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# On the black hole mass decomposition in nonlinear electrodynamics

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ABSTRACT

In the weak field limit of nonlinear Lagrangians for electrodynamics, i.e. theories in which the electric fields are much smaller than the scale (threshold) fields introduced by the nonlinearities, a generalization of the Christodoulou–Ruffini mass formula for charged black holes is presented. It proves that the black hole outer horizon never decreases. It is also demonstrated that reversible transformations are, indeed, fully equivalent to constant horizon solutions for nonlinear theories encompassing asymptotically flat black hole solutions. This result is used to decompose, in an analytical and alternative way, the total mass-energy of nonlinear charged black holes, circumventing the difficulties faced to obtain it via the standard differential approach. It is also proven that the known first law of black hole thermodynamics is the direct consequence of the mass decomposition for general black hole transformations. From all the above we finally show a most important corollary: for relevant astrophysical scenarios nonlinear electrodynamics decreases the extractable energy from a black hole with respect to the Einstein–Maxwell theory. Physical interpretations for these results are also discussed.



## 1. Introduction

Black hole solutions to the Einstein equations have always attracted the attention of researchers, not only due to their unusual properties, but also from the discovery that they could be one of the most abundant sources of energy in the Universe.

From conservation laws, R. Penrose [1] showed how energy could be extracted from a black hole [2]. D. Christodoulou [3] and D. Christodoulou and R. Ruffini [4], through the study of test particles in Kerr and Kerr–Newman spacetimes [5], quantified the maximum amount of energy that can be extracted from a black hole. These works deserve some comments. First, this maximum amount of energy can be obtained only by means of the there introduced, *reversible processes*. Such processes are the only ones in which a black hole can be brought back to its initial state, after convenient interactions with test particles. Therefore, reversible transformations constitute the most efficient processes of energy extraction from a black hole. Furthermore, it was also introduced in Refs. [3,4] the concept of *irreducible mass*. This mass can never be diminished by any sort of processes and hence would constitute an intrinsic property of the system, namely the *fundamental energy state of a black hole*. This is exactly the case of Schwarzschild black holes. From this irreducible mass, one can immediately verify that the area of a black hole never decreases after any infinitesimal transformation performed on it. Moreover, one can write down the total energy of a black hole in terms of this quantity [4].

Turning to effective nonlinear theories of electromagnetism, their conceptual asset is that they allow the insertion of desired effects such as quantum-mechanical, avoidance of singular solutions, and others e.g. via classical fields [6]. As a first approach, all of these theories are built up in terms of the two local invariants constructed out of the electromagnetic fields [7,8]. Notice that the field equations of nonlinear theories have the generic problem of not satisfying their hyperbolic conditions for all physical situations (see e.g. [9,10]). The aforementioned invariants are assumed to be functions of a four-vector potential in the same functional way as their classic counterparts, being therefore also gauge independent invariants. We quote for instance the Born–Infeld Lagrangian [11], conceived with the purpose of solving the problem of the infinite self-energy of an electron in the classic theory of electromagnetism. The Born–Infeld Lagrangian has gained a renewal of interest since the effective Lagrangian of string theory in its low energy limit has an analog form to it [12]. It has also been minimally coupled to general relativity, leading to an exact

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solution [13,14] and this coupling has been studied in a variety of problems [15–17]. Another worthwhile example of nonlinear electromagnetic theory is the Euler–Heisenberg Lagrangian [18,19]. This Lagrangian allows one to take effectively into account one-loop corrections from the Maxwellian Lagrangian coming from Quantum Electrodynamics (QED), and it has been extensively studied in the literature [6]. Nonlinear theories of electromagnetism have also been investigated in the context of astrophysics. For instance, they could play an important role in the description of the motion of particles in the neighborhood of some astrophysical systems [20], as a simulacrum to dark energy and as a simulacrum of dark energy [21].

In connection with the above discussion, the thermodynamics of black holes [22] in the presence of nonlinear theories of electromagnetism has also been investigated. The zeroth and first laws (see Section 7) have been studied in detail [12,23], allowing the raise of other important issues. We quote for example the difficulty in generalizing the so-called Smarr mass [24,25] (a parametrization of the Christodoulou–Ruffini mass [4]) for nonlinear theories [12]. Many efforts have been pursued in this direction, through the suggestion of systematic ways to write down this mass, which has led to some inconsistencies (see e.g. Ref. [26]). For some specific nonlinear Lagrangians, this problem has been circumvented [27].

We first deal with static spherically symmetric electrovacuum solutions to the Einstein equations minimally coupled to Abelian nonlinear theories of electromagnetism, i.e. nonlinear charged black holes, for electric fields that are much smaller than the scale fields introduced by the nonlinearities, i.e. *weak field* nonlinear Lagrangians. We decompose the total mass-energy of a charged black hole in terms of its characteristic parameters: charge, irreducible mass, and nonlinear scale parameter. We also show the constancy of the black hole outer horizon in the case of reversible transformations. We then generalize the previous results for a generic nonlinear theory leading to an asymptotically flat black hole solution. As an immediate consequence of this general result, we show that the first law of black hole thermodynamics (or mechanics) in the context of nonlinear electrodynamics [12] is a by-product of this mass decomposition. These results also allow us to investigate the extraction of energy from charged black holes in the framework of nonlinear theories of electromagnetism.

The article is organized as follows. In Section 2 the notation is established and the field equations are stated and solved formally in the spherically symmetric case for nonlinear electromagnetic theories that lead to null fields at infinity. In Section 3, reversible transformations are investigated. In Section 4 the field equations are solved for the weak field limit of nonlinear theories of electromagnetism. Section 5 is devoted to the deduction of the total mass-energy of a charged black hole in terms of irreducible and extractable quantities, when reversible transformations are taken into account. In Section 6 variations of the outer horizon associated with the capture of test particles in nonlinear theories of electromagnetism are analyzed. In Section 7, we shall present the way to decompose the energy of a black hole within nonlinear theories of electromagnetism and show that it leads automatically to the first law of black hole mechanics. Finally, in Section 8 we discuss the results of this work. We use geometric units with $c = G = 1$, and metric signature $-2$.

## 2. Field equations

The minimal coupling between gravity and nonlinear electrodynamics that depends only on the local invariant $F$ can be stated mathematically through the action

$$S = \int d^4x\sqrt{-g}\left(-\frac{R}{16\pi} + \frac{L_{em}(F)}{4\pi}\right) \doteq S_g + \frac{S_{em}}{4\pi}, \tag{1}$$

where $F \doteq F^{\mu\nu}F_{\mu\nu}$, $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu$ is the electromagnetic four-potential, $R$ is the Ricci scalar, $S_g$ is the action for the gravitational field, $S_{em}$ is the action of the electromagnetic theory under interest, and $g$ the determinant of the metric $g_{\mu\nu}$ of the spacetime. Under the variation of Eq. (1) with respect to $g^{\mu\nu}$, and applying the principle of least action, one obtains (see e.g. Ref. [8])

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T^{(em)}_{\mu\nu}, \tag{2}$$

with $R_{\mu\nu}$ the Ricci tensor and $T^{(em)}_{\mu\nu}$ the energy–momentum tensor of the electromagnetic field, defined as

$$4\pi T^{(em)}_{\mu\nu} \doteq \frac{2}{\sqrt{-g}}\frac{\delta S_{em}}{\delta g^{\mu\nu}} = 4L^{(em)}_F F_{\mu\alpha}F_{\nu\rho}g^{\alpha\rho} - L^{(em)}g_{\mu\nu}, \tag{3}$$

where $L^{(em)}_F \doteq \partial L^{(em)}/\partial F$.

Application of the principle of least action in Eq. (1) with respect to $A_\mu(x^\beta)$ gives

$$\nabla_\mu\left(L^{(em)}_F F^{\mu\nu}\right) = 0, \tag{4}$$

since we are interested in solutions to general relativity in the absence of sources.

In the static spherically symmetric case, it is possible to solve the Einstein equations minimally coupled to nonlinear electromagnetism theories [see Eqs. (2) and (4)] and due to the form of the energy–momentum tensor in this case the metric must be of the form

$$ds^2 = e^\nu dt^2 - e^{-\nu}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2, \tag{5}$$

with [8,23]

$$e^\nu = 1 - \frac{2M}{r} + \frac{8\pi}{r}\int_r^\infty r'^2 T_0{}^0(r')dr', \tag{6}$$

where the integration constant $M$ stands for the total mass-energy of the black hole as measured by observers at infinity.

Eq. (4) in this special spherically symmetric case reduce to

$$L^{(em)}_F E_r r^2 = -\frac{Q}{4}, \tag{7}$$

where $Q$ is an arbitrary constant representing physically the total charge of the black hole.

If one defines

$$E_r \doteq -\frac{\partial A_0}{\partial r} \quad \text{and} \quad \frac{\partial \mathcal{F}}{\partial r} \doteq -L^{(em)}r^2, \tag{8}$$

and take into account Eqs. (4), (5) and (7), then Eq. (6) can be rewritten as

$$e^\nu = 1 - \frac{2M}{r} + \frac{2QA_0}{r} - \frac{2\mathcal{F}}{r}, \tag{9}$$

where it has been imposed a gauge such that the scalar potential $A_0$ goes to zero when the radial coordinate goes to infinity, which also holds to $\mathcal{F}$. These conditions guarantee that the associated nonlinear black holes are asymptotically flat (Minkowskian). In this work we are not interested in Lagrangian densities which do not fulfill this condition.

The black hole horizons are given by the solutions to

$$g_{00}(r_h) = e^{\nu(r_h)} = 0. \tag{10}$$

## 3. Reversible and irreversible transformations

A way to investigate the motion of test particles in a static spherically symmetric spacetime is through the solution to the Hamilton–Jacobi equation. The trajectories of the test particles can be obtained in the traditional way (see e.g. Ref. [28]) through the particle constants of motion (energy $E$, orbital angular momentum $L$, and the Carter constant) [5,29]. The energy of the test particle is given by [3–5,29]

$$E = qA_0 + \sqrt{\frac{e^{\nu}}{r^2}\left[r^4\left(p^{\theta}\right)^2 + \frac{L^2}{\sin^2\theta} + m^2r^2\right] + \left(p^r\right)^2}, \tag{11}$$

where $p^{\mu} \doteq m dx^{\mu}/d\tau$, $\tau$ an affine parameter along the worldline of the particle and $q$ its charge. The "+" sign has been chosen in Eq. (11), because we are interested just in particles traveling to the future [29,30].

If the worldline of an arbitrary test particle intersects the outer horizon, i.e. the largest solution to Eq. (10), then the changes in the energy and charge of the black hole (which lead to another black hole configuration infinitesimally close to the initial one) reads: $\delta M = E$ and $\delta Q = q$ [29], respectively.

From Eq. (11), one can see that the only way to apply a reversible transformation in the sense of Christodoulou–Ruffini [3,4,6] to a black hole interacting with a test particle is by demanding that its square root term is null. It guarantees that a nonlinear black hole can always be restored to its initial configuration, as demanded by reversible transformations, see Section 1, after a test particle has crossed the horizon. Hence, from Eq. (11) and the aforementioned conservation laws, reversible transformations select geodesics whose changes to the black hole masses are minima and are given by

$$\delta M_{min} = qA_0(r_+) = \delta Q\, A_0(r_+). \tag{12}$$

Clearly, Eq. (12) is the mathematical expression for the physical case where $|p^r(r_+)|$ is much smaller than $|qA_0(r_+)|$, that is, when irreversible processes are negligible. Reversible transformations are important processes since like the internal energy of a thermodynamical system, the energy $M$ of a black hole is assumed to be an exact differential. This therefore allows one to describe intrinsic properties of the spacetime by using test particles; see Eq. (12).

For the sake of completeness, in the case of general black hole transformations one has

$$\delta M \geq qA_0(r_+). \tag{13}$$

## 4. Weak field nonlinear Lagrangians

An interesting and convenient limit for investigating nonlinear properties of Lagrangians is when their electric fields are small compared to their scale or threshold fields, set defined off by the nonlinearities [21]. In this limit, one expects that their leading term be the Maxwell Lagrangian [23]. In this line, assuming the nonexistence of magnetic charges, let us first investigate Lagrangian densities given by

$$L^{(em)} = -\frac{F}{4} + \frac{\mu}{4}F^2, \tag{14}$$

where $\mu$ is related to the scale field of the theory under interest, and as a necessary condition to avoid any violation of the most experimentally tested physical theory, the Maxwell theory, this nonlinear term is assumed such that it must be much smaller than the Maxwell one. This means we are generically interested in electric fields that satisfy

$$E_r \ll \frac{1}{\sqrt{\mu}}. \tag{15}$$

Physically speaking, the second term of Eq. (14) is a first order correction to the Maxwell theory. For instance, in the case of the Euler–Heisenberg Lagrangian, the nonlinearities are related to quantum corrections, whose scale field is $E_c = m_e^2c^3/(e\hbar) \approx 10^{18}$ $V/m$, where $m_e$ is the electron rest-mass, $e$ is the fundamental charge, and $\hbar$ is the reduced Planck constant (see e.g. [6], and references therein). Hence, in virtue of this limit a perturbative analysis could be carried out. The sign of $\mu$ in principle could be arbitrary. Nevertheless, from the inspection of the Euler–Heisenberg Lagrangian, for instance, this constant turns out to be positive [19]. The same behavior happens if one expands perturbatively the Born–Infeld Lagrangian [6,11–15,21]. It is worth to stress that the weak field analysis is however not very restrictive in terms of the strength of the fields. Note for instance that for the Euler–Heisenberg and Born–Infeld theories, our analysis is indeed meaningful for electric fields $E_r \sim 10^{18}$ V/m (see Section 8).

When one interprets nonlinear Lagrangians as the ones related to effective media [11], then one expects that their associated electric field solutions should be reduced. This constrains the sign of $\mu$, as we shall show below. Nevertheless, it is not ruled out in principle Lagrangians where the associated electric field could increase.

By substituting Eq. (14) into Eq. (7) and the first term of Eq. (8), solving exactly and then expanding perturbatively (or by directly working perturbatively), one can easily show that

$$E_r(r) = \frac{Q}{r^2}\left(1 - \frac{4\mu Q^2}{r^4}\right), \qquad A_0(r) = \frac{Q}{r}\left(1 - \frac{4\mu Q^2}{5r^4}\right). \tag{16}$$

Expressions (16) are just meaningful if the characteristic distances of the system are such that

$$r \gg r_c, \qquad r_c^4 = |\mu| M^2 \xi^2, \qquad \xi \doteq \frac{Q}{M}. \tag{17}$$

As we pointed out before, when $\mu > 0$, the modulus of the electric field diminishes in comparison to the pure Maxwellian case, while the opposite happens when $\mu < 0$. The former case is exactly what happens in usual media [31], while the latter could happen in the so-called metamaterials (see e.g. Refs. [32,33]).

From Eq. (14), the second term of Eq. (8) and Eq. (16), and assuming that the constraint in Eq. (17) is fulfilled, it is also readly shown that

$$\mathcal{F} = \frac{Q^2}{2r}\left(1 - \frac{6\mu Q^2}{5r^4}\right). \tag{18}$$

When Eqs. (16) and (18) are substituted in the expression for the $g_{00}$ component of the metric, Eq. (9), one obtains

$$e^{\nu} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{2\mu Q^4}{5r^6}. \tag{19}$$

The above result agrees with the one obtained in Ref. [34], for the Euler–Heisenberg Lagrangian density, in the corresponding units. Notice that when $\mu = 0$, i.e., for the Maxwell Lagrangian [see Eq. (14)], Eq. (19) gives the well-known Reissner–Nordström solution (see e.g. Ref. [29]).

The outer horizon can be found perturbatively from Eqs. (10) and (19) and the result is

$$r_+ = \mathcal{R}_+\left(1 + \frac{\mu Q^4}{5(\mathcal{R}_+)^5\sqrt{M^2 - Q^2}}\right), \tag{20}$$

where we defined

$$\mathcal{R}_+ \doteq M + \sqrt{M^2 - Q^2}, \tag{21}$$

functionally the same as the outer horizon in the Reissner–Nordström solution. Besides, in Eq. (20), it was understood that the second term in the parenthesis is much smaller than one. These latter equations are not valid in the case $Q = M$, and up to what extent the above perturbative analysis is meaningful in the proximity of this limit is dictated by the value $\mu/M^2$. Namely, the smaller the $\mu/M^2$, the closer one can approach $Q = M$ using perturbative theory. For the sake of reference, in Euler–Heisenberg and standard Born–Infeld theories, $\mu \sim 10^{-33}$ (e.s.u)$^{-2}$ [6,11], hence for objects of masses around $M \sim 10^5 M_\odot$, $\mu/M^2$ when brought to geometrical units ($\mu$ [cm$^2$] $= \mu$ [(e.s.u)$^{-2}$]$c^4/G$ and $M$ [cm] $= M$ [g]$G/c^2$), would be approximately $10^{-4}$. In this specific example, the limit $Q = M$ can be therefore approached with a precision of up to four decimals within the perturbative analysis presented here.

For the classic extreme value $Q = M$, the perturbative solution to the outer horizon is

$$r_+^{Q=M} = M\left\{1 + \sqrt{\frac{2\mu}{5M^2}} - \frac{4\mu}{5M^2} + \mathcal{O}\left[\left(\frac{\mu}{M^2}\right)^{\frac{3}{2}}\right]\right\}, \tag{22}$$

and there exist inner horizons given by

$$r_-^{Q=M} = M\left\{1 - \sqrt{\frac{2\mu}{5M^2}} - \frac{4\mu}{5M^2} + \mathcal{O}\left[\left(\frac{\mu}{M^2}\right)^{\frac{3}{2}}\right]\right\} \tag{23}$$

$$r_{ncl}^{Q=M} = M\left\{\left(\frac{2\mu}{5M^2}\right)^{\frac{1}{4}} + \left(\frac{\mu}{10M^2}\right)^{\frac{1}{2}} + \mathcal{O}\left[\left(\frac{\mu}{M^2}\right)^{\frac{3}{4}}\right]\right\}, \tag{24}$$

where $r_-^{Q=M}$ in Eq. (23) is the nonlinear version of the inner horizon in Reissner–Nordström solution, and the solution given by Eq. (24) has no classical (ncl) counterpart, being intrinsically due to corrections to the Maxwell theory, e.g. quantum. Notice that when $\mu \neq 0$ the inner and outer horizons are never equal in an arbitrary nonlinear theory given by Eq. (14) in the case $Q = M$. Hence, as we expect, when corrections are added to Maxwell theory, the degeneracy in the case $Q = M$, is broken. Nevertheless, due to the continuity of the metric, there always exists a value of $|\xi|$ where the horizons degenerate, depending now on $\mu/M^2$. We stress that Eq. (24) is just a mathematical solution to Eqs. (19) and (10), being physically meaningless, as the following analysis shows. Assume that the charge of the black hole is comparable with its mass (minimum value for being relevant the *nonclassical horizon*), that is $Q^2 \sim M^2$. From Eq. (24), however, one has $r_{ncl}^{Q=M} \sim (\mu M^2)^{1/4} = r_c$. Since just distances much larger than $r_c$ are physically meaningful in the realm of our perturbative calculations; see Eq. (17), it is proved that $r_{ncl}^{Q=M}$ is not physically relevant. The above reasoning implies that perturbative changes in the Maxwell Lagrangian just lead to corrections in the Reissner–Nordström horizons. This means that naked singularities still rise in such theories, but now for values of $|\xi|$ sightly larger or smaller than one, depending upon the sign and absolute value of $\mu/M^2$.

## 5. The weak field black hole mass decomposition

Assume a test particle being captured by a black hole under a reversible transformation. In mathematical terms, this means that the equality in Eq. (13) is to be taken into account and the changes can be considered as infinitesimals. By taking into account the second term in Eq. (16) and Eq. (20), one ends up to first order approximation with

$$\frac{dM}{dQ} = \frac{Q}{\mathcal{R}_+} - \frac{\mu Q^3}{5(\mathcal{R}_+)^5}\left[\frac{Q^2}{\mathcal{R}_+\sqrt{M^2 - Q^2}} + 4\right]. \tag{25}$$

Since we are supposing that the second term of the above equation is much smaller than the first one, the method of successive approximations can be used. We shall suppose that

$$M(Q) = M^{(0)}(Q) + \mu M^{(1)}(Q), \tag{26}$$

where the second term of the above expression is thought of as a perturbation. At the zeroth order approximation, $M^{(0)}$ satisfies the differential equation

$$\frac{dM^{(0)}}{dQ} = \frac{Q}{M^{(0)} + \sqrt{(M^{(0)})^2 - Q^2}}. \tag{27}$$

As it is known, the solution to the above equation is [4]

$$M^{(0)}(Q) = M_{irr} + \frac{Q^2}{4M_{irr}}, \tag{28}$$

where $M_{irr}$ is a constant of integration known as the irreducible mass and it accounts for the total energy of the system when the charge of the black hole is zero. Expression (28) is the Christodoulou–Ruffini black hole mass formula valid for a classical spherically symmetric charged black hole. By substituting this expression into Eq. (21) one obtains $\mathcal{R}_+ = 2M_{irr}$ and then it follows that $Q^2/(2\mathcal{R}_+) \leq M/2$, where the equality is valid in the case $Q = M$. Hence, up to 50% of the total mass of a black hole is due to the electromagnetic energy contribution $Q^2/(4M_{irr})$.

Substituting Eq. (26) into Eq. (25) and working now up to first order approximation, using Eqs. (27) and (28) we have

$$\frac{dM^{(1)}}{dQ} = -\frac{Q}{2M_{irr}[M_{irr} - Q^2/(4M_{irr})]}\left[M^{(1)} + \frac{Q^4}{160M_{irr}^5}\right] - \frac{Q^3}{40M_{irr}^5} \tag{29}$$

from which we obtain

$$M^{(1)}(Q) = -\frac{Q^4}{160M_{irr}^5}. \tag{30}$$

The above equation is obtained by imposing $M^{(1)}(0) = 0$, which is physically clear from our previous considerations. Since energy could be extracted from black holes only when it is charged [see Eq. (13)], the extractable energy, $M_{ext}$, or the *blackholic energy* [6], in weak fields nonlinear theories of electromagnetism given by Eq. (14) is

$$M_{ext}(Q) = \frac{Q^2}{4M_{irr}} - \frac{\mu Q^4}{160M_{irr}^5}. \tag{31}$$

As it can be checked easily, the above equation is exactly the electromagnetic energy $E^{(em)}$ [35,36] stored in the electric field in the spacetime given by Eq. (19) viz.,

$$E^{(em)} = 4\pi\int_{r_+}^{\infty} T_0{}^0 r^2 dr = \int_{r_+}^{\infty}\int_{2\pi}^{0}\int_{\pi}^{0} T_0{}^0\sqrt{g}\,d\theta\, d\varphi\, dr, \tag{32}$$

where $g$ is the determinant of the metric, that in Schwarzschild coordinates is given by $r^2\sin^2\theta$; see Eq. (5). Notice that even in the case where corrections to the Maxwell Lagrangian are present (e.g. quantum), $r_+ = 2M_{irr}$, as is clear from Eqs. (20), (21), (26), (28) and (30).

From Eq. (31), one clearly sees that the total amount energy that can be extracted from a nonlinear charged black hole is reduced if $\mu > 0$, in relation to the Maxwell counterpart. The positiveness of $\mu$ is valid both to the Euler–Heisenberg effective nonlinear Lagrangian to one-loop QED as well as to the standard Born–Infeld Lagrangian, as we pointed out earlier. Hence, in these theories, the extractable energy is always smaller than 50% of the total energy. More precisely, from Eqs. (20), (21), (26), (28) and (30),

$$M_{ext} \leq \frac{M}{2} - \frac{\mu Q^4}{320 M_{irr}^4 \sqrt{M^2 - Q^2}}, \tag{33}$$

the equality in this case being true only when $\mu = 0$.

## 6. Transformations in the outer horizon

Under the capture of a test particle of energy $E$ and charge $q$, one has that the black hole undergoes the (infinitesimal) given changes $\delta M = E$ and $\delta Q = q$, satisfying Eq. (13). Since the outer horizon of this black hole is dependent upon $M$ and $Q$, it would also undergo a change. Such a change can be obtained in the scope of the perturbative description we are carrying out and the basic equation for doing so is Eq. (20).

By using Eqs. (20), (21), (13) and the second term of Eq. (16), one can easily show that

$$\delta r_+ \geq -\frac{\mu Q^4 \delta \mathcal{R}_+}{5(\mathcal{R}_+)^5 (M^2 - Q^2)} \left[\mathcal{R}_+ + 3\sqrt{M^2 - Q^2}\right]. \tag{34}$$

As it can be seen from Eqs. (21), (26), (28) and (30), $\delta \mathcal{R}_+ \sim \mathcal{O}(\mu)$, then, up to first order in $\mu$, we have $\delta r_+ \geq 0$. This result can be easily understood if one notices that up to first order approximation in $\mu$, based on the two last sections, $r_+ = 2M_{irr}$. Under irreversible transformations, however, Eq. (34) shows that the outer horizon increases. Notice that the above results are just valid for $Q/M < 1$.

Another way of realizing whether or not there is an increase of the outer horizon due to the capture of a test particle is to search for the solutions to Eqs. (10) and (19) when one performs the changes $M \to M + \delta M$ and $Q \to Q + \delta Q$, satisfying Eq. (13). If one defines generally $r_+$ as the largest solution to Eqs. (10) and (19), then it is simple to verify that $\delta r_+ = 0$ for reversible transformations. For irreversible transformations, $\delta r_+ > 0$. Hence, generically, one has $\delta r_+ \geq 0$ for an arbitrary infinitesimal transformation undergone by the black hole in nonlinear weak field electromagnetism.

## 7. Energy decomposition for asymptotically flat nonlinear black holes

Weak field nonlinear Lagrangians suggest that the outer horizon of spherically symmetric $L(F)$ theories are $r_+ = 2M_{irr}$ when reversible transformations are considered, for any range of the electric field, and not only for the one where $E_r \ll 1/\sqrt{\mu}$. Now we shall show that indeed this is the case. This means that is possible to obtain the total mass-energy of spherically symmetric, asymptotically flat, nonlinear black holes in an algebraic way, overcoming the problems in solving differential equations coming from the thermodynamical approach. Also, it gives us the extractable energy from nonlinear black holes.

Assume that the invariant $F = -2E_r^2$ is such that $F = F(r, Q)$. From Eqs. (7)–(9), it follows that

$$Q \frac{\partial A_0}{\partial Q} = \frac{\partial \mathcal{F}}{\partial Q}. \tag{35}$$

Assume now that $r_+ = C = \text{constant}$, that is, the outer horizon is an intrinsic property of the system. From Eqs. (10) and (35), one shows immediately that

$$\delta M = \delta Q\, A_0|_{r_+ = C}. \tag{36}$$

It can be checked that the above equation is valid only when $r_+ = C$. We recall that we assumed Eq. (36) as the law for reversible transformations (energy conservation). Thereby, we showed that reversible transformations are fully equivalent to having constant horizons in spherically symmetric black hole solutions to general relativity. Since Eq. (36) is valid for any stage of the sequence of reversible transformations for any theory satisfying the conditions mentioned before, it is even so when $Q = 0$ and hence, $C = 2M_{irr}$. So, horizons for reversible transformations are dependent just upon the fundamental energy states black holes, $2M_{irr}$. Even more remarkable is that we already know the solution to Eq. (36), which from Eqs. (6), (10) and (9) is

$$\begin{aligned} M &= M_{irr} + Q A_0|_{r=2M_{irr}} - \mathcal{F}|_{r=2M_{irr}} \\ &= M_{irr} + 4\pi \int_{2M_{irr}}^{\infty} r'^2 {T_0}^0(r') dr'. \end{aligned} \tag{37}$$

The above equation is the generalization of the Christodoulou–Ruffini black hole mass decomposition formula to $L(F)$ theories that do not depend upon $M$. If this is not the case, one then have an algebraic equation to solve. The extractable energy $M - M_{irr}$ from $L(F)$ can be read off immediately from Eq. (37) and as we expect, it is the same as Eq. (32); it can also be checked that Eq. (37) is in total agreement with the results for the weak field Lagrangians in terms of the differential approach.

It is worth to notice that Eq. (37) could be also obtained from Eq. (30) of Ref. [23], by replacing there the relation $r_h = 2M_{irr}$. However, following the purely mathematical approach in [23], this latter assumption does not find a clear physical justification. Our approach in this work is completely different from [23]: it is based on physical requirements of energy and charge conservation laws and reversible transformations. As a consequence of these physical requirements, we actually showed that the horizon is indeed a constant of integration, hence an independent quantity.

Since in the present case the horizon area is $A = 4\pi r_+^2$, Eq. (37) can as well be written in terms of it. As we showed above, for reversible transformations the outer horizon must be kept constant and the mass change must be given by Eq. (36). Nevertheless, intuitively, one would expect the total mass of a given black hole to have a definite meaning. In this sense, Eq. (37) in terms of the black hole area should be the expression for the mass even in the case $A$ changes. Such a general statement is reinforced by the fact that it is true for black holes described by the Maxwell Lagrangian.[1] As we show now, this is precisely the case also in nonlinear electrodynamics. Initially we recall that the surface gravity [22] in spherically symmetric solutions in the form [37]

$$\kappa = \frac{(e^{\nu})'|_{r_+}}{2} \tag{38}$$

where the prime means differentiation with respect to the radial coordinate and from Eqs. (9) and (10) the above equation can be cast as

$$\kappa = \frac{1}{2r_+}\left[1 + 2Q \frac{\partial A_0}{\partial r_+} - 2\frac{\partial \mathcal{F}}{\partial r_+}\right]. \tag{39}$$

From Eqs. (10) and (35), one can see in the general case that

$$\delta M = A_0 \delta Q + \frac{\kappa}{8\pi} \delta A, \tag{40}$$

[1] This can be seen in Refs. [24,25] when one works with its final mass expression, $M$, and check it is exactly the same as Eq. (2) of Ref. [4] in the context of reversible transformations.

where Eq. (39) was used. Nevertheless, this is nothing but the generalized first law of black hole mechanics for nonlinear electrodynamics [12]. Since $M$ as given in Eq. (37) was derived from Eqs. (10) and (35), it is assured its variation satisfies Eq. (40). Hence, it is the generalization under the physical approach of the parametrization done by Smarr [24,25] for the classical Christodoulou–Ruffini black hole mass formula in the context of nonlinear electrodynamics. We would like to stress that all the previous reasoning is a direct consequence of having $M$ as an exact differential. Besides, Eq. (37) can be written in the suggestive way as

$$M = Q\,A_0(r_+) + \frac{A}{8\pi r_+}\left[1 - 2\frac{\mathcal{F}(r_+)}{r_+}\right]. \tag{41}$$

From Eq. (39), we see that in general the term in the square brackets of the above equation does not coincide with $2\kappa r_+$. This could be easily seen in the scope of weak field nonlinear theories described by Eq. (14) analyzed previously. Nevertheless, for the case of the Maxwell Lagrangian, the term inside the square brackets of Eq. (41) is exactly $2\kappa r_+$. It implies that the generalized Christodoulou–Ruffini black hole mass formula does not keep the same functional form in nonlinear electrodynamics as in the classic Maxwellian case.

## 8. Discussion

As we have shown above, in the weak field limit of nonlinear Lagrangians, a generalization of the Christodoulou–Ruffini black hole mass decomposition formula can always be obtained; see Eqs. (26), (28) and (30).

Indeed, the weak field limit of nonlinear theories of electromagnetism lead to the constancy of the outer horizon when reversible transformations are taken into account ($2M_{irr}$, exactly as the horizon in the Schwarzschild theory). For irreversible transformations, it always increases. We have also shown that these results actually are valid for any nonlinear asymptotically flat black hole, once it is the only way to lead to the equation coming from the laws of energy and charge conservation for reversible transformations; see the equality in Eq. (13). As a by-product, it allowed us to write down the total mass and the extractable energy (upper limit) of nonlinear spherically symmetric black holes in terms of their charge, outer horizon areas and the scale parameters coming from the electrodynamic theory under interest. When irreversible transformations are present, for each transformation, $\delta r_+ > 0$ iff $(1-8\pi {T_0}^0|_{r_+} r_+^2) > 0$, as it can be seen from Eq. (6). From the same equation, it can be checked that this is always valid when there exists an outer horizon. Hence, for $L(F)$, the areas of the outer horizons never decrease for irreversible processes. With this generalized Christodoulou–Ruffini black hole mass formula, one can notice that the known first law of black hole mechanics [12] is just its direct consequence and hence one could say that it defines such a law. In general such a mass is not functionally the same as the one obtained in the case of the classic Maxwell electrodynamics. If the entropy of a black hole is proportional to its horizon area, the approach of reversible and irreversible transformations lead to the conclusion it can never decrease even in the context of nonlinear electrodynamics.

Turning to astrophysics, it is important to discuss the specific sign of the nonlinear correction parameter, $\mu$. Its positiveness is indeed in agreement with very well-founded nonlinear Lagrangians, such as the Euler–Heisenberg and the Born–Infeld Lagrangians. We have shown that $\mu > 0$ implies that the extractable energy of a black hole described by weak field nonlinear Lagrangian is always smaller than the one associated with the Maxwell Lagrangian; see Eqs. (31) and (33). Hence, due to a continuity argument, we are led to a most important corollary of this work: nonlinear theories of the electromagnetism reduce the amount of extractable energy from a black hole with respect to the classical Einstein–Maxwell case. It means that the extractable energy from nonlinear black holes are always smaller than half of their total mass, which is the largest amount of extractable energy obtained from the Christodoulou–Ruffini black hole mass formula. This result might, in principle, be relevant in the context of gamma-ray bursts (see e.g. [38] and references therein) since their energy budget, as shown by Damour & Ruffini [39], comes from the electromagnetic energy of the black hole extractable by the electron–positron pair creation process á la Sauter–Heisenberg–Euler–Schwinger. However, it is important to keep in mind that for quantitative estimates the perturbative analysis presented in this work is valid only if the condition (15) is satisfied. In the case of the Euler–Heisenberg Lagrangian ($1/\sqrt{\mu} \approx 200E_c$) and for a black hole mass $M \sim 3M_\odot$, as expected from the gravitational collapse of a neutron star to a black hole, $\mu/M^2 \approx 8.2 \times 10^5$, so for a charge to mass ratio $\xi = 5 \times 10^{-4}$ (at the outer horizon $r = r_+$, $E_r/E_c \approx 21$), the reduction of the extractable energy is of only 0.5% with respect to the Maxwell case. For supermassive black holes in active galactic nuclei, e.g. $M \sim 10^9 M_\odot$ ($\mu/M^2 \approx 7.4 \times 10^{-12}$), we obtain for $\xi = 0.9999$ (at $r_+$, $E_r/E_c \approx 5 \times 10^{-4}$) an extractable energy reduced only by $10^{-8}$% with respect to the Maxwell case.

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