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An algebraic study of exactness in partial contexts

Tommaso Moraschini

Department of Philosophy, Università degli Studi di Milano, Italy

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ABSTRACT

DMF's are the natural algebraic tool for modelling reasoning with Körner's partial predicates. We provide two representation theorems for DMF's which give rise to two adjunctions, the first between DMF and the category of sets and the second between DMF and the category of distributive lattices with minimum. Then we propose a logic $\mathcal{L}_{\{1\}}$ for dealing with exactness in partial contexts, which belongs neither to the Leibniz, nor to the Frege hierarchies, and carry on its study with techniques of abstract algebraic logic. Finally a fully adequate and algebraizable Gentzen system for $\mathcal{L}_{\{1\}}$ is given.

Partial predicates were introduced by the neokantian philosopher Körner and studied from a logical point of view for example in [4] and [11]. Here we will talk about partial sets instead of partial predicates since we are concerned with the extensional point of view. Let X be a set, as classical sets over X are subsets of X, a *partial set* over X is a pair $\langle A, B\rangle$ such that $A, B \subseteq X$ and $A \cap B = \emptyset$, where A represents the set of individuals of X for which we are sure that the predicate holds and B the set of individuals for which we are sure of the contrary. In this sense partial sets deal with exactness in partial contexts. We let $\mathfrak{D}(X) = \{\langle A, B\rangle \mid \langle A, B\rangle \text{ is a partial set over } X\}$. Conjunctions, disjunctions and complements of partial sets are defined in a very nice way: for pick $\langle A, B\rangle, \langle C, D\rangle \in \mathfrak{D}(X)$, we let $\langle A, B\rangle \cap \langle C, D\rangle = \langle A \cap C, B \cup D\rangle$, $\langle A, B\rangle \cup \langle C, D\rangle = \langle A \cup C, B \cap D\rangle$ and $\neg\langle A, B\rangle = \langle B, A\rangle$. We say that a $\mathcal{S} \subseteq \mathfrak{D}(X)$ is a *field of partial sets* over X if $\mathcal{S}$ is closed under partial conjunctions, disjunctions, complements and contains $\langle \emptyset, \emptyset\rangle$.

The relation between partial sets and exactness in partial contexts can be explained in terms of rough set theory, which originates with Pawlak [12]. In order to do this recall that rough sets arise from *approximation spaces* $\langle X, \theta\rangle$ where X is a set and θ is an equivalence relation on X. Equivalence classes in $\{x/\theta \mid x \in X\}$ are thought as representing sets of indiscernible elements of X. Given a property $A \subseteq X$ of individuals of X it is usual to think of its *lower approximation* $\mathcal{L}^{\theta}(A) = \{x/\theta \mid x/\theta \subseteq A\}$ as representing the elements of X which, despite our rough vision, surely enjoy the property A. Accordingly, if we are given a family of properties $\mathcal{A} \subseteq \mathcal{P}(X)$ we let $\mathcal{L}^{\theta}(\mathcal{A}) = \{\mathcal{L}^{\theta}(A) \mid A \in \mathcal{A}\}$.

If we pick a field of partial sets $\mathcal{S}$ on X we can define a relation $\theta_{\mathcal{S}}$ on X as follows: for every $x, y \in X$ we let $\langle x, y\rangle \in \theta_{\mathcal{S}}$ if and only if $(x \in A \Leftrightarrow y \in A)$ and $(x \in B \Leftrightarrow y \in B)$ for every $\langle A, B\rangle \in \mathcal{S}$. It is easy to prove that $\langle X, \theta\rangle$ is an approximation space. Moreover if we consider the family of classical predicates $\mathcal{A}_{\mathcal{S}} = \{A \mid \langle A, B\rangle \in \mathcal{S} \text{ for some } B \subseteq X\}$ on X it is easy to prove that $\mathcal{K}(\mathcal{A}_{\mathcal{S}}) = \{\langle \mathcal{L}^{\theta_{\mathcal{S}}}(A), \mathcal{L}^{\theta_{\mathcal{S}}}(B)\rangle \mid A, B \in \mathcal{A}_{\mathcal{S}} \text{ such that } A \cap B = \emptyset\}$ is a field of partial sets such that $\mathcal{S} \subseteq \mathcal{K}(\mathcal{A}_{\mathcal{S}})$, i.e. $\mathcal{S}$ comes from the lower approximation of $\mathcal{A}_{\mathcal{S}}$ in $\langle X, \theta\rangle$. We believe this fact justifies the choice of taking partial predicates as the correct framework to carry on a discourse about exactness in partial contexts.

Negri proved that the algebraic structure of fields of partial sets can be abstracted, yielding the class of DMF algebras which enjoys a strong connection with the original intuitive framework of partial sets. More precisely, he proved

E-mail address: tommaso.moraschini@gmail.com.

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a representation theorem which says that they are essentially the same [11, Theorem 3.1]. We consider that this fact confirms the intuition that the variety of DMF algebras is a good framework where to develop a logical study of exactness in partial contexts. It is worth remarking that in [11] DMF algebras are assumed to be bounded, whereas we will work with their possibly unbounded version, which we call DMF lattices.

The structure of the paper is as follows: Section 1 contains a brief survey of the main concepts from the field of abstract algebraic logic we will make use of along the paper. In Section 2 we provide some two representation theorems for DMF lattices, the first one telling us that every DMF lattice is a subdirect powers of a three-element chain [Theorem 2.4] and the second one that it is isomorphic to a filed of partial sets [Theorem 2.6]. This tools allows us to develop the connection between partial sets and DMF lattices in two adjunctions: the first one between the category of DMF lattices and the one of sets [Theorem 2.7] and the second one between the category of DMF lattices and the one of distributive lattices with a minimum element [Theorem 2.8]. As a consequence a characterisation of free DMF lattices in terms of partial sets is obtained [Theorem 2.10]. In Section 3 we define a concrete logic $\mathcal{L}_{\{1\}}$ which preserves exact truth in partial contexts and study its algebraic semantics. We begin by proving that its algebraic counterpart is the class of DMF lattices [Theorem 3.2]. But the more involved characterisation is the one of Leibniz reduced models, whose algebraic component enjoys a nice local behaviour [Theorem 3.5] and which are strictly connected with the reduced models of the $\{\wedge, \vee, \bot\}$-fragment of classical propositional logic [Corollary 3.6]. Finally, Section 4 focuses on the syntactic characterisation of our logic $\mathcal{L}_{\{1\}}$. The main result from this point of view is the presentation of a fully adequate Gentzen system for it [Corollary 4.4] which is algebraizable with algebraic semantics the class of DMF lattices [Theorem 4.5].

1. Preliminaries

Here we present a brief survey of the main definitions and results of abstract algebraic logic we will make use of along the article, a systematic exposition can be found for example in [1,2,5,9,8]. We begin by the definition of logic. In order to do this recall that a **closure operator** over a set A is a monotone function $C: \mathcal{P}(A) \to \mathcal{P}(A)$ such that $X \subseteq C(X) = CC(X)$ for every $X \in \mathcal{P}(A)$ and that a **closure system** on A is a family $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections such that $A \in \mathcal{C}$. It is well-known that the closed sets (fixed points) of a closure operator form a closure system and that given a closure system $\mathcal{C}$ one can construct a closure operator C by letting $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ for every $X \in \mathcal{P}(A)$. These transformations are indeed one inverse to the other.

Then, fixed an algebraic type $\mathscr{L}$, we denote by Fm the set of formulas over $\mathscr{L}$ built up with countably many variables and by $\boldsymbol{Fm}$ the corresponding absolutely free algebra with countably many free generators. By a **logic** $\mathcal{L}$ we mean a closure operator $C_{\mathcal{L}}: \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ which is structural in the sense that $\sigma C_{\mathcal{L}} \subseteq C_{\mathcal{L}}\sigma$ for every endomorphism (or, equivalently, substitution) $\sigma: \boldsymbol{Fm} \to \boldsymbol{Fm}$. It is worth remarking that finitarity of the closure operator is not assumed. Given $\Gamma \cup \{\varphi\} \subseteq Fm$ we write $\Gamma \vdash_{\mathcal{L}} \varphi$ to denote the fact that $\varphi \in C_{\mathcal{L}}(\Gamma)$. Given two logics $\mathcal{L}$ and $\mathcal{L}'$, we will write $\mathcal{L} \leqslant \mathcal{L}'$ if $C_{\mathcal{L}}(\Gamma) \subseteq C_{\mathcal{L}'}(\Gamma)$ for every $\Gamma \subseteq Fm$. In this case we also say that $\mathcal{L}'$ is an extension of $\mathcal{L}$.

Now we turn to describe how to build models for a logic out of algebras $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \ldots$ (respectively with universes $A, B, C \ldots$). This is done in a very natural way by considering the elements of the algebras as truth values and by selecting some of them, which are taken as representants of truth. More precisely, given a logic $\mathcal{L}$ and an algebra $\boldsymbol{A}$, we say that a set $F \subseteq A$ is a **deductive filter** of $\mathcal{L}$ over $\boldsymbol{A}$ if the following condition holds

$$\text{if } \Gamma \vdash_{\mathcal{L}} \varphi, \text{ then for every homomorphism } h: \boldsymbol{Fm} \to \boldsymbol{A}$$
$$\text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. We denote by $\mathcal{F}i_{\mathcal{L}}(\boldsymbol{A})$ the set of deductive filters of $\mathcal{L}$ over $\boldsymbol{A}$, which turn out to form a complete lattice ordered under the set-theoretical inclusion. We say that a pair $\langle \boldsymbol{A}, F\rangle$ is a **matrix** if $\boldsymbol{A}$ is an algebra and $F \subseteq A$ and that a matrix $\langle \boldsymbol{A}, F\rangle$ is a **model** of a logic $\mathcal{L}$ if $F \in \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A})$.

Matrices of the form $\langle \boldsymbol{A}, F\rangle$ are associated with congruences over $\boldsymbol{A}$ in a way independent from any logic. In order to explain how, let us fix some notation. Given an algebra $\boldsymbol{A}$, we will denote by $\mathrm{Co}(\boldsymbol{A})$ its lattice of congruences. Then we say that $\theta \in \mathrm{Co}(\boldsymbol{A})$ is **compatible** with the set $F \subseteq A$ if it satisfies the following condition:

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta, \text{ then } b \in F$$

for every $a, b \in A$. It is easy to prove that given any $F \subseteq A$, there exists a largest congruence on $\boldsymbol{A}$ compatible with F. We denote this congruence by $\boldsymbol{\Omega} F$ and refer it as the **Leibniz congruence** of F on $\boldsymbol{A}$. The definition of Leibniz congruence gives rise to a map $\boldsymbol{\Omega}: \mathcal{P}(A) \to \mathrm{Co}(\boldsymbol{A})$, called **Leibniz operator**, whose behaviour over deductive filters of the logic captures interesting facts concerning the definability of truth and that of equivalence. The Leibniz congruence allows us to associate to a logic $\mathcal{L}$ a special class of models and of algebras:

$$\boldsymbol{Mod}^{*}\mathcal{L} = \{\langle \boldsymbol{A}, F\rangle: F \in \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}) \text{ and } \boldsymbol{\Omega} F = \Delta\}$$
$$\boldsymbol{Alg}^{*}\mathcal{L} = \{\boldsymbol{A}: \text{ there is } F \in \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}) \text{ such that } \boldsymbol{\Omega} F = \Delta\}.$$

We will refer to elements of $\boldsymbol{Mod}^{*}\mathcal{L}$ as to **Leibniz-reduced model**s and to the ones of $\boldsymbol{Alg}^{*}\mathcal{L}$ as to **Leibniz-reduced algebras** of $\mathcal{L}$.

Another way of building algebraic models for logics is achieved with generalised matrices or g-matrices for short. The idea that lies behind this construction is that of modelling the consequence operator of the logic $C_{\mathcal{L}}$ instead of some notion of truth. More precisely, a **g-matrix** is a pair $\langle \boldsymbol{A}, \mathcal{C}\rangle$ where $\boldsymbol{A}$ is an algebra and $\mathcal{C}$ a closure system on it. We say that a g-matrix $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a **g-model** of a logic $\mathcal{L}$, if $F \in \mathcal{F}i_{\mathcal{L}}(\boldsymbol{A})$ for every $F \in \mathcal{C}$.

As we expect, g-matrices of the form $\langle \boldsymbol{A}, \mathcal{C}\rangle$ are associated with congruences over $\boldsymbol{A}$ since it is easy to prove that, given a closure system $\mathcal{C}$ on A, the intersection $\bigcap_{F\in\mathcal{C}} \boldsymbol{\Omega} F$ is the largest congruence compatible with every $F \in \mathcal{C}$. We denote this congruence by $\widetilde{\boldsymbol{\Omega}}\mathcal{C}$ and refer to it as to the **Tarski congruence** of $\mathcal{C}$ on $\boldsymbol{A}$. It is worth remarking that, given a closure system $\mathcal{C}$ on A, there is a strong connection between its Tarski congruence and its **Frege relation** $\boldsymbol{\Lambda}\mathcal{C} = \{\langle a, b\rangle \in A \times A : C\{a\} = C\{b\}\}$, since $\widetilde{\boldsymbol{\Omega}}\mathcal{C}$ is the largest congruence on $\boldsymbol{A}$ below $\boldsymbol{\Lambda}\mathcal{C}$. We say that a logic $\mathcal{L}$ is **selfextensional** if $\boldsymbol{\Lambda}\mathcal{L}$ is a congruence too. In this case we clearly have that $\boldsymbol{\Lambda}\mathcal{L} = \widetilde{\boldsymbol{\Omega}}\mathcal{L}$.

The Tarski congruence also allows us to associate to a logic $\mathcal{L}$ a special class of models and of algebras:

$$\boldsymbol{GMod^*}\mathcal{L} = \big\{\langle \boldsymbol{A}, \mathcal{C}\rangle \;\big|\; \langle \boldsymbol{A}, \mathcal{C}\rangle \text{ is a g-model of } \mathcal{L} \text{ and } \widetilde{\boldsymbol{\Omega}}\mathcal{C} = \Delta\big\}$$
$$\boldsymbol{Alg}\,\mathcal{L} = \big\{\boldsymbol{A} \;\big|\; \text{there is a g-model } \langle \boldsymbol{A}, \mathcal{C}\rangle \text{ of } \mathcal{L} \text{ such that } \widetilde{\boldsymbol{\Omega}}\mathcal{C} = \Delta\big\}.$$

We will refer to $\boldsymbol{Alg}\,\mathcal{L}$ as to the **algebraic counterpart** of $\mathcal{L}$. It is easy to prove that also in this case $\mathcal{L}$ is complete with respect to $\boldsymbol{GMod^*}\mathcal{L}$ in the sense that for every $\Gamma \cup \{\varphi\} \subseteq Fm$

$$\Gamma \vdash_{\mathcal{L}} \varphi \iff h(\varphi) \in C\big(h[\Gamma]\big) \quad \text{for every } \langle \boldsymbol{A}, \mathcal{C}\rangle \in \boldsymbol{GMod^*}\mathcal{L} \text{ and every homomorphism } h : \boldsymbol{Fm} \to \boldsymbol{A}.$$

The inclusions which hold in general between the two classes of algebras associated to $\mathcal{L}$ we considered are $\boldsymbol{Alg^*}\mathcal{L} \subseteq \boldsymbol{Alg}\,\mathcal{L}$ and it is easy to prove that $\mathbb{P}_{\mathrm{SD}}\boldsymbol{Alg^*}\mathcal{L} = \boldsymbol{Alg}\,\mathcal{L}$.

2. Representation theorems

In the introduction we mentioned that DMF lattices are the algebraic abstraction of fields of partial sets. We believe that this fact motivates the choice of using them to model reasoning in partial contexts. But before defining a concrete logic on them, let us spend few words about how their relation with partial predicates can be expressed from a purely algebraic point of view. An algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, n\rangle$ is a **DMF lattice** if it is a Kleene lattice with one fixed point for negation, i.e. a distributive lattice that satisfies the following equations:

$$x \vee y = \neg(\neg x \wedge \neg y) \qquad x \wedge y = \neg(\neg x \vee \neg y)$$
$$\neg\neg x = x \qquad \neg n = n$$
$$x \wedge \neg x \leqslant y \vee \neg y.$$

For a general study of Kleene lattices we refer to [10]. We will denote by DMF the class of DMF lattices and call **normality** the condition $x \wedge \neg x \leqslant y \vee \neg y$. Examples of DMF's are easy to construct: for every set X, the field of partial sets over it, $\mathfrak{D}(X)$, is a DMF. We list below some basic properties of $\boldsymbol{A} \in \mathrm{DMF}$. In particular, condition (iv) of Lemma 2.1 tells us that n is the only fixed point of negation in $\boldsymbol{A}$ and condition (v) of Lemma 2.1 that the identity of two elements of $\boldsymbol{A}$ can be decided just looking at its positive cone.

Lemma 2.1. *In* DMF *the following conditions hold*:

(i) $x \wedge \neg x \leqslant n \leqslant y \vee \neg y$;
(ii) $x \leqslant y \Rightarrow \neg y \leqslant \neg x$;
(iii) $x < n \Rightarrow n < \neg x$;
(iv) $x = \neg x \Rightarrow x = n$;
(v) $(x \vee n = y \vee n \text{ and } \neg x \vee n = \neg y \vee n) \Rightarrow x = y$.

The first goal of this section will be to provide some representation theorems for DMF's analogous to the ones of [11] but with different tools. Corollary 2.5 is a consequence of Lemma 2 of [10], but we prefer to present the whole construction, since it will play a interesting role in our logical study. In order to do that let us fix some notation: we let $\boldsymbol{Z}_3 = \langle\{-1, n, 1\}, \wedge, \vee, \neg, n\rangle$ be the three-element chain ordered as $-1 < n < 1$, such that $\neg x = -x$ for every $x \in \{-1, 1\}$ and $\neg n = n$. We write $\boldsymbol{Z}_{2m+1}$ to denote finite odd algebras defined in the same way for every $m \in \mathbb{N}$. It is straightforward to prove that they are DMF's too.

The first step in the direction of a representation in terms of subdirect powers is to individuate the elements which will play a role analogous to that of ultrafilters in the case of Boolean algebras. Let $\boldsymbol{A} \in \mathrm{DMF}$, we say that $F \subseteq A$ is a **partial filter** if it is a prime[1] filter such that $n \notin F$. Given $\boldsymbol{A} \in \mathrm{DMF}$ we let $\mathfrak{P}(\boldsymbol{A}) = \{F \subseteq A \mid F \text{ is a partial filter}\}$. Since $n \notin F$ it is easy to

[1] By asking for primality we require partial filters to be non-empty and proper.

prove that if $a \in F$, then $\neg a \notin F$ for every $F \in \mathfrak{P}(\boldsymbol{A})$ and $a \in A$. Notice it is possible that $a, \neg a \notin F$ for some $a \in A$. As in the case of Boolean algebras there is a bijection between partial filters and epimorphisms onto a particular DMF, namely $\boldsymbol{Z}_3$. Pick $\boldsymbol{A} \in \mathsf{DMF}$, we let $\beta : \mathfrak{P}(\boldsymbol{A}) \to \{f : \boldsymbol{A} \to \boldsymbol{Z}_3 \mid f \text{ is epi}\}$ be defined as

$$\beta(F)(a) = \begin{cases} 1 & \text{if } a \in F \\ n & a \notin F \text{ and } \neg a \notin F \\ -1 & \text{if } \neg a \in F. \end{cases}$$

Lemma 2.2. *Let $\boldsymbol{A} \in \mathsf{DMF}$. $\beta : \mathfrak{P}(\boldsymbol{A}) \to \{f : \boldsymbol{A} \to \boldsymbol{Z}_3 \mid f$ is epi$\}$ is a bijection.*

Proof. It is straightforward to check β is well defined. We turn to prove it is injective. Let $F, G \in \mathfrak{P}(\boldsymbol{A})$ such that $F \neq G$. Then we can assume without loss of generality that there is $a \in F$ such that $a \notin G$. This is to say that $\beta(F)(a) = 1$ and $\beta(G)(a) \leqslant n$, therefore we are done. Finally we check that β is surjective. Let $f : \boldsymbol{A} \to \boldsymbol{Z}_3$ be an epi. Since inverse images of prime filters under homomorphisms are prime filters, $f^{-1}\{1\}$ is a prime filter and clearly $n \notin f^{-1}\{1\}$, therefore $f^{-1}\{1\} \in \mathfrak{P}(\boldsymbol{A})$. Since $\beta(f^{-1}\{1\}) = f$ we are done. □

In other words, there is a bijection between $\mathfrak{P}(\boldsymbol{A})$ and congruences on $\boldsymbol{A}$ which yield a quotient isomorphic to $\boldsymbol{Z}_3$. This result allows us to recover a partial version of the Prime Filter Theorem.

Lemma 2.3. *Let $\boldsymbol{A} \in \mathsf{DMF}$ and $a, b \in A$. If $a \neq b$, there is $F \in \mathfrak{P}(\boldsymbol{A})$ such that $\beta(F)(a) \neq \beta(F)(b)$.*

Proof. Let $\boldsymbol{A} \in \mathsf{DMF}$ and $a, b \in A$ such that $a \neq b$. By the Prime Filter Theorem we can assume without loss of generality that there is a prime filter P such that $a \in P$ and $b \notin P$. Let $f : \boldsymbol{A} \to \boldsymbol{Z}_3$ be defined as

$$f(c) = \begin{cases} 1 & \text{if } c \in P \text{ and } \neg c \notin P \\ n & \text{if } c, \neg c \in P \text{ or } c, \neg c \notin P \\ -1 & \text{if } c \notin P \text{ and } \neg c \in P \end{cases}$$

for every $c \in A$. Using primality and normality it is easy to prove f is a homomorphism. Surjectiveness follows from the fact that prime filters are neither trivial, nor prime. Now we turn to prove $f(a) \neq f(b)$. To do this suppose towards a contradiction $f(a) = f(b)$, then $a, \neg a \in P$ and $b, \neg b \notin P$. Since $a \wedge \neg a \leqslant n \leqslant b \vee \neg b$, by primality we have that $b \in P$ or $\neg b \in P$ against the assumption. By Lemma 2.2 we conclude that there is $F \in \mathfrak{P}(\boldsymbol{A})$ such that $\beta(F)(a) \neq \beta(F)(b)$. □

It is now straightforward to obtain a subdirect representation of DMF's into direct powers of $\boldsymbol{Z}_3$ indexed by partial filters. We believe this yields some deeper philosophical understanding of DMF's in the sense that if we pick partial filters as representing exact information (as they do not include contradictions), we can think of the value n as the denotation of propositions which are neither exactly true, nor exactly false. In order to do this, for every $\boldsymbol{A} \in \mathsf{DMF}$ we consider the map $\alpha : \boldsymbol{A} \to \prod_{F \in \mathfrak{P}(\boldsymbol{A})} \boldsymbol{Z}_{3_F}$ such that $\alpha(a)(F) = \beta(F)(a)$ for every $a \in A$ and every $F \in \mathfrak{P}(\boldsymbol{A})$, where $\boldsymbol{Z}_{3_F}$ is a copy of $\boldsymbol{Z}_3$.

Theorem 2.4. *Let $\boldsymbol{A} \in \mathsf{DMF}$ non-trivial. $\alpha : \boldsymbol{A} \to \prod_{F \in \mathfrak{P}(\boldsymbol{A})} \boldsymbol{Z}_{3_F}$ is a subdirect embedding.*

Proof. By Lemma 2.2 α is a homomorphism surjective on each component. The fact it is injective follows from Lemma 2.3. □

Corollary 2.5. *$\mathbb{V}(\boldsymbol{Z}_3) = \mathsf{DMF}$ and $\boldsymbol{Z}_3$ is the only subdirectly irreducible member of* DMF.

This is all for what concerns the representation in subdirect powers. However we can gain a more concrete representation theorem, analogous to Theorem 3.1 of [11], which explicits the relation between DMF's and Körner's partial predicates. In order to do this pick $\boldsymbol{A} \in \mathsf{DMF}$ and let $\gamma : \boldsymbol{A} \to \mathfrak{D}(\mathfrak{P}(\boldsymbol{A}))$ be defined as

$$\gamma(a) = \big\langle \{F \mid a \in F\}, \{F \mid \neg a \in F\} \big\rangle$$

for every $a \in A$. The following representation theorem tells us that every DMF has indeed the structure of a field of partial sets. This fact can be seen as stating that DMF's are indeed the correct algebraic tool for modelling reasoning with partial predicates.

Theorem 2.6. *Let $\boldsymbol{A} \in \mathsf{DMF}$. $\{\gamma(a) \mid a \in A\}$ is a field of partial sets over $\mathfrak{P}(\boldsymbol{A})$ isomorphic to $\boldsymbol{A}$ via γ.*

Proof. The proof that $\{\gamma(a) \mid a \in A\}$ is a field of partial sets coincides with that of that γ is a homomorphism. Let $a, b \in A$, using the primality of F we have that $\gamma(a) \cap \gamma(b) = \langle \{F \mid a \in F\}, \{F \mid \neg a \in F\} \rangle \cap \langle \{F \mid b \in F\}, \{F \mid \neg b \in F\} \rangle = \langle \{F \mid a \wedge b \in F\}, \{F \mid \neg a \vee \neg b \in F\} \rangle = \{F \mid a \wedge b \in F\}, \{F \mid \neg(a \wedge b) \in F\} = \gamma(a \wedge b)$. The proof that $\{\gamma(a) \mid a \in A\}$ is closed under $\cup$ and that γ

preserves $\vee$ is analogous. Moreover we have that $\sim\gamma(a) = \sim\langle\{F \mid a \in F\}, \{F \mid \neg a \in F\}\rangle = \langle\{F \mid \neg a \in F\}, \{F \mid a \in F\}\rangle = \gamma(\neg a)$. The same happens for n since $\gamma(n) = \langle\emptyset, \emptyset\rangle$. We conclude that $\{\gamma(a) \mid a \in A\}$ is a field of partial sets and that γ is a homomorphism. Clearly γ is surjective. The fact that it is injective follows from Lemma 2.3. □

The connection between partial filters and DMF's can be now stated in a categorical fashion as a dual adjunction between the category of DMF's with homomorphisms between them, which we will denote by DMF, and the category SET of sets and functions between them. In order to do this given any $\boldsymbol{A} \in \mathsf{DMF}$ we let $\mathcal{F}(\boldsymbol{A}) = \mathfrak{P}(\boldsymbol{A}) \cup \{\emptyset\}$ and given a homomorphism $f: \boldsymbol{A} \to \boldsymbol{B}$ with $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{DMF}$ we let $\mathcal{F}(f): \mathcal{F}(\boldsymbol{B}) \to \mathcal{F}(\boldsymbol{A})$ be the function defined as

$$\mathcal{F}(f)(F) = f^{-1}[F]$$

for every $F \in \mathcal{F}(\boldsymbol{B})$. It is easy to prove that $\mathcal{F}: \mathsf{DMF} \to \mathsf{SET}$ is a contravariant functor. Moreover given $X \in \mathsf{SET}$ we let $\mathcal{D}(X) = \mathfrak{D}(X)$ and given a function $f: X \to Y$ with $X, Y \in \mathsf{SET}$ we let $\mathcal{D}(f): \mathcal{D}(Y) \to \mathcal{D}(X)$ be defined as

$$\mathcal{D}(f)\langle A, B\rangle = \langle f^{-1}[A], f^{-1}[B]\rangle$$

for every $\langle A, B\rangle \in \mathcal{D}(Y)$. As we expect $\mathcal{D}: \mathsf{SET} \to \mathsf{DMF}$ is a contravariant functor too.

Theorem 2.7. *$\mathcal{F} \dashv \mathcal{D}$ is a dual adjunction.*

Proof. We have to define two natural transformations. Pick $X \in \mathsf{SET}^{op}$ and let $\varepsilon_X: \mathcal{F}\mathcal{D}(X) \to X$ be the arrow in SET^{op} defined as $f(x) = \{\langle A, B\rangle \in \mathfrak{D}(X) \mid x \in A\}$ for every $x \in X$. It is easy to prove that $\varepsilon: \mathcal{F}\mathcal{D} \to 1_{\mathsf{SET}^{op}}$ is a natural transformation. Now we turn to define the second natural transformation. Pick $\boldsymbol{A} \in \mathsf{DMF}$, we let $\eta_{\boldsymbol{A}}: \boldsymbol{A} \to \mathcal{D}\mathcal{F}(\boldsymbol{A})$ be defined as $\eta_{\boldsymbol{A}}(a) = \gamma(a)$ for every $a \in A$ where γ is the function defined in Theorem 2.6. By the same theorem we know that $\eta_{\boldsymbol{A}}$ is a homomorphism. Now let $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{DMF}$, $f: \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism and $a \in A$, we have that $\mathcal{D}\mathcal{F}(f)\eta_{\boldsymbol{A}}(a) = \langle\{F \mid a \in f^{-1}F\}, \{F \mid \neg a \in f^{-1}F\}\rangle = \langle\{F \mid f(a) \in F\}, \{F \mid f(\neg a) \in F\}\rangle = \eta_{\boldsymbol{B}} f(a)$. We conclude that $\eta: 1_{\mathsf{DMF}} \to \mathcal{D}\mathcal{F}$ is a natural transformation.

Finally we turn to prove the adjunction conditions. It is easy to prove that $\varepsilon_{\mathcal{F}(\boldsymbol{A})} \circ \mathcal{F}(\eta_{\boldsymbol{A}}) = 1_{\mathcal{F}(\boldsymbol{A})}$ for every $\boldsymbol{A} \in \mathsf{DMF}$ and $F \in \mathcal{F}(\boldsymbol{A})$. In order to conclude, let $X \in \mathsf{SET}^{op}$ and $\langle A, B\rangle \in \mathfrak{D}(X)$, we have that $\mathcal{D}(\varepsilon_X)\eta_{\mathcal{D}(X)}\langle A, B\rangle = \mathcal{D}(\varepsilon_X)\langle\{F \mid \langle A, B\rangle \in F\}, \{F \mid \langle B, A\rangle \in F\}\rangle = \langle\{a \in X \mid a \in A\}, \{b \in X \mid b \in B\}\rangle = \langle A, B\rangle$. We conclude that $\mathcal{D}(\varepsilon_X) \circ \eta_{\mathcal{D}(X)} = 1_{\mathcal{D}(X)}$. □

In order to gain more information about the structure of DMF's we need to introduce a new concept: given $\boldsymbol{A} \in \mathsf{DMF}$, we let $\uparrow n = \{a \in A \mid n \leqslant a\}$ be its **positive cone**. The construction of DMF's as fields of partial sets presented before can be abstracted to a lattice theoretical context. In [11] a way to construct bounded DMF's starting from bounded distributive lattices is described. This idea generalises naturally to arbitrary DMF's giving rise to an adjunction (see [3] for a general construction of this kind). In order to make it clearer, let us fix some notation: we denote by $\mathsf{DL}_\bot$ the category of distributive lattices with minimum $\bot$ and lattice homomorphisms which preserve $\bot$ as arrows. Now we construct a functor $\pi: \mathsf{DL}_\bot \to \mathsf{DMF}$. Pick $\boldsymbol{L} \in \mathsf{DL}_\bot$, we let

$$\pi(\boldsymbol{L}) = \langle\{\langle a, b\rangle \in L^2 \colon a \wedge b = \bot\}, \wedge, \vee, \neg, n\rangle$$

where $\langle a, b\rangle \wedge \langle c, d\rangle = \langle a \wedge c, b \vee d\rangle$, $\langle a, b\rangle \vee \langle c, d\rangle = \langle a \vee c, b \wedge d\rangle$, $\neg\langle a, b\rangle = \langle b, a\rangle$, $n = \langle\bot, \bot\rangle$ for every $\langle a, b\rangle, \langle c, d\rangle \in \pi(\boldsymbol{A})$. For every arrow $f: \boldsymbol{L} \to \boldsymbol{M}$ in $\mathsf{DL}_\bot$ we let $\pi(f): \pi(\boldsymbol{L}) \to \pi(\boldsymbol{M})$ be defined as

$$\pi(f)\langle a, b\rangle = \langle f(a), f(b)\rangle$$

for every $\langle a, b\rangle \in \pi(\boldsymbol{L})$. It is easy to prove π is indeed a functor. The way back from DMF to $\mathsf{DL}_\bot$ is pretty natural: we pick positive cones and restrictions of homomorphisms to them. More precisely given $\boldsymbol{A} \in \mathsf{DMF}$ we let $\uparrow(\boldsymbol{A}) = \langle\uparrow n, \wedge, \vee, n\rangle$ and given a DMF arrow $f: \boldsymbol{A} \to \boldsymbol{B}$ we let $\uparrow(f): \uparrow(\boldsymbol{A}) \to \uparrow(\boldsymbol{B})$ be the restriction of f to $\uparrow(\boldsymbol{A})$. Clearly $\uparrow: \mathsf{DL}_\bot \to \mathsf{DMF}$ is a functor.

Theorem 2.8. *$\uparrow \dashv \pi$ is an adjunction.*

Proof. First we have to define two special natural transformations. To do this pick $\boldsymbol{L} \in \mathsf{DL}_\bot$, we let $\varepsilon_{\boldsymbol{L}}: \uparrow\pi(\boldsymbol{L}) \to \boldsymbol{L}$ defined as $\varepsilon_{\boldsymbol{L}}\langle a, \bot\rangle = a$ for every $\langle a, \bot\rangle \in \uparrow\pi(\boldsymbol{L})$. It is easy to prove that $\varepsilon: \uparrow\pi \to 1_{\mathsf{DL}_\bot}$ is a natural transformation. For what concerns the other natural transformation, given $\boldsymbol{A} \in \mathsf{DMF}$, we let $\eta_{\boldsymbol{A}}: \boldsymbol{A} \to \pi\uparrow(\boldsymbol{A})$ defined as $\eta_{\boldsymbol{A}}(a) = \langle a \vee n, \neg a \vee n\rangle$ for every $a \in A$. Using Theorem 2.3 of [11] it is easy to prove that $\eta: \pi\uparrow \to 1_{\mathsf{DMF}}$ is indeed a natural transformation.

It only remains to prove the adjunction conditions. This is easy for let $\boldsymbol{A} \in \mathsf{DMF}$ and $a \in \uparrow(\boldsymbol{A})$, we have that $\varepsilon_{\uparrow(\boldsymbol{A})} \circ \uparrow(\eta_{\boldsymbol{A}})(a) = \varepsilon_{\uparrow(\boldsymbol{A})}\langle a, n\rangle = a$. Therefore we conclude that $\varepsilon_{\uparrow(\boldsymbol{A})} \circ \uparrow(\eta_{\boldsymbol{A}}) = 1_{\uparrow(\boldsymbol{A})}$. Now let $\boldsymbol{L} \in \mathsf{DL}_\bot$ and $\langle a, b\rangle \in \pi(\boldsymbol{L})$, we have that $\pi(\varepsilon_{\boldsymbol{L}}) \circ \eta_{\pi(\boldsymbol{L})}\langle a, b\rangle = \pi(\varepsilon_{\boldsymbol{L}})\langle\langle a, b\rangle \vee \langle\bot, \bot\rangle, \neg\langle a, b\rangle \vee \langle\bot, \bot\rangle\rangle = \pi(\varepsilon_{\boldsymbol{L}})\langle\langle a \vee \bot, b \wedge \bot\rangle, \langle b \vee \bot, a \wedge \bot\rangle\rangle = \pi(\varepsilon_{\boldsymbol{L}})\langle\langle a, \bot\rangle, \langle b, \bot\rangle\rangle = \langle\varepsilon_{\boldsymbol{L}}\langle a, \bot\rangle, \varepsilon_{\boldsymbol{L}}\langle b, \bot\rangle\rangle = \langle a, b\rangle$. We conclude that $\pi(\varepsilon_{\boldsymbol{L}}) \circ \eta_{\pi(\boldsymbol{L})} = 1_{\pi(\boldsymbol{L})}$. □

The functor π constructs DMF's starting from positive cones. However it is easy to prove that in general DMF's are not determined up to isomorphism by their positive cones as shown in Fig. 1, were we present three DMF's which coincide

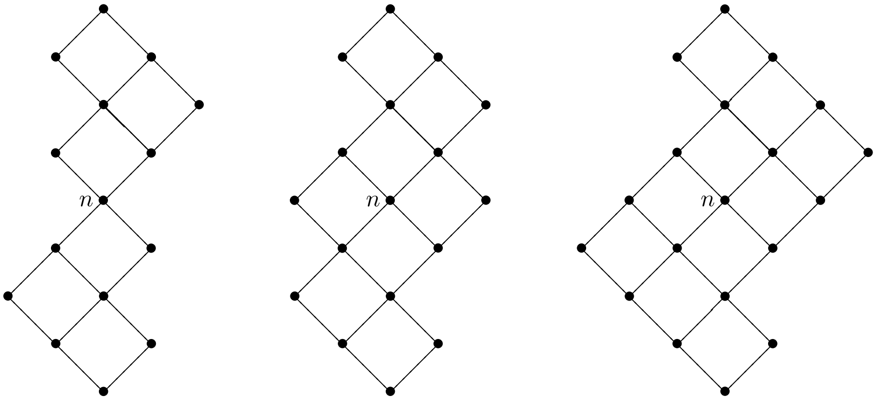

Fig. 1. Some DMF's with the same positive cone.

on the positive cone, but differ for what concerns elements incomparable with n. More precisely we have that DMF's determined up to isomorphism by their positive cones are just the ones where n is $\wedge$-prime as we record below.

Corollary 2.9. *Let $\boldsymbol{A} \in \mathsf{DMF}$. The following are equivalent*:

(i) *if* $\uparrow(\boldsymbol{A}) \cong \uparrow(\boldsymbol{B})$, *then* $\boldsymbol{A} \cong \boldsymbol{B}$, *for every* $\boldsymbol{B} \in \mathsf{DMF}$;
(ii) *n is $\wedge$-prime.*

Proof. (i) $\Rightarrow$ (ii) Since $\boldsymbol{A}$ is determined up to isomorphism by its positive cone, we know that each one of its elements is comparable with n. Moreover we claim that n is $\wedge$-prime. For pick $a, b \in A$ such that $a \wedge b = n$. Since by assumption $\boldsymbol{A} \cong \pi \uparrow (\boldsymbol{A})$, we have that $\langle a, b\rangle$ is comparable with $\langle n, n\rangle$. Suppose without loss of generality that $\langle n, n\rangle \leqslant \langle a, b\rangle$. We conclude that $b = n$ and therefore that n is $\wedge$-prime.

(ii) $\Rightarrow$ (i) Let n be $\wedge$-prime. First we prove that $\boldsymbol{A} \cong \pi \uparrow (\boldsymbol{A})$. By Theorem 2.8 we know that $\eta_{\boldsymbol{A}} : \boldsymbol{A} \to \pi \uparrow (\boldsymbol{A})$ is an embedding. Then pick $\langle a, b\rangle \in \pi \uparrow (\boldsymbol{A})$, we know that $a \wedge b = n$. Since n is $\wedge$-prime this is to say that $a = n$ or $b = n$. Suppose without loss of generality that $a = n$, then $\langle a, b\rangle = \langle n, b\rangle = \langle \neg b \vee n, b \vee n\rangle = \eta_{\boldsymbol{A}} = \neg b$. This proves that $\boldsymbol{A} \cong \pi \uparrow (\boldsymbol{A})$. Now suppose that there is an isomorphism $f : \uparrow(\boldsymbol{B}) \to \uparrow(\boldsymbol{A})$ for some $\boldsymbol{B} \in \mathsf{DMF}$. By Theorem 2.8 there is an embedding $\hat{f} : \boldsymbol{B} \to \boldsymbol{A}$ such that

$$\hat{f}(b) = \begin{cases} f(b) & \text{if } b \geqslant n \\ \neg f(b) & \text{otherwise.} \end{cases}$$

Since $\hat{f}$ is forced to be surjective this ends the proof that $\boldsymbol{A} \cong \boldsymbol{B}$. $\square$

The relation between DMF's and partial sets is reflected also in the structure of free DMF's, which enjoy a nice partial behaviour. Let $\mathbb{X}$ be an arbitrary set of variables, for every $x \in \mathbb{X}$ we let

$$\bar{x} = \Big\langle \Big\{ \langle A, B\rangle \in \mathfrak{D}(\mathbb{X}) \Bigm| x \in A \Big\}, \Big\{ \langle A, B\rangle \in \mathfrak{D}(\mathbb{X}) \Bigm| x \in B \Big\} \Big\rangle.$$

Then we let $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ be the field of partial sets over $\mathfrak{D}(\mathbb{X})$ generated by $\{\bar{x}\}_{x \in \mathbb{X}}$.

Theorem 2.10. *Let $\mathbb{X}$ be a set of variables. $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ is the free algebra over* DMF *with free generators $\{\bar{x}\}_{x \in \mathbb{X}}$, where $\bar{x} \neq \bar{y}$ for every $x, y \in \mathbb{X}$ such that $x \neq y$.*

Proof. Since DMF is a variety we know that there is $\boldsymbol{A} \in \mathsf{DMF}$ which is up to isomorphism the free algebra on DMF with free generators $\mathbb{X}$. By Theorem 2.6 we know that $\boldsymbol{A} \cong \{\gamma(a) \mid a \in A\}$. We will construct an isomorphism between $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ and $\{\gamma(a) \mid a \in A\}$.

In order to do this observe that since $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ is a field of partial sets clearly $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X}) \in \mathsf{DMF}$. Now we let $f : \mathfrak{P}(\boldsymbol{A}) \to \mathfrak{D}(\mathbb{X}) \setminus \{\langle \emptyset, \emptyset\rangle\}$ defined as

$$f(F) = \big\langle \{x \in \mathbb{X} \mid x \in F\}, \{x \in \mathbb{X} \mid \neg x \in F\} \big\rangle$$

for every $F \in \mathfrak{P}(\boldsymbol{A})$. Our aim will be to prove that f is indeed a bijection. We begin by checking its injectivity. Let $F, G \in \mathfrak{P}(\boldsymbol{A})$ such that $F \neq G$. We know by Lemma 2.2 that $\beta(F) \neq \beta(G)$. Since $\beta(F), \beta(G) : \boldsymbol{A} \to \boldsymbol{Z}_3$ and $\boldsymbol{A}$ is generated by $\mathbb{X}$, this yields that there is $x \in \mathbb{X}$ such that $\beta(F)(x) \neq \beta(G)(x)$. Without loss of generality we may reduce to the following cases:

1. $\beta(F)(x) = 1$ and $\beta(G)(x) = n$.
2. $\beta(F)(x) = 1$ and $\beta(G)(x) = -1$.
3. $\beta(F)(x) = n$ and $\beta(G)(x) = -1$.

Applying once again Lemma 2.2 these cases translate to the following:

1. $x \in F$ and $x, \neg x \notin G$.
2. $x \in F$ and $\neg x \in G$.
3. $x, \neg x \notin F$ and $\neg x \in G$.

This easily implies that $f(F) \neq f(G)$, therefore we are done. Now we check f is surjective. Let $\langle A, B\rangle \in \mathfrak{D}(\mathbb{X})$ and consider the unique homomorphism $h : \boldsymbol{A} \to \boldsymbol{Z}_3$ such that

$$h(x) = \begin{cases} 1 & \text{if } x \in A \\ n & \text{if } x \notin A \cup B \\ -1 & \text{if } x \in B \end{cases}$$

for every $x \in \mathbb{X}$. It is easy to check that $h^{-1}\{1\} \in \mathfrak{P}(\boldsymbol{A})$ and finally that $f(h^{-1}\{1\}) = \langle A, B\rangle$. We conclude that f is a bijection.

Now, it is easy to figure out which candidate we shall choose for our isomorphism, namely the unique homomorphism $\lambda : \{\gamma(a) \mid a \in A\} \to \mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ defined as $\lambda\gamma(x) = \bar{x}$ for every $x \in \mathbb{X}$ (observe that $\{\gamma(a) \mid a \in A\}$ is freely generated by $\{\gamma(x) \mid x \in \mathbb{X}\}$, since $\boldsymbol{A}$ is freely generated by $\mathbb{X}$). Since $\mathbf{F}_{\mathsf{DMF}}(\mathbb{X})$ is generated by $\{\bar{x}\}_{x\in\mathbb{X}}$ we know that λ is an epimorphism. It only remains to prove it is an embedding. In order to do this one can check, using the fact that f is a bijection, that for every $a \in A$

$$\lambda\gamma(a) = \bigl\langle \bigl\{ f(F) \bigm| a \in F \bigr\}, \bigl\{ f(F) \bigm| \neg a \in F \bigr\} \bigr\rangle.$$

This is done as usual by induction on terms. The fact that λ is injective follows now from the injectivity of f. □

3. A logic for exact truth

Now we turn to define a logic for partial predicates which preserves exact truth. Since we have seen, both in the intuitive explanation and in the representation theorems, that partial sets are intrinsically three valued, the natural choice is to define a logic trough the matrix $\langle \boldsymbol{Z}_3, \{1\}\rangle$. More precisely we let

$$\Gamma \vdash \varphi \quad \Longleftrightarrow \quad \text{if } h[\Gamma] \subseteq \{1\}, \quad \text{then } h(\varphi) = 1$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and every homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{Z}_3$. We will call this logic $\mathcal{L}_{\{1\}}$ since it is intended to represent exact truth in partial contexts. Observe that $\mathcal{L}_{\{1\}}$ is finitary since it is defined by a finite matrix and that it has no theorems since $\{n\}$ is the universe of a subalgebra of $\boldsymbol{Z}_3$. The first problem we would like to solve is to individuate the algebraic counterpart of this logic. In order to do this we need to state explicitly the connection between equations in DMF and the Frege relation of $\mathcal{L}_{\{1\}}$, i.e., its interderivability relation.

Lemma 3.1. *Let $\alpha, \beta \in Fm$. The following conditions are equivalent*:

(i) $\boldsymbol{Z}_3 \vDash \alpha \approx \beta$;
(ii) $\mathsf{DMF} \vDash \alpha \approx \beta$;
(iii) $\alpha \dashv\vdash \beta$ *and* $\neg\alpha \dashv\vdash \neg\beta$.

The fact that the Frege relation is strictly connected to the equations holding in DMF allows us to gain informations about the Tarski congruence for g-models which behave especially well. This allows us to easily prove that the algebraic counterpart of $\mathcal{L}_{\{1\}}$ coincides with DMF. In order to do this we need to introduce some properties typical of g-matrices. Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a g-matrix. We say that it has the property of the **conjunction** (PC) if for every $a, b \in A$ it holds $\mathcal{C}\{a \wedge b\} = \mathcal{C}\{a, b\}$, the property of the **disjunction** (PDI) if for every $X \cup \{a, b\} \subseteq A$ it holds $\mathcal{C}\{X, a \vee b\} = \mathcal{C}\{X, a\} \cap \mathcal{C}\{X, b\}$, the property of **double negation** (PDN) if for every $a \in A$ it holds $\mathcal{C}\{\neg\neg a\} = \mathcal{C}\{a\}$, the property of **De Morgan** (PDM) if for every $a, b \in A$ it holds $\mathcal{C}\{\neg(a \wedge b)\} = \mathcal{C}\{\neg a \vee \neg b\}$ and $\mathcal{C}\{\neg(a \vee b)\} = \mathcal{C}\{\neg a \wedge \neg b\}$ and finally the property of **normality** (N) if for every $a \in A$ it holds $a \in \mathcal{C}\{n \vee \neg n\}$ and $a \in \mathcal{C}(X) \Rightarrow n \in \mathcal{C}\{X, \neg a\}$. We are now ready to state the desired result.

Theorem 3.2. $\boldsymbol{Alg}\,\mathcal{L}_{\{1\}} = \mathsf{DMF}$.

Proof. Consider $\boldsymbol{Z}_3$, it is easy to prove that $\boldsymbol{\Omega}\{1\} = \Delta$ and therefore that $\boldsymbol{Z}_3 \in \boldsymbol{Alg}\,\mathcal{L}_{\{1\}}$. Since $\boldsymbol{Alg}\,\mathcal{L}_{\{1\}} = \mathbb{P}_{\mathsf{SD}}\boldsymbol{Alg}^*\mathcal{L}_{\{1\}}$, by Corollary 2.5 we have that

$$\mathsf{DMF} = \mathbb{P}_{\mathsf{SD}}\boldsymbol{Z}_3 \subseteq \mathbb{P}_{\mathsf{SD}}\boldsymbol{Alg}^*\mathcal{L}_{\{1\}} = \boldsymbol{Alg}\,\mathcal{L}_{\{1\}}.$$

In general we have that $\boldsymbol{Alg}\,\mathcal{L}_{\{1\}} \subseteq \mathbb{V}(\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}\mathcal{L}_{\{1\}})$. Since $\widetilde{\boldsymbol{\Omega}}\mathcal{L}_{\{1\}}$ is fully invariant we have that $\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}\mathcal{L}_{\{1\}} \vDash \alpha \approx \beta$ if and only if $\langle \alpha, \beta \rangle \in \widetilde{\boldsymbol{\Omega}}\mathcal{L}_{\{1\}}$. In Lemma 3.4.10 of [13] it is proved that if a generalised matrix $\langle \boldsymbol{A}, \mathcal{C} \rangle$ has (PC), (PDI), (PDM) and (PDN), then $\langle a, b \rangle \in \widetilde{\boldsymbol{\Omega}}\mathcal{C}$ if and only if $\mathcal{C}(a) = \mathcal{C}(b)$ and $\mathcal{C}(\neg a) = \mathcal{C}(\neg b)$ for every $a, b \in A$. Since $\langle \boldsymbol{Fm}, \mathcal{L}_{\{1\}} \rangle$ has (PC), (PDI), (PDM) and (PDN) we conclude that $\alpha \dashv\vdash \beta$ and $\neg\alpha \dashv\vdash \neg\beta$. By Lemma 3.1 this is equivalent to $\boldsymbol{Z}_3 \vDash \alpha \approx \beta$. We conclude that $\mathbb{V}(\boldsymbol{Z}_3) = \mathbb{V}(\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}\mathcal{L}_{\{1\}})$ and finally that $\boldsymbol{Alg}\,\mathcal{L}_{\{1\}} \subseteq \mathsf{DMF}$. □

Until now we characterised the algebraic component of a semantics for $\mathcal{L}_{\{1\}}$, but, in order to work as a semantics, algebras need to be equipped with some more structure. In our case this role will be played by deductive filters which give rise to matrix models for $\mathcal{L}_{\{1\}}$. The interesting fact is that deductive filters on DMF's are very close to partial filters in the sense that they do not contain contradictions and enjoy a weak form of primality. More precisely, given $\boldsymbol{A} \in \mathsf{DMF}$, we let $\mathcal{F}i_n(\boldsymbol{A})$ be the set of lattice filters of $\boldsymbol{A}$ which moreover satisfy the following conditions[2]:

($\bot$) $n \notin F$;
(P$\bot$) if $n \vee a \in F$, then $a \in F$.

Lemma 3.3. *Let* $\boldsymbol{A} \in \mathsf{DMF}$. $\mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A}) = \mathcal{F}i_n(\boldsymbol{A}) \cup \{\emptyset, A\}$. *Moreover if* $\emptyset \neq \{a_1, \ldots, a_k\} \subseteq A$, *then*

$$\mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a_1, \ldots, a_k\} = \{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}.$$

Proof. First observe that $\emptyset \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$ since the logic has no theorems and that it always holds $A \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$. We prove the inclusion $\mathcal{F}i_n(\boldsymbol{A}) \subseteq \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$. Let $F \in \mathcal{F}i_n(\boldsymbol{A})$ and pick $a \notin F$. By condition (P$\bot$) we know that $a \vee n \notin F$. Then by the Prime Filter Theorem there is a prime filter P such that $F \subseteq P$ and $a \vee n \notin P$. In particular we have that $n \notin P$ and therefore $P \in \mathfrak{P}(\boldsymbol{A})$. Since $a \notin P$ we conclude that

$$F = \bigcap\bigl\{P \in \mathfrak{P}(\boldsymbol{A}) \colon F \subseteq P\bigr\}.$$

Now consider the map $\alpha \colon \boldsymbol{A} \to \prod_{F \in \mathfrak{P}(\boldsymbol{A})} \boldsymbol{Z}_{3_F}$ defined in Theorem 2.4. Clearly $P = (\pi_P \alpha)^{-1}\{1\}$. Since inverse images of deductive filters under homomorphisms are deductive filters we conclude that $P \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$. Finally since $\mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$ is closed under arbitrary intersections we conclude that $F \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$.

Now we prove the other inclusion. Let $F \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$ such that $F \notin \{\emptyset, A\}$. Since the deductions $x, y \vdash x \wedge y$ and $x \vdash x \vee y$ hold in the logic, F is a lattice filter. Suppose towards a contradiction that ($\bot$) does not hold, i.e. that $n \in F$. Since the deduction $n \vdash x$ holds in the logic, we would have $F = A$ against the assumption. The fact that (P$\bot$) holds follows from the fact that the deduction $x \vee n \vdash x$ holds in the logic. We conclude that $\mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A}) = \mathcal{F}i_n(\boldsymbol{A}) \cup \{\emptyset, A\}$.

Now we turn to prove the second part. It is easy to prove that $\{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}$ is a lattice filter which includes $\{a_1, \ldots, a_k\}$. Moreover it is clear that $n \in \{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}$ if and only if $\{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\} = A$. Now suppose that $b \vee n \in \{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}$. We have that $a_1 \wedge \cdots \wedge a_k \leqslant (b \vee n) \vee n = b \vee n$ and therefore that $b \in \{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}$. We conclude that $\{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\} \in \mathcal{F}i_n(\boldsymbol{A}) \cup \{\emptyset, A\} = \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$. This yields that $\mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a_1, \ldots, a_k\} \subseteq \{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}$.

Now we prove the other inclusion. Let $b \in \{b \in A \mid a_1 \wedge \cdots \wedge a_k \leqslant b \vee n\}$. This yields that $a_1 \wedge \cdots \wedge a_k \leqslant b \vee n$ and therefore, since $\mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a_1, \ldots, a_k\}$ is a lattice filter, that $b \vee n \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a_1, \ldots, a_k\}$. Since $\mathcal{F}i_n(\boldsymbol{A}) \cup \{\emptyset, A\} = \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$ we conclude that $b \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a_1, \ldots, a_k\}$. □

Now that we know how deductive filters in DMF look like, it is natural to ask for a characterisation of Leibniz reduced models of $\mathcal{L}_{\{1\}}$. Since our logic is not protoalgebraic, as we remark below, this is not a trivial problem because in principle we do not know which DMF's will play the role of the algebraic components of Leibniz reduced models.

Corollary 3.4. $\mathcal{L}_{\{1\}}$ *is neither protoalgebraic, nor truth-equational, nor selfextensional.*

Proof. $\mathcal{L}_{\{1\}}$ is neither protoalgebraic, nor truth-equational since it has no theorems and is not almost inconsistent. In order to prove it is not selfextensional, we reason as follows: using the definition of the logic it is easy to prove that $x \dashv\vdash x \vee n$. But clearly $\neg x \nvdash \neg(x \vee n)$, therefore the Frege relation is not a congruence. □

Our strategy for characterising Leibniz reduced models of $\mathcal{L}_{\{1\}}$ is to make use of the fact that we know how reduced models of Belnap's four-valued logic look like, since they were studied in detail by Font in [6].

Theorem 3.5. *Let* $\boldsymbol{A}$ *be non-trivial.* $\langle \boldsymbol{A}, F \rangle \in \boldsymbol{Mod}^{*}\mathcal{L}_{\{1\}}$ *if and only if the following conditions hold*:

[2] These names derive from the intuition that the first condition tells us that n represents a contradiction and the second one that F is prime with respect to this contradiction.

(i) $\boldsymbol{A} \in \mathsf{DMF}$;
(ii) $\boldsymbol{A}$ *has a maximum* 1 *and* $F = \{1\}$;
(iii) *if* $a < b$, *then there is* $c \in A$ *such that* $a \vee c < b \vee c = 1$, *for every* $a, b \geqslant n$.

Proof. ($\Rightarrow$) Let $\langle \boldsymbol{A}, F\rangle \in \boldsymbol{Mod}^{*}\mathcal{L}_{\{1\}}$. By Theorem 3.2 we know that $\boldsymbol{A} \in \mathsf{DMF}$. Moreover by Lemma 3.3 we know that F is a lattice filter. We reason as follows: first observe that the $\{n\}$-free reduct of $\boldsymbol{A}$ is a De Morgan lattice, therefore $\langle \boldsymbol{A}, F\rangle$ is a model of Belnap's four-valued logic [6, Proposition 3.12]. Since the Leibniz congruence is not affected by the presence of constants, we conclude that $\langle a, b\rangle \in \boldsymbol{\Omega} F$ if and only if

$$(a \vee c \in F \Leftrightarrow b \vee c \in F) \quad \text{and} \quad (\neg a \vee c \in F \Leftrightarrow \neg b \vee c \in F)$$

for every $c \in A$ [6, Proposition 3.13]. Since $\boldsymbol{A}$ is not trivial, we know that $F \neq \emptyset$. Then pick $a, b \in F$ and $c \in A$. Clearly we have that $a \vee c \in F$ and $b \vee c \in F$. Now suppose that $\neg a \vee c \in F$. This yields that $(a \wedge \neg a) \vee (a \wedge c) = a \wedge (\neg a \vee c) \in F$. Since $a \wedge \neg a \leqslant n$ we conclude that $n \vee (a \wedge c) \in F$. Now, since the deduction $x \vee n \vdash x$ holds in $\mathcal{L}_{\{1\}}$, we conclude that $a \wedge c \in F$ and finally that $c \in F$, from which it follows that $\neg b \vee c \in F$. This is to say that $\langle a, b\rangle \in \boldsymbol{\Omega} F$, but since we assumed that $\boldsymbol{\Omega} F = \Delta$, we conclude that $a = b$. Therefore F is a singleton $\{1\}$ and clearly 1 is the maximum of $\boldsymbol{A}$.

Now let $a, b \geqslant n$ such that $a < b$. Therefore there is $c \in A$ such that $a \vee c < b \vee c = 1$ or $\neg b \vee c < \neg a \vee c = 1$ [6, Theorem 3.14]. Suppose towards a contradiction that there is $c \in A$ such that $\neg b \vee c < \neg a \vee c = 1$. Since $n \leqslant a$ we get that $\neg a \leqslant n$ and therefore that $n \vee c = 1$. Since the deduction $x \vee n \vdash x$ holds in $\mathcal{L}_{\{1\}}$ we conclude that $c = 1$ and therefore $\neg b \vee c = 1$ against the assumption, therefore we are done.

($\Leftarrow$) It is enough to prove that if $a < b$ then there is $c \in A$ such that $a \vee c < b \vee c = 1$ or $\neg b \vee c < \neg a \vee c = 1$ [6, Theorem 3.14]. By point (v) of Lemma 2.1 we know that the quasi-equation

$$(x \vee n = y \vee n \text{ and } \neg x \vee n = \neg y \vee n) \Rightarrow x = y$$

holds in DMF. Therefore we conclude that $a \vee n < b \vee n$ or $\neg b \vee n < \neg a \vee n$. Suppose that $a \vee n < b \vee n$ (the other case is analogous), then by assumption there is $d \in A$ such that $(a \vee n) \vee d < (b \vee n) \vee d = 1$, therefore letting $c := n \vee d$ we are done. $\square$

This result is indeed curious since it tells us that Leibniz reduced models of $\mathcal{L}_{\{1\}}$ enjoy a local behaviour, in the sense that the fact that a model is reduced depends on the structure of the positive cone of the algebra. This can be stated in a nicer way if we consider the characterisation of Leibniz reduced algebras for the $\{\wedge, \vee\}$-fragment of classical logic given by Font, Guzmán and Verdú in [7]. Actually in our case we will need to add the minimum $\bot$ which will play the role of n. For let $\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$ be the $\{\wedge, \vee, \bot\}$-fragment of classical propositional logic, it turns out that its Leibniz reduced algebras consist exactly of the positive cones of Leibniz reduced algebras for $\mathcal{L}_{\{1\}}$ as we remark below.

Corollary 3.6. *Let* $\boldsymbol{A} \in \mathsf{DMF}$. $\boldsymbol{A} \in \boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}}$ *if and only if* $\uparrow(\boldsymbol{A}) \in \boldsymbol{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$.

Proof. First of all let us characterise $\boldsymbol{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$. In order to do this we reason as follows: since classical logic is selfextensional and $\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$ is a fragment of it, we know it is selfextensional too. Now pick any of the equations $\alpha \approx \beta$ defining the variety $\mathsf{DL}_{\bot}$, it is easy to prove that in $\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$ the deduction $\alpha \dashv\vdash \beta$ holds and therefore that $\langle \alpha, \beta\rangle \in \boldsymbol{\Lambda}\mathcal{CPC}_{\{\wedge,\vee,\bot\}} = \widetilde{\boldsymbol{\Omega}}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$. Since $\widetilde{\boldsymbol{\Omega}}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$ is fully invariant this yields $\mathbf{Fm}/\widetilde{\boldsymbol{\Omega}}\mathcal{CPC}_{\{\wedge,\vee,\bot\}} \vDash \alpha \approx \beta$ and therefore $\mathbb{V}(\mathbf{Fm}/\widetilde{\boldsymbol{\Omega}}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}) \vDash \alpha \approx \beta$. We conclude that $\boldsymbol{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}} \subseteq \mathbb{V}(\mathbf{Fm}/\widetilde{\boldsymbol{\Omega}}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}) \subseteq \mathsf{DL}_{\bot}$.

Let $\boldsymbol{L} \in \mathsf{DL}_{\bot}$, we claim that $\boldsymbol{L} \in \boldsymbol{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$ if and only if $\boldsymbol{L}$ has a maximum 1 and for every $a, b \in A$ such that $a < b$ there is $c \in L$ such that $a \vee c < b \vee c = 1$. For this suppose $\boldsymbol{L} \in \mathsf{DL}_{\bot}$ then there is $F \subseteq L$ such that $\langle \boldsymbol{L}, F\rangle$ is reduced. It is clear that F is a deductive filter of the $\{\wedge, \vee\}$-fragment of classical logic too (when quitting $\bot$ from the language), therefore we are done [7, p. 127]. For the other direction it is clear that $\langle \boldsymbol{L}, \{1\}\rangle$ is a model of $\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$. To prove this, just observe that $\{1\}$ is the intersection of all prime filters of $\boldsymbol{L}$ which are deductive filters on $\boldsymbol{L}$ since they are inverse images under homomorphisms of the deductive filter $\{\top\}$ over the two-element distributive lattice with minimum $\mathbf{2}$. But $\langle \boldsymbol{L}, \{1\}\rangle$ is reduced [7, p. 127], therefore $\boldsymbol{L} \in \boldsymbol{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$. It is now straightforward that $\boldsymbol{A} \in \boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}}$ if and only if $\uparrow(\boldsymbol{A}) \in \boldsymbol{Alg}^{*}\mathcal{CPC}_{\{\wedge,\vee,\bot\}}$ for every $\boldsymbol{A} \in \mathsf{DMF}$. $\square$

It is now clear that $\boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}} \subsetneq \boldsymbol{Alg}\,\mathcal{L}_{\{1\}}$, observe for example that $\boldsymbol{Z}_3$ is the only chain in $\boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}}$ and $\mathbf{Z}_{2m+1} \in \mathsf{DMF}$ for every $m \in \mathbb{N}$. Moreover $\boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}}$ is not a quasi-variety since is not closed under the subalgebra operator. In order to prove this observe that $\boldsymbol{Z}_3 \times \boldsymbol{Z}_3 \in \boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}}$ and that $\boldsymbol{Z}_5 \in \mathbb{S}(\boldsymbol{Z}_3 \times \boldsymbol{Z}_3)$, since $\boldsymbol{Z}_5 \notin \boldsymbol{Alg}^{*}\mathcal{L}_{\{1\}}$ we are done.

4. A Gentzen system

In the previous section we went trough the study of the algebraic semantics of $\mathcal{L}_{\{1\}}$, while here we would like to concentrate on a syntactic characterisation of the logic. More precisely we look for a Gentzen system fully adequate for $\mathcal{L}_{\{1\}}$ whose search will coincide with the study of full g-models of $\mathcal{L}_{\{1\}}$ (see for example [8]). Next lemma moves a first step in this direction by individuating a set of Tarski-style conditions on g-matrices which yield Tarski reduced models in DMF.

Lemma 4.1. *Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a g-matrix. If $\langle \boldsymbol{A}, \mathcal{C}\rangle$ has the properties (PC), (PDI), (PDN), (PDM) and (N), then $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathsf{DMF}$.*

Proof. Assume that $\langle \boldsymbol{A}, \mathcal{C}\rangle$ satisfies the assumption. By Lemma 3.4.10 of [13] we have that $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}\mathcal{C}$ if and only if $\mathcal{C}\{a\} = \mathcal{C}\{b\}$ and $\mathcal{C}\{\neg a\} = \mathcal{C}\{\neg b\}$ for every $a, b \in A$ and that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C}$ is a De Morgan lattice. In order to check that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathsf{DMF}$ it only remains to prove the normality condition $x \wedge \neg x \leqslant y \vee \neg y$ and the fact that n is a fixed point of negation.

From (N), (PDI) and (PDN) it follows that $\mathcal{C}\{n\} = \mathcal{C}\{\neg n\}$ and $\mathcal{C}\{\neg n\} = \mathcal{C}\{\neg\neg n\}$ and therefore that $n/\widetilde{\boldsymbol{\Omega}}\mathcal{C} = \neg(n/\widetilde{\boldsymbol{\Omega}}\mathcal{C})$. Now we turn to check the normality condition. Let $a, b \in A$, we want to prove that $a/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \wedge \neg a/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \leqslant b/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \vee \neg b/\widetilde{\boldsymbol{\Omega}}\mathcal{C}$. This is equivalent to prove $\langle (a \wedge \neg a) \wedge (b \vee \neg b), a \wedge \neg a\rangle \in \widetilde{\boldsymbol{\Omega}}\mathcal{C}$ which amounts to the following conditions:

1. $\mathcal{C}\{(a \wedge \neg a) \wedge (b \vee \neg b)\} = \mathcal{C}\{a \wedge \neg a\}$.
2. $\mathcal{C}\{\neg[(a \wedge \neg a) \wedge (b \vee \neg b)]\} = \mathcal{C}\{\neg(a \wedge \neg a)\}$.

We begin by proving 1. By (PC) clearly $\mathcal{C}\{a \wedge \neg a\} \subseteq \mathcal{C}\{(a \wedge \neg a) \wedge (b \vee \neg b)\}$. Then we turn to prove the other inclusion. With (N) and (PDI) one can check that $c \in \mathcal{C}\{n\}$ for every $c \in A$. Since $a \in \mathcal{C}\{a\}$ by (N) and (PC) we have that $n \in \mathcal{C}\{a, \neg a\} = \mathcal{C}\{a \wedge \neg a\}$ and therefore that $c \in \mathcal{C}\{a \wedge \neg a\}$ for every $c \in A$. In particular we have that $b \vee \neg b \in \mathcal{C}\{a \wedge \neg a\}$ and therefore by (PC) that $(a \wedge \neg a) \wedge (b \vee \neg b) \in \mathcal{C}\{a \wedge \neg a\}$. Now we turn to check 2. Applying in succession (PDM), (PDI), (PDM), (PDI), (PC), (PDN) and the fact that $c \in \mathcal{C}\{b, \neg b\}$ for every $c \in A$, we have that $\mathcal{C}\{\neg[(a \wedge \neg a) \wedge (b \vee \neg b)]\} = \mathcal{C}\{a\} \cap \mathcal{C}\{\neg\neg a\}$. Since by (PDM) and (PDI) $\mathcal{C}\{\neg(a \wedge \neg a)\} = \mathcal{C}\{a\} \cap \mathcal{C}\{\neg\neg a\}$ we are done. □

We are now ready to present our characterisation of full g-models of $\mathcal{L}_{\{1\}}$ which will play the role of a completeness theorem in our quest for a Gentzen system for $\mathcal{L}_{\{1\}}$. This characterisation is obtained just adding the fact that the g-matrix is finitary and has no theorems to the conditions of the previous lemma.

Theorem 4.2. *Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a g-matrix. $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full g-model of $\mathcal{L}_{\{1\}}$ if and only if it is finitary, has no theorems and has the (PC), (PDI), (PDN), (PDM) and the (N).*

Proof. ($\Rightarrow$) Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ be a full g-model. Being finitary, having no theorems, (PC), (PDI), (PDN) and (PDM) transfer always from the logic to every full g-model. For what concerns (N), the first part transfers from the logic to every full g-model since it is expressible as the Hilbert style rule $n \vee \neg n \vdash x$. For the second part we need a slightly longer argument. Recall that since $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a full g-model, the Tarski projection $\pi_{\widetilde{\boldsymbol{\Omega}}} : \langle \boldsymbol{A}, \mathcal{C}\rangle \to \langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C}, \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})\rangle$ is a strict surjective homomorphism. This yields in particular that $a \in \mathcal{C}(X)$ if and only if $a/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(X/\widetilde{\boldsymbol{\Omega}}\mathcal{C})$ for every $X \cup \{a\} \subseteq A$. So let $a \in \mathcal{C}(X)$, using finitarity and (PC) of $\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C}, \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})\rangle$, we have $a/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{b_1/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \wedge \cdots \wedge b_k/\widetilde{\boldsymbol{\Omega}}\mathcal{C}\}$ for some $\{b_1, \ldots, b_k\} \subseteq X$. We let $b := b_1 \wedge \cdots \wedge b_k$. By (PC) of $\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C}, \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})\rangle$ we have that $(a \wedge \neg a)/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{b/\widetilde{\boldsymbol{\Omega}}\mathcal{C}, \neg a/\widetilde{\boldsymbol{\Omega}}\mathcal{C}\}$. By normality together with Lemma 3.3 we have that $n/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{b/\widetilde{\boldsymbol{\Omega}}\mathcal{C}, \neg a/\widetilde{\boldsymbol{\Omega}}\mathcal{C}\}$. Therefore we conclude that $n \in \mathcal{C}\{X, \neg a\}$.

($\Leftarrow$) Let $\langle \boldsymbol{A}, \mathcal{C}\rangle$ satisfies the assumption, by Lemma 4.1 we have that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathsf{DMF}$. It only remains to prove that $\pi_{\widetilde{\boldsymbol{\Omega}}} : \langle \boldsymbol{A}, \mathcal{C}\rangle \to \langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}\mathcal{C}, \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})\rangle$ is strict. This is equivalent of proving $b \in \mathcal{C}(X)$ if and only if $b/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(X/\widetilde{\boldsymbol{\Omega}}\mathcal{C})$ for every $X \cup \{b\} \subseteq A$ with $X \neq \emptyset$. Then pick $X \cup \{b\} \subseteq A$ with $X \neq \emptyset$. Suppose that $b \in \mathcal{C}(X)$, since $\mathcal{C}$ is finitary $b \in \mathcal{C}\{a_1, \ldots, a_k\}$ for some $\{a_1, \ldots, a_k\} \subseteq X$. Let $a := a_1 \wedge \cdots \wedge a_k$, by Lemma 3.3 we have that $b/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a/\widetilde{\boldsymbol{\Omega}}\mathcal{C}\}$ if and only if $a/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \leqslant (b \vee n)/\widetilde{\boldsymbol{\Omega}}\mathcal{C}$. By Lemma 3.4.10 of [13] this is equivalent to:

1. $\mathcal{C}\{a \vee (b \vee n)\} = \mathcal{C}\{b \vee n\}$.
2. $\mathcal{C}\{\neg[a \vee (b \vee n)]\} = \mathcal{C}\{\neg(b \vee n)\}$.

Applying in succession (PDI), (PDI), the fact that $c \in \mathcal{C}\{n\}$ for every $c \in A$, $b \in \mathcal{C}\{a\}$, the fact that $c \in \mathcal{C}\{n\}$ for every $c \in A$ and (PDI) we prove 1. To prove 2 we reason as follows: applying in succession (PDM), (PC), (PDM), (PC) and the fact that $c \in \mathcal{C}\{\neg n\}$ for every $c \in A$, we have that $\mathcal{C}\{\neg[a \vee (b \vee n)]\} = \mathcal{C}\{\neg(b \vee n)\} = A$. Since by (PDM), (PC) and the fact that $c \in \mathcal{C}\{\neg n\}$ for every $c \in A$ it follows that $\mathcal{C}\{\neg(b \vee n)\} = A$, we are done.

Now we prove the other direction. Let $b/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(X/\widetilde{\boldsymbol{\Omega}}\mathcal{C})$. Since $\mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})$ is finitary and has (PC) there are $\{a_1 \ldots, a_k\} \subseteq X$ such that $\mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a_1/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \wedge \cdots \wedge a_k/\widetilde{\boldsymbol{\Omega}}\mathcal{C}\} = \mathcal{F}i_{\mathcal{L}_{\{1\}}}(X/\widetilde{\boldsymbol{\Omega}}\mathcal{C})$. Let $a := a_1 \wedge \cdots \wedge a_k$. Clearly $b/\widetilde{\boldsymbol{\Omega}}\mathcal{C} \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}\{a/\widetilde{\boldsymbol{\Omega}}\mathcal{C}\}$ which is equivalent to $\langle a \vee (b \vee n), b \vee n\rangle \in \widetilde{\boldsymbol{\Omega}}\mathcal{C}$ by Lemma 3.3. Since $\widetilde{\boldsymbol{\Omega}}\mathcal{C} \subseteq \boldsymbol{\Lambda}\mathcal{C}$ this yields $b \vee n \in \mathcal{C}\{a \vee (b \vee n)\}$. Applying (PDI) and (N) several times we obtain that $\mathcal{C}\{b\} \subseteq \mathcal{C}\{a\}$ and therefore $b \in \mathcal{C}X$. □

As a corollary we get a logification of DMF's, in the sense that they can be characterised as algebras on which it is possible to find a closure operator which satisfies certain Tarski-style conditions and separates points in a peculiar way.

Corollary 4.3. *$\boldsymbol{A} \in \mathsf{DMF}$ if and only if there is a finitary closure system without theorems $\mathcal{C} \subseteq \mathcal{P}(\boldsymbol{A})$ which satisfies (PC), (PDI), (PDN), (PDM), (N) and satisfies the following separation condition for every $a, b \in A$:*

if $\mathcal{C}\{a\} = \mathcal{C}\{b\}$ and $\mathcal{C}\{\neg a\} = \mathcal{C}\{\neg b\}$, then $a = b$.

As we promised, drawing consequences from Theorem 4.2, we obtain a fully adequate Gentzen system for the logic $\mathcal{L}_{\{1\}}$. Since our logic is finitary and without theorems we shall consider sequents whose premises are non-empty finite sets of formulas. The desired Gentzen system arises naturally from the Tarski-style conditions which characterise full g-models:

$$\alpha \rhd \alpha\,(R) \qquad \frac{\Gamma \rhd \alpha}{\Gamma, \beta \rhd \alpha}(W) \qquad \frac{\Gamma \rhd \alpha \quad \Gamma, \alpha \rhd \beta}{\Gamma \rhd \beta}(Cut)$$

$$\frac{\Gamma, \alpha, \beta \rhd \gamma}{\Gamma, \alpha \wedge \beta \rhd \gamma}(\wedge\rhd) \qquad \frac{\Gamma \rhd \alpha \quad \Gamma \rhd \beta}{\Gamma \rhd \alpha \wedge \beta}(\rhd\wedge)$$

$$\frac{\Gamma, \alpha \rhd \gamma \qquad \Gamma, \beta \rhd \gamma}{\Gamma, \alpha \vee \beta \rhd \gamma}(\vee\rhd) \qquad \frac{\Gamma \rhd \alpha}{\Gamma \rhd \alpha \vee \beta}, \frac{\Gamma \rhd \alpha}{\Gamma \rhd \beta \vee \alpha}(\rhd\vee)$$

$$\frac{\Gamma, \alpha \rhd \beta}{\Gamma \neg\neg\alpha, \rhd \beta}(\neg\rhd) \qquad \frac{\Gamma \rhd \alpha}{\Gamma \rhd \neg\neg\alpha}(\rhd\neg)$$

$$\frac{\Gamma, \neg\alpha \rhd \gamma \qquad \Gamma, \neg\beta \rhd \gamma}{\Gamma, \neg(\alpha \wedge \beta) \rhd \gamma}(\neg \wedge \rhd) \qquad \frac{\Gamma \rhd \neg\alpha}{\Gamma \rhd \neg(\alpha \wedge \beta)}, \frac{\Gamma \rhd \neg\beta}{\Gamma \rhd \neg(\alpha \wedge \beta)}(\rhd\neg\wedge)$$

$$\frac{\Gamma, \neg\alpha, \neg\beta \rhd \gamma}{\Gamma, \neg(\alpha \vee \beta) \rhd \gamma}(\neg \vee \rhd) \qquad \frac{\Gamma \rhd \neg\alpha \qquad \Gamma \rhd \neg\beta}{\Gamma \rhd \neg(\alpha \vee \beta)}(\rhd\neg\vee)$$

$$n \vee \neg n \rhd \alpha\,(n\rhd) \qquad \frac{\Gamma \rhd \alpha}{\Gamma, \neg\alpha \rhd n}(\rhd n).$$

If we denote by $\mathfrak{G}$ this Gentzen system, we can state our strong completeness result as follows.

Corollary 4.4. *$\mathfrak{G}$ is fully adequate for $\mathcal{L}_{\{1\}}$.*

Even if $\mathcal{L}_{\{1\}}$ is not algebraizable, it turns out that it enjoys an algebraizable Gentzen system. In order to give an explicit formulation of the transformers which yield algebraizability, let us denote by *Seq* the set of sequents whose premises are non-empty finite sets of formulas and by *Eq* the set of equations. We let $\tau : \mathcal{P}(Seq) \longrightarrow \mathcal{P}(Eq) : \rho$ be the residuated mappings defined as

$$\tau(\Gamma \rhd \alpha) = \bigwedge \Gamma \leqslant \alpha \vee n \qquad \rho(\alpha \approx \beta) = \{\alpha \lhd\rhd \beta, \neg\alpha \lhd\rhd \neg\beta\}$$

for every $\Gamma \rhd \alpha \in Seq$ and $\alpha \approx \beta \in Eq$.

Theorem 4.5. *$\mathfrak{G}$ is algebraizable with equivalent algebraic semantics* DMF *via τ and ρ.*

Proof. The first condition of algebraizability consists of $\frac{\{\Gamma_i \rhd \alpha_i \mid i<k\}}{\Gamma \rhd \alpha}$ if and only if $\mathsf{DMF} \vDash (\bigwedge \Gamma_1 \leqslant \alpha_1 \vee n$ and $\ldots$ and $\bigwedge \Gamma_{k-1} \leqslant \alpha_{k-1} \vee n) \Rightarrow \bigwedge \Gamma \leqslant \alpha \vee n$ for every $k \in \mathbb{N}$. The "only if" direction is easily proved by checking rules and axioms of the system $\mathfrak{G}$. For the "if" direction suppose that $\mathsf{DMF} \nvDash (\bigwedge \Gamma_1 \leqslant \alpha_1 \vee n$ and $\ldots$ and $\bigwedge \Gamma_{k-1} \leqslant \alpha_{k-1} \vee n) \Rightarrow \bigwedge \Gamma \leqslant \alpha \vee n$. Therefore there are $\boldsymbol{A} \in \mathsf{DMF}$ and a homomorphism $f : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $f(\bigwedge \Gamma_i) \leqslant f(\alpha_i) \vee f(n)$ for every $i < k$ and $f(\bigwedge \Gamma) \nleqslant f(\alpha) \vee f(n)$. By Lemma 3.3 this is to say that $f(\alpha_i) \in \mathcal{F}i_{\mathcal{L}_{\{1\}}}(f[\Gamma_i])$ for every $i < k$ and $f(\alpha) \notin \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\Gamma)$. Since $\langle \boldsymbol{A}, \mathcal{F}i_{\mathcal{L}_{\{1\}}}(\boldsymbol{A})\rangle$ is a full g-model, by Corollary 4.4 we conclude that the rule $\frac{\{\Gamma_i \rhd \alpha_i \mid i<k\}}{\Gamma \rhd \alpha}$ is not derivable in $\mathfrak{G}$. The second condition of algebraizability consists of $\alpha \approx \beta \vDash_{\mathsf{DMF}} \tau\rho(\alpha \approx \beta)$ and $\tau\rho(\alpha \approx \beta) \vDash_{\mathsf{DMF}} \alpha \approx \beta$ and follows from an easy computation in DMF. □

5. Future work

We believe that the study of DMF's and its relation with logics of uncertainty can be enriched in several directions. Here we mention at least two: the first one consists in extending $\mathcal{L}_{\{1\}}$ (or related propositional systems) to the first order level and in studying their relation with first order fuzzy logics and the second one in developing the connection with rough sets theory and modal DMF's which can be naturally defined starting from approximation spaces.

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References

[1] W.J. Blok, D. Pigozzi, Protoalgebraic logics, Stud. Log. 45 (1986) 337–369.
[2] W.J. Blok, D. Pigozzi, Algebraizable Logics, Mem. Am. Math. Soc., vol. 396, AMS, Providence, January 1989.
[3] R. Cignoli, The class of Kleene algebras satisfying an interpolation property and Nelson algebras, Algebra Univers. 23 (3) (1986) 262–292.
[4] J.P. Cleave, A Study of Logics, Oxf. Log. Guides, vol. 18, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications.
[5] J. Czelakowski, Protoalgebraic logics, Trends in Logic—Studia Logica Library, vol. 10, Kluwer Academic Publishers, Dordrecht, 2001.
[6] J.M. Font, Belnap's four-valued logic and De Morgan lattices, Log. J. IGPL 5 (3) (1997) 413–440.
[7] J.M. Font, F. Guzmán, V. Verdú, Characterization of the reduced matrices for the $\{\wedge, \vee\}$-fragment of classical logic, Bull. Sect. Log. 20 (1991) 124–128.
[8] J.M. Font, R. Jansana, A General Algebraic Semantics for Sentential Logics, second ed., Lecture Notes in Logic, vol. 7, Springer-Verlag, 2009. Electronic version freely available through Project Euclid at www.projecteuclid.org/euclid.lnl/1235416965.
[9] J.M. Font, R. Jansana, D. Pigozzi, A survey on abstract algebraic logic, Stud. Log. (Spec. Iss. Abstr. Algebraic Log., Part II) 74 (1–2) (2003) 13–97. With an "Update" in Stud. Log. (Spec. Iss. Abstr. Algebraic Log., Part II) 91 (2009) 125–130.
[10] J.A. Kalman, Lattices with involution, Trans. Am. Math. Soc. 87 (1958) 485–491.
[11] M. Negri, DMF-algebras: representation and topological characterisation, Boll. Unione Mat. Ital. 1-B (8) (1998).
[12] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (5) (1982) 341–356.
[13] U. Rivieccio, An algebraic study of bilattice-based logics, Ph.D. Dissertation, University of Barcelona, 2010.