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A historical account on characterizations of C^1-manifolds in Euclidean spaces by tangent cones

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ABSTRACT

A historical account on characterizations of C^1-manifolds in Euclidean spaces by tangent cones is provided. Old characterizations of smooth manifold (by tangent cones), due to Valiron (1926, 1927) and Severi (1929, 1934) are recovered; modern characterizations, due to Gluck (1966, 1968) and Tierno (1997) are restated. All these results are consequences of the Four-cones coincidence theorem due to [1].

Contents

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0. Introduction

The aim of this paper is to provide a historical account as well as several geometric characterizations of C^1-manifolds by tangent cones in $\mathbb{R}^n$. We complete so the study beginning in [1]. Let us remember that, if F is an arbitrary subset of Euclidean space $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, the *lower* and the *upper tangent cones* to F at x, indicated respectively by $\operatorname{Tan}^-(F,x)$ and $\operatorname{Tan}^+(F,x)$ are defined in terms of sequences by

$$(0.1)\qquad v \in \operatorname{Tan}^-(F,x) \iff \begin{cases} \forall\{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \\ \exists\{x_m\}_m \subset F \text{ such that } \lim_m \frac{x_m - x}{\lambda_m} = v, \end{cases}$$

$$(0.2)\qquad v \in \operatorname{Tan}^+(F,x) \iff \begin{cases} \exists\{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \\ \exists\{x_m\}_m \subset F \text{ such that } \lim_m \frac{x_m - x}{\lambda_m} = v. \end{cases}$$

The *lower* and the *upper paratangent cones* to F at x, indicated respectively by $\operatorname{pTan}^-(F,x)$ and $\operatorname{pTan}^+(F,x)$ are defined in terms of sequences by

$$(0.3)\qquad v \in \operatorname{pTan}^-(F,x) \iff \begin{cases} \forall\{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \\ \forall\{y_m\}_m \subset F \text{ with } y_m \to x, \\ \exists\{x_m\}_m \subset F \text{ such that } \lim_m \frac{x_m - y_m}{\lambda_m} = v, \end{cases}$$

$$(0.4)\qquad v \in \operatorname{pTan}^+(F,x) \iff \begin{cases} \exists\{\lambda_m\}_m \subset \mathbb{R}_{++} \text{ with } \lambda_m \to 0^+, \\ \exists\{y_m\}_m \subset F \text{ with } y_m \to x, \\ \exists\{x_m\}_m \subset F \text{ such that } \lim_m \frac{x_m - y_m}{\lambda_m} = v. \end{cases}$$

In general

$$(0.5)\qquad \operatorname{pTan}^-(F,x) \subset \operatorname{Tan}^-(F,x) \subset \operatorname{Tan}^+(F,x) \subset \operatorname{pTan}^+(F,x).$$

We refer to [1] for several properties and preliminary results about tangent cones. In particular, we recall the most important result proved in [1], i.e. the characterization of C^1-manifolds by the Four-cones coincidence theorem:

Theorem 0.1 *(Four-cones coincidence theorem: global version). A non-empty subset F of $\mathbb{R}^n$ is a C^1-manifold if and only if F is locally compact and the lower and upper paratangent cones to F coincide at every point, i.e.,*

$$(0.6)\qquad \operatorname{pTan}^-(F,x) = \operatorname{pTan}^+(F,x) \quad \textit{for every } x \in F.$$

Notice that condition (0.6) amounts to the set inclusion

$$(0.7)\qquad \operatorname{pTan}^+(F,x) \subset \operatorname*{Li}_{F \ni y \to x} \operatorname{Tan}^+(F,y) \quad \text{for every } x \in F.$$

In this paper we examine the historic study of C^1-manifolds, recovering old geometrical characterizations of smooth manifolds by tangent cones due to Valiron (1926, 1927) and Severi (1929, 1934). More modern characterizations due to Gluck (1966, 1968) and Tierno (1997) are restated too. The paper is organized as follows:

Section 1: **From Fréchet problem to modern characterizations of smooth manifolds.** In 1925 Fréchet inquires into existence of non-singular continuously differentiable parametric representations of continuous curves. This problem had been a starting, motivating and reference point for subsequent research by various mathematicians. Two basic conditions for the existence of a non-singular parametrization of a set (either curve or surface) were given by Valiron:

($*$) continuously turning tangent space, and
($**$) locally injective orthogonal projections on tangent spaces.

Other conditions ensuring ($*$) and ($**$) were given by Severi by means of paratangency instead of tangency. Valiron (1926, 1927) and Severi (1929, 1934) present, in the setting of the topological manifolds, the first geometrical characterizations of C^1-manifold by tangent cones. From a historical point of view, an essential condition to a complete geometrical characterization of C^1-manifold by tangent cones, has been a solid and univocal (but not necessary unique) definition of tangency and a C^1 version of differentiability, the so-called strict differentiability. Finally, Guarreschi extended the results of Severi introducing the projection character.

Section 2: **Old characterizations of C^1-manifolds.** This section presents the characterizations obtained through the Valiron condition and the Severi simplicity condition, exposed in two subsections. In a first subsection we study Euclidean graphs as smooth manifolds.

Section 3: **Modern characterizations of C^1-manifolds.** This section is devoted to the exposition of the modern results due to Gluck and Tierno. The characterizations are presented in several subsections and proved in the light of the Four-cones coincidence theorem.

The paper is concluded by two appendices. In the first one (Appendix A) we give a geometric and analytic characterization of Strict diffeomorphisms. In the second one (Appendix B) we restate the Four-cones coincidence theorem, proving a different version than given in [1].

Remark. In the following sections, the symbols $\mathbb{R}$ and $\mathbb{N}$ will denote the real and natural numbers, respectively; and $\mathbb{N}_k := \{m \in \mathbb{N}\colon\ m \geqslant k\}$ for $k \in \mathbb{N}$, $\mathbb{R}_+ := \{x \in \mathbb{R}\colon\ x \geqslant 0\}$, $\mathbb{R}_{++} := \{x \in \mathbb{R}\colon\ x > 0\}$. If not otherwise specified, any set will be a subset of some finite dimensional Euclidean space $\mathbb{R}^n$. An open (resp. closed) ball of center $\hat{x}$ and radius ε will be denoted by $\mathrm{B}_\varepsilon(x)$ (resp. $\bar{\mathrm{B}}_\varepsilon(x)$). $\mathbb{P}(\mathbb{R}^n)$ denotes the set of all subsets of $\mathbb{R}^n$. The set of accumulation points of a given set A and its interior are denoted by $\mathrm{der}(A)$ and $\mathrm{int}(A)$, respectively.

1. From Fréchet problem to modern characterizations of smooth manifolds

In [10, (1887), vol. III, p. 587] Jordan defines a curve as a continuous image of an interval. By means of a notion of rectifiability, Jordan gives mathematical concreteness and coherence to the usage of the term "length" and, moreover, by parametrization of sets he provides fresh impetus to the study of local and global properties of sets.

Surprisingly for Jordan's epoch, continuous curves did not fit to common intuition on 1-dimensionality and null area of their *loci*. In fact, Peano in [17, (1890)] constructed a continuous curve filling a square. Clearly, Peano's curve is not simple. An example of a simple continuous curve of non-null area was given by Lebesgue [11, (1903)] and by Osgood [15, (1903)]. Nalli [13,14, (1911)] characterized the locus of simple continuous plane curves by means of *local connectedness* (a new notion, introduced by Nalli). Three years later, Mazurkiewicz [12, (1914)] and Hahn [9, (1914)] proved the celebrated theorem: "A set of Euclidean space is a continuous image of a compact interval if and only if it is a locally connected continuum".

In absence of differentiable properties, the continuity alone does not capture intuitive curve aspects. Aware of this lack, to recover geometric properties of the locus of a continuous curve, Fréchet (see [4, (1925), pp. 292–293] and [5, (1928), pp. 152–154]) proposed the following *problem*: *Find a non-singular parametric representation*[1] *of the locus of a continuous curve having tangent straight-line at every point*. Let's quote Fréchet from the first reference:

> On sait qu'une courbe continue sans point multiple et ayant une tangente déterminée en chaque point peut avoir une représentation paramétrique constituée de fonctions dérivables $x(t)$, $y(t)$, $z(t)$, mais dont les dérivées peuvent exceptionnellement s'annuler à la fois [...]
>
> Ce qui précède nous encourage à proposer la question suivante, dont la solution à première vue ne paraît pas douteuse:
>
> *Si une courbe continue est douée partout* (ou en un point) *d'une tangente, peut-on la représenter paramétriquement par des fonctions dérivables partout* (ou au point correspondant)*?* Bien entendu, dans cet énoncé, la tangente est définie géométriquement, c'est-à-dire comme limite d'une corde.

Fréchet's confidence about a solution to his problem was dampened in 1926 by Valiron [26, (1927)]. After making precise and explicit the meaning of both *tangent half-straight-line* and *tangent straight-line*, Valiron gives the following proposition.

Theorem 1.1. *(See* Valiron *[26, (1927), p. 47].) If a continuous curve admits a continuously variable oriented tangent straight-line at its points, then it has a non-singular continuously differentiable parametric representation.*[2] □

Valiron [25, (1926)] provides an analogous proposition for surfaces of ordinary 3-dimensional space. To attain this aim, he introduces the concept of *oriented tangent plane* to a surface F, takes into account continuously turning oriented tangent plane and, in addition, adopts the following condition at every point $x \in F$:

(1.1) (Valiron [25, (1926), p. 190]) *The orthogonal projection on the oriented tangent plane to F at x is injective on an open neighborhood of x in F.*

[1] Here and in the sequel, "*non-singular parametric representation*" stands for "differentiable parametric representation with everywhere non-null derivative".

[2] Using our terminology, a 2-dimensional (resp. 1-dimensional) vector space H is an *oriented tangent plane* (resp. an *oriented tangent straight-line*) to a set F at a point x, if H is equal to the upper tangent cone to F at x. In other words, F admits an oriented tangent plane (resp. an oriented tangent straight-line) at a point x if and only if the upper tangent cone at x is a 2-dimensional (resp. 1-dimensional) vector space.

Theorem 1.2. *(See* VALIRON *[25, (1926)].) Let $F \subset \mathbb{R}^3$ be homeomorphic to a 2-dimensional open connected set. If F admits a continuously variable oriented tangent plane and the condition* (1.1) *holds, then F locally coincides with the graph of a continuously differentiable function.* □

Following PAUC's counterexample [16, (1940), p. 96] to the Fréchet problem, CHOQUET, in his thesis [2, (1948), p. 170], provides necessary and sufficient conditions for the FRÉCHET supposition to hold. Like VALIRON, PAUC and CHOQUET make precise and explicit the meaning of tangent straight-line. Besides, CHOQUET considers a more general problem: *If a variety admits a linear tangent variety at every point (or has certain regularity), is there a regular parametrization (or a parametrization having an analogous degree of regularity)?* In this spirit, ZAHORSKI and CHOQUET proves the following two propositions.

Proposition 1.3 (ZAHORSKI). *(See* CHOQUET *[2, (1947), pp. 173–174].) If a continuous arc admits a tangent straight-line at all but* (*possibly*) *countably many points, then it has a differentiable parametric representation.* □

Proposition 1.4. *(See* CHOQUET *[2, (1947), p. 174].) A continuous image of a compact interval is a rectifiable curve if and only if it admits a Lipschitzian differentiable parametric representation.* □

Invoking seminal papers of FRÉCHET [4, (1925)], and VALIRON [25,26, (1926, 1927)], SEVERI looks for non-singular continuously differentiable parametric representations of a curve (resp. surface). Main ingredients of the solutions of SEVERI are *strict differentiability* and *paratangency*. Strict differentiability ensures that curves (resp. surfaces) have a continuously turning tangent straight-line (resp. plane); it is geometrically characterized in terms of paratangency (see Proposition 2.8 of [1]). On the other hand, aware of the need of VALIRON's condition (1.1), SEVERI assumes the following *simplicity condition* and, consequently, ensures VALIRON's condition by replacing VALIRON's oriented tangent plane by a paratangent plane.

Definition 1.5. (See SEVERI [21, (1929), p. 194], [18, (1930), p. 216], [19, (1931), p. 341], [20, (1934), p. 194].) A d-dimensional topological manifold F of $\mathbb{R}^n$ satisfies the *Severi simplicity condition*, if the dimension of the linear hull of the upper paratangent cone to F at every point is at most d.

Theorem 1.6. *(See* SEVERI *[20, (1934), pp. 194, 196].) If F is a topological manifold of dimension one* (*resp. two*) *satisfying Severi simplicity condition, then the upper paratangent cone at every point is a one* (*resp. two*) *dimensional vector space which varies continuously.* □

To extend this theorem, GUARESCHI [8, (1940), p. 415] introduces the projection character.

Definition 1.7. Let $F \subset \mathbb{R}^n$ and $\hat{x} \in F \cap \mathrm{der}(F)$. The GUARESCHI's projection character of F at $\hat{x}$ is the smallest natural number d such that there are an open neighborhood Ω at $\hat{x}$ and a d-dimensional vector subspace V of $\mathbb{R}^n$ such that the orthogonal projection of $F \cap \Omega$ on V is injective.

GUARESCHI shows that the projection character of a d-dimensional topological manifold F verifying the Severi simplicity condition Definition 1.5, is equal to d at its every point. And then he proves the following theorem.

Theorem 1.8. *(See* GUARESCHI *[8, (1940), p. 418].) If F is a d-dimensional topological manifold and the Guareschi projection character is equal to d at its every point, then the parantangent cone is a vector space at its every point and varies continuously.*

2. Old characterizations of C^1-manifolds

2.1. When is a graph a C^1-manifold?

We denote by $\mathbb{G}(\mathbb{R}^{d+n}, d)$ the (Grasmannian) set of all the d-dimensional subspaces of $\mathbb{R}^{d+n}$. A set-valued function $\varphi : A \to \mathcal{P}(\mathbb{R}^n)$ is said to be continuous at $\hat{x} \in A$ whenever

$$\operatorname*{Ls}_{A \ni x \to \hat{x}} \varphi(x) \subset \varphi(\hat{x}) \subset \operatorname*{Li}_{A \ni x \to \hat{x}} \varphi(x).$$

It is well-known, φ is continuous at $\hat{x}$ if and only if

$$\operatorname*{Ls}_{m \to \infty} \varphi(x_m) \subset \varphi(\hat{x}) \subset \operatorname*{Li}_{m \to \infty} \varphi(x_m) \quad \text{for every } \{x_m\}_m \subset A \text{ converging to } \hat{x}.$$

See Section 3 of [1] for more details.

Proposition 2.1. *Let $f : A \to \mathbb{R}^n$ be a function on an open subset A of $\mathbb{R}^d$. Then the following three properties are equivalent*:

(2.1) *f is a C^1-function on A*;
(2.2) (Valiron [25, (1926), p. 192]) *f is differentiable on A and the map*

$$\big(x, f(x)\big) \mapsto \operatorname{Tan}^+\big(\operatorname{graph}(f), \big(x, f(x)\big)\big)$$

from $\operatorname{graph}(f)$ *into* $\mathbb{G}(\mathbb{R}^{d+n}, d)$ *is continuous as a set-valued function*;
(2.3) (Severi [20, (1934), p. 196]) *f is strictly differentiable.*

The equivalence (2.1) $\Longleftrightarrow$ (2.3) is proved by Theorem 7.1 of [3], while (2.1) $\Longleftrightarrow$ (2.2) comes directly out of the following Lemmata 2.2 and 2.3.

Lemma 2.2. *Let V_m for $m \in \mathbb{N}$ and V be vector subspaces of $\mathbb{R}^n$.*

(2.4) *If* $\dim(V_m) = \dim(V)$ *eventually in m, then*

$$V \subset \operatorname*{Li}_{m\to\infty} V_m \quad \Longleftrightarrow \quad \operatorname*{Ls}_{m\to\infty} V_m \subset V.$$

(2.5) *If*

$$\operatorname*{Ls}_{m\to\infty} V_m \subset V \subset \operatorname*{Li}_{m\to\infty} V_m$$

then $\dim(V_m) = \dim(V)$ *eventually in m.*

Proof. We need to prove (2.5), while the first statement is already present in [1]. Obviously, the set inclusion $V \subset \operatorname{Li}_{m\to\infty} V_m$ implies that there is $\bar{m} \in \mathbb{N}$ such that $\dim(V) \leqslant \dim(V_m)$ for every $m \geqslant \bar{m}$. On the other hand, the set inclusion $\operatorname{Ls}_{m\to\infty} V_m \subset V$ implies that $\dim(V) \geqslant d$, if $\dim(V_m) \geqslant d$ for infinitely many m. In conclusion, there is $\tilde{m}$ such that $\dim(V) = \dim(V_m)$ for every $m \geqslant \tilde{m}$. $\square$

Lemma 2.3. *Let $\{L_k\}_{k\in\mathbb{N}}$ be a sequence of linear maps $L_k : \mathbb{R}^d \to \mathbb{R}^n$. The following statements are equivalent*:

(2.6) $\lim_{k\to\infty} \|L_k - L\|_\infty = 0$;
(2.7) $\operatorname{Ls}_{k\to\infty} \operatorname{graph}(L_k) \subset \operatorname{graph}(L) \subset \operatorname{Li}_{k\to\infty} \operatorname{graph}(L_k)$.

Proof. (2.6) $\Longrightarrow$ (2.7): Since $\dim(\operatorname{graph}(L_k))$ is d for every k, by the point (2.4) of Lemma 2.2 we can just prove

$$\operatorname*{Ls}_{k\to\infty} \operatorname{graph}(L_k) \subset \operatorname{graph}(L).$$

Let $v \in \mathbb{R}^d$ and $w \in \mathbb{R}^n$ such that $(v, w) \in \operatorname{Ls}_k \operatorname{graph}(L_k)$. Then there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\lim_{j\to\infty} \big(v_j, L_{k_j}(v_j)\big) = (v, w),$$

where $\{L_{k_j}\}_{j\in\mathbb{N}}$ is a subsequence of $\{L_k\}_{k\in\mathbb{N}}$. Then clearly $\lim_{j\to\infty} v_j = v$ and $\lim_{j\to\infty} L_{k_j}(v_j) = w$. Moreover

$$\begin{aligned}\big\|L_{k_j}(v_j) - L(v)\big\| &\leqslant \big\|L_{k_j}(v_j) - L(v_j)\big\| + \big\|L(v_j) - L(v)\big\| \\ &\leqslant \|L_{k_j} - L\|_\infty \cdot \|v_j\| + \|L\|_\infty \cdot \|v_j - v\|\end{aligned}$$

i.e. $\lim_{j\to\infty} L_{k_j}(v_j) = L(v)$ and therefore $w = L(v)$.

(2.7) $\Longrightarrow$ (2.6): We firstly prove that $\{\|L_k\|_\infty\colon k \in \mathbb{N}\}$ is bounded. Suppose it is false by contradiction, then there exists a subsequence $\{L_{k_j}\}_{j\in\mathbb{N}}$ such that $\lim_{k\to\infty} \|L_{k_j}\|_\infty = \infty$ and therefore there are $\{v_j\}_j \subset \mathbb{R}^d$ with $\|v_j\| = 1$ for all $j \in \mathbb{N}$ such that $\lim_{j\to\infty} \|L_{k_j}(v_j)\| = \infty$. By the compactness of the closed unit ball in $\mathbb{R}^n$, up to passing to a subsequence of $\{v_j\}_{j\in\mathbb{N}}$, we can suppose that there exists $w \in \mathbb{R}^n$ with $\|w\| = 1$ such that

$$\lim_{j\to\infty} \left(\frac{v_j}{\|L_{k_j}(v_j)\|}, \frac{L_{k_j}(v_j)}{\|L_{k_j}(v_j)\|} \right) = (0, w).$$

Then $(0, w) \in \operatorname{Ls}_k \operatorname{graph}(L_k)$, which is in contradiction with (2.7).

In order to prove (2.6) we can simply prove the following statement: every subsequence $\{L_{k_j}\}_{j\in\mathbb{N}}$ has a subsequence $\{L_{k_{j_i}}\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \|L_{k_{j_i}} - L\|_\infty = 0$. If $\{L_{k_j}\}_{j\in\mathbb{N}}$ is a subsequence, then it is bounded and therefore it has a subsequence $\{L_{k_{j_i}}\}_{i\in\mathbb{N}}$ converging to some linear map $L' : \mathbb{R}^d \to \mathbb{R}^n$. By the previous implication (2.6) $\Rightarrow$ (2.7), it holds

$$\operatorname{graph}(L') = \operatorname*{Ls}_{i\to\infty} \operatorname{graph}(L_{k_{j_i}}) \subset \operatorname*{Ls}_{k\to\infty} \operatorname{graph}(L_k) \subset \operatorname{graph}(L).$$

Hence $\operatorname{graph}(L') = \operatorname{graph}(L)$ and therefore $L = L'$. □

Theorem 2.4. *Let $f : A \to \mathbb{R}^n$ be a function on an open subset A of $\mathbb{R}^d$. Then the following two properties are equivalent*:

(2.8) $\operatorname{graph}(f) \subset \mathbb{R}^{d+n}$ *is a C^1-submanifold and* $\operatorname{Tan}^+(\operatorname{graph}(f), (t, f(t)))$ *does not contain vertical lines for all $t \in A$.*
(2.9) *f is a C^1-function on A.*

Proof. (2.8) ⇒ (2.9) comes by statement (4.5) of Proposition 4.2 of [1], Proposition 2.3 of [1] and Proposition 2.1.
(2.9) ⇒ (2.8) comes by Lemma 4.5 of [1] and Proposition 2.1. □

Proposition 2.5. *Let $f : A \to \mathbb{R}^n$ be a function on a subset A of $\mathbb{R}^d$. Then A is an open subset of $\mathbb{R}^d$ and f is a C^1-function on A if and only if the following three properties hold*:

(2.10) *A is locally compact and f is continuous,*
(2.11) $\operatorname{Tan}^+(\operatorname{graph}(f), (x, f(x)))$ *is a d-dimensional vector space of $\mathbb{R}^d \times \mathbb{R}^n$ and does not include vertical lines for every $x \in A$,*
(2.12) *the map* $(x, f(x)) \mapsto \operatorname{Tan}^+(\operatorname{graph}(f), (x, f(x)))$ *from* $\operatorname{graph}(f)$ *into* $\mathbb{G}(\mathbb{R}^{d+n}, d)$ *is continuous.*

Proof. *Necessity*. By Proposition 2.1, it is immediate. *Sufficiency*. Since the upper tangent cone $\operatorname{Tan}^+(\operatorname{graph}(f), (x, f(x)))$ is a d-dimensional vector subspace of $\mathbb{R}^d \times \mathbb{R}^n$ without vertical lines, it is the graph of a linear function from $\mathbb{R}^d$ to $\mathbb{R}^n$; hence, for every $v \in \mathbb{R}^d$ there is $w \in \mathbb{R}^n$ such that $(v, w) \in \operatorname{Tan}^+(\operatorname{graph}(f), (x, f(x)))$. Therefore

$$(*1) \qquad \operatorname{Tan}^+(A, x) = \mathbb{R}^d \quad \text{for every } x \in A.$$

Moreover, Proposition 2.3 of [1] and the continuity of f imply the differentiability of f on A. Now, in virtue of properties (1.20) and (1.21) of [1], from $(*1)$ and the local compactness of A it follows that A is open. Thus, by Proposition 2.1, condition (2.12) entails the C^1 smoothness of f on A, as required. □

2.2. Valiron

According to (1.1) we consider the

(2.13) *Valiron condition* for a set $F \subset \mathbb{R}^n$: For every point $x \in F$, the orthogonal projection on the linear upper tangent space $\operatorname{LTan}^+(F, x)$ is injective on an open neighborhood of x in F.

The following theorem extends Valiron's Theorem 1.2.

Theorem 2.6 *(Valiron theorem). A non-empty subset F of $\mathbb{R}^n$ is a d-dimensional C^1-manifold if and only if F is a d-dimensional topological manifold, Valiron condition* (2.13) *is satisfied and*

(2.14) *$x \mapsto \operatorname{Tan}^+(F, x)$ is a continuous set-valued map from F to $\mathbb{G}(\mathbb{R}^n, d)$, i.e.,*

$$\operatorname{pTan}^-(F, x) = \operatorname{LTan}^+(F, x) \quad \textit{and} \quad \dim\bigl(\operatorname{LTan}^+(F, x)\bigr) = d \quad \textit{for every } x \in F.$$ [3]

Proof. *Necessity*. By Proposition 4.2 of [1] it is obvious. *Sufficiency*. Assume F locally compact and verifying both Valiron condition and the property (2.14). Clearly

$$(*1) \qquad \operatorname{Tan}^+(F, x) \text{ is } d\text{-dimensional vector space, for every } x \in F.$$

1*st case*: $d = 0$. By property (1.22) of [1] and $(*1)$, every point of F is isolated. Hence F is a C^1-manifold of dimension zero.

2*nd case*: $d = n$. F being an n-topological manifold of $\mathbb{R}^n$, the set F is open.

3*rd case*: $0 < d < n$. Fix $\hat{x} \in F$. Since $\operatorname{Tan}^+(F, \hat{x})$ is a d-dimensional vector space, without loss of generality we assume that $\operatorname{Tan}^+(F, \hat{x})$ is generated by the first d vectors of the canonical basis of $\mathbb{R}^n$. Then, by Valiron condition, property (2.14) and Theorem 3.3 of [1] there exist an open neighborhood Ω of $\hat{x}$ in $\mathbb{R}^n$, a set $A \subset \mathbb{R}^d$ and a function $\varphi : A \to \mathbb{R}^{n-d}$ such that

$$(*2) \qquad \operatorname{graph}(\varphi) = F \cap \Omega,$$

[3] The d-dimensionality condition cannot be dropped in (2.14), as it is shown by the following example. Consider the set $F := \{(x, y, z) \in \mathbb{R}^3\colon\ (x^2 + y^2 + z^2)^2 = 4(x^2 + y^2)\}$. F is a torus generated by turning the circle $S := \{(0, y, z)\colon\ (y-1)^2 + z^2 = 1\}$ about the z-axis. Since the circle S is tangent to z-axis at $(0, 0, 0)$, the set F is a 2-dimensional C^1-manifold at every point different from $(0, 0, 0)$; while $\operatorname{pTan}^-(F, (0, 0, 0)) = \operatorname{Tan}^+(F, (0, 0, 0)) = \operatorname{LTan}^+(F, (0, 0, 0)) = \mathbb{R}e_3$ and, consequently, $\dim(\operatorname{LTan}^+(F, (0, 0, 0))) = 1$.

and

$$(*3)\qquad \operatorname{ang}\big(\mathrm{Tan}^+(F,x),\mathrm{Tan}^+(F,\hat{x})\big)<\frac{\pi}{2}\quad \text{for every } x\in F\cap\Omega.$$

The projection map $(x,\varphi(x))\mapsto x$ from $F\cap\Omega$ into $\mathbb{R}^d$ is both continuous and injective; hence, $F\cap\Omega$ and $\mathbb{R}^d$ both being d-dimensional topological manifolds, by Brouwer domain invariance theorem we have that

$(*4)$ A is open and φ is continuous.

On the other hand, since Ω is open, by $(*2)$ and (4.6) of [1] we have that

$$(*5)\qquad \mathrm{Tan}^+\big(\mathrm{graph}(\varphi),\big(t,\varphi(t)\big)\big)=\mathrm{Tan}^+\big(F,\big(t,\varphi(t)\big)\big)\quad \text{for every } t\in A.$$

Hence, by $(*1)$ and $(*3)$, the cones $\mathrm{Tan}^+(\mathrm{graph}(\varphi),(t,\varphi(t)))$ are d-dimensional vector spaces of $\mathbb{R}^d\times\mathbb{R}^{n-d}$ which, for every $t\in A$, do not include vertical lines. Moreover, by (2.14) the map $(t,\varphi(t))\mapsto \mathrm{Tan}^+(\mathrm{graph}(\varphi),(t,\varphi(t)))$ is continuous on $\mathrm{graph}(\varphi)$. Therefore, from Proposition 2.5 it follows that φ is C^1-function on the open set A. Thus F is a d-dimensional C^1-manifold at $\hat{x}$. Finally, $\hat{x}$ being an arbitrary point of F, we have that F is d-dimensional C^1-manifold of $\mathbb{R}^n$, as required. □

Corollary 2.7. *A non-empty subset F of $\mathbb{R}^n$ is a d-dimensional C^1-manifold if and only if F is a d-dimensional topological manifold, Valiron condition* (2.13) *is satisfied and, moreover, for every $x\in F$, $\dim(\mathrm{LTan}^+(F,x))\leqslant d$ and*

(2.15) *the set-valued map $x\mapsto \mathrm{Tan}^+(F,x)$ from F to $\mathbb{G}(\mathbb{R}^n,d)$ is either lower or upper semicontinuous.*

Lemma 2.8. *Let $\varphi: A\to\mathbb{R}^k$ be a continuous function from an open subset A of $\mathbb{R}^d$ such that, $\mathrm{LTan}^+(\mathrm{graph}(\varphi),(\hat{t},\varphi(\hat{t})))\subset\mathbb{R}^d\times\{0_k\}$ for a point $\hat{t}\in A$. Then*

$$(2.16)\qquad \mathrm{Tan}^+\big(\mathrm{graph}(\varphi),\big(\hat{t},\varphi(\hat{t})\big)\big)=\mathrm{LTan}^+\big(\mathrm{graph}(\varphi),\big(\hat{t},\varphi(\hat{t})\big)\big)=\mathbb{R}^d\times\{0_k\}.$$ [4]

The simple proof of this lemma is left to the reader.

Proof of Corollary 2.7. *Sufficiency*. By Valiron condition (2.13), at every point $\hat{x}\in F$ we have an open neighborhood $\Omega\subset\mathbb{R}^n$ of $\hat{x}$ such that the orthogonal projection π onto $\mathrm{LTan}^+(F,\hat{x})$ is continuous and injective on $F\cap\Omega$. Observe that $F\cap\Omega$ is a d-dimensional topological manifold and, on the other hand, $\mathrm{LTan}^+(F,\hat{x})$ is a topological manifold of dimension less than or equal to d; then by Brouwer domain invariance theorem we have that

$$(*1)\qquad \dim\big(\mathrm{LTan}^+(F,\hat{x})\big)=d,$$

and the restriction $\pi_{|F\cap\Omega}$ is a homeomorphism from $F\cap\Omega$ onto the open subset $\pi(F\cap\Omega)$ of $\mathrm{LTan}^+(F,\hat{x})$. Therefore, from Lemma 2.8 it follows that

$$(*2)\qquad \mathrm{Tan}^+(F\cap\Omega,\hat{x})=\mathrm{LTan}^+(F\cap\Omega,\hat{x}).$$

Hence, Ω being an open neighborhood of $\hat{x}$, by (4.6) of [1] we have that

$$(*3)\qquad \mathrm{Tan}^+(F,\hat{x})=\mathrm{LTan}^+(F,\hat{x})$$

for every $\hat{x}\in F$. Thus, $\mathrm{Tan}^+(F,\hat{x})$ is a d-dimensional vector space; therefore, (2.15) and Theorem 3.3 of [1] imply (2.14). Hence, by Valiron Theorem 2.6 F is a d-dimensional C^1-manifold. □

2.3. Severi

The following theorem extends Severi's theorem 1.6, by involving a simplicity condition for d-dimensional topological manifolds. Severi simplicity condition 1.5 can be restated as

$$(2.17)\qquad \dim\big(\mathrm{pLTan}^+(F,x)\big)\leqslant d\quad \text{for every } x\in F.$$

Lemma 2.9. *Let $\varphi: A\to\mathbb{R}^k$ be a continuous function from an open subset A of $\mathbb{R}^d$ such that, $\mathrm{pLTan}^+(\mathrm{graph}(\varphi),(\hat{t},\varphi(\hat{t})))\subset\mathbb{R}^d\times\{0_k\}$ for a point $\hat{t}\in A$. Then*

$$(2.18)\qquad \mathrm{Tan}^+\big(\mathrm{graph}(\varphi),\big(\hat{t},\varphi(\hat{t})\big)\big)=\mathrm{pLTan}^+\big(\mathrm{graph}(\varphi),\big(\hat{t},\varphi(\hat{t})\big)\big)=\mathbb{R}^d\times\{0_k\}.$$ [5]

[4] The zero element of $\mathbb{R}^k$ is denoted by 0_k.
[5] The zero element of $\mathbb{R}^k$ is denoted by 0_k.

The simple proof of this lemma is left for the reader.

Lemma 2.10. *Let $F \subset \mathbb{R}^n$ be a d-dimensional topological manifold verifying Severi simplicity condition (2.17). Then*

(2.19) *Valiron condition* (2.13) *holds, and*
(2.20) $\mathrm{Tan}^+(F, x) = \mathrm{pLTan}^+(F, x)$ *and* $\dim(\mathrm{pLTan}^+(F, x)) = d$*, for every* $x \in F$.

Proof. Fix a point $x \in F$. 1*st case*: $d = 0$. Since $\{0\} \subset \mathrm{Tan}^+(F, x) \subset \mathrm{pLTan}^+(F, x)$, condition (2.17) implies (2.20). On the other hand, F being 0-dimensional topological manifold, the point x is isolated in F; therefore Valiron condition holds. 2*nd case*: $d > 0$. By Severi simplicity condition, $\dim(\mathrm{pLTan}^+(F, x)) \leqslant d$; hence, without loss of generality we assume that $\mathrm{pLTan}^+(F, x)$ is a subspace of the vector space generated by the first d vectors of the canonical basis of $\mathbb{R}^n$, i.e.

(∗1) $\quad \mathrm{pLTan}^+(F, x) \subset \mathbb{R}^d \times \{0_{n-d}\}.$

Then, by property (1.23) of [1] there exist an open neighborhood Ω of x, a set $A \subset \mathbb{R}^d$ and a function $\varphi : A \to \mathbb{R}^{n-d}$ such that

(∗2) $\quad \mathrm{graph}(\varphi) = F \cap \Omega.$

The projection map $(t, \varphi(t)) \mapsto t$ from $F \cap \Omega$ into $\mathbb{R}^d$ is both continuous and injective; hence, $F \cap \Omega$ and $\mathbb{R}^d$ both being d-dimensional topological manifolds, by Brouwer domain invariance theorem we have that

(∗3) $\quad A$ is an open subset of $\mathbb{R}^d$.

Let $\hat{t}$ denote the element of A such that $x = (\hat{t}, \varphi(\hat{t}))$; then by (∗1)

(∗4) $\quad \mathrm{pLTan}^+\big(\mathrm{graph}(\varphi), (\hat{t}, \varphi(\hat{t}))\big) \subset \mathbb{R}^d \times \{0_k\}.$

Therefore, equalities (2.18) of Lemma 2.9 imply:

(∗5) $\quad \mathrm{Tan}^+(F \cap \Omega, x) = \mathrm{pLTan}^+(F \cap \Omega, x) \quad \text{and} \quad \dim\big(\mathrm{pLTan}^+(F \cap \Omega, x)\big) = d.$

Ω being an open neighborhood of x, by (4.6) of [1] we have (2.20), as required. Finally, by (2.20) $\mathrm{pLTan}^+(F, x) = \mathrm{LTan}^+(F, x)$; hence by property (1.23) of [1] the orthogonal projection onto $\mathrm{LTan}^+(F, x)$ is injective on a neighborhood of x in F; therefore, Valiron condition (2.13) holds. □

Theorem 2.11 *(Severi theorem). A non-empty subset F of $\mathbb{R}^n$ is a d-dimensional C^1-manifold if and only if F is a d-dimensional topological manifold and Severi simplicity condition holds (2.17).*

Proof. *Necessity*. By Proposition 4.2 of [1], it is obvious. From Valiron Theorem 2.6 *sufficiency* follows. In fact, F is a d-dimensional topological manifold; by (2.18) F verifies Valiron condition; by (2.20), for every $x \in F$, $\mathrm{Tan}^+(F, x)$ is a d-dimensional vector space. Moreover, by (2.20) we have $\mathrm{Tan}^+(F, x) = \mathrm{pTan}^+(F, x)$; therefore the property (1.18) of [1] implies the upper semicontinuity of the map $x \mapsto \mathrm{Tan}^+(F, x)$ from F to $\mathbb{G}(\mathbb{R}^n, d)$, i.e., for every $x \in F$

(∗1) $\quad \operatorname*{Ls}_{F \ni y \to x} \mathrm{Tan}(F, y) \subset \mathrm{Tan}(F, x).$

On the other hand, by Lemma 2.2 the upper semicontinuity (∗1) amounts to the lower semicontinuity

(∗2) $\quad \mathrm{Tan}(F, x) \subset \operatorname*{Li}_{F \ni y \to x} \mathrm{Tan}(F, y)$

for every $x \in F$. Hence the map $x \mapsto \mathrm{Tan}^+(F, x)$ is continuous, as required by Valiron theorem. □

By Proposition 2.6 of [1] Severi simplicity condition amounts to the following property:

(2.21) at every $x \in F$ there exists a d-dimensional vector space which is paratangent in traditional sense to F,

equivalently, for every $\hat{x} \in \mathrm{der}(F)$ there exists a d-dimensional vector space V such that

(2.22) $$\lim_{\substack{F \ni x, y \to \hat{x} \\ x \neq y}} \frac{\mathrm{dist}(y, x + V)}{\|x - y\|} = 0.$$

Recalling the notations of Section 3 of [1] and in particular equation (3.5) of [1], we have that $\mathrm{dist}(x, V) = \frac{\|(\bigwedge_{i=1}^d v_i) \wedge x\|}{\|\bigwedge_{i=1}^d v_i\|}$, where $\{v_i\}_{i=1}^d$ is a base of V. Hence

Corollary 2.12. *A non-empty subset F of $\mathbb{R}^n$ is a d-dimensional C^1-manifold if and only if F is a d-dimensional topological manifold and, for every $\hat{x} \in \operatorname{der}(F)$ there exists a non-null simple d-vector $\bigwedge_{i=1}^d v_i$ (where $\{v_i\}_{i=1}^d \subset \mathbb{R}^n$) such that*

$$\lim_{\substack{F \ni x, y \to \hat{x} \\ x \neq y}} \frac{\|(\bigwedge_{i=1}^d v_i) \wedge (x-y)\|}{\|x-y\|} = 0. \tag{2.23}$$

On the other hand, $\operatorname{dist}(x, V) = \| \sum_{i=1}^{n-d} \langle x, w_i \rangle w_i \|$, where $\{w_i\}_{i=1}^{n-d}$ is an orthonormal base of $V^\perp$; hence

Corollary 2.13. *A non-empty subset F of $\mathbb{R}^n$ is a d-dimensional C^1-manifold if and only if F is a d-dimensional topological manifold and, for every $\hat{x} \in \operatorname{der}(F)$ there exists an orthonormal family $\{w_i\}_{i=1}^{n-d}$ of vectors of $\mathbb{R}^n$ such that*

$$\lim_{\substack{F \ni x, y \to \hat{x} \\ x \neq y}} \frac{\sum_{i=1}^{n-d} \langle x-y, w_i \rangle^2}{\|x-y\|^2} = 0. \tag{2.24}$$

In particular, for C^1-manifolds of codimension 1, we have

Corollary 2.14. *A non-empty subset F of $\mathbb{R}^n$ is an $(n-1)$-dimensional C^1-manifold if and only if F is an $(n-1)$-dimensional topological manifold and, for every $\hat{x} \in \operatorname{der}(F)$ there exists a non-null vector $w \in \mathbb{R}^n$ such that*

$$\lim_{\substack{F \ni x, y \to \hat{x} \\ x \neq y}} \frac{\langle x-y, w \rangle}{\|x-y\|} = 0. \tag{2.25}$$

3. Modern characterizations of C^1-manifolds

3.1. Tierno

Both old and modern characterizations of C^1-manifolds can be deduced from the Four-cones coincidence theorem 0.1; as example, we state and prove the following theorem, due to Tierno (see [23, (1997)], [24, (2000)]).

Theorem 3.1 *(Tierno's two-cones coincidence theorem). A non-empty set F of $\mathbb{R}^n$ is a d-dimensional C^1-manifold if and only if F is locally compact and the upper tangent and upper paratangent cones to F coincide and are d-dimensional vector spaces[6] at every point, i.e.,*

$$\operatorname{Tan}^+(F,x) = \operatorname{pLTan}^+(F,x) \quad \text{and} \quad \dim\big(\operatorname{LTan}^+(F,x)\big) = d \quad \text{for every } x \in F. \tag{3.1}$$

This theorem provides an efficacious test for visual geometrical recognition of C^1-manifolds. In fact, it follows that F is a d-dimensional C^1-manifold if and only if

(3.2) at every point of F, the upper tangent vectors to F form a d-dimensional vector space which is paratangent in traditional sense to F.

In symbols, by Proposition 2.6 of [1] this condition becomes

$$\operatorname{Tan}^+(F,x) = \operatorname{LTan}^+(F,x), \qquad \dim\big(\operatorname{LTan}^+(F,x)\big) = d \quad \text{and} \quad \operatorname{pTan}^+(F,x) \subset \operatorname{LTan}^+(F,x) \quad \text{for every } x \in F. \tag{3.3}$$

Proof. *Necessity*. By Proposition 4.2 of [1], it is obvious. *Sufficiency*. By Theorem 0.1 it is enough to show that $\operatorname{pTan}^-(F,x) = \operatorname{pTan}^+(F,x)$ for every $x \in F$. The first equality in (3.1) means:

$$\operatorname{Tan}^+(F,x) = \operatorname{LTan}^+(F,x) = \operatorname{pTan}^+(F,x) = \operatorname{pLTan}^+(F,x) \tag{$*1$}$$

for every $x \in F$. Hence, by the properties (1.18) and (1.20) of [1] we have

$$\operatorname{pLTan}^+(F,x) \supset \operatorname*{Ls}_{F \ni y \to x} \operatorname{pLTan}^+(F,y) \tag{$*2$}$$

[6] The d-dimensionality condition cannot be dropped in (3.1). In fact, define $F := \{x \in \mathbb{R}^n\colon$ either $\|x\| = 0$ or $\frac{1}{\|x\|} \in \mathbb{N}\}$. Then $\operatorname{Tan}^+(F,0) = \operatorname{pLTan}^+(F,0) = \mathbb{R}^n$; moreover, for every $x \in F$ with $\|x\| \neq 0$, one has $\operatorname{Tan}^+(F,x) = \operatorname{pLTan}^+(F,x) = \{v \in \mathbb{R}^n\colon \langle v, x\rangle = 0\}$ and $\dim(\operatorname{pLTan}^+(F,x)) = n-1$. Notice that F is a C^1-manifold only at every $x \neq 0$.

and

$$(*3)\qquad \mathrm{pTan}^-(F,x) = \operatorname*{Li}_{F\ni y\to x} \mathrm{pLTan}^+(F,y),$$

for every $x \in F$. By (3.1) the vector spaces $\mathrm{pLTan}^+(F,x)$ and $\mathrm{pLTan}^+(F,y)$ have the same dimension; hence, from $(*2)$ and Lemma 2.2 it follows that

$$(*4)\qquad \mathrm{pLTan}^+(F,x) = \operatorname*{Li}_{F\ni y\to x} \mathrm{pLTan}^+(F,y).$$

Therefore, by $(*3)$ and $(*4)$ we obtain that $\mathrm{pTan}^-(F,x) = \mathrm{pTan}^+(F,x)$ for every $x \in F$, as required. □

Let us remark, we can use Lemma 2.2 to prove the global version of the Four-cones coincidence theorem 0.1 beginning by Tierno's Theorem 3.1.

3.2. Gluck

In his article [7, (1968), p. 33] Gluck gives two characterizations of C^1-manifolds.

The first one was actually instrumental of the main theorem and we won't need it, but we however state and prove it in Proposition 3.2 in order to show that it is a consequence of Theorem 2.6 (Valiron theorem). The second characterization will follow in Theorem 3.6.

Proposition 3.2. *(See Gluck [6, (1966), pp. 199, 202] and [7, (1968), p. 45].) A non-empty set $F \subset \mathbb{R}^n$ is a d-dimensional C^1-manifold if and only F is a d-dimensional topological manifold and there exists a continuous map $\mathrm{LTan} : F \to \mathbb{G}(\mathbb{R}^n, d)$ such that, for every $x \in F$,*

(3.4) $\mathrm{LTan}(x)$ *is tangent in traditional sense to F at x,*
(3.5) *the orthogonal projection of F on $\mathrm{LTan}(x)$ is injective on an open neighborhood of x in F.*

Proof. *Necessity*. By Proposition 4.2 of [1] it is obvious. *Sufficiency*. Let $\hat{x} \in F$. By (3.4) and Proposition 2.2 of [1],

$$(*1)\qquad \mathrm{Tan}^+(F,\hat{x}) \subset \mathrm{LTan}(\hat{x}).$$

On the other hand, F and $\mathrm{LTan}(F,\hat{x})$ both being d-dimensional topological manifolds, by Brouwer domain invariance theorem and (3.5), the orthogonal projection of F into $\hat{x} + \mathrm{LTan}(\hat{x})$ map Ω onto an open neighborhood of $\hat{x}$ in $\hat{x} + \mathrm{LTan}(\hat{x})$; hence,

$$(*2)\qquad \mathrm{Tan}^+(F,\hat{x}) = \mathrm{LTan}(\hat{x}).$$

Hence, $\hat{x}$ being an arbitrary point of F, the maps $x \to \mathrm{Tan}^+(F,x)$ and $x \to \mathrm{LTan}(F,x)$ are equal. Therefore, applying Theorem 2.6, we have that F is a C^1-manifold, as required. □

The second characterization of C^1-manifold, which is the main theorem of Gluck's paper, is very elaborated and stimulating and is based on "secant map" and "shape function". Because the case $d = 1$ is easier, we state it in a separated form, as it appears in [6, (1966), p. 200].

Theorem 3.3 *(Gluck's secant map theorem, for $d = 1$). Let F be a one-dimensional topological manifold of $\mathbb{R}^n$. Then F is a C^1-manifold if and only if the function Σ (called* secant map*) from $(F \times F) \setminus \{(x,x)\colon x \in F\}$ to $\mathbb{G}(\mathbb{R}^n, 1)$ which assigns to each pair x, y of distinct points of F the unidimensional vector space generated by $x - y$, admits a continuous extension over all $F \times F$.*

Proof. *Necessity*. Assume F is a C^1-manifold of $\mathbb{R}^n$. In order to have the required extension, it is enough to assign $\mathrm{Tan}^+(F,x)$ to every pair (x,x). *Sufficiency*. Let $\mathrm{pLTan}(F,x)$ denote the value of the extension of Σ at (x,x). Clearly $\mathrm{pLTan}(F,x)$ is a one-dimensional vector space which is paratangent in traditional sense at x. Therefore, by Theorem 2.11 F is a one-dimensional C^1-manifold. □

Before stating and proving Theorem 3.6 we need to introduce the shape and the secant maps.

Define the *secant map*

$$\Sigma : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{d+1 \text{ times}} \longrightarrow \bigcup_{k=1}^{d} \mathbb{G}\big(\mathbb{R}^n, k\big)$$

as

$$\Sigma(x_0, \ldots, x_d) := \mathrm{span}(x_1 - x_0, \ldots, x_d - x_0).$$

The map Σ is not continuous, mostly because of two phenomena: the simplex[7] $\Delta(x_0,\dots,x_d)$ can continuous fall in dimension and, even when $(x_0,\dots,x_d)$ and $(y_0,\dots,y_d)$ are near in $(\mathbb{R}^n)^{d+1}$, the vector spaces they generate can be orthogonal to each other.

In order to avoid this obstacle, Gluck introduces the *shape function*

$$\sigma : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{d+1 \text{ times}} \longrightarrow [0,1]$$

which is related to the distribution of the edges of the simplex $\Delta(x_0,\dots,x_d)$ in the projective space $\mathcal{P}(V)$, where V is the linear hull of $x_1 - x_0,\dots,x_d - x_0$. For instance, $\sigma(x_0,\dots,x_d) = 0$ if and only if $(x_0,\dots,x_d)$ generate a vector space of dimension strictly less than d; on the other side, if they generate an "equilateral" simplex, then $\sigma(x_0,\dots,x_d) = 1$.

Let's also define for any $F \subset \mathbb{R}^n$ and $\sigma_0 \in \mathbb{R}$

$$(F)^{d+1}_{\sigma_0} := \left\{(x_0,\dots,x_d) \in F^{d+1}\colon\ \sigma(x_0,\dots,x_d) > \sigma_0\right\}.$$

The following two lemmata are needed in Theorem 3.6 and we refer to the article of Gluck for their proofs.

Lemma 3.4. *(See* Gluck *[7, (1968), Theorem 4.4, p. 38].) Let $Q \subset \mathbb{R}^n$ be a d-plane and let $\sigma_0 > 0$. For every $\epsilon > 0$ there exists $\delta > 0$ such that if $(x_0,\dots,x_d) \in (\mathbb{R}^n)^{d+1}_{\sigma_0}$ with $\operatorname{ang}(x_i - x_j, Q) < \delta$ then $\operatorname{ang}(\Sigma(x_0,\dots,x_d), Q) < \epsilon$.*

Lemma 3.5. *(See* Gluck *[7, (1968), Theorem 8.2, p. 44].) Let F be a topological submanifold in $\mathbb{R}^n$ of dimension d. Let U be a connected open subset of F and $x_0, x_1 \in U$ be such that $\|x_1 - x_0\| < \operatorname{diam} U$. Then there exist $x_2,\dots,x_d \in U$ such that*

$$\sigma(x_0,x_1,x_2,\dots,x_d) > \frac{1}{\sqrt{d}}.$$

The first consequence of this statement is that, for every $0 \leqslant \sigma_0 < \frac{1}{\sqrt{d}}$, the diagonal $D := \{(x,\dots,x)\colon\ x \in F\}$ lies in the closure of $(F)^{d+1}_{\sigma_0}$.

Theorem 3.6. *(See* Gluck *[7, (1968), p. 34].) Let F be a d-dimensional topological manifold in $\mathbb{R}^n$, and let $0 < \sigma_0 < \frac{1}{\sqrt{d}}$ be a real number. Then F is a C^1-manifold in $\mathbb{R}^n$ if and only if the secant map*

$$\Sigma : (F)^{d+1}_{\sigma_0} \longrightarrow \mathbb{G}\big(\mathbb{R}^n, d\big)$$

admits a continuous extension on $(F)^{d+1}_{\sigma_0} \cup D$.

Proof. *Necessity*. Suppose $F \subset \mathbb{R}^n$ is a C^1-manifold and let's define $\Sigma(p,p,\dots,p) := \operatorname{pLTan}^+(F,p)$ for every $p \in F$. We have to prove the continuity at the points on the diagonal D. Fix $p \in F$ and let $\{(x_0^{(m)},\dots,x_d^{(m)})\}_{m\in\mathbb{N}} \subset (F)^{d+1}_{\sigma_0}$ be a sequence converging to $(p,\dots,p)$.

Let $\epsilon > 0$. Then let $\delta > 0$ be as in Lemma 3.4. Because $\operatorname{pLTan}^+(F,p)$ is paratangent in traditional sense to F, there exists $N \in \mathbb{N}$ such that

$$\operatorname{ang}\big(x_i^{(m)} - x_0^{(m)}, \operatorname{pLTan}^+(F,p)\big) < \delta \quad \text{for } i \in \{1,\dots,d\} \text{ and } m > N.$$

Thanks to Lemma 3.4 we obtain $\operatorname{ang}(\Sigma(x_0^{(m)},\dots,x_d^{(m)}), \operatorname{pLTan}^+(F,p)) < \epsilon$ for all $m > N$.

Sufficiency. In order to apply the Severi Theorem 2.11, we shall prove $\operatorname{pTan}^+(F,p) \subset \Sigma(p,\dots,p)$ for all $p \in F$. Let $v \in \operatorname{pTan}^+(F,p)$. Then there are $\{\lambda_m\}_m \subset (0,+\infty)$ and $\{x_m, y_m\}_m \subset F$ such that $x_m, y_m \to p$, $\lambda_m \to 0^+$ and $\frac{x_m - y_m}{\lambda_m} \to v$.

Fix $\epsilon > 0$. By the continuity of the secant map, there is a neighborhood $U \subset M$ of p such that $\operatorname{ang}(\Sigma(z_0,\dots,z_d), \Sigma(p,\dots, p)) < \epsilon$ for every $(z_0,\dots,z_d) \in (U)^{d+1}_{\sigma_0}$. Let $N \in \mathbb{N}$ such that $x_m, y_m \in U$ and $\|x_m - y_m\| < \operatorname{diam}(U)$ for all $m > N$. Thanks to Lemma 3.5, for every $m > N$ there are $z_2^m,\dots,z_d^m \in U$ such that $(x_m, y_m, z_2^m,\dots,z_d^m) \in (U)^{d+1}_{\sigma_0}$. Therefore for all $m > N$

$$\operatorname{ang}\big(x_m - y_m, \Sigma(p,\dots,p)\big) \leqslant \operatorname{ang}\big(\Sigma\big(x_m, y_m, z_2^m,\dots,z_d^m\big), \Sigma(p,\dots,p)\big) < \epsilon.$$

It follows $v \in \Sigma(p,\dots,p)$. $\square$

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[7] We denote by $\Delta(x_0,\dots,x_d)$ the convex hull of the set $\{x_,\dots,x_d\} \subset \mathbb{R}^n$ and we call it *simplex*.

Appendix A. Strict diffeomorphisms: An analytic characterization

Let A be a non-empty subset of $\mathbb{R}^n$. A function $f : A \to \mathbb{R}^d$ is said to be *strictly differentiable* on A, if $A \subset \operatorname{der}(A)$ and f is strictly differentiable at every point $x \in A$. Moreover, f is said to be a *strict diffeomorphism* on A, if f is injective and both f and f^{-1} are strictly differentiable.[8]

Theorem A.1. *Let A be a non-empty subset of $\mathbb{R}^n$. A function $f : A \to \mathbb{R}^d$ is a strict diffeomorphism on A if and only if the following two properties hold*:

(A.1) *f is strictly differentiable homeomorphism on A,*
(A.2) *for every $x \in A$, a strict differential of f at x is injective on $\mathrm{pTan}^+(A, x)$.*

Proof. *Necessity*: Let L be a strict diffeomorphism of f at $x \in A$, we have only to prove that L is injective on $\mathrm{pTan}^+(A, x)$ for every $x \in A$. By Lemma 4.4 of [1] we have

$$\mathrm{pTan}^+\big(\operatorname{graph}(f), \big(x, f(x)\big)\big) = \big\{\big(v, L(v)\big):\ v \in \mathrm{pTan}^+(A, x)\big\},$$
$$\mathrm{pTan}^+\big(\big(\operatorname{graph}\big(f^{-1}\big)\big)^{-1}, \big(x, f(x)\big)\big) = \big\{\big(L^{-1}\big(f(x)\big)(w), w\big):\ w \in \mathrm{pTan}^+\big(f(A), x\big)\big\}$$

where it is clear that $f(x) \in \operatorname{der}(f(A))$. Notice that $\operatorname{graph}(f) = (\operatorname{graph}(f^{-1}))^{-1}$. It follows that

$$\mathrm{pTan}^+\big(\operatorname{graph}(f), \big(x, f(x)\big)\big) = \mathrm{pTan}^+\big(\big(\operatorname{graph}\big(f^{-1}\big)\big)^{-1}, \big(x, f(x)\big)\big)$$

and therefore

$$L_{|\mathrm{pTan}^+(A,x)} = \big(L^{-1}\big(f(x)\big)_{|\mathrm{pTan}^+(f(A), f(x))}\big)^{-1}.$$

It follows that L is injective on $\mathrm{pTan}^+(A, x)$.

Sufficiency: By (A.2) L is injective on $\mathrm{pTan}^+(A, x)$, there exists a linear map $M : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$M_{|L(\mathrm{pTan}^+(A,x))} = (L_{|\mathrm{pTan}^+(A,x)})^{-1}.$$

It follows

$$\mathrm{pTan}^+\big(\big(\operatorname{graph}\big(f^{-1}\big)\big)^{-1}, \big(x, f(x)\big)\big) = \mathrm{pTan}^+\big(\operatorname{graph}(f), \big(x, f(x)\big)\big)$$
$$\subset \operatorname{graph}(L_{|\mathrm{pTan}^+(A,x)}) \subset \operatorname{graph}(L) \cap \big(\operatorname{graph}(M)\big)^{-1}.$$

In conclusion we can apply (2.14) of Proposition 2.8 of [1] and obtain the thesis. □

Appendix B. A restatement of the local version of Four-cones coincidence theorem

In [1] is stated and proved the following

Theorem B.1 *(Four-cones coincidence theorem: local version). Let $F \subset \mathbb{R}^n$ and let $\hat{x} \in F$. Then F is a C^1-manifold at $\hat{x}$ if and only if the following three properties hold*:

(B.1) *F is locally compact at $\hat{x}$,*
(B.2) $\mathrm{pTan}^-(F, \hat{x}) = \mathrm{pTan}^+(F, \hat{x})$,
(B.3) *there exists an open ball $\mathrm{B}_\delta(\hat{x})$ centered at $\hat{x}$ such that $\mathrm{pTan}^+(F, x)$ is a vector space i.e. $\mathrm{pTan}^+(F, x) = \mathrm{pLTan}^+(F, x)$ for every $x \in F \cap \mathrm{B}_\delta(\hat{x})$.*

We give here an improved formulation:

Theorem B.2 *(Four-cones coincidence theorem: a reformulation of the local version). Let $F \subset \mathbb{R}^n$ and let $\hat{x} \in F$. Then F is a C^1-manifold at $\hat{x}$ if and only if the following three properties hold*:

(B.4) *F is locally compact at $\hat{x}$,*
(B.5) $\mathrm{pTan}^+(F, \hat{x}) \subset \mathrm{Li}_{F \ni x \to \hat{x}} \mathrm{Tan}^+(F, x)$,
(B.6) $\mathrm{Ls}_{F \ni x \to \hat{x}} \mathrm{pLTan}^+(F, x) \subset \mathrm{pLTan}^+(F, \hat{x})$.

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[8] The domain of f^{-1} is the image $f(A)$.

By comparison between Theorem B.1 and Theorem B.2, observe that (B.2) and (B.5) are equivalent by Eq. (1.20) of [1]; and, in virtue of (1.18) of [1], the condition (B.6) holds, whenever (B.3) is true.

The proof of Theorem B.2 is the same as that of Theorem B.1 proved in [1], where we replace condition (B.3) with (B.6) and the inductive process is obtained by the following lemma:

Lemma B.3. *Let $A \subset \mathbb{R}^d$ with $A \subset \operatorname{der}(A)$ and let $\varphi : A \to \mathbb{R}$ be strictly differentiable on A and let $\hat{t} \in A$. If the properties* (B.4), (B.5), (B.6) *of Theorem* B.2 *hold for $F := \operatorname{graph}(\varphi)$ and $\hat{x} := (\hat{t}, \varphi(\hat{t}))$, then they hold for A at $\hat{t}$, that is*:

(B.7) *A is locally compact at $\hat{t}$,*
(B.8) $\operatorname{pTan}^+(A, \hat{t}) \subset \operatorname{Li}_{A \ni t \to \hat{t}} \operatorname{Tan}^+(A, t)$,
(B.9) $\operatorname{Ls}_{A \ni t \to \hat{t}} \operatorname{pLTan}^+(A, t) \subset \operatorname{pLTan}^+(A, \hat{t})$.

Proof. By Lemma 4.3 of [1] A is locally compact at $\hat{t}$, i.e. (B.7) holds. As we have already observed, (B.2) holds for F at $\hat{x}$. Thanks to the Lemma 4.4 of [1], (B.2) holds also for A at $\hat{t}$. Hence (B.8) follows by the property (1.20) of [1] for A at $\hat{t}$.

It remains only to prove (B.9). Firstly notice that

$$(*2)\qquad \operatorname*{Ls}_{F \ni x \to \hat{x}} \operatorname{pLTan}^+(F, x) \overset{(B.6)}{\subset} \operatorname{pLTan}^+(F, \hat{x}) = \operatorname{pTan}^+(F, \hat{x}) \overset{(B.5)}{\subset} \operatorname*{Li}_{F \ni x \to \hat{x}} \operatorname{Tan}^+(F, x) \subset \operatorname*{Li}_{F \ni x \to \hat{x}} \operatorname{pLTan}^+(F, x)$$

where the equality holds because $\operatorname{pTan}^+(F, \hat{x})$ is a vector space for (B.5) and the last inclusion follows from $\operatorname{Tan}^+(F, x) \subset \operatorname{pLTan}^+(F, x)$ for every $x \in F$. Hence, $(*2)$ and Lemma 2.2(2.5) imply that there exists $\delta > 0$ such that it holds

$$(*3)\qquad \dim\bigl(\operatorname{pLTan}^+(F, x)\bigr) = \dim\bigl(\operatorname{pLTan}^+(F, \hat{x})\bigr) \quad \text{for every } x \in F \cap B_\delta(\hat{x}).$$

Since φ is strictly differentiable on A, $\operatorname{pLTan}^+(F, x)$ is contained in the graph of any strict differential of φ at x and therefore

$$(*4)\qquad \operatorname{pLTan}^+(F, x) \text{ does not contain vertical lines}$$

for all $x \in F$. Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection along the last coordinate. By $(*4)$ it holds

$$(*5)\qquad \dim \pi\bigl(\operatorname{pLTan}^+(F, x)\bigr) = \dim\bigl(\operatorname{pLTan}^+(F, x)\bigr)$$

for all $x \in F$. Thanks to $(*2)$ and the continuity of π, it holds:

$$(*6)\qquad \pi\bigl(\operatorname{pLTan}^+(F, \hat{x})\bigr) \subset \operatorname*{Li}_{F \ni x \to \hat{x}} \pi\bigl(\operatorname{pLTan}^+(F, x)\bigr).$$

From $(*3)$, $(*5)$ and the point (2.4) of Lemma 2.2 we obtain

$$(*7)\qquad \operatorname*{Ls}_{F \ni x \to \hat{x}} \pi\bigl(\operatorname{pLTan}^+(F, x)\bigr) \subset \pi\bigl(\operatorname{pLTan}^+(F, \hat{x})\bigr).$$

Noticing that for all $t \in A$

$$\operatorname{pLTan}^+(A, t) = \pi\bigl(\operatorname{pLTan}^+\bigl(F, \bigl(t, \varphi(t)\bigr)\bigr)\bigr)$$

we finally obtain:

$$\operatorname*{Ls}_{A \ni t \to \hat{t}} \operatorname{pLTan}^+(A, t) \subset \pi\bigl(\operatorname{pLTan}^+(F, \hat{x})\bigr) = \operatorname{pLTan}^+(A, \hat{t}). \qquad \square$$

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