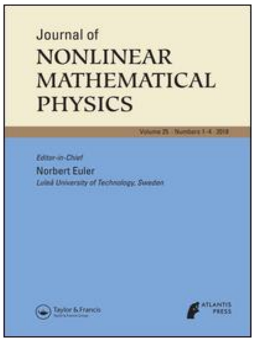

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Time-independent Hamiltonians describing systems with friction: the "cyclotron with friction"

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As is well-known, any ordinary differential equation in one dimension can be cast as the Euler–Lagrange equation of an appropriate Lagrangian. Additionally, if the initial equation is autonomous, the Lagrangian can always be chosen to be time-independent. In two dimensions, however, the situation is more complex, and there exist systems of ODEs which cannot be described by any Lagrangian. In this paper we display Hamiltonians which describe the behaviour of a charged particle moving in a plane under the combined influence of a constant electric field (in the plane) and a constant magnetic field (orthogonal to the plane) as well as a friction force proportional to the velocity ("cyclotron with friction").

1. Introduction

In the context of classical and quantum mechanics, systems with friction have been the focus of considerable interest. These systems are typically characterized by motions in which the moving particle stops asymptotically (as the time $t \to +\infty$). This stopping arises because of the coupling of the particle of interest, often referred to as the central system—but note that in this paper we only consider the motion of a single particle—to an external system, often comprising a large number of degrees of freedom, usually referred to as the bath.

It is a remarkable fact, however, that in some specific cases it is possible to rewrite the equations describing the motion with friction in terms of a—possibly time-dependent—Hamiltonian involving only the moving particle. In particular, this is always the case for motions in an arbitrary one-dimensional potential with a friction linear in the velocity. This can be seen as follows: the Hamiltonian

$$h(p,z,t) = \exp(-ct)\frac{p^2}{2} + \exp(ct)V(z) \,, \tag{1.1}$$

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with $p \equiv p(t)$ and $z \equiv z(t)$ the standard canonical variables and c a positive constant, yields via the standard Hamiltonian equations

$$\dot{z} = \partial h(p,z)/\partial p = \exp(-ct)p \, , \tag{1.2a}$$

$$\dot{p} = -\partial h(p,z)/\partial z = -\exp(ct) dV(z)/dz \, , \tag{1.2b}$$

the Newtonian equation of motion

$$\ddot{z} = -c\dot{z} - dV(z)/dz, \tag{1.3}$$

which corresponds to a (without loss of generality, unit mass) particle of coordinate $z \equiv z(t)$ moving in a potential $V(z)$ under the additional influence of a friction with coefficient c. [Note that, here and hereafter, a superimposed dot indicates differentiation with respect to the time t, and that we omit to indicate the explicit time-dependence of various quantities whenever this is unlikely to cause misunderstandings].

There is a significant amount of literature on this subject: for a review of relevant results see [1], and for some typical results see [2–10]. Note that a considerable part of this literature is devoted to the question of quantizing models with friction. To this end they describe the system with friction in Hamiltonian terms and then proceed to quantize this system in the usual manner. In this paper we limit ourselves to classical considerations, postponing the treatment of the quantal case to a separate paper.

The general question of the circumstances under which a given system of ordinary differential equations (ODEs) of second order can be obtained as the Euler–Lagrange equations for a given Lagrangian has received considerable attention in the mathematical literature. In particular, it is straightforward to show that *any* one-dimensional system can indeed be obtained via a Lagrangian. The two-dimensional case has also received a complete treatment in [11], the main result being that a system of 2 second order ODEs *cannot* generally be obtained from a Lagrangian.

A further issue of interest is, of course, that of determining when this Lagrangian can be chosen to be independent of time. Sarlet [12] showed that quite generally, if a set of equations derivable from a Lagrangian is autonomous, then this Lagrangian can be chosen to be *time-independent*. But it is generally not susceptible of explicit expression. We are in particular not aware of an *explicit* time-independent Lagrangian yielding the equation of motion (1.3) for an arbitrary potential $V(z)$. However, for $V(z) = \lambda z^2/2$, a Hamiltonian of the form

$$H(p,z) = \frac{1}{2} \ln \left[z^2 \sec^2(\omega z p) \right] - \frac{czp}{2} \, , \tag{1.4}$$

where $\omega = \sqrt{4\lambda - c^2}/2$, was given in [13] for the underdamped case. Similar expressions are also given for the other cases. While a systematic method to obtain such results is indicated in that paper, it does not extend to the case of an arbitrary potential $V(z)$.

Similarly, in the same paper [13], the Hamiltonian

$$H(p,z) = \exp(p) + cz \tag{1.5}$$

is given for the free particle moving against friction, namely according to the Newtonian equation of motion

$$\ddot{z} = -c\dot{z} \, . \tag{1.6}$$

Another simple example is given by

$$H(p,z) = \exp(p) + \frac{Ep}{c} + cz \;, \tag{1.7}$$

which generates, as can easily be verified, the dynamics of a particle moving against friction in a constant force field E, that is

$$\ddot{z} = -c\dot{z} + E \;. \tag{1.8}$$

Note moreover that these Hamiltonians can be generalized by a canonical transformation that does not modify the canonical coordinate z and replaces the canonical coordinate p with

$$\tilde{p} = p + \alpha(z) \;, \tag{1.9}$$

where $\alpha(z)$ is an *a priori* arbitrary function. Since the transformation (1.9) is canonical and reduces to the identity on the position variables, it generally does not affect the Newtonian equations. There is thus an arbitrary function $\alpha(z)$ which can be introduced at will. As an example, this transforms the Hamiltonian (1.5) to the more general Hamiltonian

$$H(p;z) = f(z)\exp(p) + cz \;. \tag{1.10}$$

Let us emphasize that in this case the function $f(z)$ can be *arbitrarily assigned* as long as it has *no real* zeros, while it does not appear at all in the Newtonian equation of motion (1.6).

Remark 1.1. If $f(z_0) = 0$, then if one takes z_0 as initial value for z, $z(t)$ does not move at all, $z(t) = z_0$, whatever the initial momentum. Hence, it will be impossible to assign a *non-zero* initial velocity, so that the system would not really be equivalent to free motion against friction, see (1.6). □

To conclude these remarks, we note that Hamiltonians featuring a nonconventional kinetic energy term—indeed resembling the Hamiltonians (1.5) and (1.7)—have been previously investigated (see for example [14–20, 22]), but to the best of our knowledge their suitability to treat also the case *with friction* had not been previously noted.

In Section 2 the extensions by complexification of some of these findings to a two-dimensional context is discussed. It is indeed thereby possible to obtain a system of ODEs describing a charged particle moving in a plane under the influence of a homogeneous magnetic field perpendicular to that plane and of a friction force proportional to the velocity, which we shall call "cyclotron with friction" (and the model also allows for the additional presence of a constant electric field lying in the plane: see below). In Section 3 some considerations relevant to the symmetries and conservation laws of this model are tersely presented. The last Section 4 ("Outlook") outlines further developments, to be pursued by ourselves and/or by others in future publications.

2. Extension by complexification to motions in the plane

Consider the two Hamiltonians (1.5) and (1.7). Their analytic nature allows to extend them straightforwardly by complexification to describe motions taking place in a plane, which will be seen to correspond to physically interesting systems: specifically, the motion against friction of a charged particle in the presence of a perpendicular constant magnetic field, or a constant electric field lying in that plane, or of both these forces.

Consider the Hamiltonian (1.5) which yields the Newtonian equations of motion (1.6) of a free particle moving against friction. If we set $c = a + \mathbf{i}b$ and go to the *complex* plane, we obtain the following pair of Poisson commuting Hamiltonians

$$\begin{aligned} H_R(p_x, p_y; x, y) &= \Re\left[H(p_x - \mathbf{i}p_y, x + \mathbf{i}y)\right] \\ &= \exp(p_x)\cos(p_y) + ax - by\ , \qquad (2.1a) \\ H_I(p_x, p_y; x, y) &= \Im\left[H(p_x - \mathbf{i}p_y, x + \mathbf{i}y)\right] \\ &= -\exp(p_x)\sin(p_y) + bx + ay\ . \qquad (2.1b) \end{aligned}$$

[Here and hereafter, $\mathbf{i}$ is the imaginary unit, $\mathbf{i}^2 = -1$; a, b are two arbitrary real constants; $q(t) \equiv x(t) + \mathbf{i}y(t)$, $p(t) \equiv p_x(t) - \mathbf{i}p_y(t)$ (note the minus sign), so that $x \equiv x(t)$, $y \equiv y(t)$ are the real canonical coordinates of the particle moving in the Cartesian xy-plane; and $p_x \equiv p_x(t)$, $p_y \equiv p_y(t)$ are the corresponding real canonical momenta]. If we similarly complexify the Newtonian equations of motion (1.6) characterizing free motion against friction, we obtain

$$\ddot{x} = -a\dot{x} + b\dot{y}\ , \qquad \ddot{y} = -b\dot{x} - a\dot{y}\ . \qquad (2.2)$$

It is readily verified that (2.2) are obtained as the Hamilton equations corresponding to the Hamiltonian $H_R(p_x, p_y; x, y)$ defined in (2.1a). It is thus seen that this Hamiltonian provides a description of the motion—against a friction characterized by the parameter a—of a particle moving in the Cartesian xy-plane in the presence ($b \neq 0$) or absence ($b = 0$) of a constant magnetic field orthogonal to that plane.

Remark 2.1. Note that the equations of motion (12a) can be reformulated in a 3-dimensional context as follow, by introducing the 3-vector $\vec{r} \equiv (x, y, 0)$ in the xy-Cartesian plane and the unit vector $\hat{z} \equiv (0, 0, 1)$ orthogonal to that plane:

$$\ddot{\vec{r}} = -a\dot{\vec{r}} + b\dot{\vec{r}} \wedge \hat{z}\ , \qquad (2.3)$$

where the symbol $\wedge$ denotes the 3-dimensional vector product. The last term in the right-hand side is the Lorentz force. □

The explicit solution of (2.2) is readily found to be

$$x(t) \pm \mathbf{i}y(t) = x(0) \pm \mathbf{i}y(0) + [\dot{x}(0) \pm \mathbf{i}\dot{y}(0)]\, \frac{1 - e^{-at}\exp(\mp\mathbf{i}bt)}{a \mp \mathbf{i}b}\ . \qquad (2.4)$$

This formula, besides displaying quite explicitly the time evolution in the Cartesian xy-plane, shall be of importance in the next Section 3 where we analyze the significance of the symmetries and conservation laws associated to this motion.

We may proceed similarly with the Hamiltonian (1.7). In this case, the *real* part H_R of the complexified Hamiltonian reads

$$H_R(p_x, p_y; x, y) = e^{p_x}\cos p_y + \frac{(aE_x + bE_y)p_x + (aE_y - bE_x)p_y}{a^2 + b^2} + ax - by\ . \qquad (2.5)$$

Here we set again $c = a + \mathbf{i}b$ and in addition $E = E_x + \mathbf{i}E_y$. The corresponding Newtonian equations of motion obtained by complexification then read

$$\ddot{x} = -a\dot{x} + b\dot{y} + E_x\ , \qquad \ddot{y} = -b\dot{x} - a\dot{y} + E_y\ . \qquad (2.6)$$

These equations represent the motion in the Cartesian xy-plane (against friction: $a > 0$) of a particle moving under the influence of crossed electric and magnetic fields. The general solution of these equations is

$$\begin{aligned} x(t) = x(0) + \frac{1}{(a^2+b^2)^2} \Big\{ & (a^2+b^2)(aE_x+bE_y)t + \\ & \left[e^{-at}\cos(bt)-1\right]\left[(a^2-b^2)E_x+2abE_y-(a^2+b^2)(a\dot{x}(0)+b\dot{y}(0))\right] + \\ & e^{-at}\sin(bt)\left[(-2abE_x+(a^2-b^2)E_y+(a^2+b^2)(b\dot{x}(0)-a\dot{y}(0))\right] \Big\} \end{aligned} \tag{2.7a}$$

$$\begin{aligned} y(t) = y(0) + \frac{1}{(a^2+b^2)^2} \Big\{ & (a^2+b^2)(-bE_x+aE_y)t + \\ & \left[e^{-at}\cos(bt)-1\right]\left[-2abE_x+(a^2-b^2)E_y-(a^2+b^2)(b\dot{x}(0)-a\dot{y}(0))\right] + \\ & e^{-at}\sin(bt)\left[(-(a^2-b^2)E_x-2abE_y+(a^2+b^2)(a\dot{x}(0)+b\dot{y}(0))\right] \Big\} \end{aligned} \tag{2.7b}$$

These solutions can be described in qualitative terms as follows: they correspond to the general solution of the equations (2.2), see (2.4), which correspond to the homogeneous part of (2.6), added to the special solutions of (2.6) given by $(v_x t, v_y t)$, where the two parameters v_x and v_y can be explicitly obtained by solving the following system of two linear equations:

$$av_x - bv_y = E_x, \qquad bv_x + av_y = E_y \; . \tag{2.8}$$

The motion thus eventually becomes asymptotically rectilinear in the remote future, whereas it becomes a spiralling motion in the remote past, with a transition between these behaviors at intermediate times given by (2.7). We may additionally point out that, in 3 dimensions, the exact solution for electric and magnetic fields in general position, that is, not necessarily orthogonal, can also be written down explicitly, though we do not know whether the corresponding dynamics can be expressed in terms of a Hamiltonian.

3. Symmetries and conservation laws

In the following, we discuss the connection between symmetries and conservation laws for the systems we have considered. Specifically, we focus on the Hamiltonian (2.1a), since it has a large group of symmetries yet represents a system *with friction*, indeed it corresponds to the physically relevant case of a "cyclotron with friction". It is thus of interest to study these symmetries and to see how Noether's theorem, for example, applies in this setting. Indeed, we do not ordinarily think of a system moving against friction as having, say, a conserved angular momentum. Here two remarks are important: first, the symmetries must correspond to symmetries of the dynamics on all of phase space; second, we must make sure that the symmetries we use can really be implemented as canonical transformations.

There are two obvious geometrical symmetries in the dynamics of the particle moving against friction in a plane in the presence of a perpendicular homogeneous magnetic field: translations (in both directions) and rotations around an arbitrary origin. The latter, however, are not symmetries of the full phase space orbit: indeed a rotation, if it is to be a canonical transformation, must operate in the same way on both momenta and positions. But the trajectory of the momentum is a straight line in a fixed direction: if we wish to rotate this trajectory, we need to change both a and b (see (2.2)).

In spite of the existence of a Hamiltonian structure as well as of an (apparent) rotational invariance, there is thus no equivalent to angular momentum conservation for this model (but see below for an alternative symmetry property).

Translations, on the other hand, do lead to interesting symmetries. The two-dimensional group of translations is generated by p_x and p_y. Generally, these do not commute with $H_R(p_x, p_y; x, y)$ (see (2.1a)), but the linear combination

$$P_+ = \frac{p_x}{a} + \frac{p_y}{b} \tag{3.1}$$

does. This is due to the fact that the translation generated by P_+ leave the Hamiltonian *invariant*, since P_+ Poisson commutes with $H_R(p_x, p_y; x, y)$ (see (2.1a)),

$$\{P_+, H_R(p_x, p_y; x, y)\} = 0 \ . \tag{3.2}$$

On the other hand p_x, for example, does *not* commute with H_R. This corresponds to the fact that the Hamiltonian grows linearly in x, so that the translation in x, while it does leave the orbits invariant, transports an orbit with one energy to an orbit with another. This leads, by standard considerations, to a *time-dependent* conservation law, of the form

$$P_-(t) = \frac{p_x(t)}{a} - \frac{p_y(t)}{b} - 2t \ , \qquad \dot{P}_-(t) = 0 \ . \tag{3.3}$$

The system has therefore two degrees of freedom, and three conserved quantities, namely: P_+, H_R and H_I (see (3.1), (2.1a) and (2.1b)). With such a number of conservation laws, the system is *maximally superintegrable*. This is, of course, unsurprising; indeed—as shown above, see (2.4)—it is even possible, for this model, to write explicitly the solution of its initial-value problem. It is nevertheless remarkable that the conservation laws can be expressed in such a simple manner.

Finally, we may discuss the meaning of these various conservation laws. Since the momenta do not have an obvious physical significance, we first express P_+ in terms of $\dot{x}$ and $\dot{y}$. Using the equations of motion

$$\dot{x} = \exp(p_x)\cos(p_y) \ , \qquad \dot{y} = -\exp(p_x)\sin(p_y) \ , \tag{3.4a}$$

$$\dot{p}_x = -a \ , \qquad \dot{p}_y = b \ , \tag{3.4b}$$

we immediately find

$$\begin{aligned} P_+ &= \frac{1}{2a}\ln\left(\dot{x}^2 + \dot{y}^2\right) + \frac{p_y}{b} \\ &= \frac{1}{2a}\ln\left(\dot{x}^2 + \dot{y}^2\right) + t + \frac{p_y(0)}{b} \ . \end{aligned} \tag{3.5}$$

The conservation of P_+ thus entails that the standard kinetic energy $\left(\dot{x}^2 + \dot{y}^2\right)/2$ of the particle decays exponentially in t at a rate $2a$ to compensate for the increase of the term linear in t.

The other two conservation laws, namely H_R and H_I, correspond to the coordinates $x(\infty)$ and $y(\infty)$ of the final position of the system as $t \to +\infty$. This follows from the fact that $p_x \to -\infty$ as $t \to +\infty$, so that

$$\mathscr{E}_R = -ax(+\infty) + by(+\infty) \ , \qquad \mathscr{E}_I = -bx(+\infty) - ay(+\infty) \ , \tag{3.6}$$

where $\mathscr{E}_R$ and $\mathscr{E}_I$ are, say, the initial values of H_R and H_I respectively. Note that these formulas suggest which symmetry corresponds to the quantity H_I (see (2.1b)). Since H_I has the same form as H_R

(see (2.1a)) up to an appropriate interchange of the parameters a and b, we may say that the dynamics corresponding to a magnetic field characterized by the parameter $-a$ and a friction characterized by the parameter b commutes with the dynamics defined by a magnetic field characterized by the parameter b and a friction characterized by the parameter a. So, it appears to be this remarkable symmetry which is the underlying feature allowing the treatment presented here of this model.

Let us complete this Section 3 by noting that the treatment presented here of a particle in a magnetic field with friction is quite different from the conventional treatment: if we consider the case without friction ($a = 0$), we see that the two coordinates of the circular orbit, obtained from $\mathscr{E}_R$ and $\mathscr{E}_I$ via (3.6), are quantities which *Poisson commute* among each other, whereas in the standard Hamiltonian treatment of the motion of a charged particle moving in the plane in the presence of an orthogonal uniform magnetic field, these two coordinates are *canonically conjugate* to each other.

4. Outlook

There are two natural developments suggested by the results reported above.

The first is the quantization of the Hamiltonian models introduced above. In the case of the one-dimensional damped harmonic oscillator, considerable work has been done to study its quantization using a Hamiltonian description for the equation of motion, see in particular [1] and references therein. While this approach is physically questionable, since it is important, in quantum mechanics, to take into account the interaction of the central system with the environment, it is nevertheless of interest, since it allows to study the possibility of describing an irreversible process by an unitary evolution. We are presently pursuing this line of research, the results of which will appear shortly [21].

The other is the treatment of some analogous "solvable" model, involving however the motion of several interacting particles rather than just a single one. An appealing candidate is the many-body "goldfish" model, see [22–24].

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