pulsar magnetosphere and the dynamical features of discrete pulses.

In § 2 we derive general relativistic equations describing electrodynamics of a rotating magnetized neutron star. We rewrite these equations in the frame of reference corotating with the neutron star surface and reduce them to the form most convenient for further applications in § 2.1. In § 2.2 we provide general relativistic expressions for the magnetic field strength and derive an equation for the field lines. In § 2.3 we present the general relativistic analog of the Goldreich-Julian charge density. We discuss the space-charge–limited current approximation and justify the assumption of relativistic motion of

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electrons (positrons) in § 2.4. In § 3 we introduce the regime of quasi–steady state particle flow within the domain of open field lines and present corresponding solutions. In § 4 we briefly discuss some consequences of this regime for pulsars and summarize our principal conclusions.

2. DERIVATION OF BASIC EQUATIONS

The effects of general relativity are very important for the electrodynamics of a neutron star: the dragging of inertial frames of reference significantly affects the electric field generated in the vicinity of a rotating magnetized neutron star, while the static part of the gravitational field results in additional enhancement of the strength of electric and magnetic fields near a star. Here we shall therefore present the derivation of general relativistic electrodynamic equations suitable for various neutron star applications.

The main system of equations describing electrodynamics around a rotating neutron star can be derived in a few different ways. In this paper we will use the geometrical point of view (see, e.g., Misner, Thorne, & Wheeler 1973). In the geometrical language the electromagnetic field is described by the antisymmetric tensor (Maxwell tensor) of the second rank or by the 2-form $\boldsymbol{F}$, the introduction of which does not require coordinates. The geometrical approach is not only simpler than that of 3 + 1 ("space plus time"; see Thorne, Price, & Macdonald 1986), but it also allows us to derive the equations in the 3 + 1 form much more easily than the 3 + 1 point of view itself does. In geometrical form the Maxwell equations read

$$\boldsymbol{dF} = 0 , \tag{1}$$

$$\boldsymbol{d^*F} = \frac{4\pi}{c}\, {}^*\boldsymbol{J} , \tag{2}$$

where $\boldsymbol{d}$ is the operator of external differentiation (see, e.g., Flanders 1963; Misner et al. 1973), $^*\boldsymbol{F}$ is the tensor dual to the tensor $\boldsymbol{F}$, with

$$^*F_{\alpha\beta} = \tfrac{1}{2}F^{\mu\nu}\epsilon_{\mu\nu\alpha\beta} , \tag{3}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is an axial Levi-Civita tensor that can be produced with the aid of the antisymmetric 4-index Levi-Civita symbol $\varepsilon_{\alpha\beta\gamma\delta}$ (we shall use the lower indices for it, with $\varepsilon_{0123} = +1$). For the spacetime with a metric $g_{\mu\nu}$ we have

$$\epsilon^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{|g|}}\,\varepsilon_{\alpha\beta\gamma\delta} , \qquad \epsilon_{\alpha\beta\gamma\delta} = \sqrt{|g|}\varepsilon^{\alpha\beta\gamma\delta} , \tag{4}$$

where $g \equiv \det g_{\mu\nu}$.

In equation (2), $\boldsymbol{J} = (\rho c, \boldsymbol{j})$ is the vector of "4-current," ρ is the electric charge density, and $\boldsymbol{j}$ is the vector of the electric current density. It is important that the form of equations (1) and (2) does not depend on the choice of a coordinate basis.

The Maxwell tensor can be represented as a sum of the antisymmetric tensor products ("external products"):

$$\boldsymbol{F} = \tfrac{1}{2}F_{\alpha\beta}\,\boldsymbol{dx}^\alpha \wedge \boldsymbol{dx}^\beta , \tag{5}$$

where $\wedge$ is the symbol for external product, and $\boldsymbol{dx}^\alpha \wedge \boldsymbol{dx}^\beta$ are the "basis 2-forms" of a given local coordinate basis. Since $\boldsymbol{F}$ is invariant, we can derive the transformation equations for $F_{\alpha\beta}$ in different coordinate systems using equation (5).

Let us consider the metric of an asymptotically flat, stationary, axially symmetric spacetime around a rotating gravitating body (see, e.g., Landau & Lifshitz 1975). In spherical polar coordinates $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$, we have

$$ds^2 = A^2(c\,dt)^2 - B^2(dr)^2 - C^2(d\theta)^2 - D^2(d\phi - \omega\,dt)^2 , \tag{6}$$

where $A = B^{-1} = (1 - r_g/r)^{1/2}$ is the so-called gravitational redshift function, $C = r$, $D = r\sin\theta$, $\omega = 2GJ/c^2r^3$, $r_g = 2GM/c^2$ is the gravitational radius of a body (neutron star) of mass M, $\boldsymbol{J}$ is the angular momentum of a neutron star, c is the speed of light, and G is the gravitational constant. The metric in equation (6) is nothing else but the approximation of the Kerr metric when the ratio J/Mcr_g is small. The presence of the nondiagonal component in the metric in equation (6) results in the well-known effect of dragging of inertial frames of reference (the Lense-Thirring effect) with the angular velocity

$$\omega = \frac{2G\boldsymbol{J}}{c^2\boldsymbol{r}^3} . \tag{7}$$

Since one can always locally transform the spacetime metric into the Minkowski metric, in any point of space one always can find the orthonormal system of basis vectors and 1-forms not necessarily coinciding with the vectors tangent to the coordinate lines and with the gradients to the coordinate surfaces, respectively. These basis vectors and 1-forms are usually denoted as $\boldsymbol{e}_{\hat{\mu}}$ and $\boldsymbol{\omega}^{\hat{\mu}}$, respectively, where the caret indicates that the system is orthonormal. Let us consider the orthonormal basis of 1-forms, corresponding to the zero angular momentum observer (ZAMO; see, e.g., Thorne et al. 1986 for the introduction and discussion) in the geometry described by the metric in equation (6):

$$\boldsymbol{\omega}^{\hat{t}} = Ac\,\boldsymbol{dt} , \qquad \boldsymbol{\omega}^{\hat{r}} = B\,\boldsymbol{dr} , \qquad \boldsymbol{\omega}^{\hat{\theta}} = C\,\boldsymbol{d\theta} , \qquad \boldsymbol{\omega}^{\hat{\phi}} = D(\boldsymbol{d\phi} - \omega\,\boldsymbol{dt}) . \tag{8}$$

The tensor $\boldsymbol{F}$ can now be expressed as

$$\boldsymbol{F} = \tfrac{1}{2}f_{\hat{\mu}\hat{\nu}}\,\boldsymbol{\omega}^{\hat{\mu}} \wedge \boldsymbol{\omega}^{\hat{\nu}} , \tag{9}$$

where $f_{\hat{\mu}\hat{\nu}}$ are the components of the Maxwell tensor in the basis displayed in equations (8). The components $f_{\hat{\mu}\hat{\nu}}$ in orthonormal bases are usually referred to as physical components, since they are reduced to ordinary components of vector analysis in a flat space. The coordinate basis $\boldsymbol{e}_\alpha$ corresponding to a chosen coordinate system (t, r, θ, ϕ) has a dual basis of 1-forms:

$$\omega^t = c\,\boldsymbol{d}t\ , \qquad \omega^r = \boldsymbol{d}r\ , \qquad \omega^\theta = \boldsymbol{d}\theta\ , \qquad \omega^\phi = \boldsymbol{d}\phi\ . \tag{10}$$

Substituting the relationships in equations (8) in equation (9) and comparing the result with equation (5), we get the relationships between the components of the Maxwell tensor in the bases of 1-forms in equations (8) and (10):

$$f_{\hat{t}\hat{r}} = A^{-1}B^{-1}(F_{tr} - \omega F_{r\phi})\ , \tag{11a}$$
$$f_{\hat{t}\hat{\theta}} = A^{-1}C^{-1}(F_{t\theta} - \omega F_{\theta\phi})\ , \tag{11b}$$
$$f_{\hat{t}\hat{\phi}} = A^{-1}D^{-1}F_{t\phi}\ , \tag{11c}$$
$$f_{\hat{r}\hat{\theta}} = B^{-1}C^{-1}F_{r\theta}\ , \tag{11d}$$
$$f_{\hat{r}\hat{\phi}} = B^{-1}D^{-1}F_{r\phi}\ , \tag{11e}$$
$$f_{\hat{\theta}\hat{\phi}} = C^{-1}D^{-1}F_{\theta\phi}\ , \tag{11f}$$
$$F_{tr} = ABf_{\hat{t}\hat{r}} + \omega BDf_{\hat{r}\hat{\phi}}, \tag{12a}$$
$$F_{t\theta} = ACf_{\hat{t}\hat{\theta}} + \omega CDf_{\hat{\theta}\hat{\phi}}\ , \tag{12b}$$
$$F_{t\phi} = ADf_{\hat{t}\hat{\phi}}\ , \tag{12c}$$
$$F_{r\theta} = BCf_{\hat{r}\hat{\theta}}\ , \tag{12d}$$
$$F_{r\phi} = BDf_{\hat{r}\hat{\phi}}\ , \tag{12e}$$
$$F_{\theta\phi} = CDf_{\hat{\theta}\hat{\phi}}\ , \tag{12f}$$

Let us calculate the tensor

$$*\boldsymbol{F} = \tfrac{1}{2}{*F}_{\mu\nu}\,\boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu\ , \tag{13}$$

where

$$*F_{\mu\nu} = \tfrac{1}{2}F^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu} = \tfrac{1}{2}g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}\,\epsilon_{\alpha\beta\mu\nu} = \tfrac{1}{2}\sqrt{|g|}\,\varepsilon_{\alpha\beta\mu\nu}\,g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}\ . \tag{14}$$

Taking into account equations (12), we get

$$*F_{tr} = f_{\hat{\theta}\hat{\phi}} + A^{-1}wf_{\hat{t}\hat{\theta}}\ , \tag{15a}$$
$$*F_{t\theta} = -r(Af_{\hat{r}\hat{\phi}} + wf_{\hat{t}\hat{r}})\ , \tag{15b}$$
$$*F_{t\phi} = Ar\sin\theta f_{\hat{r}\hat{\theta}}\ , \tag{15c}$$
$$*F_{r\theta} = -A^{-1}rf_{\hat{t}\hat{\phi}\hat{\theta}}\ , \tag{15d}$$
$$*F_{r\phi} = A^{-1}r\sin\theta f_{\hat{t}\hat{\theta}}\ , \tag{15e}$$
$$*F_{\theta\phi} = -r^2\sin\theta f_{\hat{t}\hat{r}}\ , \tag{15f}$$

where $w = |\boldsymbol{w}|$, $\boldsymbol{w} = \omega\boldsymbol{m}$, and $\boldsymbol{m} \equiv r\sin\theta\,\boldsymbol{e}_{\hat{\phi}}$ is the Killing vector corresponding to the axial symmetry.

To complete the formulation, we should present the expression for the 4-current:

$$*\boldsymbol{J} = \rho cBCD(\boldsymbol{d}r \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d}\phi - \omega\,\boldsymbol{d}r \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d}t) - A(j_{\hat{r}}\,CD\,\boldsymbol{d}\theta \wedge \boldsymbol{d}\phi \wedge \boldsymbol{d}t - j_{\hat{\theta}}\,DB\,\boldsymbol{d}r \wedge \boldsymbol{d}\phi \wedge \boldsymbol{d}t + j_{\hat{\phi}}\,CB\,\boldsymbol{d}r \wedge \boldsymbol{d}\theta \wedge \boldsymbol{d}t)\ . \tag{16}$$

Now let us consider the physical components of the Maxwell tensor $f_{\hat{\alpha}\hat{\beta}}$:

$$f_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} 0 & E_{\hat{r}} & E_{\hat{\theta}} & E_{\hat{\phi}} \\ -E_{\hat{r}} & 0 & -B_{\hat{\phi}} & B_{\hat{\theta}} \\ -E_{\hat{\theta}} & B_{\hat{\phi}} & 0 & -B_{\hat{r}} \\ -E_{\hat{\phi}} & -B_{\hat{\theta}} & B_{\hat{r}} & 0 \end{pmatrix}, \tag{17}$$

where the physical components of vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ correspond to the "ZAMO" in a local orthonormal basis in spherical geometry:

$$\boldsymbol{e}_{\hat{r}} = A\boldsymbol{e}_r = A\frac{\partial}{\partial r}\ , \qquad \boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\,\boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}\ , \qquad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\,\boldsymbol{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\ . \tag{18}$$

Inserting equation (5) in equation (1) and making use of equations (12), we get the first couple of Maxwell equations:

$$\nabla \cdot \boldsymbol{B} = 0\ , \tag{19}$$

$$\nabla \times (\alpha\boldsymbol{E}) = -\frac{1}{c}\left(\frac{\partial\boldsymbol{B}}{\partial t} + \mathscr{L}_{\omega\boldsymbol{m}}\boldsymbol{B}\right). \tag{20}$$

From equation (2), taking into account expressions (13) and (16), we get the second couple of Maxwell equations:

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho \ , \tag{21}$$

$$\nabla \times (\alpha \boldsymbol{B}) = \frac{1}{c}\left(\frac{\partial \boldsymbol{E}}{\partial t} + \mathscr{L}_{\omega \boldsymbol{m}} \boldsymbol{E}\right) + \frac{4\pi}{c}\alpha \boldsymbol{j} \ , \tag{22}$$

where $\alpha = (1 - r_g/r)^{1/2}$ ($\equiv A$, as denoted in the metric given in eq. [6]); $\rho = \sum_k q_k n_k$, q_k is the charge of a particle, n_k is the particle number density, and summation is over all species; $\mathscr{L}_{\omega \boldsymbol{m}}$ is the Lie derivative along the vector $\omega \boldsymbol{m}$.

Note that

$$\mathscr{L}_{\omega \boldsymbol{m}} \boldsymbol{B} \equiv -\nabla \times (\boldsymbol{w} \times \boldsymbol{B}) \ , \tag{23}$$

$$\mathscr{L}_{\omega \boldsymbol{m}} \boldsymbol{E} \equiv -[\nabla \times (\boldsymbol{w} \times E) - w(\nabla \cdot E)] \ . \tag{24}$$

Equations (19)–(22) are fully identical with the equations obtained within the framework of the 3 + 1 formalism (see, e.g., Thorne et al. 1986). Note that all electrodynamic quantities ($\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{j}$, and ρ) in these equations are such as measured by ZAMO. In addition, the above system of equations must be supplemented with the charge continuity equation (which is a consequence of eq. [22]):

$$\frac{\partial \rho}{\partial t} + \omega \boldsymbol{m} \cdot \nabla \rho = -\nabla \cdot (\alpha \boldsymbol{j}) \ . \tag{25}$$

Here and in what follows it is implied that the standard operators of vector analysis (gradient, divergence, and curl) should be taken in corresponding curvilinear coordinates (see the basis given by eqs. [18]).

2.1. *Electrodynamic Equations in the Frame of Reference Corotating with a Neutron Star*

For our further purposes it is convenient to work in the coordinate system corotating with a star. If a neutron star rotates with an angular velocity $\boldsymbol{\Omega}$ relative to the distant observer, then in the corotating frame, equation (20), together with equation (23), takes the form

$$\nabla \times \left[\alpha \boldsymbol{E} - \frac{1}{c}(\boldsymbol{w} - \boldsymbol{u}) \times \boldsymbol{B}\right] = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \ , \tag{26}$$

where $\boldsymbol{u} = \Omega r \sin\theta\, \boldsymbol{e}_{\hat{\phi}}$.

We assume that the magnetic field of a neutron star is stationary in the corotating frame, which implies that in equation (26) we can set $\partial \boldsymbol{B}/\partial t \equiv 0$. Then from equation (26) it follows that

$$\alpha \boldsymbol{E} - \frac{1}{c}(\boldsymbol{w} - \boldsymbol{u}) \times \boldsymbol{B} = -\nabla \Phi \ , \tag{27}$$

where Φ is a scalar electric potential.

Taking the divergence of equation (27) and making use of equation (21), we get

$$\nabla \cdot \left[\frac{1}{\alpha}\nabla \Phi + \frac{1}{\alpha c}(\boldsymbol{u} - \boldsymbol{w}) \times \boldsymbol{B}\right] = -4\pi\rho \ . \tag{28}$$

Finally, to the system of basic electrodynamic equations presented above, one should add the general relativistic equation of motion for a particle of charge q (see Thorne & Macdonald 1982; Macdonald & Thorne 1982):

$$\alpha^{-1}\frac{d}{dt}\boldsymbol{p} = \mu\gamma \boldsymbol{g} + q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) + \boldsymbol{f} \ , \tag{29}$$

where $d/dt \equiv \partial + \omega \boldsymbol{m} \cdot \nabla + \alpha \boldsymbol{v} \cdot \nabla$, $\partial/\partial t + \omega \boldsymbol{m} \cdot \nabla$ is the global time derivative along ZAMO trajectories, $\boldsymbol{p} = \gamma\mu \boldsymbol{v}$ is the momentum of a particle, $\gamma = (1 - v^2/c^2)^{-1/2}$ is the particle Lorentz factor, μ is the rest mass of a particle, $\boldsymbol{f}$ is external force other than electromagnetic, and $\boldsymbol{g}$ is the gravitational acceleration. It must be pointed out that in the present problem acceleration due to gravity may be justifiably ignored.

We shall solve equations (23) and (28) for a given structure of the external magnetic field of a neutron star. The latter should be determined from the solution of equations (19) and (22). Note that in the problem under consideration equation (22) is simply reduced to $\nabla \times (\alpha \boldsymbol{B}) = 0$, since the terms on its right-hand side are of order of $\lesssim R^{-1}(\Omega R/c)^2 B$ (here R is the radius of a neutron star) within the light-cylinder radius $R_{\rm LC} = c/\Omega$ and can be neglected. In other words, we assume that in the inner magnetosphere of a neutron star the effects of the magnetospheric currents on the external magnetic configuration of a neutron star are negligibly small.

2.2. *External Magnetic Field of a Neutron Star*

We assume that the external magnetic field of a neutron star may be well represented by the dipole component (in magnetic polar coordinates r, ϑ, and φ):

$$\boldsymbol{B}^d = -\frac{1}{2\pi r \sin\vartheta}\, \boldsymbol{e}_{\hat{\varphi}} \times \nabla[\Psi(r)\sin^2\vartheta] \ , \tag{30}$$

where

$$\Psi(r) \equiv \int \boldsymbol{B} \cdot d\boldsymbol{\Sigma} = \int_0^{\pi} \int_0^{2\pi} B_{\hat{r}} r^2 \sin \vartheta \, d\vartheta \, d\varphi \tag{31}$$

is the magnetic flux through the magnetic hemisphere of radius r.

The solution of $\nabla \times (\alpha \boldsymbol{B}) = 0$ requires the following form for the function Ψ:

$$\Psi(r) = \pi R^2 B_0 \frac{f(\eta)}{f(1)} \frac{1}{\eta}, \qquad f(\eta) = -3\left(\frac{\eta}{\varepsilon}\right)^3 \left[\ln\left(1 - \frac{\varepsilon}{\eta}\right) + \frac{\varepsilon}{\eta}\left(1 + \frac{\varepsilon}{2\eta}\right)\right], \tag{32}$$

where B_0 is the normalization value of the magnetic field strength at the stellar surface, $\varepsilon \equiv r_g/R$, and $\eta \equiv r/R$. When $\varepsilon/\eta \ll 1$, then $f(\eta) \approx 1 + (3/4)\varepsilon/\eta + (3/5)(\varepsilon/\eta)^2 + \cdots$.

Now the explicit expressions for the physical components of the magnetic field of a neutron star can be written as

$$B^d_{\hat{r}} \equiv \frac{1}{\pi} \frac{\Psi}{r^2} \cos \vartheta = B_0 \frac{f(\eta)}{f(1)} \frac{1}{\eta^3} \cos \vartheta , \tag{33}$$

$$B^d_{\hat{\vartheta}} \equiv -\frac{\alpha}{2\pi r} \frac{\partial \Psi}{\partial r} \sin \vartheta = \frac{1}{2} B_0 \alpha \left[-2 \frac{f(\eta)}{f(1)} + \frac{3}{f(1)(1 - \varepsilon/\eta)}\right] \frac{1}{\eta^3} \sin \vartheta . \tag{34}$$

Note that the solution of this form was first derived by Ginzburg & Ozernoy (1964). Later on, this solution was reproduced in a number of papers (see, e.g., Wasserman & Shapiro 1983; Muslimov & Tsygan 1986, 1990). The general solution for arbitrary multipoles is presented by Anderson & Cohen (1970) and by Muslimov & Tsygan (1986) in terms of the Legendre functions of the second order and hypergeometric functions, respectively.

Along with the variable η we shall use the variable $\xi = \vartheta/\Theta$ $(0 \le \xi \le 1)$ labeling the open field lines. Here Θ is the polar angle (magnetic colatitude) of a radius vector sliding along the last open field line. Then the family of the magnetic field lines will be described by the equation

$$\Psi(r) \sin^2 \vartheta = \Psi(R) \sin^2 \vartheta_0, \tag{35}$$

where ϑ_0 is the magnetic colatitude at the footpoint of a field line at the stellar surface. Thus, using expression (32), we can write the general relativistic formula for the open magnetic field line:

$$\vartheta \equiv \xi\Theta = \sin^{-1} \left\{\left[\eta \frac{f(1)}{f(\eta)}\right]^{1/2} \sin (\xi\Theta_0)\right\}, \tag{36}$$

where Θ_0 is the magnetic colatitude at the footpoint of a last open field line defined as follows. For the last open field line the radius vector makes an angle $\vartheta = \pi/2$ at the light cylinder, so that from equation (36) we obtain $\Theta_0 \equiv \sin^{-1} [R/R_{\rm LC} f(1)]^{1/2}$, since $f(\eta_{\rm LC}) = 1 + O(r_g/R_{\rm LC}) \approx 1$.

2.3. *Goldreich-Julian Charge Density in General Relativity*

Equation (28) can be rewritten as (cf. Muslimov & Tsygan 1992)

$$\nabla \cdot \left(\frac{1}{\alpha} \nabla\Phi\right) = -4\pi(\rho - \rho_{\rm GJ}) , \tag{37}$$

where

$$\rho_{\rm GJ} \equiv -\frac{1}{4\pi c} \nabla \cdot \left[\frac{1}{\alpha} (\boldsymbol{u} - \boldsymbol{w}) \times \boldsymbol{B}\right] = -\frac{1}{4\pi c} \nabla \cdot \left[\frac{1}{\alpha}\left(1 - \frac{\kappa}{\eta^3}\right)\boldsymbol{u} \times \boldsymbol{B}\right] \tag{38}$$

is the general relativistic analog of the Goldreich-Julian (Goldreich & Julian 1969) charge density. Here the parameter $\kappa \equiv \varepsilon\beta$, where $\beta \equiv I/I_0$, I is the moment of inertia of a neutron star, and $I_0 = MR^2$.

In this paper the analysis will be limited (the natural limit may be set up, e.g., by the process of production of an electron-positron plasma) by the region of open field lines lying well within the light-cylinder radius, and we can therefore use *a small-angle approximation*, $\vartheta \ll 1$. In this approximation, we get

$$\rho_{\rm GJ} \simeq -\frac{\Omega B_0}{2\pi c \alpha \eta^3} \frac{f(\eta)}{f(1)} \left[\left(1 - \frac{\kappa}{\eta^3}\right) \cos \chi + \frac{3}{2} H(\eta)\xi\Theta \sin \chi \cos \varphi\right], \tag{39}$$

where $H(\eta) = \varepsilon/\eta - \kappa/\eta^3 + (1 - 3\varepsilon/2\eta + \kappa/2\eta^3)/[f(\eta)(1 - \varepsilon/\eta)]$, χ is the angle between the magnetic and spin axes of a neutron star, and φ is the magnetic azimuthal angle.

2.4. *Space-Charge–limited Current Approximation*

In most studies of a pulsar magnetosphere allowing for the flow of charged particles in the domain of open field lines, the so-called space-charge–limited current (flow) approximation is exploited. The adequacy of this approximation was additionally reinforced by the finding that the work function of a neutron star surface with the canonical value of the magnetic field strength of 10^{12} G is $\sim$100 eV (Jones 1985, 1986; Neuhauser, Langanke, & Koonin 1986; Neuhauser, Koonin, &

Langanke 1987) rather than ~ 1 keV (Ruderman 1971), which implies significantly facilitated ejection of charges from the surface. The regime of the space-charge–limited current, well known from the pioneering studies of diodes, has been introduced in the theory of pulsars by Sturrock (1971), and then extensively discussed by many authors (see, e.g., Michel 1974; Tademaru 1974; Cheng & Ruderman 1977; Fawley, Arons, & Scharlemann 1977; Arons & Scharlemann 1979; Arons 1981). It is worthwhile to recall here the physical basis for space-charge limitation of current (see, e.g., Pederson, Studer, & Whinnery 1966). It is known from the operation of a diode that, as the temperature of a cathode (emitter) is increased, electrons are emitted and a negative space charge appears in the region between cathode and anode. This space charge depresses the potential, and for increased electron emission the space charge becomes sufficient to lower the electric field at the cathode to zero. Thus, because of the negative charge above the cathode, only a limited maximum current can flow for a given voltage. This regime of operation is called space-charge limitation of current. Within the context of pulsar electrodynamics this regime implies that at the stellar surface (at $r = R$) the condition $E_{\|} = 0$ holds, where $E_{\|}$ is the electric field component parallel to the magnetic field. Also, within this approximation the charged particles are treated as a cold, dissipation-free gas.

In this paper we investigate the regime where the current has a steady state component and, superimposed on it, a small-amplitude alternating component. This regime still implies the space-charge limitation of current and zero steady state electric field at the stellar surface. The electric potential Φ at the stellar surface is equal to zero (see § 3.2). However, the value of the alternating component of the electric field parallel to the magnetic field may not vanish at the stellar surface and is determined by the dynamics of ejection of charges from the surface. The new and important feature of this regime is that it allows the non–steady state ejection of charged particles from the stellar surface and their modulated flow in the domain of open field lines.

Now let us estimate the characteristic height h_c above which electrons become relativistic. Near the very surface the electric potential $\Phi \sim \Phi_0(h/R)^2$, where $\Phi_0 = (\Omega R^2/c)B_0$ and h is the height above the stellar surface. From the condition $|e|\Phi \sim \mu_e c^2$, we find that $h_c \sim 3(P/B_{12})^{1/2}$ cm, where P is the pulsar spin period in seconds and $B_{12} = B_0/10^{12}$ G. This estimate means that we can adequately treat electrons (positrons) as moving relativistically from the very surface of a neutron star (cf. Sturrock 1971).

3. QUASI–STEADY STATE REGIME OF CHARGED PARTICLE FLOW

Consider the flow of relativistic particles in the region of open field lines. We assume that the density of charged particles depends on time, and that the time dependence is determined by the nonstationarity of the very process of injection of particles into the region of open field lines. The general expression for the particle space charge density can be written as

$$\rho(\eta, \xi, t) = \bar{\rho}(\eta, \xi)[1 + F(\xi, \eta, t)] , \tag{40}$$

where $\bar{\rho}$ is the "steady state" part of the solution. We will assume $F(\xi, \eta, t) = a(\xi)x(\eta, t)$, where a and x are some functions to be specified at $\eta = 1$. Separability in equation (40) means that the perturbation of particle supply along the field lines occurs coherently, which may not be the case in general. Note that the function $a(\xi)$ characterizes the distribution of particle charge density across the polar cap, and the function $x(\eta, t)$ characterizes time-dependent distribution of particle charge density along each individual magnetic field line.

From the charge conservation equation (25) we get

$$\bar{\rho} a \frac{\partial x}{\partial t} + c\boldsymbol{B} \cdot \nabla \left\{ \frac{\alpha}{B} \bar{\rho}[1 + a(\xi)x(\eta, t)] \right\} = 0 . \tag{41}$$

The steady state solution of equation (25) implies that on a field line we have (see eqs. [30] and [35])

$$\frac{\alpha \bar{\rho}}{B} = \mathscr{F}(\xi) , \tag{42}$$

where $\mathscr{F}$ is some function of ξ alone and is determined from the boundary condition at the stellar surface (e.g., a linear function, $a_1 + a_2 \xi$, where a_1 and a_2 are some constants). Then, in a small-angle approximation, equation (41) reduces to

$$\frac{\partial x}{\partial t} + \alpha^2 \left(\frac{c}{R}\right) \frac{\partial x}{\partial \eta} = 0 , \tag{43}$$

where additional α comes from the radial gradient in curvilinear coordinates (see eqs. [18]). The solution of this equation is an arbitrary function of the argument (*characteristics*)

$$t' = t - \frac{R}{c} \int_1^{\eta} \alpha^{-2} \, d\eta' = t - \frac{R}{c} \left[\eta - 1 + \varepsilon \ln\left(\frac{\eta - \varepsilon}{1 - \varepsilon}\right) \right] .$$

As a special illustrative case, let us assume the harmonic functional dependence of t', say, $\cos(\omega_0 t')$, where ω_0 is the angular frequency with which charged particles are ejected along each field line from the stellar surface. Then for the solution (40) we can write

$$\rho = \bar{\rho}(\eta, \xi)[1 + a(\xi) \cos(\omega_0 t')] , \tag{44}$$

where $a(\xi)$ is a function specified by the process of the non–steady state supply of plasma into the region of open magnetic field lines in the inner magnetosphere. Note that in the most general case the above expression should contain a Fourier integral over ω_0.

Using equations (33) and (39) for guidance, we shall search for the solution for the function ρ in the following form:

$$\rho(\eta, \xi, t') = \frac{\Omega B_0}{2\pi c} \frac{1}{\alpha\eta^3} \frac{f(\eta)}{f(1)} \left\{[-A_1(\xi) - A_2(\xi) \cos(\omega_0 t')] \cos\chi - \left(\frac{3}{2}\right)[D_1(\xi) + D_2(\xi) \cos(\omega_0 t')] \sin\chi \cos\varphi\right\} . \quad (45)$$

Here $A_1(\xi)$ and $D_1(\xi)$ are the functions characterizing the steady state solution, while $A_2(\xi)$ and $D_2(\xi)$ are, in general, unknown functions characterizing the amplitudes of the non–steady state components of the charge density. The form of these functions, however, may be reasonably constrained under some simplifying assumptions (see below).

Let us introduce the auxiliary function $\Upsilon = \eta\Phi/\Phi_0$ [where $\Phi_0 = (\Omega R/c)RB_0$]; then in a small-angle approximation from equation (37) we get

$$\frac{\partial^2\Upsilon}{\partial\eta^2} + \sigma^2(\eta)\frac{1}{\xi}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{1}{\xi}\frac{\partial^2}{\partial\varphi^2}\right]\Upsilon = -\frac{2}{\eta^2(1-\varepsilon/\eta)}\frac{f(\eta)}{f(1)}\left\{\left[1 - \frac{\kappa}{\eta^3} - A_1(\xi) - A_2(\xi)\cos(\omega_0 t')\right]\cos\chi + \frac{3}{2}[H(\eta)\xi\Theta(\eta) - D_1(\xi) - D_2(\xi)\cos(\omega_0 t')]\sin\chi\cos\varphi\right\}, \quad (46)$$

where $\sigma(\eta) \equiv [\eta\Theta(\eta)(1 - \varepsilon/\eta)^{1/2}]^{-1}$.

We shall look for the solution of this equation in the form

$$\Upsilon(\eta, \xi, t) = P(\eta, \xi, t)\cos\chi + Q(\eta, \xi, t)\sin\chi\cos\varphi , \quad (47)$$

where P and Q are the functions to be obtained from the following equations:

$$\left[\frac{\partial^2}{\partial\eta^2} + \sigma^2(\eta)\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)\right]P = -\frac{2}{\eta^2(1-\varepsilon/\eta)}\frac{f(\eta)}{f(1)}\left[1 - \frac{\kappa}{\eta^3} - A_1(\xi) - A_2(\xi)\cos(\omega_0 t')\right], \quad (48)$$

$$\left\{\frac{\partial^2}{\partial\eta^2} + \sigma^2(\eta)\frac{1}{\xi}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) - \frac{1}{\xi}\right]\right\}Q = -\frac{3}{\eta^2(1-\varepsilon/\eta)}\frac{f(\eta)}{f(1)}[H(\eta)\xi\Theta(\eta) - D_1(\xi) - D_2(\xi)\cos(\omega_0 t')] . \quad (49)$$

3.1. *Boundary Conditions and Method of Solution*

We shall solve the system of equations (48) and (49) subject to the appropriate boundary conditions. First, following Goldreich & Julian (1969), we assume that the surface of a polar cap and that formed by the last open field lines can be treated as electric equipotentials, and we therefore adopt the condition $\Phi = 0$ at these surfaces. Second, we should require (cf. the condition imposed by the space-charge limitation) that the steady state component of the electric field parallel to the magnetic field vanishes at the polar cap surface, $\bar{E}_{\parallel}(r = R) = 0$. The oscillating component of the electric field parallel to the magnetic field, $E_{\parallel}^{\rm osc}(r = R)$, is a time-dependent function determined by the charged particle ejection from the surface.

The system of equations (48) and (49) can be solved by representing the functions P and Q as Fourier-Bessel series:

$$P(\eta, \xi, t) = \sum_{i=1}^{\infty} P_i(\eta, t)J_0(k_i\xi) , \qquad P_i(\eta, t) \equiv \frac{2}{[J_1(k_i)]^2}\int_0^1 \xi P(\eta, \xi, t)J_0(k_i\xi)d\xi , \quad (50)$$

$$Q(\eta, \xi, t) = \sum_{i=1}^{\infty} Q_i(\eta, t)J_1(\tilde{k}_i\xi) , \qquad Q_i(\eta, t) \equiv \frac{2}{[J_2(\tilde{k}_i)]^2}\int_0^1 \xi Q(\eta, \xi, t)J_1(\tilde{k}_i\xi)d\xi , \quad (51)$$

where k_i and $\tilde{k}_i$ are the positive zeros of the functions J_0 and J_1, respectively, with $k_{i+1} > k_i$ and $\tilde{k}_{i+1} > \tilde{k}_i$. The functions P and Q then automatically satisfy the condition of equipotentiality at the surface formed by the last open field lines (at $\xi = 1$).

The equations for the Fourier-Bessel components, P_i and Q_i, are

$$\left[\frac{d^2}{d\eta^2} - \gamma_i^2(\eta)\right]P_i = -\frac{2}{\eta^2(1-\varepsilon/\eta)}\frac{f(\eta)}{f(1)}\left\{\left(1 - \frac{\kappa}{\eta^3}\right)\left[\frac{2}{k_i J_1(k_i)}\right] - A_{1i} - A_{2i}\cos(\omega_0 t')\right\}, \quad (52)$$

$$\left[\frac{d^2}{d\eta^2} - \tilde{\gamma}_i^2(\eta)\right]Q_i = -\frac{3}{\eta^2(1-\varepsilon/\eta)}\frac{f(\eta)}{f(1)}\left\{\left[\frac{2}{\tilde{k}_i J_2(\tilde{k}_i)}\right]H(\eta)\Theta(\eta) - D_{1i} - D_{2i}\cos(\omega_0 t')\right\}, \quad (53)$$

where $\gamma_i = k_i\sigma$ and $\tilde{\gamma}_i = \tilde{k}_i\sigma$ (σ is defined just following eq. [46]).

3.2. *The Solution near the Stellar Surface, at $z = \eta - 1 \ll 1$*

Near the stellar surface, when $z = \eta - 1 \ll 1$, we can linearize and easily solve equations (52) and (53) using appropriate boundary conditions. The conditions of equipotentiality of the stellar surface and zero steady state electric field at $r = R$ imply that

$$P_i(z = 0) = Q_i(z = 0) = 0 , \quad (54)$$

and

$$\bar{P}_i(z = 0) \equiv \left(\frac{d\bar{P}_i}{dz}\right)\bigg|_{z=0} = 0 , \qquad \bar{Q}_i(z = 0) \equiv \left(\frac{d\bar{Q}_i}{dz}\right)\bigg|_{z=0} = 0 . \quad (55)$$

Here the overbars denote the time-averaged part.

Then the solution of equations (52) and (53) is

$$P_i(z) = \frac{6\kappa}{1-\varepsilon}\frac{1}{\gamma_i^3(1)}\frac{2}{k_i J_1(k_i)}\left[e^{-\gamma_i(1)z} + \gamma_i(1)z - 1\right] + \frac{2}{1-\varepsilon}\frac{A_{2i}}{\Gamma_i^2}\left[e^{-\gamma_i(1)z}\cos(\omega_0 t) - \cos(\omega_0 t - vz)\right], \quad (56)$$

$$Q_i(z) = \frac{3}{1-\varepsilon}\Theta_0\,\delta(1)H(1)\frac{1}{\tilde{\gamma}_i^3(1)}\frac{2}{\tilde{k}_i J_2(\tilde{k}_i)}\left[e^{-\tilde{\gamma}_i(1)z} + \tilde{\gamma}_i(1)z - 1\right] - \frac{3}{1-\varepsilon}\frac{D_{2i}}{\tilde{\Gamma}_i^2}\left[e^{-\tilde{\gamma}_i(1)z}\cos(\omega_0 t) - \cos(\omega_0 t - vz)\right], \quad (57)$$

where

$$\delta \equiv d\ln(H\Theta)/d\eta = \{-(2\varepsilon - 4\kappa/\eta^2)/\eta^2 + 3[(\varepsilon - \kappa/\eta^2)/\eta - (4/3 - \varepsilon/\eta - 3/2f)(1 - 3\varepsilon/2\eta + \kappa/2\eta^3)/(1 - \varepsilon/\eta)]/[(\eta - \varepsilon)f]\}/\mathrm{H}(\eta),$$

$$v = \omega_0 R/[c(1-\varepsilon)],$$

$$\Gamma_i^2 \equiv \gamma_i^2(1) + v^2 = f(1)(c/\Omega R)\{k_i^2 + (\Omega R/c)^3(\omega_0/\Omega)^2\varepsilon^2/[f(1)(1-\varepsilon)]\}/(1-\varepsilon),$$

and

$$\tilde{\Gamma}_i^2 \equiv \tilde{\gamma}_i^2(1) + v^2 = f(1)(c/\Omega R)\{\tilde{k}_i^2 + (\Omega R/c)^3(\omega_0/\Omega)^2\varepsilon^2/[f(1)(1-\varepsilon)]\}/(1-\varepsilon).$$

The γ_i and $\tilde{\gamma}_i$ are defined after equation (53).

For the sake of illustration, in this section we shall assume that $\Gamma_i \approx \gamma_i(1)$ and $\tilde{\Gamma}_i \approx \tilde{\gamma}_i(1)$, which means that we will restrict ourselves to the case where $\omega_0 \ll \Omega(R_{\mathrm{LC}}/R)^{3/2}$, i.e. where the particles ejection from the surface is modulated with the frequency $\nu_0 \ll (P/1\ \mathrm{s})^{1/2}$ MHz. Also, in what follows we shall assume that $A_2(\xi) = a =$ constant and $D_2(\xi) = d\xi$, where $d =$ constant. Thus our choice of functional dependences of $A_2(\xi)$ and $D_2(\xi)$ means that the non–steady state and steady state components [since $A_1(\xi) =$ constant, and $D_1(\xi) =$ constant $\times\ \xi$; see eqs. (58) and (59)] of the charged particle supply into the region of open field lines have similar profiles as a function of ξ across this region. In other words, the non–steady state particle supply along the magnetic field lines occurs in proportion to the intensity of the steady state particle flow. Finally, from the solutions presented below (see, e.g., eq. [71]) the constants a and d can be defined as the amplitudes of the variation of the charge density about the Goldreich-Julian charge density at the stellar surface, normalized by the quantity $\Omega B_0/[2\pi c(1-\varepsilon)^{1/2}]$. Now we can calculate

$$A_{2i} \equiv \frac{2}{[J_1(k_i)]^2}\int_0^1 \xi A_2(\xi)J_0(k_i\xi)d\xi = \frac{2}{k_i J_1(k_i)}\,a$$

and

$$D_{2i} \equiv \frac{2}{[J_2(\tilde{k}_i)]^2}\int_0^1 \xi D_2(\xi)J_1(\tilde{k}_i\xi)d\xi = \frac{2}{\tilde{k}_i J_2(\tilde{k}_i)}\,d.$$

The steady state amplitudes A_1 and D_1 are, respectively,

$$A_1(\xi) \equiv \sum_{i=1}^{\infty} A_{1i}J_0(k_i\xi) \approx 1 - \kappa \quad (58)$$

and

$$D_1(\xi) \approx H(1)\Theta_0\xi. \quad (59)$$

By performing inverse Fourier-Bessel transformations, we get the following expressions for the functions P and Q:

$$\begin{aligned} P(z,\xi,t) \equiv \sum_{i=1}^{\infty} P_i(z)J_0(k_i\xi) = \frac{2}{1-\varepsilon}\Bigg\{6\kappa\Bigg[&(1-\varepsilon)^{3/2}\Theta_0^3\sum_{i=1}^{\infty} e^{-\gamma_i(1)z}\frac{J_0(k_i\xi)}{k_i^4 J_1(k_i)} \\ &+ (1-\varepsilon)\Theta_0^2 z\sum_{i=1}^{\infty}\frac{J_0(k_i\xi)}{k_i^3 J_1(k_i)} - (1-\varepsilon)^{3/2}\Theta_0^3\sum_{i=1}^{\infty}\frac{J_0(k_i\xi)}{k_i^4 J_1(k_i)}\Bigg] \\ &+ \frac{1}{4}\Theta_0^2(1-\varepsilon)a\Bigg[\cos(\omega_0 t)\sum_{i=1}^{\infty} e^{-\gamma_i(1)z}\frac{8J_0(k_i\xi)}{k_i^3 J_1(k_i)} - (1-\xi^2)\cos(\omega_0 t - vz)\Bigg]\Bigg\}, \end{aligned} \quad (60)$$

$$\begin{aligned} Q(z,\xi,t) \equiv \sum_{i=1}^{\infty} Q_i(z)J_1(\tilde{k}_i\xi) = \frac{6}{1-\varepsilon}\Bigg\{&\delta(1)H(1)\Theta_0^3(1-\varepsilon)\Bigg[\Theta_0(1-\varepsilon)^{1/2} \\ &\times\sum_{i=1}^{\infty} e^{\tilde{\gamma}_i(1)z}\frac{J_1(\tilde{k}_i\xi)}{\tilde{k}_i^4 J_2(\tilde{k}_i)} + z\sum_{i=1}^{\infty}\frac{J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)} - \Theta_0(1-\varepsilon)^{1/2}\sum_{i=1}^{\infty}\frac{J_1(\tilde{k}_i\xi)}{\tilde{k}_i^4 J_2(\tilde{k}_i)}\Bigg] \\ &- \frac{1}{16}\Theta_0^2(1-\varepsilon)d\Bigg[\cos(\omega_0 t)\sum_{i=1}^{\infty} e^{-\tilde{\gamma}_i(1)z}\frac{16J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)} - \xi(1-\xi^2)\cos(\omega_0 t - vz)\Bigg]\Bigg\}. \end{aligned} \quad (61)$$

Now we can present the general expression for the number density of charged particles,

$$\rho(\eta, \xi, t') = \frac{\Omega B_0}{2\pi c} \frac{1}{\alpha\eta^3} \frac{f(\eta)}{f(1)} \left\{[-A_1(\xi) - A_2(\xi)\cos(\omega_0 t')]\cos\chi - \frac{3}{2}[D_1(\xi) + D_2(\xi)\cos(\omega_0 t')]\sin\chi\cos\varphi\right\}$$
$$= -\frac{\Omega B_0}{2\pi c} \frac{1}{\alpha\eta^3} \frac{f(\eta)}{f(1)} \left\{[1 - \kappa + a\cos(\omega_0 t')]\cos\chi + \frac{3}{2}[\Theta_0 H(1)\xi + d\xi\cos(\omega_0 t')]\sin\chi\cos\varphi\right\}. \quad (62)$$

Thus,

$$E_\parallel \equiv \bar{E}_\parallel + E_\parallel^{\rm osc} = -\frac{1}{\alpha}\nabla_\parallel \Phi = -\frac{\Phi_0}{R}\left[\frac{\partial}{\partial\eta}\left(\frac{P}{\eta}\right)\cos\chi + \frac{\partial}{\partial\eta}\left(\frac{Q}{\eta}\right)\sin\chi\cos\varphi\right]_{|\eta=1+z}, \quad (63)$$

where $\bar{E}_\parallel$ and $E_\parallel^{\rm osc}$ are the steady state and oscillating components, respectively. These components read

$$\bar{E}_\parallel = -\frac{\Phi_0}{R}\left\{\Theta_0^2\left[12\kappa \sum_{i=1}^{\infty} \frac{1 - e^{-\gamma_i(1)(\eta-1)}}{k_i^3 J_1(k_i)} J_0(k_i\xi)\right]\cos\chi + 6\left[\Theta_0^3 \delta(1) H(1) \sum_{i=1}^{\infty} \frac{1 - e^{-\tilde{\gamma}_i(1)(\eta-1)}}{\tilde{k}_i^3 J_2(\tilde{k}_i)} J_1(\tilde{k}_i\xi)\right]\sin\chi\cos\varphi\right\}, \quad (64)$$

$$E_\parallel^{\rm osc} = \frac{\Phi_0}{R}\Theta_0^2\left\{a\left[\cos(\omega_0 t)\sum_{i=1}^{\infty} e^{-\gamma_i(1)(\eta-1)} \frac{8J_0(k_i\xi)}{k_i^3 J_1(k_i)} - (1-\xi^2)\cos[\omega_0 t - \upsilon(\eta-1)] + \upsilon(1-\xi^2)\sin[\omega_0 t - \upsilon(\eta-1)]\right.\right.$$
$$\left. + \upsilon\cos(\omega_0 t)\sum_{i=1}^{\infty} e^{-\gamma_i(1)(\eta-1)}\frac{8J_0(k_i\xi)}{k_i^3 J_1(k_i)}\right]\cos\chi + \frac{3}{4}d\left[\cos(\omega_0 t)\sum_{i=1}^{\infty} e^{-\tilde{\gamma}_i(1)(\eta-1)}\frac{16J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)}\right.$$
$$- \xi(1-\xi^2)\cos[\omega_0 t - \upsilon(\eta-1)] + \upsilon\xi(1-\xi^2)\sin[\omega_0 t - \upsilon(\eta-1)]$$
$$\left.\left. + \upsilon\cos(\omega_0 t)\sum_{i=1}^{\infty} e^{-\tilde{\gamma}_i(1)(\eta-1)}\frac{16J_1(\tilde{k}_i\xi)}{\tilde{k}_i^3 J_2(\tilde{k}_i)}\right]\sin\chi\cos\varphi\right\}. \quad (65)$$

At the stellar surface,

$$E_\parallel^{\rm osc}(\eta = 1) = \frac{1}{2}\frac{\Phi_0}{R}\Theta_0^2\upsilon\left\{a\left[(1-\xi^2)\sin(\omega_0 t) + \cos(\omega_0 t)\sum_{i=1}^{\infty}\frac{8J_0(k_i\xi)}{k_i^2 J_1(k_i)}\right]\cos\chi\right.$$
$$\left. + \frac{3}{4}d\left[\xi(1-\xi^2)\sin(\omega_0 t) + \cos(\omega_0 t)\sum_{i=1}^{\infty}\frac{16J_1(\tilde{k}_i\xi)}{\tilde{k}_i^2 J_2(\tilde{k}_i)}\right]\sin\chi\cos\varphi\right\}. \quad (66)$$

3.3. *The Solution at Distances Greater than the Polar Cap Size and Smaller than the Light-Cylinder Radius*

Let us now consider the solution in the case where $\Theta_0 \ll \eta - 1 \ll R_{\rm LC}/R$, so that $|d^2P_i/d\eta^2| \ll \gamma_i^2(\eta)|P_i|$ and $|d^2Q_i/d\eta^2| \ll \tilde{\gamma}_i^2(\eta)|Q_i|$. Neglecting $\nabla_\parallel \cdot E_\parallel$ compared with $\nabla_\perp \cdot E_\perp$ in Poisson's equation, we can write

$$P_i(\eta, t') = \frac{2}{\eta^2(1-\varepsilon/\eta)}\frac{f(\eta)}{f(1)}\frac{1}{\gamma_i^2(\eta)}\left[\left(1 - \frac{\kappa}{\eta^3}\right)\frac{2}{k_i J_1(k_i)} - A_{1i} - A_{2i}\cos(\omega_0 t')\right] \approx 2\eta\Theta_0^2\left[\kappa\left(1 - \frac{1}{\eta^3}\right) - a\cos(\omega_0 t')\right]\frac{2}{k_i^3 J_1(k_i)}, \quad (67)$$

$$P(\eta, \xi, t') \equiv \sum_{i=1}^{\infty} P_i(\eta, \xi) J_0(k_i\xi) = \frac{1}{2}\eta\Theta_0^2\left[\kappa\left(1 - \frac{1}{\eta^3}\right) - a\cos(\omega_0 t')\right](1-\xi^2), \quad (68)$$

and

$$Q_i(\eta, t') = 3\eta\Theta_0^2\left\{[H(\eta)\Theta(\eta) - H(1)\Theta_0]\frac{2}{\tilde{k}_i^3 J_2(\tilde{k}_i)} - \frac{2}{\tilde{k}_i^3 J_2(\tilde{k}_i)} d\cos(\omega_0 t')\right\}, \quad (69)$$

$$Q(\eta, \xi, t') \equiv \sum_{i=1}^{\infty} Q_i(\eta, \xi) J_1(k_i\xi) = \tfrac{3}{8}\eta\Theta_0^2[H(\eta)\Theta(\eta) - H(1)\Theta_0 - d\cos(\omega_0 t')]\xi(1-\xi^2). \quad (70)$$

Note that the obtained solutions (60) and (61) match with solutions (68) and (70) at $\eta - 1 \sim 1$.

From equations (39) and (62) one can see that at $r = R$ the effective charge density is

$$\rho_{\rm eff} \equiv \rho - \rho_{\rm GJ} = -\frac{\Omega B_0}{2\pi c(1-\varepsilon)^{1/2}}(a\cos\chi + d\xi\sin\chi\cos\varphi)\cos(\omega_0 t). \quad (71)$$

From this equation we see that $\rho_{\rm eff} \neq 0$ at $r = R$, while we have assumed the constant electric potential and the zero steady state component of the electric field $\bar{E}_\parallel = 0$ at $r = R$. The value of the oscillating component of the electric field at $r = R$ (see eq. [66]) is determined by the oscillating component of charge density at $r = R$ (see eq. [62]). This is one of the principal results of the present analysis and is a consequence of our solutions, which explicitly imply the dynamical regime of the space-charge limitation of current. That is, the solutions describe the response of the potential drop above the surface to a change in supply of charge at the surface. This response is not instantaneous, as assumed in previous steady state treatments, but has a delay due to the travel time of the charges. Thus, it is not surprising that oscillations in the input charge density at $r = R$ will manifest themselves as a similar modulation of the electric potential at $r > R$. Physically this means that the charge density of particles supplied from the stellar surface into the domain of open field lines may fluctuate, with the total density of charges at any distance between the stellar surface and a pair formation front being limited (owing to the negative feedback between the electric potential drop and a net space charge) by the local Goldreich-Julian charge density. It is important that the condition at the stellar surface may now modulate (owing to the time retardation) the distribution of the space charge in the domain of open field lines. From the above solutions for any obliquity, one can see that, in the inner magnetosphere of a neutron star, the electric potential $\Phi = \alpha\pi r^2\Theta^2(r)(\rho - \rho_{GJ})(1 - \xi^2)$, so that the accelerating electric field and hence the energies of primary particles may indeed become modulated by the "surface." Our solutions, after necessary adjustment to the more or less realistic physical situation of a pulsar, have potential for probing the dynamical effects in the innermost magnetosphere of a neutron star.

For $\Theta_0 \ll \eta - 1 \ll R_{LC}/R$ the electric field component parallel to the magnetic field can be expressed as

$$E_{\|} = -R\frac{\partial\Phi}{\partial\eta}\bigg|_{\xi=\text{constant}} = -E_{\text{vac}}\,\Theta_0^2\left\{\left[\frac{3\kappa}{2\eta^4} - \frac{1}{4}\left(\frac{\omega_0 R}{c}\right)a \sin(\omega_0 t')\right](1 - \xi^2)\cos\chi \right.$$
$$\left. + \frac{1}{8}\left[3H(\eta)\Theta(\eta)\delta(\eta) - \frac{\omega_0 R}{c}\,d \sin(\omega_0 t')\right]\xi(1 - \xi^2)\sin\chi\cos\varphi\right\}, \tag{72}$$

where $E_{\text{vac}} \equiv (\Omega R/c)B_0$ is the characteristic value of the electric field generated near the surface of a neutron star rotating in vacuum (see Deutsch 1955).

4. DISCUSSION AND CONCLUSIONS

We have considered general relativistic electrodynamics of an inner magnetosphere of a neutron star, allowing for the possibility of a quasi-steady supply of charged particles into the region of open field lines. In this paper we do not discuss the effects associated with the pair formation above the stellar surface, and it will be analyzed in detail in a separate publication. The general relativistic treatment is essential for the problem under discussion because the effect of *dragging of inertial frames of reference* results in a significant contribution to the magnitude of an accelerating electric field. Our analysis demonstrates the principal possibility of a quasi-steady regime, and it may be potentially used for a number of applications in the theory of pulsars (e.g., in the consistent treatment of the back reaction of electron-positron pairs on the accelerating electric field; see Daugherty & Harding 1996).

Let us calculate the total power carried away by relativistically moving primary particles:

$$L_p = 2\left(-c\int \overline{\rho\Phi}\,dS\right), \tag{73}$$

where the overbar denotes the averaging over time $T_0 = 2\pi/\omega_0$, and the integration is taken over the spherical surface bounded by the last open field lines.

For the nearly aligned rotator ($\chi \approx 0$), the "general relativistic" part of the solution dominates, and, using the above solutions, we can write

$$-\overline{\rho\Phi} \approx \frac{1}{4\pi}\left(\frac{\Omega B_0}{c}\right)^2 \frac{R^2\Theta_0^2}{\alpha\eta^3}\frac{f(\eta)}{f(1)}\left[\kappa(1 - \kappa) - \frac{1}{2}a^2\right](1 - \xi^2). \tag{74}$$

Thus, we get

$$L_p \approx \frac{1}{4}\frac{\Omega^4 B_0^2 R^6}{c^3 f^2(1)}\left[\kappa(1 - \kappa) - \frac{1}{2}a^2\right]. \tag{75}$$

In the steady state regime when $a^2 \ll 1$ we get

$$(L_p)_{\max} = \frac{1}{4}\frac{\Omega^4 B_0^2 R^6}{c^3 f^2(1)}\kappa(1 - \kappa) = \frac{3}{2}\kappa(1 - \kappa)\dot{E}_{\text{rot}}, \tag{76}$$

where

$$\dot{E}_{\text{rot}} \equiv \frac{1}{6}\frac{\Omega^4 B_0^2 R^6}{c^3 f^2(1)} = \frac{1}{f^2(1)}(\dot{E}_{\text{rot}})_{\text{SR}}. \tag{77}$$

Here $(\dot{E}_{\text{rot}})_{\text{SR}}$ is the standard expression for the magneto-dipole losses in special relativity. Given $\kappa \approx 0.15 I_{45}/R_6^3$ (where $I_{45} = I/10^{45}$ g cm^{-2}, $R_6 = R/10^6$ cm), we estimate that

$$(L_p)_{\max} \approx 0.22 I_{45} R_6^{-3}(1 - 0.15 I_{45} R_6^{-3})\dot{E}_{\text{rot}}. \tag{78}$$

The regime of quasi-steady supply of charged particles into the inner magnetosphere reduces the efficiency of particle acceleration which is consistent with the regime of space-charge limitation of current. Now let us compare the relative contributions of the general relativistic and classic terms in the above solutions. For the orthogonal rotator ($\chi = \pi/2$), when

$\eta \gg 1$ we can write

$$L_p \approx \frac{9}{32}\left(\frac{H(1)}{\sqrt{f(1)}}\frac{\Omega R}{c}\eta^{1/2} - d^2\right)\dot{E}_{\rm rot}. \tag{79}$$

At the small distances from the star (e.g., at $\eta \sim$ 10–30) we have

$$\frac{L_p(\chi = 0)}{L_p(\chi = \pi/2)} \approx \frac{8}{\pi}\frac{\sqrt{f(1)}}{H(1)}\kappa(1-\kappa)\frac{10^4}{\eta^{1/2}}P \approx 5\,\frac{10^3}{\eta^{1/2}}P\ , \tag{80}$$

where P is the pulsar spin period.

We may also write that

$$\frac{L_p(\chi = 0)_{\rm max}}{L_p(\chi = \pi/2)_{\rm max}} \approx \frac{16\sqrt{3}}{3\sqrt{2\pi}}\frac{\sqrt{f(1)}}{H(1)}\kappa(1-\kappa)10^2 P^{1/2} \approx 70 P^{1/2}\ . \tag{81}$$

In the derivation of expressions (78), (80), and (81) we used the values of $\kappa \approx 0.15$ ($I_{45} = 1$ and $R_6 = 1$), $\varepsilon \approx 0.4$, and $f(1) \approx 1.44$ ($M = 1.4 M_\odot$).

Thus our estimates clearly show that the general relativistic contribution is dominant for a relatively wide range of parameters. For example, for a spin period of 0.1 s, the general relativistic term dominates the classic term for the inclination angles $0 \leq \chi \lesssim 85°$.

Our solutions for the electric field, in the steady state, have some significant implications for pulsar polar cap models. These models assume that charged particles are accelerated above the polar caps, initiating pair cascades through one-photon pair creation of photons from curvature radiation (Daugherty & Harding 1982) or inverse Compton radiation (Dermer & Sturner 1994). Using estimates of the accelerating potential derived by Arons (1983), due to field line curvature in flat space, the total energy gained by the particles was not sufficient to account for the observed flux from many γ-ray pulsars unless very small emission solid angles were assumed. Furthermore, the acceleration occurred only over half of the polar cap, on those (favorably curved) field lines that curved toward the rotation axis. The electric field induced by inertial frame dragging operates on all field lines and is much higher than the field due to field line curvature alone. Thus, the potential drop at the pair formation front, and the total energy gained by particles in the open field region, is larger (Harding & Muslimov 1997). The efficiency is straightforward to calculate, being simply the total power gained by the primary charges (with the potential in eq. [73] replaced by the potential at the pair formation front) as a fraction of the spin-down power (Harding 1981). Since the general relativistic contribution to the electric field depends on cos χ, pulsars having smaller obliquity χ will have larger accelerating potential drops and thus may be favored for γ-ray pulsar emission (Muslimov 1995). This would support the single-pole γ-ray pulsar models (Daugherty & Harding 1994, 1996; Dermer & Sturner 1994), which require small obliquity.

Our general conclusions from solutions for the quasi-steady regime can be summarized as follows:

1. The quasi-steady regime of particle ejection from the stellar surface reduces the total power carried away by relativistic primary particles relative to the steady regime.
2. The fluctuation of the charge density of particles ejected from the stellar surface modulates the particle energy along a field line.
3. The inhomogeneity of the physical conditions at the stellar surface may substantially affect the global electrodynamics within the inner magnetosphere of a neutron star.

In a future paper (Harding & Muslimov 1997), we plan to explore the location of the pair formation front, the distance above the polar cap where electron-positron pair production shorts out the parallel electric field and thus determines the potential drop. The solutions presented in this paper also allow an investigation of the stability of the acceleration of charged particles above the polar caps.

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